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CATEGORIAL GENERALIZATION OF ALGEBRAIC RECURSION THEORY

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CATEGORIAL GENERALIZATION OF ALGEBRAIC RECURSION THEORY*

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0.INTRODUCTORY REMARKS

Recently there have been developed several attempts to formulate a natural generalization of recursion theory in a categorial framework. We present here another such generalization which seems to be interesting also from purely categorial point of view and arises in a natural way from a well established and point-free algebraic generalization of recursion theory - so called 'algebraic recursion theory' ([1],[7]) .

0.1. Algebraic recursion theory can be explained as an algebraic theory of least fixed points as follows: Suppose we are given a partially ordered universal algebra A, i.e. a poset A with several monotonic on each argument operations in it. From basic operations in A we can construct new monotonic operations by means of explicit expressions; let us call last operations 'explicitly definable in A'. On other hand we can construct still other monotonic operations by means of least fixed points or in other words by means of some (abstract) inductive definitions. Namely, let $f_i(x_1,\dots,x_n,y_1,\dots,y_m)$ ($i = 1,\dots,m$) be explicitly definable n+m - ary operations in A and suppose the system of inequalities

$$f_i(x_1,\dots,x_n,y_1,\dots,y_m) \le y_i \qquad , i = 1,\dots,m \tag{1}$$

has least solution $(g_1(x_1,\dots,x_n),\dots,g_m(x_1,\dots,x_n)) \in A^m$ for all $(x_1,\dots,x_n) \in A^n$. Then we have new operations g_i ; we shall call operations arising in this way inductively definable in A. The algebra A will be called inductively complete if all systems of the form (1) have least solutions and all inductively definable operations in A are explicitly definable. Nontrivial

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examples of inductively complete algebras arise in recursion theory and computer science. First example of such algebras of abstract kind - so called iterative combinatory spaces - was constructed and studied by Skordev [9] and a nice generalization of recursion theory arose from this. To Skordev is due also the general question of existence of other interesting kinds of inductively complete algebras (see [7] where they are called 'fixed-point complete algebras') and the proposition of their systematical study. Later on other kinds of such algebras were introduced by other authors ([1],[9],[10]).

In general, all known nontrivial inductively complete algebras arise as inductive completion of a suitable poalgebra A in which all systems of the form (1) have least solutions (the last is usually easily verified by some continuity arguments). Here by an inductive completion of a poalgebra A we mean an inductively complete enrichment of A with a set B of inductively definable in A operations. The problem of finding of a (simple) inductive completion of a given poalgebra A, or more precisely: to find a set B of inductively definable operations in A s.t. all systems of the form (1) have least solutions with components explicitly expressible by means of parameters $x_1,...,x_n$, basic operations in A and those from B , is what we shall call below 'the problem of inductive completion of A' ; it is a slight modification of the original Skordev's formulation of the problem.

0.2. The problem of inductive completion of poalgebras has an obvious generalization for categories: instead of a poset A we take a category C and instead of basic operations we take a set of (covariant) multiendofunctors in C (that is bi-,tri- and so on endofunctors as well as constant objects, considered as functors of zero arguments, and usual endofunctors of one argument) which we call basic endofunctors. Instead of systems of the form (1) we consider functors

$$F: C^{n+m} \longrightarrow C^{m} \qquad (2)$$

which are explicitly expressible by basic multiendofunctors in an obvious sense. Taking <u>least</u> <u>fixed</u> <u>points</u> (in the sense of Lambek [3]; see section 1.3 below) of such functors we get new multiendofunctors in C and call them inductively definable. The problem is to find a simple set B of inductively definable multiendofunctors, s.t. for every functor F of the form (2): (a) the least fixed point of F exists (the existence part of the problem), and (b) is explicitly expressible by means of basic multiendofunctors and those from B (the expressibility part of the problem).

0.3. In present paper we are going to solve the last problem for a special kind of enriched categories which were called DM-categories in [6]. DM-categories are the categorial analog of operative spaces of Ivanov [1] - one of the most important algebraic systems for which algebraic recursion theory is developed. Our choice of this system is motivated rather by a chance - it seemed at the beginning of present investigations that operative spaces will be the most simple case to begin with. There are no reasons to expect that the categorial generalization of algebraic recursion theory in other algebraic systems like combinatory spaces [7] or cartesian linear combinatory algebras [11] does not hold. However, strictly speaking, the problem is open at present for other kinds of enriched categories.

0.4. The present paper is an improved version of our first publication on DM-categories [6]. The chief improvement is in replacement of the notion of <u>iteratively closed DM-category</u> by that of <u>iterative DM-category</u>. The last notion is more general and helps to simplify some examples, (the example in section 4.3 below). Moreover, it is a generalization of a possible version of the concept of <u>iterativity</u> for operative spaces [1], while the previous notion of iteratively closed DM-category was not. But the last notion was helpful in euristical sense: it helped to discover the right generalization of the concept of iterativity. We should mention that the version of the concept of iterativity as generalized by the notion of iterative DM-category is different from the original version of Ivanov [1] but more convenient for the categorial case. That is because of the existence part of the problem of inductive completion for categories, which is rather trivial in the usual case of posets. A generalization of Ivanov's concept of iterativity for categories would require the existence part of the problem to be solved independently but this is not the case with the notion of iterative DM-category in the present paper.

0.5. An important question to be answered with respect to the theory of DM-categories as described above is about the scope of the theory: what is the variety of models of it, i.e. iterative DM-categories. Various models of the theory of iterative operative spaces, which are up to secondary details mentioned above in 0.4 a special case of iterative DM-categories, were studied before (see [1],[7]) so the question is rather following one: what we can expect from properly categorial (not degenerated to preorders) models of the theory of DM-categories. Such question is to be answered by examples; but for detailed exposition examples require, as it seems, separate papers. In present paper we give three examples in a brief exposition, leaving straightforward constructions and most of the proofs to the reader. The first of those examples is of rather general character; but it seems, on the ground of the analogy with the usual operative spaces and theirs connections with other generalizations of recursion theory, that some examples of that kind could be interpreted as proper categorial generalizations of some well known generalizations of the usual recursion theory like that of Moschovakis. The other two examples in the present paper suggest connections with proving correctness of programs.

1.DEFINITIONS.

1.1. DM-categories.

A DM-category is a category $\mathcal{F}$ with two bifunctors $\mathbb{M}:\mathcal{F}^2 \longrightarrow \mathcal{F}$ and $\mathbb{D}:\mathcal{F}^2 \longrightarrow \mathcal{F}$, three objects I,L,R, and six natural isomorphisms $\underline{\alpha},\underline{\lambda},\underline{\rho},\underline{l},\underline{r},\underline{i}$ satisfying conditions (DM1) – (DM8) below. We shall call $\mathbb{M}$ 'multiplication functor' and we shall write xy for $\mathbb{M}(x,y)$ where x and y are objects or arrows in $\mathcal{F}$. Similarly, we shall call $\mathbb{D}$ 'cartesian functor', and we shall write (x,y) for $\mathbb{D}(x,y)$. Composition of arrows f,g in $\mathcal{F}$ will be denoted by $f \circ g$. Conditions defining a DM-category are following ones:

(DM1) $\underline{\alpha}$ is an isomorphism $\underline{\alpha}(\varphi,\psi,\chi)$: $(\varphi\psi)\chi \cong \varphi(\psi\chi)$ natural in φ,ψ,χ ;

(DM2) $\underline{\lambda}$ is an isomorphism $\underline{\lambda}(\varphi)$: $I\varphi \cong \varphi$ natural in φ ;

(DM3) $\underline{\rho}$ is an isomorphism $\underline{\rho}(\varphi)$: $\varphi I \cong \varphi$ natural in φ ;

(DM4) $\underline{l}$ is an isomorphism $\underline{l}(\varphi,\psi)$: $(\varphi,\psi)L \cong \varphi$ natural in φ,ψ ;

(DM5) $\underline{r}$ is an isomorphism $\underline{r}(\varphi,\psi)$: $(\varphi,\psi)R \cong \psi$ natural in φ,ψ ;

(DM6) $\underline{i}$ is an isomorphism $\underline{i}(\varphi,\psi,\chi)$: $\varphi(\psi,\chi) \cong (\varphi\psi,\varphi\chi)$ natural in φ,ψ,χ ;

(DM7) for all $\varphi,\psi,\chi,\vartheta \in \mathcal{F}$ we have:

$$\underline{\alpha}(\varphi,\psi,\chi\vartheta)\circ\underline{\alpha}(\varphi\psi,\chi,\vartheta) = (1_{\varphi}\underline{\alpha}(\psi,\chi,\vartheta))\circ\underline{\alpha}(\varphi,\psi\chi,\vartheta)\circ(\underline{\alpha}(\varphi,\psi,\chi)1_{\vartheta}) \quad ;$$

(DM8) for all $\varphi,\psi,\chi,\vartheta \in \mathcal{F}$ we have:

$$\underline{i}(\varphi\psi,\chi,\vartheta)\circ\bar{\alpha}(\varphi,\psi,(\chi,\vartheta)) = (\bar{\alpha}(\varphi,\psi,\chi),\bar{\alpha}(\varphi,\psi,\vartheta))\circ\underline{i}(\varphi,\psi\chi,\psi\vartheta)\circ(1_{\varphi}\underline{i}(\psi,\chi,\vartheta)) ,$$

where $\bar{\alpha}$ is $\underline{\alpha}^{-1}$.

