Institute for Logic, Language and Computation
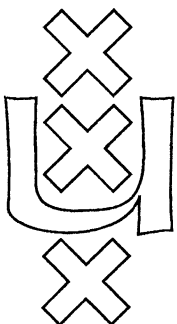
THE CONJOINABILITY RELATION IN LAMBEK CALCULUS AND LINEAR LOGIC

Mati Pentus

University of Amsterdam

The ILLC Prepublication Series

1990

Logic, Semantics and Philosophy of Language

LP-90-01 Jaap van der Does — A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar
LP-90-03 Renate Bartsch — Concept Formation and Concept Composition
LP-90-04 Aarne Ranta — Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn — Nominal Tense Logic
LP-90-06 Gennaro Chierchia — The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia — Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks — Flexible Montague Grammar
LP-90-09 Paul Dekker — The Scope of Negation in Discourse, towards a Flexible Dynamic Montague grammar
LP-90-10 Theo M.V. Janssen — Models for Discourse Markers
LP-90-11 Johan van Benthem — General Dynamics
LP-90-12 Serge Lapierre — A Functional Partial Semantics for Intensional Logic
LP-90-13 Zhisheng Huang — Logics for Belief Dependence
LP-90-14 Jeroen Groenendijk, Martin Stokhof — Two Theories of Dynamic Semantics
LP-90-15 Maarten de Rijke — The Modal Logic of Inequality
LP-90-16 Zhisheng Huang, Karen Kwast — Awareness, Negation and Logical Omniscience
LP-90-17 Paul Dekker — Existential Disclosure, Implicit Arguments in Dynamic Semantics

Mathematical Logic and Foundations

ML-90-01 Harold Schellinx — Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten — A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema — Relational Games
ML-90-04 Maarten de Rijke — Unary Interpretability Logic
ML-90-05 Domenico Zambella — Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten — Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke — A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx — Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani — Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen — Randomness in Set Theory
ML-90-11 Paul C. Gilmore — The Consistency of an Extended NaDSet

Computation and Complexity Theory

CT-90-01 John Tromp, Peter van Emde Boas — Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette — A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar — Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Edith Spaan, Leen Torenvliet — Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast — Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas — Dynamic Data Structures on Multiple Storage Media, a Tutorial
CT-90-07 Kees Doets — Greatest Fixed Points of Logic Programs
CT-90-08 Fred de Geus, Ernest Rotterdam, Sieger van Denneheuvel, Peter van Emde Boas — Physiological Modelling using RL
CT-90-09 Roel de Vrijer — Unique Normal Forms for Combinatory Logic with Parallel Conditional, a case study in conditional rewriting

Other Prepublications

X-90-01 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke — Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev — On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04 — Annual Report 1989
X-90-05 Valentin Shehtman — Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy — Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov — The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev — Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov — On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas — An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone — Provable Fixed points in $I\Delta_0 + \Omega_1$, revised version
X-90-12 Maarten de Rijke — Bi-Unary Interpretability Logic
X-90-13 K.N. Ignatiev — Dzhaparidze's Polymodal Logic: Arithmetical Completeness, Fixed Point Property, Craig's Property
X-90-14 L.A. Chagrova — Undecidable Problems in Correspondence Theory
X-90-15 A.S. Troelstra — Lectures on Linear Logic

1991

Logic, Semantics and Philosophy of Langauge

LP-91-01 Wiebe van der Hoek, Maarten de Rijke — Generalized Quantifiers and Modal Logic
LP-91-02 Frank Veltman — Defaults in Update Semantics
LP-91-03 Willem Groeneveld — Dynamic Semantics and Circular Propositions
LP-91-04 Makoto Kanazawa — The Lambek Calculus enriched with Additional Connectives
LP-91-05 Zhisheng Huang, Peter van Emde Boas — The Schoenmakers Paradox: Its Solution in a Belief Dependence Framework
LP-91-06 Zhisheng Huang, Peter van Emde Boas — Belief Dependence, Revision and Persistence
LP-91-07 Henk Verkuyl, Jaap van der Does — The Semantics of Plural Noun Phrases
LP-91-08 Víctor Sánchez Valencia — Categorial Grammar and Natural Reasoning
LP-91-09 Arthur Nieuwendijk — Semantics and Comparative Logic
LP-91-10 Johan van Benthem — Logic and the Flow of Information

Mathematical Logic and Foundations

ML-91-01 Yde Venema — Cylindric Modal Logic
ML-91-02 Alessandro Berarducci, Rineke Verbrugge — On the Metamathematics of Weak Theories
ML-91-03 Domenico Zambella — On the Proofs of Arithmetical Completeness for Interpretability Logic
ML-91-04 Raymond Hoofman, Harold Schellinx — Collapsing Graph Models by Preorders
ML-91-05 A.S. Troelstra — History of Constructivism in the Twentieth Century
ML-91-06 Inge Bethke — Finite Type Structures within Combinatory Algebras
ML-91-07 Yde Venema — Modal Derivation Rules
ML-91-08 Inge Bethke — Going Stable in Graph Models
ML-91-09 V.Yu. Shavrukov — A Note on the Diagonalizable Algebras of PA and ZF
ML-91-10 Maarten de Rijke, Yde Venema — Sahlqvist's Theorem for Boolean Algebras with Operators
ML-91-11 Rineke Verbrugge — Feasible Interpretability
ML-91-12 Johan van Benthem — Modal Frame Classes, revisited

Computation and Complexity Theory

CT-91-01 Ming Li, Paul M.B. Vitányi — Kolmogorov Complexity Arguments in Combinatorics
CT-91-02 Ming Li, John Tromp, Paul M.B. Vitányi — How to Share Concurrent Wait-Free Variables
CT-91-03 Ming Li, Paul M.B. Vitányi — Average Case Complexity under the Universal Distribution Equals Worst Case Complexity
CT-91-04 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence
CT-91-05 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence for Constraint Sets
CT-91-06 Edith Spaan — Census Techniques on Relativized Space Classes
CT-91-07 Karen L. Kwast — The Incomplete Database
CT-91-08 Kees Doets — Levationis Laus
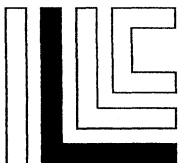
THE CONJOINABILITY RELATION IN LAMBEK CALCULUS AND LINEAR LOGIC

Mati Pentus
Department of Mathematical Logic
Faculty of Mechanics and Mathematics
Moscow State University

The Conjoinability Relation
in Lambek Calculus and Linear Logic

Mati Pentus*

Department of Mathematical Logic
Faculty of Mechanics and Mathematics
Moscow State University
119899, Moscow, Russia

Abstract

In 1958 J. Lambek introduced a calculus L of syntactic types and defined an equivalence relation on types: "$x \equiv y$ means that there exists a sequence $x = x_1, \ldots, x_n = y$ $(n \geq 1)$, such that $x_i \to x_{i+1}$ or $x_{i+1} \to x_i$ $(1 \leq i < n)$". He pointed out that $x \equiv y$ if and only if there is join z such that $x \to z$ and $y \to z$.

This paper gives an effective characterization of this equivalence for the Lambek calculi L and LP, and for the multiplicative fragments of Girard's and Yetter's linear logics. Moreover, for the non-directed Lambek calculus LP and the multiplicative fragment of Girard's linear logic, we present linear time algorithms deciding whether two types are equal, and finding a join for them if they are.

Introduction

In [4] Joachim Lambek introduced a calculus L for deriving reduction laws of syntactic types, and studied an equivalence relation on types defined as follows:
$a \sim b$ iff there is a natural number n and there exist types $c_1, \ldots, c_n$ such that $a = c_1$, $b = c_n$ and $\forall i < n$ $c_i \Rightarrow c_{i+1}$ or $c_{i+1} \Rightarrow c_i$. He also proved that this notion coincides with another one which was later called conjoinability. In [1] J. van Benthem pointed out that the question of decidability of this relation was still open in 1991.

In this paper we shall show that the equivalence of types a and b can be characterized in terms of some decidable invariants earlier introduced as necessary conditions for derivability of a sequent $a \Rightarrow b$. These invariants are primitive type counts $\#_p$ in

1

the non-directed calculus LP and the "geometric invariant" or "balance" in the directed calculus L [1].

Since LP and L are fragments of the ordinary linear logic LL [2] and the cyclic linear logic CLL [6] respectively, we shall also study similar equivalence relation in both these logics. Let MLL and $MCLL$ denote the multiplicative fragments of LL and CLL. Characterization of the equivalence in these fragments involves a new invariant defined by

$$\natural p = 0, \quad \natural(a{\cdot}b) = \natural a + \natural b, \quad \natural(a^{\perp}) = 1 - \natural a,$$

where $\cdot$ and $(\ )^{\perp}$ denote the linear conjunction and negation.

Main results of this paper are the following:

- $a \sim b$ in L iff the sequent $a \Rightarrow b$ is balanced (Section 4.1).

