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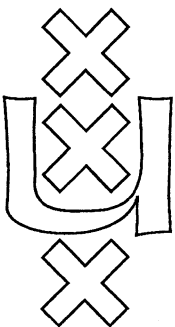## Institute for Logic, Language and Computation

MODAL QUANTIFICATION OVER STRUCTURED DOMAINS

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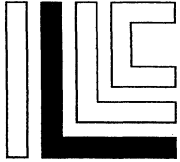
MODAL QUANTIFICATION

OVER STRUCTURED DOMAINS

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Modal Quantification over Structured Domains

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1   Generalized Quantifiers as Modal Operators

The Tarskian truth condition for the existential quantifier reads as follows:

$$M, [\bar{d}/\bar{y}] \models \exists x \varphi(x, \bar{y}) \quad \Leftrightarrow \quad \exists d \in D : M, [d/x, \bar{d}/\bar{y}] \models \varphi(x, \bar{y})$$

This may be viewed as a special case of a more general schema, when the element d is required in addition to stand in some relation R to $\bar{d}$ - where R is a finitary relation of "(in)dependence" structuring the individual domain D:

$$M, [\bar{d}/\bar{y}] \models \Diamond_x \varphi(x, \bar{y}) \quad \Leftrightarrow \quad \exists d \in D : \ R(d, \bar{d}) \ \& \ M, [d/x, \bar{d}/\bar{y}] \models \varphi(x, \bar{y})$$

This broader idea has emerged in the work of van Lambalgen 1991 on the generalized quantifiers "many", "uncountably many" and "almost all", where R stands for an independence relation. But we can think of more general applications too, with domains being arranged in different levels of accessibility, or with procedures drawing objects in possible dependencies upon one another. Ordinary predicate logic then becomes the special case of flat individual domains admitting of "random access", whose R is the universal relation.

This semantics has some clear analogies with Modal Logic, with an existential generalized quantifier as an existential modality over some domain with, not a binary, but an arbitrary finitary "accessibility relation". As a consequence, we can apply standard ideas concerning modal completeness and correspondence to understand this broader concept of quantification. The purpose of this paper is to show how this works more concretely.

The language of the logic $EL(\exists, \Diamond)$ with a generalized quantifier is the ordinary language of first-order predicate logic with equality (without functional symbols) plus an existential generalized quantifier $\Diamond$. The notion of a

w.f.f. is extended as follows: if φ is a w.f.f., then so is $\Diamond_x \varphi$. A universal dual of $\Diamond$ is defined as usual: $\Box_x \varphi =_{df} \neg \Diamond_x \neg \varphi$. We shall refer to the sublanguage without ordinary quantifiers as $EL(\Diamond)$.

$M = (D, R, V)$ is a *model* for $EL(\exists, \Diamond)$ if D and V are an ordinary domain and interpretation for first-order logic, and R is a binary relation between $d \in D$ and finite sequences $\bar{d}$ from D. For many purposes, we can reduce this to a relation $R(d, D_0)$ between individual objects and finite *sets* D_0 of such objects - but we need not make this assumption in general.

Terms (individual variables and constants) are interpreted given M and a variable assignment function v as follows:

- $[x_i]_{M,v} = v(x_i)$,

- $[c_i]_{M,v} = V(c_i)$.

The relation $M, v \models \varphi$ ("φ is *true* in M under assignment v") is defined as follows:

- $M, v \models P_i^n(t_1 \ldots t_n) \Leftrightarrow\; < [t_1]_{M,v} \ldots [t_n]_{M,v} > \in V(P_i^n)$;

- $M, v \models \neg\varphi \Leftrightarrow M, v \not\models \varphi$;

- $M, v \models \varphi \wedge \psi \Leftrightarrow M, v \models \varphi$ and $M, v \models \psi$;

- $M, v \models \exists x \psi(x) \Leftrightarrow$ there exists a variable assignment v' which differs from v at most in its assignment of a value to x ($v' =_x v$) such that $M, v' \models \psi(x)$;

- $M, v \models \Diamond_x \psi(x, \bar{t}) \Leftrightarrow$ there exists $v' =_x v$ such that $R(v'(x), [\bar{t}]_{M,v})$ and $M, v' \models \psi(x, \bar{t})$ ($\bar{t} = \{y_1, \ldots, y_n, c_1, \ldots, c_k\}$, where $\bar{y}$ are all (and just the) free variables of $\Diamond_x \psi$ and $\bar{c}$ are all the constants occurring in ψ).

  It is easy to see that

- $M, v \models \Box_x \psi(x, \bar{t}) \Leftrightarrow$ if for all $v' =_x v$: $R(v'(x), [\bar{t}]_{M,v})) \Rightarrow M, v' \models \psi(x, \bar{t})$.

We say that $M \models \varphi$ iff $M, v \models \varphi$ for all variable assignments v.

Let us define a *frame* (analogously with modal logic) $F = (D, R)$ as the underlying structure of a set of models with all possible interpretations of the predicate letters and constants. $F, v \models \varphi$ if $M, v \models \varphi$ for all models M on F. The formula φ is (globally) valid in F if, for all v, $F, v \models \varphi$ ("$F \models \varphi$").

This system resembles first-order logic in many respects, but no standard property can be taken for granted any more:

Monotonicity is restricted. Let for all variable assignements v $M, v \models \varphi(x_1, x_2) \rightarrow \psi(x_1, x_3)$ and for some assignment v $M, v \models \Diamond_{x_1} \varphi(x_1, x_2)$ (there exists $v' = x_1 v$ such that $R(v'(x_1), v(x_2))$ and $M, v' \models \varphi(x_1, x_2)$). But it does not follow that $M, v \models \Diamond_{x_1} \psi(x_1, x_3)$, because although $M, v' \models \psi(x_1, x_3)$, it is not necessary that $R(v'(x_1), v(x_3))$ holds. Indeed, the general monotonicity rule

$$\frac{\Sigma \vdash \varphi(x, \bar{t}) \rightarrow \psi(x, \bar{s})}{\Sigma \vdash \Diamond_x \varphi(x, \bar{t}) \rightarrow \Diamond_x \psi(x, \bar{s}),}$$

with x not free in Σ, is invalid. We can accept only **Restricted Monotonicity**, where φ and ψ have the same parameters.

Extensionality is also restricted. Properties which hold for exactly the same objects, are no longer identical. Consider a property P which holds for a single object a: $\forall x(P(x) \equiv x = a)$. Let $R(a, \emptyset)$ and $\neg R(a, a)$. Then, $\Diamond_x P(x)$ is true and $\Diamond_x x = a$ is false.

Substitution therefore should also be restricted: only formulas with the same parameters can be substituted. We do not have in general that

$$D, R, V, v \models \varphi[\alpha / P] \quad \Leftrightarrow \quad D, R, V[P := [\alpha]_{M,v}], v \models \varphi.$$

2   Axiomatics and Completeness

We shall now develop the basic deductive calculus for our modal quantifier logic.

Definition 1 *The minimal logic for $EL(\exists, \Diamond)$ is a calculus of sequents $\Sigma \vdash \varphi$ satisfying the usual rules for first-order logic, including all Boolean principles, as well as the following quantifier rules:*

Restricted Monotonicity plus Distribution

$$\frac{\Sigma \vdash \varphi(x, \bar{t}) \rightarrow \bigvee_{i=1}^{i=n} \psi_i(x, \bar{t})}{\Sigma \vdash \Diamond_x \varphi(x, \bar{t}) \rightarrow \bigvee_{i=1}^{i=n} \Diamond_x \psi_i(x, \bar{t})}$$

where x is not free in Σ, and parameters (free variables and constants) are exactly those displayed (only x does not necessarily occur free in ψ_i). The convention here is that an empty disjunction is a falsum.

3

Alphabetic Variants

$$\vdash\ \Diamond_x\varphi(x,\bar{t}) \equiv \Diamond_z\varphi(z,\bar{t})$$

where z does not occur (free or bound) in $\varphi(x,\bar{t})$.