Condition (DM7) is the pentagonal diagram in the definition of a monoidal category (see [2]); we shall call (DM7) and (DM8) 'coherence axioms'.

For posets $\mathcal{F}$ the notion of DM-category coincides with that of operative space [1]. Properly categorial examples of DM-categories will appear below in section 4.

1.2. Some notational conventions.

By $\mathcal{F}$ in this paper we shall denote always a DM-category; $\varphi,\psi,\chi,\xi,\eta$ etc. will be objects, and f,g,h,x,y etc. – arrows in $\mathcal{F}$. In expressions involving arrows we shall usually write φ for 1_{φ} , so if $f:\varphi\longrightarrow\psi$, then $f\varphi = f = \psi f$, and since $\mathbb{M}$ is a functor we have

$$(\varphi' g)\circ(f\psi) = fg = (f\psi')\circ(\varphi g) \qquad (1)$$

for all $f \in \mathcal{F}(\varphi,\varphi')$ and $g \in \mathcal{F}(\psi,\psi')$. We shall usually omit brackets in expressions like (1) , so in this sense multiplication is treated as binding stronger then composition $\circ$. An expression constructed by means of $\mathbb{D}$, $\mathbb{M}$ and objects of $\mathcal{F}$ defines a functor for both objects and arrows uniformly, the object constants φ being interpreted as 1_{φ} , so in the sequel we shall write such definitions for objects only. We shall write $\bar{\alpha},\bar{\lambda},\bar{\rho},\bar{l},\bar{r},\bar{i}$ for $\underline{\alpha}^{-1},\underline{\lambda}^{-1},\underline{\rho}^{-1},\underline{l}^{-1}$, $\underline{r}^{-1},\underline{i}^{-1}$ respectively and we shall usually omit expressions in brackets after $\underline{\alpha},\underline{\lambda}$ etc. , so conditions (DM1) – (DM8) can be written shortly as follows:

$$\underline{\alpha}\circ(fg)h = f(gh)\circ\underline{\alpha} \qquad (2)$$

$$\underline{\lambda}\circ If = f\circ\underline{\lambda} \qquad (3)$$

$$\underline{\rho}\circ fI = f\circ\underline{\rho} \qquad (4)$$

$$\underline{l}\circ(f,g)L = f\circ\underline{l} \tag{5}$$

$$\underline{r}\circ(f,g)R = g\circ\underline{r} \tag{6}$$

$$\underline{i}\circ f(g,h) = (fg,fh)\circ\underline{i} \tag{7}$$

$$\underline{\alpha}\circ\underline{\alpha} = \varphi\underline{\alpha}\circ\underline{\alpha}\circ\underline{\alpha}\vartheta \tag{8}$$

$$\underline{i}\circ\bar{\alpha} = (\bar{\alpha},\bar{\alpha})\circ\underline{i}\circ\varphi\underline{i} \tag{9}$$

where $f \in \mathcal{F}(\varphi,\varphi')$, $g \in \mathcal{F}(\psi,\psi')$, $h \in \mathcal{F}(\chi,\chi')$. Define

$$(X_0,\ldots,X_{n-1}) = (X_0,(X_1,\ldots(X_{n-2},X_{n-1})\ldots)) ,$$

where $X_0,\ldots,X_{n-1}$ are objects in $\mathcal{F}$ or arrows as well, and for $n = 0$ let $(X_0,\ldots,X_{n-1}) = I$ (respectively $(X_0,\ldots,X_{n-1}) = 1_I$), and for $n = 1$ let $(X_0,\ldots,X_{n-1}) = X_0$.

1.3. Least fixed points.

Let $\mathcal{C}$ be a category and let $F: \mathcal{C} \longrightarrow \mathcal{C}$ be an endofunctor. By $(F \Rightarrow \mathcal{C})$ we shall denote the category of pairs $(X;x)$ s.t. $X \in \mathcal{C}$ and $x \in \mathcal{C}(F(X),X)$; arrows $f: (X;x) \longrightarrow (Y;y)$ in $(F \Rightarrow \mathcal{C})$ are the arrows $f: X \longrightarrow Y$ in $\mathcal{C}$ s.t. $f\circ x = y\circ F(f)$. Then <u>least</u> <u>fixed</u> <u>point</u> (l.f.p.) of F is (according to [3]) an initial object $(M;m)$ of $(F \Rightarrow \mathcal{C})$. Following elementary properties of l.f.p. partially appear in [3]; we shall use them below sometimes without a reference:

(i) Suppose an endofunctor $F(A)$ in $\mathcal{C}$ depends on a parameter $A \in \mathcal{A}$, where $\mathcal{A}$ is a category, i.e. F is a functor from $\mathcal{A}\times\mathcal{C}$ to $\mathcal{C}$, and $F(A): \mathcal{C} \longrightarrow \mathcal{C}$ is the functor defined by $F(A)(X) = F(A,X)$ for $X \in \mathcal{C}$ and $F(A)(f) = F(1_A,f)$ for $f \in \mathcal{C}(X,Y)$. Let $(M(A);m(A))$ be l.f.p. of $F(A)$ for all $A \in \mathcal{A}$. Then M is a functor from $\mathcal{A}$ to $\mathcal{C}$, where $M(a)$ for $a \in \mathcal{A}(A,B)$ is determined uniquely by

$$M(a)\circ m(A) = m(B)\circ F(a,M(a)) ,$$

since $(M(B);m(B)\circ F(a,M(B))) \in (F(A) \Rightarrow \mathcal{C})$. Moreover

$$m(A)\colon F(A,M(A)) \longrightarrow M(A)$$

is an isomorphism natural in A .

(ii) If $(M;m)$ and $(N;n)$ are two l.f.p. of $F\colon \mathcal{C} \longrightarrow \mathcal{C}$, then there is an isomorphism $(M;m) \cong (N;n)$; if the functor F depends on a parameter $A \in \mathcal{A}$, then the last isomorphism is natural in A .

(iii) If $\underline{n}\colon G \cong F$ is a natural isomorphism between two endofunctors G and F in $\mathcal{A}$, then $(M;m)$ is l.f.p. of F iff $(M;m\circ\underline{n}(M))$ is l.f.p. of G .

Now let $\mathcal{D}$ be a category and let $P\colon \mathcal{D} \longrightarrow \mathcal{C}$ be a functor. Then an object $(M;m)$ of $(F \Rightarrow \mathcal{C})$ for a given functor $F\colon \mathcal{C} \longrightarrow \mathcal{C}$ will be called <u>local least fixed point</u> (shortly l.l.f.p.) <u>of</u> F <u>in</u> $(\mathcal{D},P)$ iff for any endofunctor $G\colon \mathcal{D} \longrightarrow \mathcal{D}$ s.t. $P\circ G = F\circ P$, there is unique object $(\tilde{M};\tilde{m}) \in (G \Rightarrow \mathcal{D})$, s.t. $P(\tilde{M}) = M$ and $P(\tilde{m}) = m$.

We shall use the last notion only in such cases when $\mathcal{D}$ is a comma category $(H{\downarrow}Z)$ where $H\colon \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor and $Z \in \mathcal{C}'$, and P is the usual forgetful functor from $(H{\downarrow}Z)$ to $\mathcal{C}$. In those cases it is useful to have a characterization of l.l.f.p.. in $(\mathcal{D},P)$ like that in Lemma 1 below. Let us recall that the category $(H{\downarrow}Z)$ consists of pairs $(X,\underline{x})$ where $\underline{x}\colon H(X) \longrightarrow Z$ is an arrow in $\mathcal{C}'$ and arrows $f\colon (X,\underline{x}) \longrightarrow (Y,\underline{y})$ in $(H{\downarrow}Z)$ are the arrows $f\colon X \longrightarrow Y$ in $\mathcal{C}$ for which $\underline{x} = \underline{y}\circ H(f)$.