- $a \sim b$ in LP iff $\#_p a = \#_p b$ for all literals p (Section 4.2).

- $a \sim b$ in MLL iff $\natural a = \natural b$ and $\#_p a = \#_p b$ for all literals p (Section 4.3).

- $a \sim b$ in $MCLL$ iff $\natural a = \natural b$ and the sequent $a \Rightarrow b$ is balanced (Section 4.3).

- if $b \sim c$ in LP, then there exists a type d such that $LP \vdash b \Rightarrow d$, $LP \vdash c \Rightarrow d$, and $\mid d \mid \leq 3(\mid b \mid + \mid c \mid) + 2$ (Section 5.1).

- if $b \sim c$ in MLL, then there exists a formula d such that $LP \vdash b \Rightarrow d$, $LP \vdash c \Rightarrow d$, and $\mid d \mid \leq 3(\mid b \mid + \mid c \mid)$ (Section 5.2).

Here $\mid d \mid$ denotes the total number of occurrences of literals in the formula d.

Earlier proofs of the first four of these results have been presented in [5] and [3]. In this paper we give new and shorter proofs.

1 Preliminaries

1.1 Lambek calculus

In [4, p. 165] J. Lambek introduced a formal system for deriving reduction laws of syntactic types. We shall consider this system (denoted here by L) and some variants of it.

The language of the Lambek calculus includes a non-empty denumerable set **Atom** of literals for primitive types, and three binary connectives $\cdot$, $\backslash$, $/$, called *product, left implication* and *right implication*. For the purpose of readability we shall omit parentheses whenever product occurs in the scope of a connective $\backslash$ or $/$. For example, $p{\cdot}s/q{\cdot}r$ means $(p{\cdot}s)/(q{\cdot}r)$.

Let $p, q, r, \ldots$ stand for elements of **Atom**. The letters from the beginning of the alphabet $a, b, c, \ldots$ denote types[1] built from literals with the help of product, left implication, and right implication. Let capitals $X, Y, Z, \ldots$ range over finite sequences of types,

---

[1]Throughout this paper we shall use the terms "type" and "formula" as synonyms.

possibly empty sequences. The concatenation of sequences X and Y will be denoted by X,Y.

First, we introduce the Gentzen style sequent calculus L^*. Sequents are of the form $X \Rightarrow b$ where b is a type and X is a sequence of types. The order of types in X is essential.

The only axiom scheme is $b \Rightarrow b$, where b is any type. Rules of inference are the following:

$$\frac{X \Rightarrow a \quad Y , b , Z \Rightarrow c}{Y , X , a\backslash b , Z \Rightarrow c} \; (\; \backslash \Rightarrow) \qquad\qquad \frac{a, X \Rightarrow b}{X \Rightarrow a\backslash b} \; (\; \Rightarrow \backslash)$$

$$\frac{X \Rightarrow a \quad Y , b , Z \Rightarrow c}{Y , b/a , X , Z \Rightarrow c} \; (\; / \Rightarrow) \qquad\qquad \frac{X, a \Rightarrow b}{X \Rightarrow b/a} \; (\; \Rightarrow /)$$

$$\frac{X , a , b , Y \Rightarrow c}{X , a{\cdot}b , Y \Rightarrow c} \; (\; {\cdot} \Rightarrow) \qquad\qquad \frac{X \Rightarrow a \quad Y \Rightarrow b}{X,Y \Rightarrow a{\cdot}b} \; (\; \Rightarrow {\cdot})$$

$$\frac{X \Rightarrow a \quad Y,a, Z \Rightarrow b}{Y,X,Z \Rightarrow b} \; (\; \text{CUT} \;)$$

We write $L^* \vdash X \Rightarrow b$ for "the sequent $X \Rightarrow b$ is derivable in L^* ".

Given a formal system T, we shall write $a \overset{T}{\cong} b$ iff $T \vdash a \Rightarrow b$ and $T \vdash b \Rightarrow a$. It appears that if $b \overset{T}{\cong} c$ then replacing an instance of type b in a sequent by type c does not have any effect on derivability in T. In particular $(a{\cdot}b){\cdot}c \overset{L^*}{\cong} a{\cdot}(b{\cdot}c)$ and $(a\backslash b)/c \overset{L^*}{\cong} a\backslash(b/c)$ for any types a, b and c. So, we can omit parentheses in types $(a{\cdot}b){\cdot}c$ and $(a\backslash b)/c$.

Adding to L^* the permutation rule

$$\frac{X,a,b,Y \Rightarrow c}{X,b,a,Y \Rightarrow c} \; (\; \text{P} \;)$$

we obtain the non-directed Lambek calculus L^*P. In L^*P we have $b\backslash c \overset{L^*P}{\cong} c/b$ for any types b and c, whence left and right implications fall into one connective, which is often denoted by $b{\rightarrow}c$.

The original calculi L and LP are obtained from L^* and L^*P respectively, by adding the constraint that all sequents must have non-empty antecedents. In other words, in L and LP the rules $(\Rightarrow \backslash)$ and $(\Rightarrow /)$ may be applied only if the sequence X is not empty.

Evidently the following inclusions hold:

$$
\begin{array}{ccc}
L & \subset & LP \\
\cap & & \cap \\
L^* & \subset & L^*P
\end{array}
$$

Now we shall define a notion of duality, which often allows to cut by half proofs about derivability in Lambek calculi.

$$
\begin{aligned}
\text{dual } (p) &\rightleftharpoons p \\
\text{dual } (a{\bullet}b) &\rightleftharpoons \text{dual } (b){\bullet}\, \text{dual } (a) \\
\text{dual } (a{\backslash}b) &\rightleftharpoons \text{dual } (b)/\, \text{dual } (a) \\
\text{dual } (a/b) &\rightleftharpoons \text{dual } (b){\backslash}\, \text{dual } (a) \\
\text{dual } (a_1,\ldots,a_n \Rightarrow b) &\rightleftharpoons \text{dual } (a_n),\ldots, \text{dual } (a_1) \Rightarrow \text{dual } (b)
\end{aligned}
$$

In any of the calculi considered here, a sequent is derivable if and only if its dual is derivable.

1.2   Multiplicative fragments of ordinary and cyclic linear logics

First we introduce MLL — the multiplicative fragment of ordinary (commutative) linear logic. We shall denote linear negation by $(\ )^{\perp}$, linear implication by $-\!\circ$, linear conjunction and disjunction by $\bullet$ and $+$, and corresponding units by $\mathbf{1}$ and $\mathbf{0}$.

The formulas of linear logic are defined as follows:

- $\mathbf{0}$ and $\mathbf{1}$ are formulas,

- if $p \in \mathbf{Atom}$, then p and $p^{\perp}$ are formulas,

- if a and b are formulas, then $a{\bullet}b$ and $a{+}b$ are also formulas.

We introduce linear implication and negation as abbreviations defined as:

$$
a{-}\!\circ b \;\rightleftharpoons\; (a)^{\perp}{+}b
$$

$$
\begin{aligned}
(\mathbf{1})^{\perp} &\rightleftharpoons \mathbf{0} \\
(\mathbf{0})^{\perp} &\rightleftharpoons \mathbf{1} \\
(p^{\perp})^{\perp} &\rightleftharpoons p \\
(a{\bullet}b)^{\perp} &\rightleftharpoons (a)^{\perp}{+}(b)^{\perp} \\
(a{+}b)^{\perp} &\rightleftharpoons (a)^{\perp}{\bullet}(b)^{\perp} \\
(a{-}\!\circ b)^{\perp} &\rightleftharpoons a{\bullet}(b)^{\perp}
\end{aligned}
$$

Derivable objects of linear logic are sequents $\Rightarrow a_1;\ldots;a_n$, where $a_1, \ldots, a_n$ are formulas. We shall interpret $\Rightarrow a_1;\ldots;a_n$ as $a_1{+}\cdots{+}a_n$.

4

The axiom scheme is $\Rightarrow a^\perp; a$. The inference rules of MLL are the following:

$$\frac{\Rightarrow X \; ; \; a \; ; \; b \; ; \; Y}{\Rightarrow X \; ; \; a{+}b \; ; \; Y}\;(+)$$

$$\frac{\Rightarrow X;a \qquad \Rightarrow b;Y}{\Rightarrow X \; ; \; a{\bullet}b \; ; \; Y}\;(\bullet)$$

$$\frac{\Rightarrow X;a \qquad \Rightarrow a^\perp \; ; \; Y}{\Rightarrow X;Y}\;(\text{CUT})$$

$$\frac{\Rightarrow X;a;b;Y}{\Rightarrow X;b;a;Y}\;(\text{P})$$

Now we shall consider $MCLL$ — the multiplicative fragment of the cyclic (non-commutative) linear logic presented by Yetter in [6]. The connectives are the same as in MLL except that in MCLL there are two linear implications $\backslash$ and $/$ instead of $\multimap$.