Here are some derivations in this system, corresponding to obvious validities given the above existential truth condition for the quantifier $\Diamond$:

1. $\vdash\ \bot \to \bot$

   $\vdash\ \Diamond_x\bot \to \bot$

   $\vdash\ \neg\Diamond_x\bot$

2. $\neg\varphi(\bar{y}) \vdash\ \varphi(\bar{y}) \to \bot$

   $\neg\varphi(\bar{y}) \vdash\ \Diamond_x\varphi(\bar{y}) \to \bot$

   $\vdash\ \Diamond_x\varphi(\bar{y}) \to \varphi(\bar{y})$, provided that x is not among the $\bar{y}$

3. Suppose that $\vdash \varphi \to \psi$ with x not free in ψ:

   Then:

   $\neg\psi \vdash\ \neg\varphi$

   $\neg\psi \vdash\ \varphi \to \bot$

   $\neg\psi \vdash\ \Diamond_x\varphi \to \bot$

   $\neg\psi \vdash\ \neg\Diamond_x\varphi$

   whence $\vdash \Diamond_x\varphi \to \psi$.

4. Application of (3):

   $\vdash\ \Diamond_x\varphi \to \Diamond_x\varphi$

   $\vdash\ \Diamond_x\Diamond_x\varphi \to \Diamond_x\varphi$

5. Also,

   $\vdash\ \varphi \to \exists x\varphi$

   $\vdash\ \Diamond_x\varphi \to \exists x\varphi$

6. As a final illustration, we prove a useful principle for later reference:
   $\vdash \neg\Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y}))$ (where z, x are not among $\bar{y}$, x is free for z in $\psi(z,\bar{y})$):

4

$$\vdash \psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y}) \to \psi(z,\bar{y})$$

$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y})) \to \Diamond_z\psi(z,\bar{y})$$

$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y})) \to \Diamond_x\psi(x,\bar{y})$$

and

$$\vdash \psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y}) \to \neg\Diamond_x\psi(x,\bar{y})$$

$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y})) \to \neg\Diamond_x\psi(x,\bar{y})$$

(the latter step is as in example (3) above). Therefore,

$$\vdash \Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y})) \to \bot$$

$$\vdash \neg\Diamond_z(\psi(z,\bar{y}) \wedge \neg\Diamond_x\psi(x,\bar{y}))$$

Constant Lemma. *If a constant d does not occur in Σ, and z is a new variable, then*

$$\frac{\Sigma \vdash \varphi(d)}{\Sigma \vdash \varphi(z)}$$

Proof. Induction on the length of derivations. For Restricted Monotonicity and Distribution: d occurred among the $\bar{t}$; since z is a new variable, it cannot become bound in φ or ψ_i; so, all variable conditions remain satisfied. For Alphabetic Variants, a similar argument will work. $\square$

Theorem 1 *The minimal logic is complete for universal validity.*

Proof. By a standard Henkin construction. The key point, as usual, is to create a maximally consistent set of formulas Σ - this time, adding suitable witnesses for accepted formulas $\Diamond_x\varphi$:

If Σ_n is consistent with $\Diamond_x\varphi(x,\bar{d})$,

then add a *new* individual constant d with

1. $\varphi(d,\bar{d})$,

2. $\{\psi(d,\bar{d}) \to \Diamond_x\psi(x,\bar{d}) |$ for all formulas $\psi\}$.

5

Claim. *This extension is consistent.*

Proof. Suppose it were inconsistent. Then, by standard reasoning (using the Constant Lemma), for some fresh variable z and some finite disjunction of formulas ψ_i:

$$\Sigma_n \vdash \varphi(z, \bar{d}) \rightarrow \bigvee_i (\psi_i(z, \bar{d}) \wedge \neg\Diamond_x\psi_i(x, \bar{d})).$$

Then also

$$\Sigma_n \vdash \Diamond_z\varphi(z, \bar{d}) \rightarrow \bigvee_i \Diamond_z(\psi_i(z, \bar{d}) \wedge \neg\Diamond_x\psi_i(x, \bar{d})).$$

Therefore, since $\vdash \Diamond_x\varphi(x, \bar{d}) \equiv \Diamond_z\psi(z, \bar{d})$ (by Alphabetic Variants), $\{\Sigma_n, \Diamond_x\varphi(x, \bar{d})\}$ must be consistent with some

$$\Diamond_z(\psi_i(z, \bar{d}) \wedge \neg\Diamond_x\psi_i(x, \bar{d})).$$

But this contradicts the earlier derivability of the formula

$$\neg\Diamond_z(\psi(z, \bar{d}) \wedge \neg\Diamond_x\psi(x, \bar{d})).$$

□

Now construct the Henkin model as usual, and set

$$R(d, \bar{d}) \Leftrightarrow_{df} \forall\varphi : \varphi(d, \bar{d}) \in \Sigma \Rightarrow \Diamond_x\varphi(x, \bar{d}) \in \Sigma$$

This may be compared with the usual introduction of the alternative relation R in completeness proofs for Modal Logic. To demonstrate the adequacy of the present Henkin model, all we have to prove is the following decomposition:

$$\Diamond_x\varphi(x, \bar{d}) \in \Sigma \quad \text{iff} \quad \exists d : R(d, \bar{d}) \,\&\, \varphi(d, \bar{d}) \in \Sigma$$

From left to right, this is guaranteed by the above construction of Σ (through the addition of all formulas of the second kind). From right to left, this is a trivial consequence of the definition of R.   □

If we look at the above completeness proof (and earlier examples of derivabilities), we see that no structural contraction rule or ordinary quantifier rules have been used. This observation (which is quite analogous with the situation in the minimal modal logic) motivates the following

Conjecture *Minimal logic without ordinary quantifiers is decidable.*

6

3 Model Theory

Now, to illustrate the semantical properties of modal quantifiers, we shall consider an analogue to the basic model-theoretic invariance relation of modal logic.

Definition 2 *A* bisimulation $\mathcal{B}$ *between two models* $M_1 = < D_1, R_1, V_1 >$ *and* $M_2 = < D_2, R_2, V_2 >$ *is a family of partial isomorphisms* π *with the following properties:*

1 π *is a partial bijection with* $dom(\pi) \subseteq D_1$ *and* $ran(\pi) \subseteq D_2$;

2 If $\{d_1, \ldots, d_n\} \subseteq dom(\pi)$, *then for all predicate letters*

$$< d_1, \ldots, d_n > \in V_1(P^n) \quad \Leftrightarrow \quad < \pi(d_1), \ldots, \pi(d_n) > \in V_2(P^n)$$

($d_1, \ldots, d_n$ are not necessarily distinct.)

3a If $D \subseteq dom(\pi)$ *and* $R_1(d, D)$, *then there exists an element* d' *in* D_2 *such that* $R_2(d', \pi[D])$ *and* $\{< d, d' >\} \cup \pi \in \mathcal{B}$.

3b If $D' \subseteq ran(\pi)$, $D' = \pi[D]$, *and* $R_2(d', D')$, *then there exists an element* d *in* D_1 *such that* $R_1(d, D)$ *and* $\{< d, d' >\} \cup \pi \in \mathcal{B}$. [1]

Invariance Lemma *If* φ *is a formula of* $EL(\Diamond)$ *(that is, without ordinary quantifiers) with the set of terms* $TERM(\varphi) \subseteq \{t_1, \ldots, t_n\}$, *for all* t_i *($1 \leq i \leq n$)* $[t_i]_{M_1, v_1} \in dom(\pi)$ *and* $[t_i]_{M_2, v_2} = \pi[t_i]_{M_1, v_1}$ *then*

$$M_1, v_1 \models \varphi \quad \Leftrightarrow \quad M_2, v_2 \models \varphi$$

Proof. By induction on the length of φ.

- φ is a k-place predicate letter. - By clause (2) in the definition of bisimulation.