Lemma 1. *Let* F *be an endofunctor in* $\mathcal{C}$ *and suppose that in notations above we have a mapping assigning to each object* $(X,\underline{x}) \in (H{\downarrow}Z)$ *an arrow*

$$\underline{x}^G(X)\colon H(F(X)) \longrightarrow Z \qquad .$$

Then the equalities $G(X,\underline{x}) = (F(X),\underline{x}^G(X))$ *and* $G(f) = F(f)$ *for an arrow* $f\colon (X,\underline{x}) \longrightarrow (Y,\underline{y})$ *in* $(H{\downarrow}Z)$ *define an endofunctor in* $(H{\downarrow}Z)$ *iff for every such arrow we have*

$$\vdash\!\!\!\vdash \underline{x}^G(X) = \underline{y}^G(Y)\circ H(F(f)) \tag{1}$$

Moreover $P\circ G = F\circ P$ *and every endofunctor* $G\colon (H{\downarrow}Z) \longrightarrow (H{\downarrow}Z)$ *s.t.* $P\circ G = F\circ P$ *is of that kind. An object* $(M;m) \in (F \Rightarrow \mathcal{C})$ *is l.l.f.p. of* F *in* $((H{\downarrow}Z),P)$

iff for every mapping $(X,\underline{x}) \mapsto \underline{x}^G(X)$ *satisfying* (1) *there is unique arrow* $u\colon H(M) \longrightarrow Z$ *s.t.*

$$u \circ H(m) = u^G(M) .$$

Proof. Straightforward.□

In notations of the last Lemma, we shall use to write $\underline{x}^G$ for $\underline{x}^G(X)$ below.

Lemma 2. *If* $H\colon \mathscr{C} \longrightarrow \mathscr{C}$ *is the identity functor, then* $(M;m) \in (F \Rightarrow \mathscr{C})$ *is l.f.p. of* $F\colon \mathscr{C} \longrightarrow \mathscr{C}$ *if* $(M;m)$ *is l.l.f.p. of* F *in* $((H{\downarrow}Z),P)$ *for all* $Z \in \mathscr{C}$ *.*

Proof. Easy from definitions: let $(M;m)$ be l.l.f.p. of F in $((H{\downarrow}Z),P)$ for all $Z \in \mathscr{C}$ and let $f\colon F(Z) \longrightarrow Z$ be an arrow in $\mathscr{C}$; define an endofunctor $G\colon (H{\downarrow}Z) \longrightarrow (H{\downarrow}Z)$ by $G(X,\underline{x}) = (F(X), f \circ F(\underline{x}))$ for objects and $G(f) = F(f)$ for arrows; using Lemma 1 we see that G is an endofunctor for which $P \circ G = F \circ P$, and by Lemma 1 again it follows that there is unique $u\colon M \longrightarrow Z$ s.t. $u \circ m = f \circ F(m)$ i.e. $(M;m)$ is l.f.p. of F .□ The reverse of Lemma 2 also holds and is included in Lemma 4 below.

Lemma 3. *If two functors* $H\colon \mathscr{C} \longrightarrow \mathscr{C}'$ *and* $H'\colon \mathscr{C} \longrightarrow \mathscr{C}'$ *are naturally isomorphic then for all* $Z \in \mathscr{C}'$ *:* $(M;m)$ *is l.l.f.p. of* F *in* $((H{\downarrow}Z),P)$ *iff* $(M;m)$ *is l.l.f.p. of* F *in* $((H'{\downarrow}Z),P)$.

Proof. Left to the reader.□

Lemma 4. *Let* $(M;m)$ *be l.f.p. of a functor* $F\colon \mathscr{C} \longrightarrow \mathscr{C}$ *and a functor* $H\colon \mathscr{C} \longrightarrow \mathscr{C}'$ *has a right adjoint* $H^*\colon \mathscr{C}' \longrightarrow \mathscr{C}$ *. Then* $(M;m)$ *is l.l.f.p. of* F *in* $((H{\downarrow}Z),P)$ *for all* $Z \in \mathscr{C}'$ *.*

Proof. Let $G\colon (H{\downarrow}Z) \longrightarrow (H{\downarrow}Z)$ be an endofunctor s.t. $P \circ G = F \circ P$. Then by Lemma 1 $G(X,\underline{x}) = (F(X),\underline{x}^G)$ and (1) holds. Let $Z^* = H^*(Z)$. Then there is an universal arrow $\underline{z}\colon H(Z^*) \longrightarrow Z$, i.e. $(Z^*;\underline{z})$ is a terminal object of $(H{\downarrow}Z)$. From $\underline{z}^G\colon H(F(Z^*)) \longrightarrow Z$ it follows that there is unique arrow $f\colon F(Z^*) \longrightarrow Z^*$, s.t. $\underline{z}^G = \underline{z} \circ H(f)$. Since $(M;m)$ is l.f.p. of F , there is unique $w\colon M \longrightarrow Z^*$ s.t. $w \circ m = f \circ F(w)$. Then defining $u = \underline{z} \circ H(w)\colon H(M) \longrightarrow Z$ we have:

$$u^G = \underline{z}^G \circ H(F(w)) = \underline{z} \circ H(f) \circ H(F(w)) = \underline{z} \circ H(w) \circ H(m) = u \circ H(m) \quad .$$

Conversely - suppose that $v\colon H(M) \longrightarrow Z$ and $v^G = v \circ H(m)$. Since the arrow $\underline{z}$ is universal, there is unique $g\colon M \longrightarrow Z^*$, s.t. $v = \underline{z} \circ H(g)$. We shall show that $g = w$ whence it will follow that $v = u$ and by Lemma 1 the proof will be completed. Since $(M;m)$ is l.f.p., m is an isomorphism. Let $g' = f \circ F(g) \circ m^{-1}$. Then

$$\underline{z} \circ H(g') = \underline{z} \circ H(f) \circ H(F(g)) \circ H(m^{-1}) = \underline{z}^G \circ H(F(g)) \circ H(m^{-1}) = (\underline{z} \circ H(g))^G \circ H(m^{-1})$$
$$= v^G \circ H(m^{-1}) = v \circ H(m) \circ H(m^{-1}) = v \quad .$$

Thence by the uniquity of g we have $g = g'$ i.e. $g \circ m = f \circ F(g)$ and by the uniquity of the arrow w we have $w = g$.□

1.4. Iteration and iterative DM-categories.

Let $\mathcal{F}$ be a DM-category. A <u>normal</u> <u>functor</u> (for $\mathcal{F}$) will be called a functor $H\colon \mathcal{F} \longrightarrow \mathcal{F}^N$ of the form $H(\xi) = \lambda i \in N.\varphi(\xi\nu_i)$, where N is an arbitrary finite or countable set, and $\nu_i \in \mathcal{F}_0$ for all $i \in N$, where $\mathcal{F}_0$ is the set of all objects of $\mathcal{F}$, produced from $\{I,L,R\}$ by means of the multiplication functor. Every pair of the form $((H{\downarrow}\tilde{\psi}),P)$, where $H\colon \mathcal{F} \longrightarrow \mathcal{F}^N$ is a normal functor, $\tilde{\psi}$ is an object of $\mathcal{F}^N$ and $P\colon (H{\downarrow}\tilde{\psi}) \longrightarrow \mathcal{F}$ is the usual forgetful functor, will be called a <u>normal</u> <u>projection</u> over $\mathcal{F}$. A <u>standart</u> <u>endofunctor</u> will be called any endofunctor $\Gamma\colon \mathcal{F} \longrightarrow \mathcal{F}$ of the form $\Gamma(\xi) = (I,\xi)\varphi$ and the object φ in this case will be called parameter of Γ . If Γ is a standard endofunctor with parameter φ then an object $(\mu;m) \in (\Gamma \Rightarrow \mathcal{F})$ i.e. an arrow $m\colon \Gamma(\mu) \longrightarrow \mu$ will be called <u>iteration</u> of φ , iff $(\mu;m)$ is l.l.f.p.. of Γ in every normal projection over $\mathcal{F}$. If $(\mu;m)$ is iteration of φ then by Lemmas 1.3.2 and 1.3.3 it is l.f.p. of the standard endofunctor with parameter φ and therefore it is unique up to isomorphism in $(\Gamma \Rightarrow \mathcal{F})$. The category $\mathcal{F}$ will be called <u>iterative</u> iff for every object φ of $\mathcal{F}$ there is an iteration of φ . If $\mathcal{F}$ is iterative, then by 1.3.(i) and 1.3.(ii) there is an endofunctor $\mathbb{I}$ in $\mathcal{F}$ s.t. for every $\varphi \in \mathcal{F}$ the pair $(\mathbb{I}(\varphi);\mathbf{i}(\varphi))$ for suitable arrow $\mathbf{i}(\varphi)$ is an iteration of φ , and the functor $\mathbb{I}$ is unique up to natural isomorphism. We shall fix $\mathbb{I}$ to denote such an endofunctor in $\mathcal{F}$, when $\mathcal{F}$ is an iterative DM-category. The functor $\mathbb{I}$ will be called 'iteration functor'; by 1.3.(i) we have an isomorphism

$$(\mathrm{I},\mathbb{I}(\varphi))\varphi \cong \mathbb{I}(\varphi)$$

which is natural in φ .