The formulas are defined in the same way as in MLL, but the abbreviations are different.

$$a\backslash b \;\rightleftharpoons\; (a)^\perp{+}b$$
$$b/a \;\rightleftharpoons\; b{+}(a)^\perp$$

$$(1)^\perp \;\rightleftharpoons\; 0$$
$$(0)^\perp \;\rightleftharpoons\; 1$$
$$(p^\perp)^\perp \;\rightleftharpoons\; p$$
$$(a{\bullet}b)^\perp \;\rightleftharpoons\; b^\perp{+}a^\perp$$
$$(a{+}b)^\perp \;\rightleftharpoons\; b^\perp{\bullet}a^\perp$$
$$(a\backslash b)^\perp \;\rightleftharpoons\; b^\perp{\bullet}a$$
$$(b/a)^\perp \;\rightleftharpoons\; a{\bullet}b^\perp$$

Again, the axiom scheme is $\Rightarrow a^\perp; a$. The inference rules of $MCLL$ are the following:

$$\frac{\Rightarrow X \; ; \; a \; ; \; b \; ; \; Y}{\Rightarrow X \; ; \; a{+}b \; ; \; Y}\;(+)$$

$$\frac{\Rightarrow X;a \qquad \Rightarrow b;Y}{\Rightarrow X \; ; \; a{\bullet}b \; ; \; Y}\;(\bullet)$$

$$\frac{\Rightarrow X;a \qquad \Rightarrow a^\perp \; ; \; Y}{\Rightarrow X;Y}\;(\text{CUT})$$

$$\frac{\Rightarrow X;Y}{\Rightarrow Y;X}\;(\text{ROT})$$

Obviously $MCLL \subset MLL$.

It is convenient to have in linear logics a notion of entailment similar to $a \Rightarrow b$ in Lambek calculi. We shall write $MLL \vdash a \Rightarrow b$ iff $MLL \vdash \;\Rightarrow a^\perp; b$, and $MCLL \vdash a \Rightarrow b$ iff $MCLL \vdash \;\Rightarrow a^\perp; b$.

Again, if $b \cong c$, then the formulas b and c may be replaced by each other. Immediate verification shows that $a{\bullet}(b{\bullet}c) \cong (a{\bullet}b){\bullet}c$ and $a{+}(b{+}c) \cong (a{+}b){+}c$. Hence we may omit

5

parentheses in these expressions. We shall also omit parentheses in $a{+}(b{\cdot}c)$, but not in $(a{+}b){\cdot}c$.

The phenomenon of duality is present in linear logics as well.

$$
\begin{aligned}
\text{dual } (p) &\rightleftharpoons p \\
\text{dual } (p^{\perp}) &\rightleftharpoons p^{\perp} \\
\text{dual } (a{\cdot}b) &\rightleftharpoons \text{dual } (b){\cdot}\text{ dual } (a) \\
\text{dual } (a{+}b) &\rightleftharpoons \text{dual } (b){+}\text{ dual } (a) \\
\text{dual } (\Rightarrow a_1 ; \ldots ; a_n) &\rightleftharpoons \Rightarrow \text{dual } (a_n) ; \ldots ; \text{ dual } (a_1)
\end{aligned}
$$

Cut-elimination holds in all the calculi considered in this paper (cf. [4] [1] [6]).

Remark. We note that $MCLL$ is a conservative extension of L^* and MLL is a conservative extension of L^*P. This is easily proved with the help of cut-elimination and the invariant $\natural$, which will be introduced in Section 2.4.

2 Soundness results

In this section we extend the definitions of balance and primitive type counts from Lambek calculi (cf. [1]) to the case of linear logic. We present balance in group-theoretic terms.

A numerical model, called the *count of negations*, is introduced. Both ordinary and cyclic linear logics are sound with respect to this additional invariant.

All the lemmas in this section are proved by straightforward induction. We omit the proofs of most of them.

2.1 Balance in Lambek calculus

Let FG stand for the free group generated by literals from **Atom**. We shall denote the unit of FG by Λ. There is a mapping from the Lambek calculus formulas to FG, associating to each formula b its algebraic interpretation $[\![b]\!]$, defined in the natural way:

$$
\begin{aligned}
[\![p]\!] &\rightleftharpoons p \\
[\![a{\cdot}b]\!] &\rightleftharpoons [\![a]\!] \cdot [\![b]\!] \\
[\![a{\backslash}b]\!] &\rightleftharpoons [\![a]\!]^{-1} \cdot [\![b]\!] \\
[\![b{/}a]\!] &\rightleftharpoons [\![b]\!] \cdot [\![a]\!]^{-1}
\end{aligned}
$$

Definition. A Lambek calculus sequent $a_1, \ldots, a_n \Rightarrow b$ is *balanced* iff $[\![a_1]\!] \cdots [\![a_n]\!] \overset{FG}{=} [\![b]\!]$, where $\overset{FG}{=}$ denotes the equality in the free group FG.

It is easy to see that this definition of balance is equivalent to the one given in [1].

6

Example 1 The sequent $p/q, q \Rightarrow p$ is balanced, since $[\![(p/q) \cdot q]\!] \equiv pq^{-1}q$, $[\![p]\!] \equiv p$, and $pq^{-1}q \overset{FG}{=} p$.

Lemma 1 (cf. [1]) *If $L^* \vdash X \Rightarrow b$ then $X \Rightarrow b$ is balanced.*

PROOF. Induction on derivations. We omit the trivial cases of axioms and the rule ($\bullet \Rightarrow$).

CASE ($\Rightarrow \bullet$) : If $[\![X]\!] \overset{FG}{=} [\![a]\!]$ and $[\![Y]\!] \overset{FG}{=} [\![b]\!]$ then $[\![X]\!][\![Y]\!] \overset{FG}{=} [\![a]\!][\![b]\!]$.

CASE ($\Rightarrow \backslash$) : Multiplying the equality $[\![a]\!][\![X]\!] \overset{FG}{=} [\![b]\!]$ by $[\![a]\!]^{-1}$ on the left, one obtains $[\![X]\!] \overset{FG}{=} [\![a]\!]^{-1}[\![b]\!]$ as desired.

CASE ($\backslash \Rightarrow$) : If $[\![X]\!] \overset{FG}{=} [\![a]\!]$ then $[\![X]\!][\![a]\!]^{-1} \overset{FG}{=} \Lambda$.
In turn, $[\![Y]\!][\![b]\!][\![Z]\!] \overset{FG}{=} [\![c]\!]$ entails $[\![Y]\!][\![X]\!][\![a]\!]^{-1}[\![b]\!][\![Z]\!] \overset{FG}{=} [\![c]\!]$.

CASE ($\Rightarrow /$) and ($/ \Rightarrow$) : These rules are treated similarly to their duals ($\Rightarrow \backslash$) and ($\backslash \Rightarrow$). ∎

2.2 Balance in linear logic

The formulas of linear logic are mapped into FG as follows:

$$
\begin{aligned}
[\![\mathbf{1}]\!] &\rightleftharpoons \Lambda \\
[\![\mathbf{0}]\!] &\rightleftharpoons \Lambda \\
[\![p]\!] &\rightleftharpoons p \\
[\![a^\perp]\!] &\rightleftharpoons [\![a]\!]^{-1} \\
[\![a \bullet b]\!] &\rightleftharpoons [\![a]\!] \cdot [\![b]\!] \\
[\![a+b]\!] &\rightleftharpoons [\![a]\!] \cdot [\![b]\!] \\
[\![a \backslash b]\!] &\rightleftharpoons [\![a]\!]^{-1} \cdot [\![b]\!] \\
[\![b/a]\!] &\rightleftharpoons [\![b]\!] \cdot [\![a]\!]^{-1}
\end{aligned}
$$

Definition. A sequent of the cyclic linear logic $\Rightarrow a_1 ; \ldots ; a_n$ is *balanced* iff $[\![a_1]\!] \cdot \ldots \cdot [\![a_n]\!] \overset{FG}{=} \Lambda$.

Lemma 2 (i) *If $MCLL \vdash \Rightarrow X$ then $\Rightarrow X$ is balanced.*

(ii) *In particular, if $MCLL \vdash a \Rightarrow b$ then $[\![a]\!] \overset{FG}{=} [\![b]\!]$.*

2.3 Primitive type counts

Definition. For any literal $p \in \mathbf{Atom}$, the *p-count* ($\#_p$) is the following mapping from types to integer numbers.

$$\#_p p \;\rightleftharpoons\; 1$$

$$\#_p q \;\rightleftharpoons\; 0 \;\text{, if } p \text{ and } q \text{ are distinct literals}$$
$$\#_p(a{\cdot}b) \;\rightleftharpoons\; \#_p a + \#_p b$$
$$\#_p(a\backslash b) \;\rightleftharpoons\; \#_p b - \#_p a$$
$$\#_p(b/a) \;\rightleftharpoons\; \#_p b - \#_p a$$

Lemma 3 (cf. [1]) *If $L^*P \vdash a_1, \ldots, a_n \Rightarrow b$*
then $\#_p a_1 + \cdots + \#_p a_n = \#_p b$ for any $p \in \mathbf{Atom}$.