- $\varphi = (t_1 = t_2)$. $M_1, v_1 \models t_1 = t_2$ if and only if $[t_1]_{M_1, v_1} = [t_2]_{M_2, v_2}$. Since π is a function, and $[t_1]_{M_1, v_1}, [t_2]_{M_2, v_2} \in dom(\pi)$, $\pi[t_1]_{M_1, v_1} = \pi[t_2]_{M_1, v_1}$, that is, $[t_1]_{M_2, v_2} = [t_2]_{M_2, v_2}$. Backwards: the same argument, using the fact that π^{-1} is also a (partial) function.

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[1] Alternatively, we could restrict clause 3 to R-successors of the whole domain and range, while adding a further clause closing $\mathcal{B}$ under restrictions.

- $\varphi = \neg \psi$: by the inductive hypothesis,

$$M_1, v_1 \models \psi \;\Leftrightarrow\; M_2, v_2 \models \psi$$

and hence

$$M_1, v_1 \models \neg\psi \;\Leftrightarrow\; M_2, v_2 \models \neg\psi.$$

- $\varphi = \psi_1 \wedge \psi_2$. Again, by the inductive hypothesis,

$$M_1, v_1 \models \psi_1 \;\Leftrightarrow\; M_2, v_2 \models \psi_1$$

$$M_1, v_1 \models \psi_2 \;\Leftrightarrow\; M_2, v_2 \models \psi_2$$

and so,

$$M_1, v_1 \models \psi_1 \wedge \psi_2 \;\Leftrightarrow\; M_2, v_2 \models \psi_1 \wedge \psi_2$$

- $\varphi = \Diamond_x \psi(x, \bar{t})$. Assume $M_1, v_1 \models \Diamond_x \psi(x, \bar{t})$. By the semantic truth definition, there exists an assignment v_1' which differs from v_1 at most in its assignment of value to x, such that $R(v_1'(x), [\bar{t}]_{M_1,v_1})$ and $M_1, v_1' \models \psi(x, \bar{t})$. By assumption, $\{[t_1]_{M_1,v_1}, \ldots, [t_n]_{M_1,v_1}\} \subseteq dom(\pi)$. By clause 3a, there is $d' \in D_2$ with $R(d', \pi[\bar{t}]_{M_1,v_1})$, i.e. $R(d', [\bar{t}]_{M_2,v_2})$, and $\{< d, d' >\} \cup \pi \in \mathcal{B}$. Put $v_2' =_x v_2$, $v_2'(x) = d'$. Then, for the $\pi' \in \mathcal{B}$ which consists of π and the pair $< d, d' >$, $[x]_{M_1,v_1'} = \pi'[x]_{M_2,v_2'}$, and for all t_i, $[t_i]_{M_1,v_1'} = \pi'[t_i]_{M_2,v_2'}$. By the inductive hypothesis, $M_2, v_2' \models \psi(x, \bar{t})$. But then $M_2, v_2 \models \Diamond_x \psi(x, \bar{t})$. The same argument works backwards. $\square$

Claim 1 *If φ does contain $\forall$ or $\exists$, bisimulation does not preserve truth.*

Proof. Let M_1 and M_2 be as follows:

$M_1 = < D_1, R_1, V_1 >$: $D_1 = \{d, d'\}$, $R_1 = \emptyset$, $V_1(P) = \{< d, d' >\}$;

$M_2 = < D_2, R_2, V_2 >$: $D_2 = \{e\}$, $R_2 = \emptyset$, $V_2(P) = \emptyset$.

Then $M_1, [d/x] \models \exists y P(x, y)$ and $M_2, [e/x] \not\models \exists y P(x, y)$. But at the same time, a bisimulation between these two models exists: $\mathcal{B} = \{< d, e >\}$. $\square$

Claim 2 *If φ is an arbitrary $EL(\exists, \Diamond)$ formula with $TERM(\varphi) \subseteq \{t_1, \ldots, t_n\}$, $[t_i]_{M_1,v_1} \in dom(\pi)$, $[t_i]_{M_2,v_2} = \pi[t_i]_{M_1,v_1}$, and the bisimulation has in addition the property*

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4 For every $d \in D_1$ there exists $d' \in D_2$ such that $\{< d, d' >\} \cup \pi \in \mathcal{B}$, and vice versa

then $M_1, v_1 \models \varphi \Leftrightarrow M_2, v_2 \models \varphi$.

Proof. First, note that clause (4) does not imply (3), as it may not enforce the right dependency on R. Now, all inductive steps for the Invariance Lemma still apply here, and we just have to add the following clause for ordinary quantifiers:

- $\varphi = \exists x \psi$. Let M_1, v_1 and M_2, v_2 be as in the Invariance Lemma, and $M_1, v_1 \models \exists x \psi$. Then there is an assignment $v_1' =_x v_1$ such that $M_1, v_1' \models \psi$. Let $v_1'(x) = d$. By (4), there is an element e in M_2 such that $\{< d, e >\} \cup \pi \in \mathcal{B}$. Define $v_2' =_x v_2$ with $v_2'(x) = e$. Since now for all terms t_i of ψ, including x, $[t_i]_{M_1, v_1'} = \pi'[t_i]_{M_2, v_2}$ $(\pi' = \{< d, e >\} \cup \pi)$, we can conclude by the inductive hypothesis that $M_2, v_2' \models \psi$. But then $M_2, v_2 \models \exists x \psi$. The same argument will work backwards. $\square$

Continuing the analogy with modal logic, we define a translation of $EL(\exists, \Diamond)$ formulas into the appropriate first-order logic, which is our original base language enriched with a dependence predicate R. The *standard translation ST* is defined as follows:

- $ST(P_i^n(t_1 \ldots t_n)) := P_i^n(t_1 \ldots t_n)$;

- $ST(t_1 = t_2) := (t_1 = t_2)$;

- ST commutes with classical connectives and quantifiers;

- $ST(\Diamond_x \varphi(x, \bar{t})) := \exists x (R(x, \bar{t}) \wedge ST(\varphi(x, \bar{t})))$.

Claim 3 *If φ is a formula of $EL(\exists, \Diamond)$, then*

$$M, v \models \varphi \Leftrightarrow M', v \models ST(\varphi),$$

for the classical model $M' = (D, V')$, where V' extends V to interpret the predicate R as R_M.

Definition 3 *The modal formulas (being those formulas which are standard translations of $EL(\Diamond)$ formulas) are the least set X of first-order formulas such that*

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- *atomic formulas belong to X,*

- *if ψ_1 and ψ_2 are in X, then so are $\neg\psi_1$ and $\psi_1 \wedge \psi_2$,*

- *if $\varphi(x,\bar{t}) \in X$, then $\exists x(R(x,\bar{t}) \wedge \varphi(x,\bar{t}))$ is in X too.*

Theorem 2 *A first-order formula φ is equivalent to a modal formula if and only if it is preserved under bisimulation (with the above properties 1 – 3).*

Proof. The direction from left to right follows from Invariance Lemma above. For the converse, let φ be a first-order formula with variables $x_1, \ldots, x_n$, preserved under bisimulation. We want to prove that it is equivalent to a modal formula.

Define the set $CONS_\Diamond(\varphi)$ as $\{\alpha : \alpha$ is a modal formula, $\varphi \models \alpha$ and the free variables of α are among $x_1, \ldots, x_n\}$. If we can prove that

$$(*) \quad CONS_\Diamond(\varphi) \models \varphi,$$

then we are done. For, by compactness, there will be some finite subset $\alpha_1, \ldots, \alpha_m$ of $CONS_\Diamond(\varphi)$ with $\alpha_1, \ldots, \alpha_m \models \varphi$. By definition, $\varphi \models \alpha_1, \ldots, \alpha_m$. So, then φ is equivalent to $\alpha_1 \wedge \ldots \wedge \alpha_n$, which is a conjunction of standard translations of $EL(\Diamond)$ formulas, i.e. a standard translation of the conjunction of those formulas.