Theorem 1. *If $\mathcal{F}$ is a* DM-*category in which every standard endofunctor has l.f.p. and every normal functor for $\mathcal{F}$ has right adjoint, then $\mathcal{F}$ is iterative.*

Proof. Immediately from Lemma 1.3.4.□

DM-categories satisfying conditions of Theorem 1 were called in [6] iteratively closed DM-categories. The last theorem shows that iteratively closed DM-categories are special case of iterative DM-categories and therefore the results of the present paper extend those of [6].

Theorem 2. *Let $\mathcal{F}$ be a* DM-*category in which there is an initial object* O *s.t. $\varphi\mathrm{O} \cong \mathrm{O} \cong \mathrm{OL} \cong \mathrm{OR}$ for all $\varphi \in \mathcal{F}$, and all direct limits of sequences of the form $\varphi_0 \longrightarrow \varphi_1 \longrightarrow \dots$ exist in $\mathcal{F}$ and commute with folloiwng functors: $\mathbb{M}_\varphi(\xi) = \varphi\xi$, $\mathbb{M}^\varphi(\xi) = \xi\varphi$ and $\mathbb{P}(\xi) = (\mathrm{I},\xi)$ for all $\varphi \in \mathcal{F}$. Then $\mathcal{F}$ is iterative.*

Proof. Let Γ be a standard endofunctor in $\mathcal{F}$. Define $\gamma_n = \Gamma^n(\mathrm{O})$ and $g_n = \Gamma^n(g_0)$ where g_0 is the unique arrow $\mathrm{O} \longrightarrow \gamma_1$. Then we have a sequence

$$\gamma_0 \xrightarrow{g_0} \gamma_1 \xrightarrow{g_1} \dots$$

and let its direct limit in $\mathcal{F}$ be $(\mu, \lambda n.\bar{g}_n)$ i.e. $\bar{g}_n\colon \gamma_n \longrightarrow \mu$, $\bar{g}_{n+1}\circ g_n = \bar{g}_n$ for all n and for every sequence of arrows $h_n\colon \gamma_n \longrightarrow \beta$ in $\mathcal{F}$, s.t. $h_{n+1}\circ g_n = h_n$ for all n, there is unique $h\colon \mu \longrightarrow \beta$, s.t. $h_n = h\circ\bar{g}_n$ for all n. By the conditions of the theorem the last limit commutes with Γ and since $\Gamma(\gamma_n) = \gamma_{n+1}$ there is unique $m\colon \Gamma(\mu) \longrightarrow \mu$ s.t.

$$\bar{g}_{n+1} = m\circ\Gamma(\bar{g}_n) \qquad (1)$$

for all n, and this arrow is isomorphic. To show that $(\mu;m)$ is l.l.f.p. of Γ in $((H\downarrow\tilde{\psi}),P)$ for arbitrary normal projection $((H\downarrow\tilde{\psi}),P)$ over $\mathcal{F}$ suppose that $G: (H\downarrow\tilde{\psi}) \longrightarrow (H\downarrow\tilde{\psi})$ is an endofunctor s.t. $P\circ G = \Gamma\circ P$. By Lemma 1.3.1 G has the form

$$G(\xi;x) = (\Gamma(\xi);x^G)$$

and 1.3.(1) holds. By the definition of a normal functor and the conditions of the theorem it follows that the object $H(O)$ is initial in the category $\mathcal{F}^N$ where $H: \mathcal{F} \longrightarrow \mathcal{F}^N$. Then there is unique arrow $u_0: H(O) \longrightarrow \tilde{\psi}$ and define by induction $u_{n+1} = u_n^G$. Then we have $u_n: H(\gamma_n) \longrightarrow \tilde{\psi}$ and by induction on n we can see that for all n

$$u_{n+1}\circ H(g_n) = u_n \quad .$$

Indeed, for $n = 0$ this is obvious since $H(O)$ is an initial object, and by the induction hypothesis:

$$u_{n+1} = u_n^G = (u_{n+1}\circ H(g_n))^G = u_{n+1}^G\circ H(\Gamma(g_n)) = u_{n+2}\circ H(g_{n+1}) \quad .$$

But from the conditions of the theorem and the definition of a normal functor it follows that $(H(\mu),\lambda n.H(\bar{g}_n))$ is a limit of the sequence

$$H(\gamma_0) \xrightarrow{H(g_0)} H(\gamma_1) \xrightarrow{H(g_1)} \dots \quad .$$

Therefore there is unique arrow $u:H(\mu) \longrightarrow \tilde{\psi}$ s.t. $u_n = u\circ H(\bar{g}_n)$ for all n. We shall show that u is the unique arrow for which $u^G = u\circ H(m)$ and by Lemma 1.3.1 the proof will be completed. Indeed, for all n we have:

$$u_n = u^G\circ H(m^{-1})\circ H(\bar{g}_n) \quad .$$

For $n = 0$ this is immediate since $H(O)$ is an initial object, and by (1) and 1.3.(1):

$$u^G\circ H(m^{-1})\circ H(\bar{g}_{n+1}) = u^G\circ H(\Gamma(\bar{g}_n)) = (u\circ H(\bar{g}_n))^G = u_n^G = u_{n+1} \quad .$$

Thence by the uniquity of the arrow u we have $u = u^G\circ H(m^{-1})$ i.e. $u^G = u\circ H(m)$. If $v: H(\mu) \longrightarrow \tilde{\psi}$ satisfies $v^G = v\circ H(m)$ then by induction on n we have: $u_n = v\circ H(\bar{g}_n)$ for all n whence $v = u$. Indeed, for $n = 0$ this is obvious, and by the induction hypothesis:

$$u_{n+1} = u_n^{\ G} = (v\circ H(\bar{g}_n))^G = v^G\circ H(\Gamma(\bar{g}_n)) = v\circ H(m\circ\Gamma(\bar{g}_n)) = v\circ H(\bar{g}_{n+1}) \quad .\square$$

1.5. Terms and values.

Let $c_0,\ldots,c_{l-1}$ be a list of symbols called <u>parameter symbols</u>, and let we have an infinite list of variables denoted usually by x,y,z etc. The symbols $\underline{I},\underline{L},\underline{R}$ will be called basic constants, and parameter symbols and basic constants together will be called <u>constants</u>. Define <u>terms</u> inductively as follows:

a) all constants and variables are terms; they are called <u>simple</u> terms;

b) if t and s are terms then (ts) and (t,s) are terms.

If $\mathcal{X}$ is a set of variables then by <u>Term</u>($\mathcal{X}$) we shall denote the set of all terms whose variables belong to $\mathcal{X}$, and <u>Term</u> will be the set of all terms.

Let $\mathcal{F}$ be a DM-category and suppose we have an interpretation assigning to each parameter symbol c_i an object (called a parameter) γ_i of $\mathcal{F}$. This interpretation will be fixed throughout the paper. Let $\bar{x} \equiv x_0,\ldots,x_{n-1}$ be a list of distinct variables. Then each term $t \in$ <u>Term</u>($\{\bar{x}\}$) defines a functor $[\lambda\bar{x}.t]\colon \mathcal{F}^n \longrightarrow \mathcal{F}$ called value of t in an obvious way, namely:

1) if t is a variable x_i, $i<n$, then $[\lambda\bar{x}.t](\bar{\xi}) = \xi_i$;

2) if t is a parameter symbol c_i, $i<l$, then $[\lambda\bar{x}.t](\bar{\xi}) = \gamma_i$;

3) if t is $\underline{I}$, $\underline{L}$ or $\underline{R}$ then $[\lambda\bar{x}.t](\bar{\xi})$ is I, L or R respectively;

4) if $t \equiv (sr)$ then $[\lambda\bar{x}.t](\bar{\xi}) = [\lambda\bar{x}.s](\bar{\xi})[\lambda\bar{x}.r](\bar{\xi})$;

5) if $t \equiv (s,r)$ then $[\lambda\bar{x}.t](\bar{\xi}) = ([\lambda\bar{x}.s](\bar{\xi}),[\lambda\bar{x}.r](\bar{\xi}))$;

where $\bar{\xi}$ is an arbitrary object of $\mathcal{F}^n$, and for arrows the definition of the functor $[\lambda\bar{x}.t]$ is the same replacing γ_i,I,L,R with $1_{\gamma_i},1_I,1_L,1_R$ respectively.

Sometimes we shall write $t_0t_1\ldots t_n$ for $(\ldots(t_0t_1)\ldots t_{n-1})t_n$, where $t_0,\ldots,t_n$ are terms.

1.6. Coherence properties.

A formal expression of one of the following two forms

(a) $$t(sr) \longrightarrow (ts)r$$

(i) $$t(s,r) \longrightarrow (ts,tr) \quad ,$$

where t,s,r are terms, will be called a contraction. As usual the notion of contraction gives rise to a reduction notion: we shall write $t \mapsto s$ for 's is obtained by replacing of an occurrence in t of the left hand side of a contraction with the corresponding occurrence of the right hand side of the same contraction' and $\Mapsto$ for the reflexive transitive closure of the relation $\mapsto$. A term t will be called <u>normal</u>, iff $t \mapsto s$ is impossible for any s ; s will be called normal form of t iff $t \Mapsto s$ and s is normal.