Definition. For linear logic we define the *p-count* as follows:

$$\#_p \mathbf{0} \;\rightleftharpoons\; 0$$
$$\#_p \mathbf{1} \;\rightleftharpoons\; 0$$
$$\#_p p \;\rightleftharpoons\; 1$$
$$\#_p q \;\rightleftharpoons\; 0 \;\text{, if } p \text{ and } q \text{ are distinct literals}$$
$$\#_p(a^\perp) \;\rightleftharpoons\; -\#_p a$$
$$\#_p(a{\cdot}b) \;\rightleftharpoons\; \#_p a + \#_p b$$
$$\#_p(a{+}b) \;\rightleftharpoons\; \#_p a + \#_p b$$
$$\#_p(a{-\!\circ}\, b) \;\rightleftharpoons\; \#_p b - \#_p a$$

Lemma 4 (i) *If $MLL \vdash \; \Rightarrow a_1 ; \; \ldots \; ; \; a_n$*
    then $\#_p a_1 + \cdots + \#_p a_n = 0$ for any $p \in \mathbf{Atom}$.

(ii) *In particular, if $MLL \vdash a \Rightarrow b$ then $\#_p a = \#_p b$ for any $p \in \mathbf{Atom}$.*

2.4   The count of negations

Definition. The *count of negations* is the following mapping $\natural$ from formulas of linear logic to integer numbers.

$$\natural p \;\rightleftharpoons\; 0$$
$$\natural p^\perp \;\rightleftharpoons\; 1$$
$$\natural \mathbf{1} \;\rightleftharpoons\; 0$$
$$\natural \mathbf{0} \;\rightleftharpoons\; 1$$
$$\natural(a{\cdot}b) \;\rightleftharpoons\; \natural a + \natural b$$
$$\natural(a{+}b) \;\rightleftharpoons\; \natural a + \natural b - 1$$

The following equalities are easy consequences of the definitions of the linear negation and implications.

$$\natural(a^{\perp}) \;\rightleftharpoons\; 1 - \natural a$$
$$\natural(a \multimap b) \;\rightleftharpoons\; \natural b - \natural a$$
$$\natural(a \backslash b) \;\rightleftharpoons\; \natural b - \natural a$$
$$\natural(a/b) \;\rightleftharpoons\; \natural a - \natural b$$

Lemma 5 (i) *If* $MLL \vdash \; \Rightarrow a_1 \; ; \; \ldots \; ; \; a_n$ *then* $\natural(a_1 + \cdots + a_n) = 0$.

(ii) *In particular, if* $MLL \vdash a \Rightarrow b$ *then* $\natural a = \natural b$.

3 Equivalent formulas

Definition. Given a formal system T, we say that two formulas a and b are *equivalent* in theory T (we write $a \overset{T}{\sim} b$) iff there is a natural number n and there exist formulas $c_1, \ldots, c_n$ such that $a = c_1$, $b = c_n$, and

$$T \vdash c_i \Rightarrow c_{i+1} \quad \text{or} \quad T \vdash c_{i+1} \Rightarrow c_i \quad \text{for any} \; i < n.$$

Remark. In other words, $\overset{T}{\sim}$ is the reflexive, symmetric, transitive closure of the relation $\overset{T}{\leq}$, where $a \overset{T}{\leq} b$ means that the sequent $a \Rightarrow b$ is derivable in the theory T.

Definition. We say that type c is a *join* for a set of formulas $\{a_1, \ldots, a_n\}$ in a theory T iff $T \vdash a_i \Rightarrow c$ for any $i \leq n$.

Definition. (cf. [1]) Two formulas a and b are *conjoinable* in a theory T iff there is a join for $\{a, b\}$ in T.

Example 2 Let c and d be any types of the Lambek calculus. Then $c\backslash c$ and d/d are conjoinable in the pure Lambek calculus L.

The following derivations show that $c\backslash c \bullet d/d$ is a consequent of both $c\backslash c$ and d/d in L.

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{c \Rightarrow c \quad d \Rightarrow d}{c \, , \, d \Rightarrow c \bullet d} \; (\Rightarrow \bullet)
    }{c \Rightarrow c \bullet d/d} \; (\Rightarrow /) \qquad c \Rightarrow c
  }{c \, , \, c\backslash c \Rightarrow c \bullet d/d} \; (\backslash \Rightarrow)
}{c\backslash c \Rightarrow c\backslash c \bullet d/d} \; (\Rightarrow \backslash)
$$

$$
\cfrac{
  \cfrac{
    \cfrac{
      d \Rightarrow d \qquad \cfrac{c \Rightarrow c \quad d \Rightarrow d}{c \, , \, d \Rightarrow c \bullet d} \; (\Rightarrow \bullet)
    }{c \, , \, d/d \, , \, d \Rightarrow c \bullet d} \; (/ \Rightarrow)
  }{c \, , \, d/d \Rightarrow c \bullet d/d} \; (\Rightarrow /)
}{d/d \Rightarrow c\backslash c \bullet d/d} \; (\Rightarrow \backslash)
$$

The next two lemmas, belonging to J. Lambek [4], show that conjoinability in L coincides with the equivalence relation $\overset{L}{\sim}$.

Lemma 6 (Diamond property) *Let a and b be any types of L. Then the following two assertions are equivalent.*

(i) *There exists a type c such that $L \vdash a \Rightarrow c$ and $L \vdash b \Rightarrow c$, i.e. a and b are conjoinable in L.*

(ii) *There exists a type d such that $L \vdash d \Rightarrow a$ and $L \vdash d \Rightarrow b$.*

$$
\begin{array}{ccc}
 & a & \\
\nearrow & & \searrow \\
d & & c \\
\searrow & & \nearrow \\
 & b &
\end{array}
$$
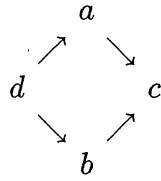
In other words, we can find any of the types c or d, indicated on the figure, if the other three types are given.

PROOF. We give a proof slightly shorter than in [4].

CASE (i) $\rightarrow$ (ii) : We put $d = (a/c){\bullet}c{\bullet}(c\backslash b)$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{b \Rightarrow c \qquad a \Rightarrow a}{c \Rightarrow c \qquad a/c \, , \; b \Rightarrow a}(/ \; \Rightarrow \,)}{a/c \, , \; c \, , \; c\backslash b \Rightarrow a}(\backslash \; \Rightarrow \,)}{(a/c){\bullet}c{\bullet}(c\backslash b) \Rightarrow a}({\bullet} \Rightarrow \,)}{}
$$

The sequent $(a/c){\bullet}c{\bullet}(c\backslash b) \Rightarrow b$ is derived dually.

CASE (ii) $\rightarrow$ (i) : We put $c = (d/a)\backslash d/(b\backslash d)$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{d \Rightarrow b \qquad d \Rightarrow d}{a \Rightarrow a \qquad d \, , \; b\backslash d \Rightarrow d}(\backslash \; \Rightarrow \,)}{d/a \, , \; a \, , \; b\backslash d \Rightarrow d}(/ \; \Rightarrow \,)}{a \, , \; b\backslash d \Rightarrow (d/a)\backslash d}(\Rightarrow \backslash)}{a \Rightarrow (d/a)\backslash d/(b\backslash d)}(\Rightarrow /)}{}
$$

The sequent $b \Rightarrow (d/a)\backslash d/(b\backslash d)$ is derived similarly. ∎

Lemma 7 *Two types a and b are equivalent if and only if the assertion* (i) *(and consequently also* (ii)*) from the previous lemma holds.*

10

PROOF. The 'if' part is obvious. To prove the 'only if' part, we assume that there are n types $e_1, \ldots, e_n$ such that $e_i \Rightarrow e_{i+1}$ or $e_{i+1} \Rightarrow e_i$ for any $i < n$. Now the assertion (i) of Lemma 6 follows from (ii) $\rightarrow$ (i) and

$$\frac{e_i \Rightarrow e_{i+1} \qquad e_{i+1} \Rightarrow e_{i+2}}{e_i \Rightarrow e_{i+2}} \text{ (CUT)}$$

by induction on n. ∎

Corollary 8 *A finite set of types has a join in L if and only if the types are pairwise conjoinable in L.*

Lemma 9 *The relation $\overset{L}{\sim}$ is a congruence on types.*

PROOF. Follows immediately from the admissibility of the rules

$$\frac{a \Rightarrow b}{a{\cdot}c \Rightarrow b{\cdot}c} \qquad \frac{a \Rightarrow b}{c\backslash a \Rightarrow c\backslash b} \qquad \frac{a \Rightarrow b}{b\backslash c \Rightarrow a\backslash c}$$

∎

Corollary 10 (Replacement property of the equivalence) *Let c_d be a type, containing a subtype d, and let c_b come from c_d by replacing one occurrence of d by a type b. If $d \sim b$, then $c_d \sim c_b$.*

4 Completeness of the equivalence with respect to group interpretations

In this section we prove that the notion of equivalence of two types in the directed Lambek calculus coincides with the balance, and in the undirected Lambek calculus it coincides with the equality of primitive type counts. Similar characterization of equivalence in linear logics involves in addition the count of negations.