Now we start proving $(*)$. Assume that for some model $M, v \models CONS_\Diamond(\varphi)$. We show that $M, v \models \varphi$. Let us denote the set of all modal formulas true in M and having free variables among $x_1, \ldots, x_n$ as X_M. This is consistent with φ (by the truth of $CONS_\Diamond(\varphi)$ in M, v) and therefore there should be a model N for $\varphi \cup X_M$: say, $N, v' \models \varphi \cup X_M$.

Let $v(x_1) = d_1, \ldots, v(x_n) = d_n$ in M and $v'(x_1) = d_1', \ldots, v'(x_n) = d_n'$ in N. Now, take ω-saturated elementary extensions $\mathcal{M}$ and $\mathcal{N}$ of M and N. We define a relation of bisimulation between $\mathcal{M}$ and $\mathcal{N}$ as follows:
$(**)$ $\mathcal{B}$ is the family of partial mappings π such that $\pi = \{(e_1, \pi(e_1)), \ldots, (e_n, \pi(e_n))\}$ if for all modal formulas ψ with at most free variables $x_1, \ldots, x_n$ and any two assignments v, v' with $v(x_i) = e_i$, $v'(x_i) = \pi(e_i)$ $(1 \leq i \leq n)$,

$$\mathcal{M}, v \models \psi \quad \Leftrightarrow \quad \mathcal{N}, v' \models \psi$$

To prove that $(**)$ indeed defines a bisimulation relation, we must check whether the properties (1)–(3) hold for $\mathcal{B}$. Here, (1) is trivial. Case (2) is immediate, since atomic formulas are also standard translations of (atomic) formulas in $EL(\Diamond)$). Next, we check the zigzag clause 3a. Assume that

$e_1, \ldots, e_k \in dom(\pi)$ and $R(e, e_1, \ldots, e_k)$. We must prove that there exists e' in $\mathcal{N}$ such that $R(e', \pi(e_1), \ldots, \pi(e_k))$ and $\{< e, e' >\} \cup \pi \in \mathcal{B}$. Take the set Ψ of all modal formulas with variables interpreted as $e, e_1, \ldots, e_k$ which are true in $\mathcal{M}$ under variable assignment v. We need an element e' in $\mathcal{N}$ such that all formulas in Ψ are true in $\mathcal{N}$ under v' when e' is assigned to the variable which was assigned e in $\mathcal{M}$. By saturation, it suffices to find such an e' for each finite subset Ψ_0 of Ψ. But these must exist, because the modal formula $ST(\Diamond_x \bigwedge \Psi_0(x, e_1, \ldots, e_k))$ holds in $\mathcal{M}$ and hence $ST(\Diamond_x \bigwedge \Psi_0(x, \pi(e_1), \ldots, \pi(e_k)))$ holds in $\mathcal{N}$. The appropriate check for the converse direction 3b is proved analogously.

We must also show that $\{< d_1, d_1' >, \ldots, < d_n, d_n' >\} \in \mathcal{B}$. But this is so because for all modal formulas ψ with variables interpreted as $d_1, \ldots, d_n$ in M,

$$M, v \models \psi \;\Leftrightarrow\; N, v' \models \psi$$

(by the construction of N), and hence

$$\mathcal{M}, v \models \psi \;\Leftrightarrow\; \mathcal{N}, v' \models \psi.$$

Finally, since φ is invariant under bisimulation and $\{< d_1, d_1' >, \ldots, < d_n, d_n' >\} \in \mathcal{B}$, $\mathcal{N} \models \varphi(d_1', \ldots, d_n')$ will now imply $\mathcal{M} \models \varphi(d_1, \ldots, d_n)$. Since $\mathcal{M}$ is an elementary extension of M, $M \models \varphi(d_1, \ldots, d_n)$, that is, $M, v \models \varphi(x_1, \ldots, x_n)$, and we are done. $\square$

4 Frame Correspondence

If a formula φ of $EL(\exists, \Diamond)$ is valid in a frame F (under an assignment v), then classically

$$F, v \models \forall P_i^n \ldots \forall P_l^m ST(\varphi),$$

where $P_i^n, \ldots, P_l^m$ are the predicate letters in φ. If this second-order formula has a first-order equivalent (containing only R and $=$), φ is called *first-order definable*. This means that if φ is true in all models over F, then R has the property defined by φ, and vice versa. Additional quantifier principles added to the minimal logic will now express special conditions on the relation R. One bunch of examples arises if we look at some properties of the standard existential quantifier $\exists$:

Unrestricted Distribution $\qquad \Diamond_x(\varphi \vee \psi) \leftrightarrow \Diamond_x \varphi \vee \Diamond_x \psi.$

In one direction, this gives us unrestricted "Monotonicity" for $\Diamond$:

$$\Diamond_x \varphi \rightarrow \Diamond_x(\varphi \vee \psi).$$

This corresponds to the frame condition of

Upward Monotonicity $\qquad R(x, \bar{y}) \rightarrow R(x, \bar{y}, \bar{z})$

Proof. Suppose that $R(x, \bar{y})$. Define the following predicate:

$$P(u, \bar{v}) := u = x \wedge \bar{v} = \bar{y}.$$

We have $R(x, \bar{y}) \wedge P(x, \bar{y})$, whence $\Diamond_x P(x, \bar{y})$ holds. Therefore,

$$\Diamond_x(P(x, \bar{y}) \vee \perp(\bar{z}))$$

(where $\perp(\bar{z})$ is any contradiction involving $\bar{z}$): i.e., there exists d with $R(d, \bar{y}, \bar{z})$ and $P(d, \bar{y}) \vee \perp(\bar{z})$: the latter must be because $P(d, \bar{y})$: i.e. $d = x$, and hence $R(x, \bar{y}, \bar{z})$. $\quad \Box$

By a similar kind of argument, again making an appropriate substitution for the two predicates involved, the opposite direction

$$\Diamond_x(\varphi \vee \psi) \rightarrow \Diamond_x \varphi \vee \Diamond_x \psi$$

corresponds to the frame condition of

Downward Monotonicity $\qquad R(x, \bar{y}, \bar{z}) \rightarrow R(x, \bar{y})$

Together, these reduce the finitary relation R to an essentially *unary* "restriction" to the subdomain of all objects d satisfying the condition $R(d)$. It would also be of interest to see whether we can stop short of this, with quantifiers merely reducing the finitary relation R to a compound of *binary* ones (as happens in the generalized modal semantics for program operators proposed in van Benthem 1992).

Remark. Classical analogies may be slightly misleading here. E.g., the implication

$$\Diamond_x(\varphi \wedge \psi) \rightarrow \Diamond_x \varphi$$

(cf. $\Diamond(\varphi \wedge \psi) \rightarrow \Diamond \varphi$) expresses Downward Monotonicity, rather than the Upward Monotonicity of

$$\Diamond_x \varphi \rightarrow \Diamond_x(\varphi \vee \psi)$$

(cf. $\Diamond\varphi \rightarrow \Diamond(\varphi\vee\psi)$), even though the latter is equivalent with it in standard modal logic. Thus, it should in fact imply unlimited distribution - as may be seen using the available distribution in our minimal logic. In the latter calculus, "limited distribution" sanctions

1. $\Diamond_x(\varphi(x,\bar{y}) \vee \psi(x,\bar{z})) \rightarrow \Diamond_x((\varphi(x,\bar{y}) \wedge \top(\bar{z})) \vee (\psi(x,\bar{z}) \wedge \top(\bar{y})))$

2. $\Diamond_x((\varphi(x,\bar{y}) \wedge \top(\bar{z})) \vee (\psi(x,\bar{z}) \wedge \top(\bar{y}))) \rightarrow$
   $\rightarrow \Diamond_x(\varphi(x,\bar{y}) \wedge \top(\bar{z})) \vee \Diamond_x(\psi(x,\bar{z}) \wedge \top(\bar{y}))$,

3. from which the unlimited version $\Diamond_x(\varphi \vee \psi) \rightarrow \Diamond_x\varphi \vee \Diamond_x\psi$ follows by the above implication, passing to the appropriate conjuncts.