Lemma 1. *For every term* t *there is unique normal form* t^b *of* t *.*

<u>Proof.</u> Indeed, for any term t let <u>lh</u>(t) be the length of t and define a number $\delta(t)$ by induction on t as follows:

(i) if t is simple, then $\delta(t) = 0$;

(ii) if $t = (rs)$, then $\delta(t) = \delta(r) + \underline{lh}(s)$;

(iii) if $t = (r,s)$, then $\delta(t) = \delta(r) + \delta(s) + 1$.

Then using induction on $\delta(t)$ we see that following equalities define uniquely a total operation on terms denoted by t^b for a term t:

$$s^b = s \tag{1}$$

$$(ps)^b = p^b s \tag{2}$$

$$(p(qr))^b = ((pq)r)^b \tag{3}$$

$$(p(q,r))^b = ((pq)^b,(pr)^b) \tag{4}$$

$$((q,r))^b = (q^b,r^b) \tag{5}$$

where s is simple and p,q,r are arbitrary terms. Again, we have for all terms t and s : t^b is normal; $t \Mapsto t^b$; and if $t \mapsto s$ then $t^b = s^b$. The former two are obvious and the last is shown straightforwardly by induction on $\delta(t)$. □

Now we shall define for every term $t \in \underline{\mathrm{Term}}(\{x_0,\ldots,x_{n-1}\})$ an isomorphism

$$\underline{b}_t(\bar{\xi}): [\lambda\bar{x}.t](\bar{\xi}) \cong [\lambda\bar{x}.t^b](\bar{\xi})$$

natural in $\bar{\xi}$, where $\bar{\xi} = (\xi_0,\ldots,\xi_{n-1})$ and $\bar{x} = (x_0,\ldots,x_{n-1})$. Writing for short $\underline{b}(t)$ for $\underline{b}_t(\bar{\xi})$ and $t*$ for $[\lambda\bar{x}.t](\bar{\xi})$ for any $t \in \underline{\mathrm{Term}}(\{x_0,\ldots,x_{n-1}\})$, define $\underline{b}_t(\bar{\xi})$ as follows:

(b1) if t is normal then $\underline{b}(t) = t*$;

(b2) if s is simple and t is not normal then $\underline{b}(ts) = \underline{b}(t)s*$;

(b3) if $s = pq$ is normal then $\underline{b}(ts) = \underline{b}(tp)q*\circ\bar{\alpha}$;

(b4) if $s = (s_0,s_1)$ is normal then $\underline{b}(ts) = (\underline{b}(ts_0),\underline{b}(ts_1))\circ\underline{i}$;

(b5) if s is not normal then $\underline{b}(ts) = \underline{b}(ts^b)\circ t*\underline{b}(s)$;

(b6) if (t_0,t_1) is not normal then $\underline{b}((t_0,t_1)) = (\underline{b}(t_0),\underline{b}(t_1))$.

Note that (b2) and (b6) hold for any terms t,t_0,t_1, and (b5) holds also for any t,s .

Lemma 2. *For all terms* t,r *and every normal* s *we have:*

$$\underline{b}(ts) = \underline{b}(t^b s)\circ\underline{b}(t)s* \tag{6}$$

and

$$\underline{b}(t(rs)) = \underline{b}((tr)s)\circ\bar{\alpha} \tag{7}.$$

<u>Proof.</u> Induction on s for both (6) and (7) . If s is simple, then

$$\underline{b}(t^b s)\circ\underline{b}(t)s* = (t^b)*s*\circ\underline{b}(t)s* = \underline{b}(ts) .$$

If $s = pq$ then q is simple since s is normal, and by (b3), 1.2.(2) and the induction hypothesis for p :

$$\underline{b}(t^b s)\circ\underline{b}(t)s* = \underline{b}(t^b p)q*\circ\bar{\alpha}\circ\underline{b}(t)(p*q*) = \underline{b}(t^b p)q*\circ\underline{b}(t)p*q*\circ\bar{\alpha}$$

$$= \underline{b}(tp)q*\circ\bar{\alpha} = \underline{b}(ts) .$$

If $s = (s_0,s_1)$ then similarly

$$\underline{b}(t^b s)\circ\underline{b}(t)s* = (\underline{b}(t^b s_0),\underline{b}(t^b s_1))\circ\underline{i}\circ\underline{b}(t)(s_0*,s_1*)$$

$$= (\underline{b}(t^b s_0)\circ\underline{b}(t)s_0*,\underline{b}(t^b s_1)\circ\underline{b}(t)s_1*)\circ\underline{i}$$

$$= (\underline{b}(ts_0),\underline{b}(ts_1))\circ\underline{i} = \underline{b}(ts) \quad .$$

This proves (6). If rs is normal then (7) is immediate from (b2) and (b3). Suppose rs is not normal. Then by (b5) we have

$$\underline{b}(t(rs)) = \underline{b}(t(rs)^b)\circ t*\underline{b}(rs) \qquad (8).$$

Consider cases for s . If s is simple, then

$$\underline{b}(t(rs)) = \underline{b}(t(rs)^b)\circ t*(\underline{b}(r)s*) = \underline{b}(t(r^b s))\circ t*(\underline{b}(r)s*)$$

$$= \underline{b}(tr^b)s*\circ\bar{\alpha}\circ t*(\underline{b}(r)s*) = \underline{b}(tr^b)s*\circ(t*\underline{b}(r))s*\circ\bar{\alpha} = \underline{b}(tr)s*\circ\bar{\alpha}$$

$$= \underline{b}((tr)s)\circ\bar{\alpha} \quad .$$

If $s = pq$ then q is simple, and using (8), the induction hypothesis for p, and 1.2.(8), we have :

$$\underline{b}(t(rs)) = \underline{b}(t((rp)^b q))\circ t*(\underline{b}(rp)q*)\circ t*\bar{\alpha}$$

$$= \underline{b}(t(rp)^b)q*\circ\bar{\alpha}\circ t*(\underline{b}(rp)q*)\circ t*\bar{\alpha} = \underline{b}(t(rp)^b)q*\circ(t*\underline{b}(rp))q*\circ\bar{\alpha}\circ t*\bar{\alpha}$$

$$= \underline{b}(t(rp))q*\circ\bar{\alpha}\circ t*\bar{\alpha} = \underline{b}((tr)p)q*\circ\bar{\alpha}q*\circ\bar{\alpha}\circ t*\bar{\alpha} = \underline{b}((tr)p)q*\circ\bar{\alpha}\circ\bar{\alpha}$$

$$= \underline{b}((tr)s)\circ\bar{\alpha} \quad .$$

Finally, if $s = (s_0,s_1)$, then s_0 and s_1 are normal, and using (8), the induction hypothesis for s_0 and s_1, and 1.2.(9), we have:

$$\underline{b}(t(rs)) = \underline{b}(t((rs_0)^b,(rs_1)^b))\circ t*(\underline{b}(rs_0),\underline{b}(rs_1))\circ t*\underline{i}$$

$$= (\underline{b}(t(rs_0)^b),\underline{b}(t(rs_1)^b))\circ\underline{i}\circ t*(\underline{b}(rs_0),\underline{b}(rs_1))\circ t*\underline{i}$$

$$= (\underline{b}(t(rs_0)),\underline{b}(t(rs_1)))\circ\underline{i}\circ t*\underline{i} = (\underline{b}((tr)s_0)\circ\bar{\alpha},\underline{b}((tr)s_1)\circ\bar{\alpha})\circ\underline{i}\circ t*\underline{i}$$

$$= (\underline{b}((tr)s_0),\underline{b}((tr)s_1))\circ(\bar{\alpha},\bar{\alpha})\circ\underline{i}\circ t*\underline{i} = (\underline{b}((tr)s_0),\underline{b}((tr)s_1))\circ\underline{i}\circ\bar{\alpha}$$

$$= \underline{b}((tr)s)\circ\bar{\alpha} \ .\square$$

Equalities (6) and (7) will be referred below as 'coherence properties'.

2. RECURSION THEORY IN DM-CATEGORIES.

2.1. The coding theorem.

Let $\mathcal{F}$ be a DM-category. A <u>term system</u> in $\mathcal{F}$ is a pair $(\bar{s};\bar{x})$ where $\bar{x}$ and $\bar{s}$

are strings $\alpha_0,\ldots,\alpha_{n-1}$ and $s_0,\ldots,s_{n-1}$ of variables α_i and terms s_i from $\underline{\mathrm{Term}}(\{\bar{\alpha}\})$ respectively. Each term system $S \equiv (\bar{s};\bar{\alpha})$ defines a functor $S^*: \mathcal{F}^n \longrightarrow \mathcal{F}^n$ by $S^*(\bar{\xi}) = (S_0(\bar{\xi}),\ldots,S_{n-1}(\bar{\xi}))$ where $S_i = [\lambda\bar{\alpha}.s_i]$ for all $i<n$. A term system $S = (\bar{s};\bar{\alpha})$ will be called <u>normal</u>, iff all terms $s_0,\ldots,s_{n-1}$ are normal.