4.1 Pure Lambek calculus

Theorem 1 *For any types a and b, the following three clauses are equivalent.*

(i) $a \overset{L}{\sim} b$

(ii) $a \overset{L^*}{\sim} b$

(iii) $[\![a]\!] \overset{FG}{=} [\![b]\!]$

11

PROOF. (i) → (ii)

Follows from $L \subset L^*$.

(ii) → (iii)

Let $L^* \vdash a \Rightarrow c$ and $L^* \vdash b \Rightarrow c$. By Lemma 1 this entails $[\![a]\!] \overset{FG}{=} [\![c]\!]$ and $[\![b]\!] \overset{FG}{=} [\![c]\!]$. Hence $[\![a]\!] \overset{FG}{=} [\![b]\!]$.

(iii) → (i)

We prove that if $[\![a]\!] \overset{FG}{=} [\![b]\!]$ then $[a]_\sim = [b]_\sim$, where $[\ ]_\sim$ denotes the equivalence class of a type with respect to the relation $\overset{L}{\sim}$.

First, we note that all types of the form $c\backslash c$ and d/d belong to one equivalence class, which we shall denote by $\mathbf{1}_\sim$ (cf. Example 2).

Now we verify that the equivalence classes form a group with unit $\mathbf{1}_\sim$ and operators

$$[c]_\sim \cdot [d]_\sim \rightleftharpoons [c{\bullet}d]_\sim \quad , \quad [c]_\sim^{-1} \rightleftharpoons [c\backslash c/c]_\sim.$$

The associativity law is obvious. The property $[c]_\sim \cdot \mathbf{1}_\sim = [c]_\sim$ is evident from $L \vdash c{\bullet}(c\backslash c) \Rightarrow c$. Similarly, $[c]_\sim^{-1} \cdot [c]_\sim = \mathbf{1}_\sim$ follows from $L \vdash (c\backslash c/c){\bullet}c \Rightarrow c\backslash c$. The dual laws hold also. Hence the classes $[c]_\sim$ form a group.

Let us consider the mapping $[\![p]\!] \mapsto [p]_\sim$ from the generators of the free group FG into the group of the congruence classes of types. We extend this mapping to a homomorphism h. All that we need is to prove that, for any type c, h maps $[\![c]\!]$ to $[c]_\sim$. This is done by induction on the construction of the type c.

CASE 1: $h([\![c{\bullet}d]\!]) = h([\![c]\!]) \cdot h([\![d]\!]) = [c]_\sim \cdot [d]_\sim = [c{\bullet}d]_\sim$

CASE 2: $h([\![c\backslash d]\!]) = h([\![c]\!]^{-1} \cdot [\![d]\!]) = [c]_\sim^{-1} \cdot [d]_\sim$

It is sufficient to prove that $[c]_\sim^{-1} \cdot [d]_\sim = [c\backslash d]_\sim$. We have

$$(c\backslash(c/c)){\bullet}d \overset{L}{\sim} (c\backslash(d/d)){\bullet}d \overset{L}{\cong} ((c\backslash d)/d){\bullet}d.$$

Evidently $L \vdash ((c\backslash d)/d){\bullet}d \Rightarrow c\backslash d$. Hence $(c\backslash c/c){\bullet}d \overset{L}{\sim} c\backslash d$.

CASE 3: Similarly to the previous case, we verify that $h([\![c/d]\!]) = [c/d]_\sim$. ■

4.2   Lambek calculus with permutation

Theorem 2 *For any types a and b, the following three clauses are equivalent.*

(i) $a \overset{LP}{\sim} b$

(ii) $a \overset{L^*P}{\sim} b$

(iii) $\#_p a = \#_p b$ *for any $p \in \mathbf{Atom}$.*

PROOF. We note that "$\#_p a = \#_p b$ for any $p \in \mathbf{Atom}$" means that $[\![a]\!]$ and $[\![b]\!]$ are equal in the free <u>Abelian</u> group generated by **Atom**. Now the theorem is proved by a trivial modification of the proof of Theorem 1. ■

12

Remark. If we extend the language of the calculus L^* with the constant **1**, add the axiom $\Rightarrow \mathbf{1}$ and the rule

$$\frac{X\ ,\ Y \Rightarrow a}{X\ ,\ \mathbf{1}\ ,\ Y \Rightarrow a}\ (\mathbf{1} \Rightarrow)$$

we obtain the calculus L_1. In the similar way, LP_1 is obtained from L^*P. Theorem 1 and Theorem 2 hold also in calculi L_1 and LP_1 respectively, if we extend the definitions of $[\![\ ]\!]$ and $\#_p$ by

$$[\![\mathbf{1}]\!] \rightleftharpoons \Lambda \quad \text{and} \quad \#_p\mathbf{1} \rightleftharpoons 0.$$

4.3   Linear logics

In the full linear logic any two formulas a and b are trivially equivalent, because

$$a \Rightarrow a \oplus b \quad \text{and} \quad b \Rightarrow a \oplus b\ ,$$

where $a \oplus b$ stands for the non-linear (additive) disjunction, sometimes also denoted by $a \sqcup b$. The situation is different in the multiplicative fragment.

Theorem 3 *Let a and b be any two formulas of MCLL. Then $a \overset{MCLL}{\sim} b$ if and only if $\natural a = \natural b$ and $[\![a]\!] \overset{FG}{=} [\![b]\!]$.*

PROOF. The 'if' part is an easy consequence of soundness with respect to $\natural$ and balance (cf. Lemma 5 (ii) and Lemma 2 (ii)).

To prove the 'only if' part we introduce a new group G, generated by elements of **Atom** together with **0**, and satisfying the identity $\mathbf{0} \cdot c = c \cdot \mathbf{0}$ for any $c \in G$. Let Λ stand for the group unit of G. By $[\![\ ]\!]_G$ we denote the following mapping from the set of formulas of linear logic to the group G.

$$
\begin{aligned}
[\![\mathbf{1}]\!]_G &\rightleftharpoons \Lambda \\
[\![\mathbf{0}]\!]_G &\rightleftharpoons \mathbf{0} \\
[\![p]\!]_G &\rightleftharpoons p \\
[\![c^{\perp}]\!]_G &\rightleftharpoons [\![c]\!]_G^{-1} \cdot \mathbf{0} \\
[\![c\bullet d]\!]_G &\rightleftharpoons [\![c]\!]_G \cdot [\![d]\!]_G \\
[\![c{+}d]\!]_G &\rightleftharpoons [\![c]\!]_G \cdot [\![d]\!]_G \cdot \mathbf{0}^{-1} \\
[\![c\backslash d]\!]_G &\rightleftharpoons [\![c]\!]_G^{-1} \cdot [\![d]\!]_G \\
[\![c/d]\!]_G &\rightleftharpoons [\![c]\!]_G \cdot [\![d]\!]_G^{-1}
\end{aligned}
$$

Obviously, $[\![a]\!]_G \overset{G}{=} [\![b]\!]_G$ if and only if $\natural a = \natural b$ and $[\![a]\!] \overset{FG}{=} [\![b]\!]$.

Similarly to Theorem 1, we prove that if $[\![a]\!]_G \overset{G}{=} [\![b]\!]_G$ then $[a]_\sim = [b]_\sim$, where $[\ ]_\sim$ denotes now the equivalence class with respect to the relation $\overset{MCLL}{\sim}$.

To extend the mapping $[\![p]\!]_G \mapsto [p]_\sim$ to a homomorphism we need two facts in addition to what has already been proved in Theorem 1.

13

(1) $[\mathbf{0}]_\sim \cdot [c]_\sim = [c]_\sim \cdot [\mathbf{0}]_\sim$

   Evidently $MCLL \vdash c{\bullet}c^\perp \Rightarrow \mathbf{0}$ and $MCLL \vdash c^\perp{\bullet}c \Rightarrow \mathbf{0}$.

   Hence $\mathbf{0}{\bullet}c \overset{MCLL}{\sim} (c{\bullet}c^\perp){\bullet}c \overset{MCLL}{\sim} c{\bullet}(c^\perp{\bullet}c) \overset{MCLL}{\sim} c{\bullet}\mathbf{0}$.

(2) $[c\backslash c/c]_\sim \cdot [\mathbf{0}]_\sim = [c^\perp]_\sim$

   This follows from

$$(c\backslash c/c){\bullet}(c{\bullet}c^\perp) \overset{MCLL}{\sim} (((c\backslash c)/c){\bullet}c){\bullet}c^\perp \overset{MCLL}{\sim} (c\backslash c){\bullet}c^\perp \overset{MCLL}{\sim} c^\perp$$

∎

Theorem 4 *Let a and b be any two formulas of MLL. Then $a \overset{MLL}{\sim} b$ if and only if $\natural a = \natural b$ and $\#_p a = \#_p b$ for any $p \in$ Atom.*

PROOF. We note that $\natural a = \natural b$ and $\#_p a = \#_p b$ for any $p \in$ Atom if and only if $[a]_G$ and $[b]_G$ are equal in the free <u>Abelian</u> group. Hence the theorem is proved by a trivial modification of the proof of Theorem 3. ∎

Remark. If we leave out the units $\mathbf{1}$ and $\mathbf{0}$ we obtain the constant-free fragments of $MCLL$ and MLL. Theorems 3 and 4 hold also for these fragments.