$\square$

Finally, the above unary relation gets trivialized to universality by the principle of

Instantiation $\qquad \varphi \rightarrow \Diamond_x\varphi$

This corresponds to the frame condition $\forall x\forall yR(x,y)$ (provided that we assume non-empty individual domains, that is). The idea is this: let x,y be arbitrary, and let $P(x,y)$ hold of just these. We must have that $\Diamond_xP(x,y)$: i.e., some object d exists with $R(d,y)$ and $P(d,y)$: whence $R(x,y)$. $\square$

Another source of examples is the analysis of various properties of the standard quantifier $\exists$ which are all lumped together as being "valid" in ordinary predicate logic, but which now become distinguishable as different properties of dependence. To be sure, such differences also become visible in other more sensitive semantics, such as those for intuitionistic predicate logic, or the logic of polyadic generalized quantifiers. Indeed, one concrete interpretation of the above structured domains would be

individuals   pairs <world, individual>
dependence $(w,x)R(v,y)$ iff $w \subseteq v$ & $y = x$,

reflecting the standard possible worlds semantics for intuitionistic logic. We continue with one example of this kind:

Prenex operations $\qquad \Diamond_x(\varphi \vee \psi) \leftrightarrow \varphi \vee \Diamond_x\psi$, where x not free in φ.

The direction $\rightarrow$ here turns out universally valid in case $\psi, \varphi \vee \psi$ have the same free variables, and hence derivable:

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$$\neg\varphi \vdash (\varphi \lor \psi) \to \psi$$

$$\neg\varphi \vdash \Diamond_x(\varphi \lor \psi) \to \Diamond_x\psi$$

$$\vdash \Diamond_x(\varphi \lor \psi) \to \Diamond_x\psi \lor \varphi,$$

Otherwise, it will enforce the earlier Downward Monotonicity: $R(x, \bar{y}, \bar{z}) \to R(x, \bar{y})$. The direction $\leftarrow$ corresponds to the conjunction of $R(x, \bar{y}) \to R(x, \bar{y}, \bar{z})$ and $\exists x R(x, \bar{y})$. $\quad\square$

A comparison of quantifier axioms and similar modal axioms can also provide some interesting correspondences. For example, how would one write a quantifier version of the well-known K4-axiom: as

$$\Diamond_x\varphi \to \Diamond_x\Diamond_x\varphi$$

or with the more complex decoration

$$\Diamond_x\varphi \to \Diamond_y\Diamond_x\varphi?$$

The first one is universally valid, the second one defines

$$R(x, y\bar{z}) \to R(y, \bar{z})$$

(in case y is free in φ). Another direction is also possible: which quantifier principles correspond to well known properties of Kripke frames? Well-known examples are the three defining properties of equivalence relations:

Reflexivity $R(x, x)$

Transitivity $R(y, x) \land R(z, y) \to R(z, x)$

Symmetry $R(x, y) \to R(y, x)$

Fact. *These principles are definable in $EL(\exists, \Diamond)$:*

- *Reflexivity corresponds to $\Diamond_y x = y$;*

- *Transitivity corresponds to $\Diamond_y(\top(x) \land \Diamond_z(\top(y) \land P(z))) \to \Diamond_z(\top(x) \land P(z))$*

- *Symmetry corresponds to $\forall x \Box_y P(x, y) \to \forall y \Box_x P(x, y)$.*

(Proofs will be given in Section 5 below.)

Some negative results concerning definability of first-order properties in $EL(\Diamond)$ can be obtained using frame constructions familiar from modal logic.

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Definition 4 *Let $F = <D, R>$ be a frame and $d_1, \ldots, d_n \in D$. A subframe $F' = <D', R'>$ of F is generated by $d_1, \ldots, d_n$ if*

- *D' is the smallest subdomain of D containing $d_1, \ldots, d_n$ which is closed under accessibility, and*

- *R' is the restriction of R to D'.*

Theorem 3 *Let F' be a generated subframe of F, v a valuation restricted to the elements of D', and φ a formula of $EL(\Diamond)$. Then*

$$F, v \models \varphi \;\Leftrightarrow\; F', v \models \varphi$$

(in other words, $EL(\Diamond)$-formulas are invariant for generated subframes).

Proof. For any pair of models $M = <F, V>$ and $M' = <F', V>$ the identity map from D' to D gives an obvious bisimulation, and we can apply our invariance results from Section 3. □

Examples (Modal undefinability).

- $\exists x \neg R(x, x)$ is not definable by an $EL(\Diamond)$ formula. Consider

$$F = \; < \{d_1, d_2\}, \{<d_1, d_1>\} >,$$

where it holds, and the generated subframe

$$F' = \; < \{d_1\}, \{<d_1, d_1>\} >,$$

where it fails.

- $\forall x \forall y (x \neq y \;\Rightarrow\; R(x, y))$ is not definable in $EL(\Diamond)$. Consider the same two frames, but now in the opposite direction. □

The language of $EL(\exists, \Diamond)$ with ordinary quantifiers added is much more powerful. For instance, $\exists x \neg R(x, x)$ is definable as $\exists x \neg \Diamond_y x = y$, and $\forall x \forall y (x \neq y \Rightarrow R(x, y))$ as $\exists y (x \neq y \land P(x, y)) \rightarrow \Diamond_y P(x, y)$. Of course, a great deal of expressive power is due to the presence of identity in this language. Here is a more general result demonstrating this.

Theorem 4 *Every purely universal R-condition is $\Diamond, \exists$-definable.*

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Proof. (Cf. Proposition 2.4 in de Rijke (1992a)). Consider any R-condition of the following form:

$$\forall y_1 \ldots \forall y_n \; BOOL(R, =, y_1, \ldots, y_n),$$

where "$BOOL$" is a purely Boolean condition. Introduce a predicate P_{y_i} for every universally quantified variable y_i, which holds exactly for y_i: $\exists! x P_{y_i}(x)$. Define a translation $*$ of first-order formulas with R into $EL(\exists, \Diamond)$, such that $*$ commutes with Boolean connectives and $=$, where

$$(R(y, \bar{z}))^* = \Diamond_u(P_y(u) \wedge \top(\bar{z})).$$

Then the $EL(\exists, \Diamond)$ equivalent of the R-property will be

$$\exists! x P_{y_1}(x) \wedge \ldots \wedge \exists! x P_{y_n}(x) \to (BOOL(R, =, y_1, \ldots, y_n))^*$$

$\square$

Open problem *Are all first-order properties of R $\exists, \Diamond$-definable?*

We conjecture that the answer to this question is negative. A possible counterexample is the first-order formula $\exists x \forall y R(x, y)$.

There is a more general theory behind these various observations. The above axioms whose frame correspondences were analysed all had "Sahlqvist forms" in a suitably general sense, and the proof method depends on finding suitable "minimal valuations". In the next Section, we make this precise.

5 A Sahlqvist Theorem

Theorem 5 *All formulas of the "Sahlqvist form" $\varphi \to \psi$, where*

1. *φ is constructed from*

   - *atomic formulas, possibly prefixed by $\Box_x$, $\forall$;*
   - *formulas in which predicate letters occur only negatively*

   using $\wedge, \vee, \Diamond_x, \exists$

2. *in ψ all predicate letters occur only positively*

are first-order definable.