It was yet mentioned in the introduction that principal object of present paper are l.f.p. of endofunctors of the form S^* where S is a term system. From 1.3.(iii) and results of 1.6 it follows that without a loss of generality we may restrict ourselves with normal systems S . Now we are going to state our main result (Theorem 1 below) about such endofunctors. In order to do this we need some additional definitions.

Let $S = (\bar{s};\bar{\alpha})$ be a normal term system in $\mathcal{F}$. Then a set K of normal terms will be called <u>closed</u> under S iff following conditions hold:

(i) $\alpha_i \in K$ and $s_i \in K$ for all $i<n$;

(ii) if c is a constant and $pc \in K$, then $p \in K$;

(iii) if $(t,r) \in K$, then $t \in K$ and $r \in K$;

(iv) if $p\alpha_i \in K$ where $i<n$, then $(ps_i)^b \in K$;

and with every function $k:K \longrightarrow \mathrm{Ob}\mathcal{F}$ we shall associate a function $k^S: K \longrightarrow \mathrm{Ob}\mathcal{F}$ defined as follows:

$$k^S(t) = \begin{cases} L & \text{if } t \text{ is a constant ;} \\ k(p) & \text{if } t = pc \text{ where } c \text{ is a constant;} \\ (k(q),k(r)) & \text{if } t = (q,r) \text{ ;} \\ k(s_i) & \text{if } t = \alpha_i \text{ where } i<n \text{ ;} \\ k((ps_i)^b) & \text{if } t = p\alpha_i \text{ where } i<n \text{ .} \end{cases}$$

Define for each term t an endofunctor $F_t: \mathcal{F} \longrightarrow \mathcal{F}$ by following conditions:

(F1) $F_t(\xi) = L\gamma$, if t is a constant with value γ;

(F2) $F_t(\xi) = R(\xi\gamma)$, if $t \equiv pc$, c is a constant with value γ and t is normal;

(F3) $F_t(\xi) = R\xi$ in all other cases.

Finally, here is the main definition in present section:

Definition. A <u>coding</u> for a normal term system S in $\mathcal{F}$ (w.r.t. a given interpretation of the parameters) is a triple (K,k,σ) , s.t.: K is a set of normal terms closed under S , $k\colon K \longrightarrow \mathcal{F}_0$ is a mapping into the set $\mathcal{F}_0$ defined in 1.4, $\sigma \in \mathcal{F}$, and for all $t \in K$ we have:

$$\sigma k(t) \cong F_t(k^S(t)) \qquad (1).$$

Theorem 1. *Let* $\mathcal{F}$ *be an iteratively closed* DM*-category and let* $S = (\bar{s};\bar{x})$ *be a normal term system in* $\mathcal{F}$ *where* $\bar{x} \equiv (x_0,\ldots,x_{n-1})$. *Suppose* (K,k,σ) *is a coding for* S *in* $\mathcal{F}$ *and* $(\omega;m)$ *is an iteration of* σ . *Then there is an arrow* $\bar{w}$ *in* $\mathcal{F}^n$ *s.t.* $(\omega k(x_0),\ldots,\omega k(x_{n-1});\bar{w})$ *is l.f.p. of* $S*$ *in* $\mathcal{F}^n$.

The proof of Theorem 1 which we give in section 3 below is rather long, so we prefer to consider first its corollaries. We shall obtain from it the fundamental facts of recursion theory in DM-categories, especially the inductive completeness of iterative DM-categories with translation functor (see definition of translation functor and Corollary 2 below). We shall restrict ourselves with the principal corollaries of Theorem 1 , but let us note that there are other applications of this Theorem using special kinds of codings (for more details about this see [6]). The exposition in the present section is not essentially different from that for the special case of operative spaces in the sense of Ivanov [1], the main difference between the last case and the general one for DM-categories being in the proof of Theorem 1. Therefore our proofs here will be less detailed than those in the next section and the reader may get additional information from the book [1].

Everywhere in 2.2 and 2.3 $\mathcal{F}$ will be an iterative DM-category.

2.2. Representation of natural numbers and translation functors.

Define for every natural number n an object $n^+ \in \mathcal{F}$ inductively as follows: $0^+ = L$; $(n+1)^+ = Rn^+$. Then by (DM4) and (DM5) for all $i < n$ we have an isomorphism

$$(\alpha_0,\dots,\alpha_{n-1})i^+ \cong \alpha_i$$

natural in $\alpha_0,\dots,\alpha_{n-1} \in \mathcal{F}$.

Definition. A functor $T\colon \mathcal{F} \longrightarrow \mathcal{F}$ is called a translation functor (this term is adopted from Ivanov [1]) iff for every object $\varphi \in \mathcal{F}$ and for each natural n

$$T(\varphi)n^+ \cong n^+\varphi \tag{1}$$

and the last isomorphism is natural in φ .

Lemma 1. *Let $\Psi(\varphi)$ be an endofunctor in $\mathcal{F}$ defined by $\Psi(\varphi)(\xi) = (L\varphi,R\xi)$ and for all $\varphi \in \mathcal{F}$ the functor $\Psi(\varphi)$ has l.f.p. $(T(\varphi);\underline{t}(\varphi))$ in $\mathcal{F}$. Then T is a translation functor in $\mathcal{F}$.*

Indeed, by 1.3.(i) we have a natural isomorphism $(L\varphi,RT(\varphi)) \cong T(\varphi)$ whence by induction on n we see that $T(\varphi)n^+ \cong n^+\varphi$.□ We shall call a translation functor obtained from Lemma 1 a <u>standard</u> translation functor.

Corollary 1. *If the category $\mathcal{F}$ satisfies conditions of Theorem* 1.4.2 *then there is a standard translation functor in $\mathcal{F}$.* □

Every translation functor T gives rise to a bifunctor

$$\mathbb{R}(\varphi,\psi) = (I,\mathbb{I}(T(\psi)))T(\varphi)$$

(see Ivanov [1]), called <u>a</u> T-<u>primitive</u> <u>recursion</u>. The functor $\mathbb{R}$ satisfies following natural (in φ,ψ) isomorphisms:

$$\mathbb{R}(\varphi,\psi)0^+ \cong \varphi \tag{2}$$

and

$$\mathbb{R}(\varphi,\psi)(n+1)^+ \cong \psi(\mathbb{R}(\varphi,\psi)n^+) \tag{3}$$

which follow from $\mathbb{R}(\varphi,\psi)n^+ \cong \psi\dots\psi\varphi$ (n times), the last being proved by induction on n. Objects $\varphi \in \mathcal{F}$, produced from constants L, R, I (respectively $L,R,I,\gamma_0,\dots,\gamma_{1-1}$) by means of functors $\mathbb{R}$, $\mathbb{M}$ and $\mathbb{D}$ are called T-primitive recursive (respectively T-primitive recursive in $\{\gamma_0,\dots,\gamma_{1-1}\}$).

Theorem 2. *If* T *is a translation functor in* $\mathcal{F}$ *, then for every primitive recursive function* f *there is a* T*-primitive recursive* $\varphi \in \mathcal{F}$ *, s.t.*

$$\varphi n^{+} \cong (f(n))^{+}$$

for all natural n .

Proof. Identify isomorphic objects and apply corresponding results of [5]. The proof of Proposition 8.1 in [1] can also be straightforwardly adopted. □

2.3. Universal codings and the recursion theorem.

Definition. (i) An object $\alpha \in \mathcal{F}$ is called recursive (in parameters $\gamma_0,\ldots,\gamma_{l-1}$), iff there is a system of terms $S = (\bar{s};\bar{x})$ $(s_0,\ldots,s_{n-1};\bar{x})$ and l.f.p. $(\bar{\xi};\bar{x}) \equiv (\xi_0,\ldots;\bar{x})$ of the functor S* , s.t. $\xi_0 \cong \alpha$.

(ii) Writing Nterm and Syst for the set of all normal terms and all normal systems of terms respectively, an universal coding is a pair (k,σ) , s.t. k: Syst×Nterm $\longrightarrow \mathcal{F}_0$ is a function, and for every $S \in$ Syst the triple (Nterm, $\lambda t.k(S,t)$, σ) is a coding for S .

Lemma 2. *Suppose* T *is a translation functor in* $\mathcal{F}$ *and* $S \in$ Syst *where* $S = (\bar{s};x_0,\ldots)$. *Then there is an universal coding* (k,σ) *, s.t.* $k(S,x_0) = L$ *and*

$$\sigma = (I,T(\gamma_0),\ldots,T(\gamma_{l-1}))\alpha \qquad (4)$$

for a suitable T*-primitive recursive object* $\alpha \in \mathcal{F}$.