4.4   Complete description of derivability invariants

Theorem 5 *Let Δ be an arbitrary set. Suppose that a mapping ϕ from formulas to Δ is a MLL-derivability invariant, i.e., $\phi(a) = \phi(b)$ whenever $MLL \vdash a \Rightarrow b$. Then $\phi(b)$ is actually a function of $\#_p b$ and $\natural b$.*

PROOF. We have to show that, if $\natural a = \natural b$ and $\#_p a = \#_p b$ for any $p \in$ Atom, then $\phi(a) = \phi(b)$. This is an immediate consequence of the 'if' part of Theorem 4 and the definition of equivalent formulas. ∎

Example 3 Let $\#_\bullet(b)$ denote the number of occurrences of the connective $\bullet$ in a formula b. Similarly, let $\#_+(b)$ count the occurrences of the connective $+$. It is easy to verify that $\phi(b) \rightleftharpoons \#_\bullet(b) - \#_+(b)$ is a derivability invariant for the calculus MLL.

According to Theorem 5, the function ϕ must be definable with the help of $\#_p$ and $\natural$. Really, straightforward induction shows that

$$\phi(b) = \sum_{p\in\textbf{Atom}} \#_p(b) + 2 \cdot \natural(b) - 1.$$

Remark. Similar complete sets of derivability invariants in other calculi are

1. $[\![\ ]\!]$ for L and L^*,

2. $\#_p$ for LP and L^*P,

3. $[\![\ ]\!]$ and $\natural$ for $MCLL$.

5   Linear length joins

In this section we prove that in the Lambek calculus with permutation and in the ordinary linear logic any pair of equivalent formulas $a \sim b$ has a linear length join c such that $a \Rightarrow c$ and $b \Rightarrow c$.

We define the length of a formula b (denoted by $\mid b \mid$) as the total number of occurrences of literals and constants $\mathbf{0}$ and $\mathbf{1}$.

5.1   Lambek calculus with permutation LP

First we introduce special counts for positive and negative occurrences of literals.

$$\#_p^+ p \;\rightleftharpoons\; 1$$
$$\#_p^+ q \;\rightleftharpoons\; 0, \quad \text{where } p \text{ and } q \text{ are distinct literals}$$
$$\#_p^- q \;\rightleftharpoons\; 0, \quad \text{where } p \text{ and } q \text{ are any literals (possibly coinciding)}$$
$$\#_p^+ (a \bullet b) \;\rightleftharpoons\; \#_p^+ a + \#_p^+ b$$
$$\#_p^- (a \bullet b) \;\rightleftharpoons\; \#_p^- a + \#_p^- b$$
$$\#_p^+ (a \to b) \;\rightleftharpoons\; \#_p^- a + \#_p^+ b$$
$$\#_p^- (a \to b) \;\rightleftharpoons\; \#_p^+ a + \#_p^- b$$

Evidently $\#_p a = \#_p^+ a - \#_p^- a$ for any type a and any literal p.

Let q be a fixed literal. Our aim is to show that any pair of equivalent types has a join of the form $(c_1 \bullet \ldots \bullet c_l \bullet q) \to q$, where, for any $i \leq l$, c_i is either $q \bullet (q \to p)$ or $p \bullet q \to q$ for some literal p.

Lemma 11 *Let a be any formula of LP containing only literals $p_1, \ldots, p_n$. Let $\#_{p_i}^+ a = k_i$ and $\#_{p_i}^- a = m_i$. Then*

(i) $LP \vdash (q \bullet (q \to p_1))^{k_1}, \ldots, (q \bullet (q \to p_n))^{k_n}, (p_1 \bullet q \to q)^{m_1}, \ldots, (p_n \bullet q \to q)^{m_n} \Rightarrow a,$

(ii) $LP \vdash a, (q \bullet (q \to p_1))^{m_1}, \ldots, (q \bullet (q \to p_n))^{m_n}, (p_1 \bullet q \to q)^{k_1}, \ldots, (p_n \bullet q \to q)^{k_n}, q \Rightarrow q.$

Here b^k denotes $\underbrace{b, \ldots, b}_{k \text{ times}}.$

PROOF. Induction on the construction of the formula a.
CASE 1: $a = p_i$
(i)

$$\cfrac{\cfrac{q \Rightarrow q \quad p_i \Rightarrow p_i}{q\,, \; q \to p_i \Rightarrow p_i}\;(\to \Rightarrow)}{q \bullet (q \to p_i) \Rightarrow p_i}\;(\bullet \Rightarrow)$$

15

(ii)

$$\dfrac{\dfrac{p_i \Rightarrow p_i \qquad q \Rightarrow q}{p_i \,,\ q \Rightarrow p_i\bullet q}\ (\Rightarrow \bullet) \qquad\qquad q \Rightarrow q}{\dfrac{p_i \,,\ q \,,\ p_i\bullet q \to q \Rightarrow q}{p_i \,,\ p_i\bullet q \to q \,,\ q \Rightarrow q}\ (\mathrm{P})}\ (\to \Rightarrow)$$

CASE 2: $a = b\bullet c$

By induction hypothesis there are LP-derivable sequents

$$X \Rightarrow b \quad,\quad b, Y, q \Rightarrow q \quad,\quad Z \Rightarrow c \quad,\quad c, W, q \Rightarrow q$$

of the special form given in the formulation of the lemma.

(i)

$$\dfrac{X \Rightarrow b. \quad Z \Rightarrow c}{X \,,\ Z \Rightarrow b\bullet c}\ (\Rightarrow \bullet)$$

(ii)

$$\dfrac{\dfrac{b \,,\ Y \,,\ q \Rightarrow q \quad c \,,\ W \,,\ q \Rightarrow q}{b \,,\ c \,,\ Y \,,\ W \,,\ q \Rightarrow q}\ (\mathrm{CUT})}{b\bullet c \,,\ Y \,,\ W \,,\ q \Rightarrow q}\ (\bullet \Rightarrow)$$

CASE 3: $a = b\to c$

Obviously there exists $i \le n$ such that $\#^{+}_{p_i} c > 0$. Therefore we may write the induction hypothesis as

$$X \Rightarrow b \quad,\quad b, Y, q \Rightarrow q \quad,\quad q\bullet(q \to p_i), Z \Rightarrow c \quad,\quad c, W, q \Rightarrow q.$$

(i)

$$\dfrac{\dfrac{\dfrac{b \,,\ Y \,,\ q \Rightarrow q \quad q \to p_i \Rightarrow q \to p_i}{b \,,\ Y \,,\ q \,,\ q \to p_i \Rightarrow q\bullet(q \to p_i)}\ (\Rightarrow \bullet) \qquad q\bullet(q \to p_i) \,,\ Z \Rightarrow c}{\dfrac{b \,,\ Y \,,\ q \,,\ q \to p_i \,,\ Z \Rightarrow c}{\dfrac{b \,,\ Y \,,\ q\bullet(q \to p_i) \,,\ Z \Rightarrow c}{Y \,,\ q\bullet(q \to p_i) \,,\ Z \Rightarrow b\to c}\ (\Rightarrow \to)}\ (\bullet \Rightarrow)}\ (\mathrm{CUT})}{}$$

(ii)

$$\dfrac{\dfrac{X \Rightarrow b \quad c \,,\ W \,,\ q \Rightarrow q}{\dfrac{X \,,\ b\to c \,,\ W \,,\ q \Rightarrow q}{b\to c \,,\ X \,,\ W \,,\ q \Rightarrow q}\ (\mathrm{P})}\ (\to \Rightarrow)}{}$$

∎

Lemma 12 *If $k' - k = m' - m \geq 0$ and*

$$LP \vdash X \ , \ (q \bullet (q \to p))^k \ , \ (p \bullet q \to q)^m \ \Rightarrow \ q$$

then

$$LP \vdash X \ , \ (q \bullet (q \to p))^{k'} \ , \ (p \bullet q \to q)^{m'} \ \Rightarrow \ q.$$

PROOF. Induction on $k' - k$.
For the induction step we verify that $LP \vdash q \bullet (q \to p) \ , \ p \bullet q \to q \ , \ X \Rightarrow q$
whenever $LP \vdash X \Rightarrow q$.