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Proof. First we translate $\varphi \to \psi$ into second-order logic:

$$\forall P_1^n \ldots \forall P_l^m (ST(\varphi) \to ST(\psi)),$$

where $P_1^n \ldots P_l^m$ are all the predicates in $\varphi \to \psi$. Then we remove all "empty" quantifiers (those binding variables not occurring in their scope), and rename bound individual variables in such a way that every quantifier gets its own variable which is distinct from any free variable occurring in the formula. Now it is possible to move all existential quantifiers occurring in positive subformulas of $ST(\varphi)$ to a prefix, using the following equivalences:

$$\exists x A(x) \vee \exists y B(y) \equiv \exists x \exists y (A(x) \vee B(y))$$

$$\exists x A(x) \wedge B \equiv \exists x (A \wedge B)$$

with the usual provisos on freedom and bondage. $ST(\varphi)$ has now been rewritten as

$$\exists x_1 \ldots \exists x_n \varphi'.$$

Since ψ does not contain $x_1, \ldots, x_n$ free, $ST(\varphi) \to ST(\psi)$ is equivalent to

$$\forall x_1 \ldots \forall x_n (\varphi' \to ST(\psi)).$$

Next, it would be convenient to get rid of the disjunctions in φ'. Let $\varphi' \equiv \phi_1 \vee \phi_2$.

$$\forall x_1 \ldots \forall x_n (\phi_1 \vee \phi_2 \to ST(\psi))$$

is equivalent to

$$\forall x_1 \ldots \forall x_n (\phi_1 \to ST(\psi)) \wedge \forall x_1 \ldots \forall x_n (\phi_2 \to ST(\psi)).$$

We can restrict attention to one of these conjuncts (if both components have a first-order equivalent, then so has their conjunction). So, assume that there are no disjunctions in the antecedent. Thus, we have a formula

$$\forall P_1^n \ldots \forall P_l^m \forall x_1 \ldots \forall x_n (\varphi' \to ST(\psi)),$$

where $P_1^n \ldots P_l^m$ are all the predicates in $\varphi' \to ST(\psi)$, and φ' is a conjunction of "blocks" which are of one of the following forms:

1. standard translations of atomic formulas possibly preceded by universal and $\square$-quantifiers,

2. R-statements,

3. formulas in which all predicate letters occur only negatively.

Next we rule out the use of negative formulas. The point is that $\varphi' \to ST(\psi)$ can always be rewritten as an implication whose antecedent does not contain negative formulas. Let $\varphi' = \phi_1 \wedge \phi_2$, where ϕ_2 is a negative formula. Then

$$\phi_1 \wedge \phi_2 \to ST(\psi)$$

is equivalent to

$$\phi_1 \to \neg\phi_2 \vee ST(\psi),$$

whose consequent contains only positive occurrences of predicate letters.

Let us denote the antecedent obtained (without negative formulas) φ^*. We shall now define the notion of *a minimal substitution* for every predicate letter in φ^*.

A predicate letter P_i^n can occur in φ^* more than once. Consider an occurrence $\bar{P}_i^n$ of P_i^n in φ^*. First we have to classify the variables of this occurrence (this is the only part where the present proof becomes different from the modal case). Let us assume that

- the variables which stand at the places $i_1, \ldots, i_m$ *in this occurrence* are existentially bound or free; let us denote them $x_1, \ldots, x_m$;

- the variables at the places $j_1, \ldots, j_k$ are universally bound by quantifiers which correspond to $\Box$-quantifiers in the original formula; let us call them $z_1, \ldots, z_k$;

- the rest of the variables is bound by ordinary universal quantifiers; let us call them $v_1, \ldots v_l$.

The minimal substitution $Sb(\bar{P}_i^n)$ for this occurrence of P_i^n in φ^* will be: $P_i^n(u_1, \ldots u_n)$ is the conjunction of

1. $u_{i1} = x_1, \ldots, u_{im} = x_m$;

2. $\top(v_1), \ldots, \top(v_l)$;

3. $R(u_{\alpha_1}, \ldots, u_{\alpha_f})$, where $u_{\alpha_1}, \ldots, u_{\alpha_f}$ are the variables standing at the places $\alpha_1, \ldots, \alpha_f$, and in φ^* for these variables some R-condition (corresponding to one of the variables $z_1, \ldots, z_k$) hold.

Here, we have to define the notion of an "R-condition" corresponding to the variable z_i:

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1. Let $\Box_{z_1}$ be the first (leftmost) generalized quantifier in the sequence of quantifiers preceding $\bar{P}_i^n$, and before $\Box z_1$ the ordinary universal quantifiers $\forall v_1, \ldots, \forall v_s$ occur. Then the R-condition corresponding to z_1 will be $R(z_1, v_1, \ldots, v_s, \bar{x})$,

2. Let $\Box_{z_i}$ be the generalized quantifier following $\Box_{z_{i-1}}$ in our sequence (with some $\forall v_p, \ldots, \forall v_r$ possibly standing in between):

$$\ldots \Box_{z_{i-1}} \forall v_p \ldots \forall v_r \Box_{z_i} \ldots \bar{P}_i^n$$

If the condition corresponding to z_{i-1} was $R(z_{i-1}, \bar{y})$, then the condition corresponding to z_i is $R(z_i, v_p, \ldots, v_r, z_{i-1}, \bar{y})$.

Finally, we define

$$Sb(P_i^n, \varphi^*) = \bigvee Sb(\bar{P}_i^n)$$

for all occurrences of P_i^n in φ^*. [2]

The result of substituting $Sb(P_i^n, \varphi^*)$ in $\forall x_1 \ldots \forall x_m(\varphi^* \to \psi')$, which we shall denote as

$$\forall x_1 \ldots \forall x_m(s(\varphi) \to s(\psi))$$

is our intended first-order equivalent, which contains no predicate symbols other than R and $=$. It is easy to see that it follows from the original Sahlqvist axiom, being an instantiation of a universal second-order formula

$$\forall P_i^n \ldots \forall P_l^m \forall x_1 \ldots \forall x_m(\varphi^* \to \psi').$$

We must prove the other direction to have an equivalence.

Assume that $\forall x_1 \ldots \forall x_m(s(\varphi) \to s(\psi))$ holds in some frame F under a variable assignment v. Assume, for some interpretation function V, that φ^* holds in $M = <F, V>$. To show that ψ' holds in the same model, we need the following two assertions:

Lemma 1 *For all M, v: $M, v \models \varphi^* \Rightarrow M, v \models s(\varphi)$*

Lemma 2 *Let $M, v \models \varphi^*$, and let $v(x_1) = d_1, \ldots, v(x_m) = d_m$. Define $V^*(P_i^n)$ as the set of all n-tuples which satisfy $Sb(P_i^n, \varphi^*)$ under v (that is, with $d_1, \ldots, d_m$ assigned to $x_1, \ldots, x_m$). Then*

$$V^*(P_i) \subseteq V(P_i).$$

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[2] Note that we do not need existential quantifiers here to deal with iterations of $\Box$, as in modal logic; instead of $R^n(x, y)$, which is short for $\exists y_1(R(x, y_1) \wedge \ldots \wedge \exists y_{n-1} R(y_{n-1}, y))$, we have, for iterated modalities, $R(y_1, x) \wedge \ldots \wedge R(y, y_{n-1}, \ldots, y_1, x)$.

From the first lemma it follows that $s(\varphi)$ also holds for V and v; and hence $s(\psi)$ holds. Since ψ' is positive, Lemma 2 (with the Monotonicity Lemma for classical logic) implies that $M, v \models \psi'$, as was to be shown.

Proof of lemma 1 φ^* has the form $\Psi \wedge \Gamma \wedge \Theta$, where Ψ is a conjunction of R-statements corresponding to the translations of $\Diamond$-quantifiers, Γ is a conjunction of atomic formulas, and Θ a conjunction of universally bound implications. It is easy to check that the two latter conjuncts turn into tautologies after substituting $Sb(P_i, \varphi^*)$ for every P_i in φ^*. It means that $\vdash s(\varphi) \equiv \Psi$, so it follows from any conjunction including Ψ.