Proof. The proof is rather standard one, using primitive recursive numeration of terms and systems. An essential role in it play two primitive recursive objects π,ρ defined as follows:

$$\pi = \mathbb{R}((LL,RL),(LR,RR)) \quad \text{and} \quad \rho = \mathbb{R}(LL,RT(R)) .$$

They satisfy following isomorphisms:

$$\pi n^{+} \cong (Ln^{+},Rn^{+}) \quad \text{and} \quad \rho n^{+} \cong n^{+}n^{+}$$

which follow from (1),(2) and (3) by induction on n . Now let we are given a primitive recursive numeration of the elements of $\underline{\text{Syst}}\times\underline{\text{Nterm}}$ and define $k(S,t) = (N(S,t))^+$ where $N(S,t)$ is the number of the pair (S,t) . The numeration can always be chosen to satisfy $N(S,x_0)=0$ for any fixed system S ; then $k(S,x_0) = L$. By Theorem 2 there are primitive recursive objects $\sigma_0,\rho_0,\rho_1,\rho_2,\rho_3,\rho_4,\rho_5$ s.t. for any system S and all terms t,p,q :

$$\sigma_0 k(S,t) \cong \begin{cases} 0^+ & \text{if } t \text{ is a constant;} \\ 1^+ & \text{if } t = pc \text{ where c is a constant;} \\ 2^+ & \text{if } t = (p,q) \ ; \\ 3^+ & \text{if } t \text{ is a variable;} \\ 4^+ & \text{if } t = px \text{ where } x \text{ is a variable;} \end{cases}$$

$$\rho_0 k(S,t) \cong \begin{cases} 0^+ & \text{if } t = \underline{I} \text{ or } t = p\underline{I} \ ; \\ 1^+ & \text{if } t = \underline{L} \text{ or } t = p\underline{L} \ ; \\ 2^+ & \text{if } t = \underline{R} \text{ or } t = p\underline{R} \ ; \\ (i+3)^+ & \text{if } t = c_i \text{ or } t = pc_i \text{ where } i < l \ ; \end{cases}$$

$\rho_1 k(S,t) \cong Rk(S,p)$, if $t = pc$ where c is a constant;

$\rho_2 k(S,t) \cong Rk(S,p)$ and $\rho_3 k(S,t) \cong Rk(S,q)$ where $t = (p,q)$;

$\rho_4 k(S,t) \cong Rk(S,s_i)$, if $S = (s_0,\dots,s_{n-1};x_0,\dots,x_{n-1})$, $t = x_i$ and $i<n$;

$\rho_5 k(S,t) \cong Rk(S,(px_i)^b)$, if $S = (s_0,\dots,s_{n-1};x_0,\dots,x_{n-1})$, $t = px_i$ and $i < n$.

(Note that the normal form function t^b on terms is primitive recursive, since the function δ defined in the proof of Lemma 1.6.1 is primitive recursive.) Next define

$$\alpha = (R'(0^+L,\dots,(l+2)^+L)\rho_0, R'\rho_0 T(\rho_1)\rho, L(\rho_2,\rho_3)\pi, L\rho_4, L\rho_5)\sigma_0\rho$$

where

$$R' = (LT(I),LT(L),LT(R),R) \quad ,$$

and define σ by (4) . Then it can be checked directly from these definitions that (k,σ) is an universal coding.□

From Lemma 2 and Theorem 2.1 we have immediately:

Corollary 2. *Suppose that* T *is a standard translation functor in* $\mathcal{F}$ *. Then:*

(i) *Every object* $\varphi \in \mathcal{F}$ *recursive in* $\{\gamma_0,\dots,\gamma_{l-1}\}$ *is naturally (in*

$\gamma_0,\dots,\gamma_{1-1}$) isomorphic to an object which can be expressed explicitly by means of $\gamma_0,\dots,\gamma_{1-1}$,I,L,R,$\mathbb{M}$,$\mathbb{D}$,$\mathbb{I}$,T .

(ii) *Any functor defined explicitly by means of the constants and the functors in* (i) *is naturally isomorphic to a functor* Γ *of the form*

$$\Gamma(\xi) = \mathbb{I}((I,T(\xi))\alpha)L ,$$

where α *is a* T*-primitive recursive in* $\{\gamma_0,\dots,\gamma_{1-1}\}$ *object of* $\mathcal{F}$.

(iii) *The set* Ob$\mathcal{F}/\cong$ *where* $\cong$ *is the relation of isomorphism is a combinatory algebra w.r.t. application* App *defined by:*

$$App(\varphi,\psi) = \mathbb{I}((I,T(\psi))\varphi)L .$$

(iv) *There is an object* $\omega \in \mathcal{F}$ *recursive in* $\{\gamma_0,\dots,\gamma_{1-1}\}$*, which is universal among all objects recursive in* $\{\gamma_0,\dots,\gamma_{1-1}\}$ *, i.e.*

(a) *for every recursive in* $\{\gamma_0,\dots,\gamma_{1-1}\}$ *object* $\varphi \in \mathcal{F}$ *there is a natural number* n *such that* $\varphi \cong \omega n^+$ *and*

(b) *there is a primitive recursive function* $s(n,m)$ *, s.t. for all natural* n,m

$$\omega(s(n,m))^+ \cong \omega n^+ m^+ .$$

3.PROOF OF THE MAIN THEOREM.

Assume the suppositions of theorem 2.1. Up to the end of the proof c will denote an arbitrary constant and γ will be the value of c in $\mathcal{F}$; the letters t,s,p,q,t_0 etc will denote terms. We shall adopt some rules for omitting brackets in long expressions, e.g. $\varphi\psi\chi\vartheta$ will be a short notation for $((\varphi\psi)\varphi)\vartheta$. This rule of 'association to the left' will apply to objects, arrows and terms as well, as mentioned before in 1.3. Let U be the standard endofunctor in $\mathcal{F}$ with parameter σ , i.e.

$$U(\xi) = (I,\xi)\sigma .$$

3.1. Definition of the arrows $\underline{m}$.

For every term $t \in K$ define an endofunctor G_t in $\mathcal{F}$ as follows:

$$G_t(\xi) = \begin{cases} \gamma & \text{if } t \equiv c \\ \xi k(p)\gamma & \text{if } t \equiv pc \text{ and } t \text{ is normal} \\ \xi k^S(t) & \text{otherwise} \end{cases} \quad . \tag{1}$$

Lemma 1. *For each* $t \in K$ *there is an isomorphism*

$$\underline{n}_t(\xi)\colon U(\xi)k(t) \cong G_t(\xi)$$

natural in ξ .

Indeed, by 1.1.(DM1) and 2.1.(1)

$$U(\xi)k(t) = ((I,\xi)\sigma)k(t) \cong (I,\xi)(\sigma k(t)) \cong (I,\xi)F_t(k^S(t))$$

Consider cases for $t \in K$:

1) $t = c$; then by 2.1.(F1), 1.1.(DM1), 1.1.(DM4) and 1.1.(DM2)

$$U(\xi)k(t) \cong (I,\xi)(L\gamma) \cong ((I,\xi)L)\gamma \cong I\gamma \cong \gamma = G_t(\xi) \ ;$$

2) $t = pc$ and t is normal; then by 2.1.(F2), 1.1.(DM1) and 1.1.(DM5)

$$U(\xi)k(t) \cong (I,\xi)((Rk(p))\gamma) \cong ((I,\xi)(Rk(p)))\gamma \cong (((I,\xi)R)k(p))\gamma$$

$$\cong (\xi k(p))\gamma = G_t(\xi) \ ;$$

3) all other cases; by 2.1.(F3), 1.1.(DM1), 1.1.(DM5) we have

$$U(\xi)k(t) \cong (I,\xi)(Rk^S(t)) \cong ((I,\xi)R)k^S(t) \cong \xi k^S(t) = G_t(\xi) \quad .\square$$

We shall write $\bar{n}_t(\xi)$ for $\underline{n}_t^{-1}(\xi)$. Since $(\omega;m)$ is l.f.p. of U , the arrow $m\colon U(\omega) \longrightarrow \omega$ is an isomorphism. Therefore by Lemma 1 we have an isomorphism

$$\underline{m}(t) = mk(t)\circ\bar{n}_t(\omega)\colon G_t(\omega) \cong \omega k(t) \tag{2}$$

and we shall write $\bar{m}(t)$ for $\underline{m}^{-1}(t)$.

3.2.Construction of the arrows M.