$$
\dfrac{
\dfrac{
X \Rightarrow q \qquad \dfrac{
\dfrac{
\dfrac{p \Rightarrow p \qquad q \Rightarrow q}{p \ , \ q \Rightarrow p \bullet q} (\Rightarrow \bullet) \qquad q \Rightarrow q
}{p \ , \ q \ , \ p \bullet q \to q \Rightarrow q} (\to \Rightarrow)
}{X \ , \ q \to p \ , \ q \ , \ p \bullet q \to q \Rightarrow q} (\to \Rightarrow)
}{
X \ , \ q \ , \ q \to p \ , \ p \bullet q \to q \Rightarrow q
} (\text{P})
}{
X \ , \ q \bullet (q \to p) \ , \ p \bullet q \to q \Rightarrow q
} (\bullet \Rightarrow)
$$

∎

Theorem 6 *If two types a and b are equivalent in LP then there is a type c such that*

$$LP \vdash a \Rightarrow c, \quad LP \vdash b \Rightarrow c \quad and \quad |c| \leq 3|a| + 3|b| + 2.$$

PROOF. We put

$$c = (q \bullet (q \to p_1))^{m'_1} \bullet \dots \bullet (q \bullet (q \to p_n))^{m'_n} \bullet (p_1 \bullet q \to q)^{k'_1} \bullet \dots \bullet (p_n \bullet q \to q)^{k'_n} \bullet q \to q,$$

where $k'_i = \max(\#^+_{p_i} a, \#^+_{p_i} b)$ and $m'_i = \max(\#^-_{p_i} a, \#^-_{p_i} b)$.
Evidently $\#_{p_i} a = \#_{p_i} b$ implies $\#^+_{p_i} b - \#^+_{p_i} a = \#^-_{p_i} b - \#^-_{p_i} a$. Hence

$$k'_i - \#^+_{p_i} a = m'_i - \#^-_{p_i} a \geq 0.$$

Now $LP \vdash a \Rightarrow c$ follows from Lemma 11 (ii) by applying Lemma 12 n times. Similarly $LP \vdash b \Rightarrow c$.
Since

$$\sum_{i=1}^{n} (\#^+_{p_i} a + \#^-_{p_i} a) = |a|,$$

we have

$$\sum_{i=1}^{n} (k'_i + m'_i) \leq |a| + |b| \quad \text{and thus} \quad |c| \leq 3|a| + 3|b| + 2.$$

∎

5.2 Linear logic MLL

Theorem 7 *For any pair of equivalent MLL-formulas b and c, there is a formula d such that*

$$MLL \vdash b \Rightarrow d, \quad MLL \vdash c \Rightarrow d, \quad and \quad |\, d\, | \le 3(|\, b\, | + |\, c\, |).$$

SKETCH OF THE PROOF.
Let k_i denote the number of positive occurrences of p_i in the formula b, and k_i' stand for the number of negative occurrences. By l and l' we denote the numbers of occurrences of $\mathbf{0}$ in b and c respectively (we assume that $\mathbf{0}$ and $\mathbf{1}$ have no negative occurrences). We put

$$d = \sum_i (\underbrace{p_i\bullet\mathbf{0}+\ldots+p_i\bullet\mathbf{0}}_{k_i+k_i' \text{ times}} + \underbrace{p_i{}^\perp\bullet\mathbf{0}+\ldots+p_i{}^\perp\bullet\mathbf{0}}_{m_i+m_i' \text{ times}}) + \underbrace{\mathbf{0}\bullet\mathbf{0}+\ldots+\mathbf{0}\bullet\mathbf{0}}_{l+l' \text{ times}} + \underbrace{\mathbf{1}+\ldots+\mathbf{1}}_{j \text{ times}},$$

where $j = 1 + \sum_i (m_i + m_i') + l + l' - \natural b$. Here j is selected so that $\natural d = \natural b$.

The necessary entailments $b \Rightarrow d$ and $c \Rightarrow d$ are verified by straightforward induction on the construction of formulas b and c. ∎

6 Product-free fragments of Lambek calculus

Let $\mathbf{Fm}(\backslash, /)$ stand for the set of all types in the language of two implications. In this section we show that a pair of types has a join in $\mathbf{Fm}(\backslash, /, \bullet)$ if and only if it has a join from $\mathbf{Fm}(\backslash, /)$. We also note that the diamond property for product-free fragment is different from the one in full the Lambek calculus.

6.1 Product-free joins

Theorem 8 (i) *Theorem 1 holds for $L(\backslash, /)$ and $L^*(\backslash, /)$.*

(ii) *Theorem 2 holds for $LP(\rightarrow)$ and $L^*P(\rightarrow)$.*

Here $L(\backslash, /)$ denotes the fragment of L without product. Similarly for $L^*(\backslash, /)$, $LP(\rightarrow)$ and $L^*P(\rightarrow)$.
PROOF. All the cases are similar. We give a proof for $L(\backslash, /)$ only. It suffices to prove that, if $a, b \in \mathbf{Fm}(\backslash, /)$ and $[\![a]\!] = [\![b]\!]$ in the free group, then in $\mathbf{Fm}(\backslash, /)$ there is a join for a and b. Using Theorem 1 we get a join $c \in \mathbf{Fm}(\backslash, /, \bullet)$. Lemma 13 gives a type $c' \in \mathbf{Fm}(\backslash, /)$ such that $L \vdash a \Rightarrow c'$ and $L \vdash b \Rightarrow c'$. Cut-elimination in the Lambek calculus involves that L is conservative over $L(\backslash, /)$. Hence, $a \sim b$ in $L(\backslash, /)$. ∎

Lemma 13 *Let $c \in \mathbf{Fm}(\backslash, /, \bullet)$. Then:*

(i) $\exists c' \in \mathbf{Fm}(\backslash, /)$ *such that* $L \vdash c \Rightarrow c'$,

(ii) $\exists X \subset \mathbf{Fm}(\backslash, /)$ *such that* $L \vdash X \Rightarrow c$.

18

PROOF. Induction on construction of type c.

CASE $c = p$: We take $c' = p$ and $X = p$.

CASE $c = a \cdot b$: Let $a \Rightarrow a'$, $b \Rightarrow b'$, $Y \Rightarrow a$, $Z \Rightarrow b$,
where $a', b' \in \mathbf{Fm}(\backslash, /)$ and $Y, Z \subset \mathbf{Fm}(\backslash, /)$.
We put $Z = X, Y$ and $c' = p/(b' \backslash (a' \backslash p))$ for some $p \in \mathbf{Atom}$.

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{
\cfrac{a \Rightarrow a' \quad p \Rightarrow p}{a \ , \ a' \backslash p \Rightarrow p} (\backslash \Rightarrow)
}{
b \Rightarrow b' \qquad a \ , \ a' \backslash p \Rightarrow p
}
\quad (\backslash \Rightarrow)
}{a \ , \ b \ , \ b' \backslash (a' \backslash p) \Rightarrow p} (\Rightarrow /)
}{a \ , \ b \Rightarrow p/(b' \backslash (a' \backslash p))}
}{a \cdot b \Rightarrow p/(b' \backslash (a' \backslash p))} (\bullet \Rightarrow)
$$

$$
\cfrac{Y \Rightarrow a \qquad Z \Rightarrow b}{Y, Z \Rightarrow a \cdot b} (\Rightarrow \bullet)
$$

CASE $c = a \backslash b$: Let $a \Rightarrow a'$, $b \Rightarrow b'$, $\hat{a}_1, \ldots, \hat{a}_n \Rightarrow a$, $\hat{b}_1, \ldots, \hat{b}_m \Rightarrow b$
for some $a', b', \hat{a}_i, \hat{b}_j \in \mathbf{Fm}(\backslash, /)$.
We put $c' = \hat{a}_n \backslash (\ldots \backslash (\hat{a}_1 \backslash b') \ldots)$ and $X = a' \backslash \hat{b}_1 \ , \ \hat{b}_2, \ldots, \hat{b}_m$.

$$
\cfrac{
\vdots
}{
\cfrac{\hat{a}_1, \ldots, \hat{a}_n \ , \ a \backslash b \Rightarrow b'}{a \backslash b \Rightarrow \hat{a}_n \backslash (\ldots \backslash (\hat{a}_1 \backslash b') \ldots)} (\Rightarrow \backslash)
}
$$

where the top is $\cfrac{\hat{a}_1, \ldots, \hat{a}_n \Rightarrow a \qquad b \Rightarrow b'}{\hat{a}_1, \ldots, \hat{a}_n \ , \ a \backslash b \Rightarrow b'} (\backslash \Rightarrow)$ then $(\Rightarrow \backslash)$

$$
\cfrac{
\cfrac{a \Rightarrow a' \qquad \hat{b}_1, \ldots, \hat{b}_m \Rightarrow b}{a \ , \ a' \backslash \hat{b}_1 \ , \ \hat{b}_2, \ldots, \hat{b}_m \Rightarrow b} (\backslash \Rightarrow)
}{a' \backslash \hat{b}_1 \ , \ \hat{b}_2, \ldots, \hat{b}_m \Rightarrow a \backslash b} (\Rightarrow \backslash)
$$

CASE $c = b/a$: Similar to the previous case. ■

6.2 Diamond property

Surprisingly, the product-free fragment of the Lambek calculus does not possess the diamond property (cf. Lemma 6). Instead of this a slightly different lemma holds.