Proof of lemma 2 (a.) Consider the case when the occurrence of P_i is in Γ. Every V which makes the formula true under v should include at least one tuple which satisfies the conditions from Ψ. Then it contains the tuple which satisfies $Sb(\bar{P}_i)$. (b.) Let $\bar{P}_i$ be in Θ. Then it is of the form

$$\forall y_1 \ldots \forall y_{k+l} (\mathcal{R}_1 \wedge \ldots \wedge \mathcal{R}_k \rightarrow P_i(\bar{y}, \bar{x})),$$

where $\mathcal{R}_1 \ldots \mathcal{R}_k$ are the R-conditions corresponding to the generalized quantifiers. If φ^* is true under V and v, then this subformula is true, too, which means that $V(P_i)$ includes at least all tuples $< d_1, \ldots, d_n >$ for which the relation R holds between $\alpha_1, \ldots, \alpha_f$th members, for each of the k R-conditions. So, again it contains all tuples which satisfy $Sb(\bar{P}_i, \varphi^*)$. But if for every occurrence of P_i, the set of tuples satisfying $Sb(\bar{P}_i, \varphi^*)$ is a subset of $V(P_i)$, then also their union is in $V(P_i)$. Thus, $V^*(P_i) \subseteq V(P_i)$. $\square$

Examples. Here is how the above Sahlqvist Algorithm works on the earlier examples of Reflexivity, Transitivity and Symmetry.

- *Reflexivity.* Consider $\Diamond_y x = y$. Its standard translation is

$$\exists y (R(y, x) \wedge x = y),$$

which is equivalent to $R(x, x)$.

- *Transitivity.* The standard translation of

$$\Diamond_y (\top(x) \wedge \Diamond_z (\top(y) \wedge P(z))) \rightarrow \Diamond_z (\top(x) \wedge P(z))$$

gives us

$$\forall P[\exists y (R(y,x) \wedge \top(x) \wedge \exists z (R(z,y) \wedge \top(y) \wedge P(z))) \rightarrow \exists u (R(u,x) \wedge \top(x) \wedge P(u))]$$

which can be rewritten in accordance with the Sahlqvist Algorithm as

$$\forall P \forall y \forall z (R(y,x) \wedge R(z,y) \wedge P(z)) \to \exists u (R(u,x) \wedge P(u)))$$

The minimal substitution for $P(u)$ is $u = z$, so we obtain

$$\forall y \forall z (R(y,x) \wedge R(z,y) \wedge z = z \to \exists u (R(u,x) \wedge u = z),$$

which is a first-order equivalent of transitivity:

$$\forall y \forall z (R(y,x) \wedge R(z,y) \to R(z,x))$$

- *Symmetry.* The formula

$$\forall x \Box_y P(x,y) \to \forall y \Box_x P(x,y)$$

is translated as

$$\forall P (\forall x \forall y (R(y,x) \to P(x,y)) \to \forall y \forall x (R(x,y) \to P(x,y)))$$

The minimal substitution for $P(u,v)$ is $\top(u) \wedge R(v,u)$:

$$\forall x \forall y (R(y,x) \to \top(x) \wedge R(y,x)) \to \forall x \forall y (R(x,y) \to \top(x) \wedge R(y,x))$$

The antecedent becomes trivial:

$$\top \to \forall x \forall y (R(x,y) \to R(y,x))$$

which can again be written more elegantly as

$$\forall x \forall y (R(x,y) \to R(y,x)).$$

Fact. *All purely universal R-conditions can be defined using Sahlqvist formulas only.*

Proof. We show that the algorithm for defining R-properties in $EL(\exists, \Diamond)$ described in the Theorem 4 produces Sahlqvist formulas. First, $\bigwedge_i \exists! x P_{y_i}(x)$ is a Sahlqvist antecedent: it can be rewritten as

$$\bigwedge_i \exists y_i P_{y_i}(y_i) \wedge \bigwedge_i \forall x \forall z (P_{y_i}(x) \wedge P_{y_i}(z) \to x = z)$$

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In the second conjunct, all predicate letters occur negatively (but when it is moved to the consequent in accordance with the Sahlqvist Algorithm, those occurrences become positive).

Next, in the consequent we have $(BOOL(R, =, y_i))^*$, where some predicate letters again can occur negatively. Rewrite it as a conjunction of disjunctions of "atomic" statements $(\Diamond_u(P_{y_i}(u) \wedge \top(\bar{z})))$ and their negations:

$$\Phi \rightarrow \Psi_1 \wedge \ldots \wedge \Psi_n$$

The above expression is equivalent to the following conjunction:

$$(\Phi \rightarrow \Psi_1) \wedge \ldots \wedge (\Phi \rightarrow \Psi_n),$$

where each of Ψ_j's is a disjunction of atomic statements and their negations. Now move negations of atomic statements to the antecedents:

$$\Phi \rightarrow \neg \Diamond_u(P_y(u) \wedge \top(\bar{z})) \vee \Psi \quad \text{becomes} \quad \Phi \wedge \Diamond_u(P_y(u) \wedge \top(\bar{z})) \rightarrow \Psi$$

As a result, there are no negative occurrences of predicate letters in the consequents. □

Example (Symmetry Revisited). Here is one more illustration of the preceding technique. Symmetry can be also defined "locally" using

$$P(x) \wedge \neg \exists x'(x' \neq x \wedge P(x')) \wedge Q(y) \wedge \neg \exists y'(y' \neq y \wedge Q(y')) \rightarrow$$

$$\rightarrow \neg \Diamond_u(P(u) \wedge \top(y)) \vee \Diamond_u(Q(u) \wedge \top(x)) :$$

The latter formula becomes

$$P(x) \wedge Q(y) \wedge \exists u(R(u, y) \wedge P(u)) \rightarrow \exists x'(x' \neq x \wedge P(x')) \vee \exists y'(y' \neq y \wedge Q(y')) \vee$$

$$\vee \exists v(R(v, x) \wedge Q(v)),$$

or

$$\forall u[P(x) \wedge Q(y) \wedge R(u, y) \wedge P(u) \rightarrow \exists x'(x' \neq x \wedge P(x')) \vee \exists y'(y' \neq y \wedge Q(y')) \vee$$

$$\vee \exists v(R(v, x) \wedge Q(v))].$$

The minimal substitutions are as follows:

$$P(z) := \quad z = x \vee z = u;$$

$$Q(z) := z = y.$$

The resulting formula will be then

$$\forall u((x = x \lor x = u) \land y = y \land R(u,y) \land (u = x \lor u = u) \to \exists x'(x' \neq x \land (x' = x \lor x' = u) \lor$$

$$\lor \exists y'(y' \neq y \land y' = y) \lor \exists v(R(v,x) \land v = y));$$

applying predicate logic gives

$$\forall u(R(u,y) \to \exists x'(x' \neq x \land x' = u) \lor R(y,x))$$

$$\forall u(R(u,y) \land \forall x'(x' = u \to x' = x) \to R(y,x),$$

which is equivalent to

$$\forall u(R(u,y) \to R(y,u))$$

$\square$

6 Limitative Results

If a formula does not have the form described in our Sahlqvist Theorem, it may lack a first-order equivalent. The proof that a combination $\Box(\ldots \lor \ldots)$ in the antecedent can be fatal, is adapted from the analogous proof for modal logic (see van Benthem 1983, lemma 10.6).

Lemma 3 $\Box_x(\Box_y(P(y) \land \top(x,z)) \lor P(x)) \to \Diamond_x(\Diamond_y(P(y) \land \top(x,z)) \land P(x))$ *is not first-order definable.*

Proof. Define a class of frames F_n as follows:

- $D_n = \{0, 1, \ldots, 2n + 1\}$;

- $R_n = \{< i, 0 >: 1 \leq i \leq 2n + 1\} \cup \{< i + 1, i, 0 >: 1 \leq i \leq 2n,\} \cup \{< 1, 2n + 1, 0 >\}.$

Here is a picture illustrating this with $R(j, i, 0)$ represented as "there is a line from 0 to i and an arrow from i to j":
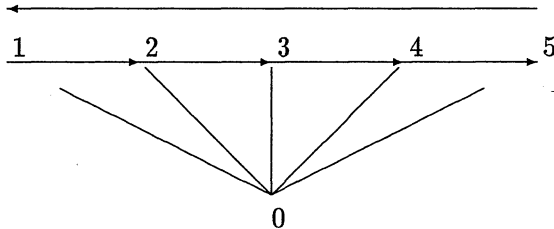
For every n and V,

$$F_n, V, [z/0] \models \Box_x(\Box_y(P(y) \land \top(x)) \lor P(x)) \rightarrow \Diamond_x(\Diamond_y(P(y) \land \top(x)) \land P(x))$$

Indeed, the antecedent is true if

$$\forall x(R(x,z) \rightarrow \forall y(R(y,x,z) \rightarrow P(y)) \lor P(x));$$

that is, if for every i with $R(i,0)$ $P(i)$ is true or P holds for each j with $R(j,i,0)$. Each such i has exactly one "successor" j with $R(j,i,0)$ and "predecessor" k with $R(i,k,0)$. They form a chain which has by definition an odd number of members. That is why, if the antecedent is true, then P should hold for some pair of neighbours in this chain. But then the consequent is also true:

$$\exists x(R(x,z) \land \exists y(R(y,x,z) \land P(y)) \land P(x)).$$

Now, assume that our formula had a first-order equivalent. For arbitrary large n, it is consistent with the following set of first-order sentences describing the frames F_n:

$$\forall x \forall y(R(x,y) \rightarrow \neg R(y,x))$$

$$\forall x \forall y \forall z(R(x,y,z) \rightarrow \neg R(y,x,z))$$

$$\exists! z \forall y R(y,z)$$

$$\forall y(\exists! x R(x,y,z) \land \exists! u R(y,u,z))$$

$$\neg \exists x_1 \ldots \exists x_{2n} \exists y(R(x_2,x_1 y) \land \ldots R(x_{2n},x_{2n-1},y) \land R(x_1,x_{2n},y)).$$

The latter formula forbids "loops" of length less than $2n+1$; that is why it is true in F_k for all $k \geq n$.

By compactness, since each finite set of these formulas has a model for suitably large n, they also have a countable model simultaneously. But in all countable models with the above properties (which are isomorphic copies of $\mathbf{Z}$ with ternary R interpreted as $R(j,i,0) := S(j,i)$ and 0 being a fixed element preceding all other elements: $R(i,0)$ for all $i \neq 0$) the formula can easily be refuted by putting $P(i)$ iff $\neg P(i-1)$ and $\neg P(i+1)$. $\quad\Box$

The same result holds for the combination $\Box_x \ldots \Diamond_y$ in the antecedent (the proof is analogous to the proof of lemma 10.2 in van Benthem 1983):

Lemma 4 $\Box_x \Diamond_y(P(y) \wedge \top(x,z)) \to \Diamond_x \Box_y(P(y) \wedge \top(x,z))$ *does not have a first-order equivalent.*

Proof. Consider the following class of models:

$$D = \{0\} \cup \{y_n : n \in N\} \cup \{y_{n_i} : n \in N, i \in \{0,1\}\} \cup \{z_f : f : N \to \{0,1\}\};$$

$$R = \{< y_n, 0 >: n \in N\} \cup \{< y_{n_i}, y_n, 0 >: n \in N, i \in \{0,1\}\} \cup \{< z_f, 0 >: f : N \to \{0,1\}\} \cup \{< y_{n_{f(n)}}, z_f, 0 >: n \in N, f : N \to \{0,1\}\}$$
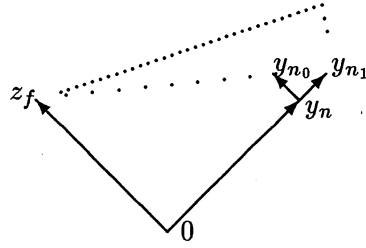
(Here an arrow from a to b describes $R(b,a)$, and the combination of arrows from a to b and from b to c - $R(c,b,a)$.)

Any model of this class validates the formula in question: assume

$$M, v = [z/0] \models \Box_x \Diamond_y(P(y) \wedge \top(x,z)).$$

This means that $\forall x(R(x,0) \to \exists y(R(y,x,0) \wedge P(y) \wedge \top(x,0))$ is true, which implies that $\forall n \exists i P(y_{n_i})$ holds. Since for every n either y_{n_0} or y_{n_1} satisfies P, we can choose f such that $P(y_{f(n)})$ for every n. Then the consequent is also true: $\exists x(R(x,0) \wedge \forall y(R(y,x,0) \to (P(y) \wedge \top(x,0))))$ (via $x = z_f$), whence

$$F, v = [z/0] \models \Box_x \Diamond_y(P(y) \wedge \top(x,z)) \to \Diamond_x \Box_y(P(y) \wedge \top(x,z))$$

M is obviously uncountable. Consider any countable elementary submodel M' of F which includes $0, y_n, y_{n_0}, y_{n_1}$ for all n. If our formula had a first-order equivalent, it would be true in M'. But it can be refuted there: since M' is countable, it does not contain some z_f. Put $y_{n_i} \in V(P)$ iff $i = f(n)$. Then the antecedent is still true (all elements which had a successor in P, still have it), but the consequent is false. $\Box$

Another limitation to the above result emerges when we try to obtain its natural generalization towards completeness of Sahlqvist logics. Here is a striking problem, due to Michiel van Lambalgen.

Example (Sahlqvist incompleteness).

Consider the following three axioms:

A1. $\Diamond_x x = x$;

A2. $\neg\Diamond_y x = y$;

A3. $\Diamond_x \varphi(x, \bar{y}) \rightarrow \Diamond_x(\varphi(x, \bar{y}) \vee \psi(x, \bar{z}))$

These properties are consistent (think of an interpretation for $\Diamond$ like "there exist at least two"). According to the Sahlqvist theorem, these axioms define the following properties of R:

R1. $\exists x R(x)$;

R2. $\neg R(x, x)$;

R3. $R(x, \bar{y}) \rightarrow R(x, \bar{y}, \bar{z})$;

    But together R1–R3 imply $\bot$:

1. $R(x)$ - R1

2. $R(x) \rightarrow R(x, x)$ - R3

3. $R(x, x)$ - 1,2

4. $\neg R(x, x)$ R2

5. $\bot$

7 Further Directions

1. *Intuitive Interpretation.* The above analysis provides a general and rather abstract picture of the behaviour of generalized quantifiers with our relational semantics. The connection between the meaning of quantifiers and some intuitive interpretation of R remains somewhat unclear. As was mentioned in the introduction, R can be viewed as a way to structure domains in different levels or subdomains, but one might also read $R(d, \bar{e})$ as

- d can be constructed using $\bar{e}$,

- d is not "too far" from the e's,

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- after you have picked up e's from the domain without replacing them, d is still available,

etc.

2. *Generalized Formats.* An important generalization of our quantifier employs a binary format. One interesting example is a binary plausibility quantifier (cf. Badaloni & Zanardo 1990):

$$\Box_x(\varphi(x, \bar{y}), \psi(x, \bar{y}))$$

to be read as "φ's are likely to be ψ's". A connection with the interpretation of R could be as follows: $R_\varphi(d, \bar{e})$ iff d is φ-typical with respect to $\bar{e}$. Different notions of typicality would correspond to different conditions on R.

3. *Generalized Frames.* The match between R-properties and qeneralized quantifier principles is not ideal, as we have seen with the above frame-incomplete logic. In van Lambalgen 1991, one may find grounds for employing a restricted version of correspondence, where only definable valuations are allowed which do not refer to the relation R, but only, say, to predicates definable via formulas of the $EL(\exists, \Diamond)$-language with the same parameters. There are precedents for such a move in standard modal logic, where "general frames" have been used with only restricted ranges for valuations (cf. Kracht 1993 for a modal correspondence theory ove the latter structures). Another possible change in format would be to retain the old models, but to change the truth definition for generalized quantifiers that we have employed so far. Various interesting options of this kind exist.

4. *Proof Theory.* The general proof theory of the above framework remains to be explored. For instance, one would like to describe the lattice of possible axiom systems for generalized quantifiers. In particular, can one find a completeness version of the above Sahlqvist Theorem, perhaps using additional modal inference rules to circumvent the above incompleteness (cf. Venema 1992).

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