Lemma 14 *Let $a_1, \ldots, a_n \in \mathbf{Fm}(\backslash, /)$. Then the following two assertions are equivalent.*

(i) *$\exists b \in \mathbf{Fm}(\backslash, /)$ such that $L(\backslash, /) \vdash a_i \Rightarrow b$ for any $i \leq n$,*

(ii) *there is a sequence X consisting of n product-free types such that*

$$L(\backslash, /) \vdash X \Rightarrow a_i \text{ for any } i \leq n.$$

Remark. For any n, there exist equivalent types $a_1, \ldots, a_n \in \mathbf{Fm}(\backslash, /)$ such that the sequence X in Lemma 14 cannot contain less than n types. Take for example $n = 2$, $a_1 = p/(p \backslash (p \backslash p))$ and $a_2 = q/(p \backslash (p \backslash q))$.
Obviously $L(\backslash, /) \vdash p, p \Rightarrow a_1$ and $L(\backslash, /) \vdash p, p \Rightarrow a_2$. Nevertheless, there is no product-free type d such that $L(\backslash, /) \vdash d \Rightarrow a_1$ and $L(\backslash, /) \vdash d \Rightarrow a_2$.

Acknowledgements

References

[1] J. van Benthem. *Language in Action*. North-Holland, Amsterdam, 1991.

[2] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[3] M.I. Kanovich and M. Pentus. *Strong Normalization for the Equivalences in Lambek Calculus and Linear Logic*. Preprint No.3 of the Department of Math. Logic, Steklov Math. Institute, Series Logic and Computer Science, Moscow, 1992.

[4] J. Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65(3):154–170, 1958.

[5] M. Pentus. *Equivalent Types in Lambek Calculus and Linear Logic*. Preprint No.2 of the Department of Math. Logic, Steklov Math. Institute, Series Logic and Computer Science, Moscow, 1992.

[6] D.N. Yetter. Quantales and noncommutative linear logic. *Journal of Symbolic Logic*, 55(1):41–64, 1990.

The ILLC Prepublication Series

CT-91-09 Ming Li, Paul M.B. Vitányi   Combinatorial Properties of Finite Sequences with high Kolmogorov Complexity
CT-91-10 John Tromp, Paul Vitányi   A Randomized Algorithm for Two-Process Wait-Free Test-and-Set
CT-91-11 Lane A. Hemachandra, Edith Spaan   Quasi-Injective Reductions
CT-91-12 Krzysztof R. Apt, Dino Pedreschi   Reasoning about Termination of Prolog Programs

Computational Linguistics
CL-91-01 J.C. Scholtes   Kohonen Feature Maps in Natural Language Processing
CL-91-02 J.C. Scholtes   Neural Nets and their Relevance for Information Retrieval
CL-91-03 Hub Prüst, Remko Scha, Martin van den Berg   A Formal Discourse Grammar tackling Verb Phrase Anaphora

Other Prepublications
X-91-01 Alexander Chagrov, Michael Zakharyaschev   The Disjunction Property of Intermediate Propositional Logics
X-91-02 Alexander Chagrov, Michael Zakharyaschev   On the Undecidability of the Disjunction Property of Intermediate Propositional Logics
X-91-03 V. Yu. Shavrukov   Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic
X-91-04 K.N. Ignatiev   Partial Conservativity and Modal Logics
X-91-05 Johan van Benthem   Temporal Logic
X-91-06   Annual Report 1990
X-91-07 A.S. Troelstra   Lectures on Linear Logic, Errata and Supplement
X-91-08 Giorgie Dzhaparidze   Logic of Tolerance
X-91-09 L.D. Beklemishev   On Bimodal Provability Logics for Π_1-axiomatized Extensions of Arithmetical Theories
X-91-10 Michiel van Lambalgen   Independence, Randomness and the Axiom of Choice
X-91-11 Michael Zakharyaschev   Canonical Formulas for K4. Part I: Basic Results
X-91-12 Herman Hendriks   Flexibele Categoriale Syntaxis en Semantiek: de proefschriften van Frans Zwarts en Michael Moortgat
X-91-13 Max I. Kanovich   The Multiplicative Fragment of Linear Logic is NP-Complete
X-91-14 Max I. Kanovich   The Horn Fragment of Linear Logic is NP-Complete
X-91-15 V. Yu. Shavrukov   Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic, revised version
X-91-16 V.G. Kanovei   Undecidable Hypotheses in Edward Nelson's Internal Set Theory
X-91-17 Michiel van Lambalgen   Independence, Randomness and the Axiom of Choice, Revised Version
X-91-18 Giovanna Cepparello   New Semantics for Predicate Modal Logic: an Analysis from a standard point of view
X-91-19 Papers presented at the Provability Interpretability Arithmetic Conference, 24-31 Aug. 1991, Dept. of Phil., Utrecht University

1992
Logic, Semantics and Philosophy of Langauge
LP-92-01 Víctor Sánchez Valencia   Lambek Grammar: an Information-based Categorial Grammar
LP-92-02 Patrick Blackburn   Modal Logic and Attribute Value Structures
LP-92-03 Szabolcs Mikulás   The Completeness of the Lambek Calculus with respect to Relational Semantics
LP-92-04 Paul Dekker   An Update Semantics for Dynamic Predicate Logic
LP-92-05 David I. Beaver   The Kinematics of Presupposition
LP-92-06 Patrick Blackburn, Edith Spaan   A Modal Perspective on the Computational Complexity of Attribute Value Grammar
LP-92-07 Jeroen Groenendijk, Martin Stokhof   A Note on Interrogatives and Adverbs of Quantification
LP-92-08 Maarten de Rijke   A System of Dynamic Modal Logic
LP-92-09 Johan van Benthem   Quantifiers in the world of Types
LP-92-10 Maarten de Rijke   Meeting Some Neighbours (a dynamic modal logic meets theories of change and knowledge representation)
LP-92-11 Johan van Benthem   A note on Dynamic Arrow Logic
LP-92-12 Heinrich Wansing   Sequent Caluli for Normal Modal Propositional Logics
LP-92-13 Dag Westerståhl   Iterated Quantifiers
LP-92-14 Jeroen Groenendijk, Martin Stokhof   Interrogatives and Adverbs of Quantification

Mathematical Logic and Foundations
ML-92-01 A.S. Troelstra   Comparing the theory of Representations and Constructive Mathematics
ML-92-02 Dmitrij P. Skvortsov, Valentin B. Shehtman   Maximal Kripke-type Semantics for Modal and Superintuitionistic Predicate Logics
ML-92-03 Zoran Marković   On the Structure of Kripke Models of Heyting Arithmetic
ML-92-04 Dimiter Vakarelov   A Modal Theory of Arrows, Arrow Logics I
ML-92-05 Domenico Zambella   Shavrukov's Theorem on the Subalgebras of Diagonalizable Algebras for Theories containing $I\Delta_0 + EXP$
ML-92-06 D.M. Gabbay, Valentin B. Shehtman   Undecidability of Modal and Intermediate First-Order Logics with Two Individual Variables
ML-92-07 Harold Schellinx   How to Broaden your Horizon
ML-92-08 Raymond Hoofman   Information Systems as Coalgebras
ML-92-09 A.S. Troelstra   Realizability
ML-92-10 V.Yu. Shavrukov   A Smart Child of Peano's

Compution and Complexity Theory
CT-92-01 Erik de Haas, Peter van Emde Boas   Object Oriented Application Flow Graphs and their Semantics
CT-92-02 Karen L. Kwast, Sieger van Denneheuvel   Weak Equivalence: Theory and Applications
CT-92-03 Krzysztof R. Apt, Kees Doets   A new Definition of SLDNF-resolution

Other Prepublications
X-92-01 Heinrich Wansing   The Logic of Information Structures
X-92-02 Konstantin N. Ignatiev   The Closed Fragment of Dzhaparidze's Polymodal Logic and the Logic of Σ_1 conservativity
X-92-03 Willem Groeneveld   Dynamic Semantics and Circular Propositions, revised version
X-92-04 Johan van Benthem   Modeling the Kinematics of Meaning
X-92-05 Erik de Haas, Peter van Emde Boas   Object Oriented Application Flow Graphs and their Semantics, revised version

1993
Mathematical Logic and Foundations
ML-93-01 Maciej Kandulski   Commutative Lambek Categorial Grammars
ML-93-02 Johan van Benthem, Natasha Alechina   Modal Quantification over Structured Domains
ML-93-03 Mati Pentus   The Conjoinablity Relation in Lambek Calculus and Linear Logic

Compution and Complexity Theory
CT-93-01 Marianne Kalsbeek   The Vanilla Meta-Interpreter for Definite Logic Programs and Ambivalent Syntax

Other Prepublications
X-93-01 Paul Dekker   Existential Disclosure, revised version