Using the condition of the theorem that $(\omega;m)$ is iteration of σ, we shall construct for all $t,s \in K$, s.t. $(ts)^b \in K$, an arrow

$$M(t,s)\colon\ \omega k(t)(\omega k(s)) \longrightarrow \omega k((ts)^b) \qquad (1)\ .$$

Fix $t \in K$ and denote by K' the set $\{s \in K \mid (ts)^b \in K\}$. Define a normal functor $H\colon \mathcal{F} \longrightarrow \mathcal{F}^{K'}$ by $H(\xi) = \lambda s{\in}K'.\omega k(t)(\xi k(s))$ and an object $\tilde{\psi} \in \mathcal{F}^{K'}$ by $\tilde{\psi} = \lambda s{\in}K'.\omega k((ts)^b)$, and for each object $(\xi,x) \in (H{\downarrow}\tilde{\psi})$ define an arrow $x'\colon H(U(\xi)) \longrightarrow \tilde{\psi}$ as follows:

$$x'(s) = \begin{cases} \underline{m}(ts)\circ\omega k(t)\underline{n}_s(\xi)\ , & \text{if } s = c\ ;\\ \underline{m}((ts)^b)\circ x(p)\gamma\circ\bar{\alpha}\circ\omega k(t)\underline{n}_s(\xi)\ , & \text{if } s = pc\ ;\\ \underline{m}((ts)^b)\circ\bar{i}\circ(x(q),x(r))\circ\underline{i}\circ\omega k(t)\underline{i}\circ\omega k(t)\underline{n}_s(\xi), & \text{if } s = (q,r)\ ;\\ \underline{m}(ts)\circ x(s_i)\circ\omega k(t)\underline{n}_s(\xi)\ , & \text{if } s = \alpha_i\ ,\ i<n\ ;\\ \underline{m}((ts)^b)\circ x((ps_i)^b)\circ\omega k(t)\underline{n}_s(\xi)\ , & \text{if } s = p\alpha_i\ ,\ i<n\ . \end{cases}$$

We leave to the reader to check, using definitions in 1.6 and 3.1, that this is indeed in arrow $x'\colon H(U(\xi)) \longrightarrow \tilde{\psi}$, i.e.

$$x'(s)\colon\ \omega k(t)(U(\xi)k(s)) \longrightarrow \omega k((ts)^b)$$

for all $s \in K'$. Then we may define an endofunctor X in $((H{\downarrow}\tilde{\psi}),P)$ by $X(\xi,x) = (U(\xi),x')$. To check that X is an endofunctor it is enough to show (according to Lemma 1.3.1) that for every arrow $f\colon (\xi,x) \longrightarrow (\eta,y)$ in $(H{\downarrow}\tilde{\psi})$ we have $x' = y'\circ H(U(f))$, i.e.

$$x'(s) = y'(s)\circ\omega k(t)(U(f)k(s))$$

for all $s \in K'$. The last is done by considering five cases for s as in the definition of $x'(s)$. As an illustration we shall do this for the case $s = pc$. Then using the equality $x = y\circ H(f)$ (since f is an arrow in $(H{\downarrow}\tilde{\psi})$), 1.2.(2), 3.1.(1) and Lemma 3.1.1 we have:

$$\begin{aligned} x'(s) &= \underline{m}((st)^b)\circ y(p)\gamma\circ(\omega k(t)(fk(p)))\gamma\circ\bar{\alpha}\circ\omega k(t)\underline{n}_s(\xi)\\ &= \underline{m}((st)^b)\circ y(p)\gamma\circ\bar{\alpha}\circ\omega k(t)(fk(p)\gamma)\circ\omega k(t)\underline{n}_s(\xi)\\ &= \underline{m}((st)^b)\circ y(p)\gamma\circ\bar{\alpha}\circ\omega k(t)G_s(f)\circ\omega k(t)\underline{n}_s(\xi)\\ &= \underline{m}((st)^b)\circ y(p)\gamma\circ\bar{\alpha}\circ\omega k(t)\underline{n}_s(\eta)\circ\omega k(t)(U(f)k(s))\\ &= y'(s)\circ\omega k(t)(U(f)k(s)) \qquad . \end{aligned}$$

In the sequel we shall write $(\xi,x)^X$ for the second component of $X(\xi,x)$, i.e. $(\xi,x)^X = x'$ in notations above. Since (ω,m) is iteration of σ , (ω,m) is l.l.f.p.. of U in the normal projection $((H{\downarrow}\tilde{\psi}),P)$ where P is the standard forgetful functor. But the endofunctor X satisfies $P\circ X = U\circ P$. Thence by Lemma 1.3.1 there is unique arrow $M: H(\omega) \longrightarrow \tilde{\psi}$, s.t. $(\omega,M)^X = M\circ H(m)$. We shall write M(t,s) for M(s) to exhibit the term t that was fixed above. Then M(t,s) is the unique system of arrows (1) which satisfies the equality

$$(\omega,M)^X(t,s) = M(t,s)\circ\omega k(t)(mk(s)) \tag{2}$$

for all $s \in K'$, where $(\omega,M)^X(t,s)$ is $(\omega,\lambda s\in K'.M(t,s))^X(s)$.

3.3.Definition of the arrow $\bar{w}$.

In the sequel we shall write $\omega k(\bar{x})$ for the object $(\omega k(x_0),\dots,\omega k(x_{n-1}))$ of $\mathcal{F}^n$, and $t*(\bar{\xi})$ for $[\lambda\bar{x}.t](\bar{\xi})$ for any $\bar{\xi} \in \mathcal{F}$ and $t \in$ Term . Define by induction on $t \in K$ an arrow

$$w(t) : t*(\omega k(\bar{x})) \longrightarrow \omega k(t)$$

as follows:

$$w(t) = \begin{cases} \underline{m}(t) & \text{if } t = c \\ \underline{m}(t)\circ w(p)\gamma & \text{if } t = pc \\ \underline{m}(t)\circ \bar{i}\circ(w(t_0),w(t_1)) & \text{if } t = (t_0,t_1) \\ \omega k(t) = 1_{\omega k(t)} & \text{if } t = x_i \text{ , } i < n \\ M(p,x_i)\circ w(p)(\omega k(x_i)) & \text{if } t = px_i \text{ , } i < n \end{cases} ;$$

and define for all $i < n$:

$$w_i = \underline{m}(x_i)\circ w(s_i) \quad .$$

Then by 3.1.(2) we have

$$w_i : s_i*(\omega k(\bar{x})) \longrightarrow \omega k(x_i) \quad ,$$

i.e. $\bar{w} : S*(\omega k(\bar{x})) \longrightarrow \omega k(\bar{x})$ in $\mathcal{F}^n$, where $\bar{w} = (w_0,\dots w_{n-1})$. We shall show that $(\omega k(\bar{x});\bar{w})$ is l.f.p. of the endofunctor S* in $\mathcal{F}^n$.

3.4. Construction of the arrows v.

Let $(\bar{\varphi};\bar{a})$ be an arbitrary object of the category $(S* \Rightarrow \mathcal{F}^n)$ (defined in 1.3), i.e. $\bar{\varphi} = (\varphi_0,\dots,\varphi_{n-1}) \in \mathcal{F}^n$ and $\bar{a}$ is a string $(a_0,\dots,a_{n-1})$ of arrows $a_i: s_i*(\bar{\varphi}) \longrightarrow \varphi_i$ in $\mathcal{F}$. We shall construct for every $t \in K$ an arrow

$$v_t(\bar{\varphi};\bar{a}) : \omega k(t) \longrightarrow t*(\bar{\varphi}) \qquad (1).$$

Fix $\bar{\varphi}$ and $\bar{a}$, and consider the functor $H:\mathcal{F}\longrightarrow\mathcal{F}^K$ defined by $H(\xi)(t) = \xi k(t)$ and let $\tilde{\psi}$ be the object of $\mathcal{F}^K$ defined by $\tilde{\psi}(t) = t*(\bar{\varphi})$ for $t \in K$. Since H is naturally isomorphic to a normal functor (namely $\lambda t{\in}K.I(\xi k(t))$) it follows by Lemma 1.3.3 that (ω,m) is l.l.f.p. of U in $((H{\downarrow}\tilde{\psi}),P)$ where P is the obvious forgetful functor. Define an endofunctor Y in $((H{\downarrow}\tilde{\psi}),P)$ by

$$Y(\xi,x) = (U(\xi),(\xi,x)^Y)$$

where

$$(\xi,x)^Y(t) = \begin{cases} \underline{n}(\xi) & \text{if } t = c \\ x(p)\gamma\circ\underline{n}_t(\xi) & \text{if } t = pc \\ (x(q),x(r))\circ\underline{i}\circ\underline{n}_t(\xi) & \text{if } t = (q,r) \\ a_i\circ x(s_i)\circ\underline{n}_t(\xi) & \text{if } t = x_i\ ,\ i < n \\ p*(\bar{\varphi})a_i\circ\bar{b}(ps_i)\circ x((ps_i)^b)\circ\underline{n}_t(\xi) & \text{if } t = px_i\ ,\ i<n\ . \end{cases}$$

We leave to the reader to check that Y is an endofunctor in $((H{\downarrow}\tilde{\psi}),P)$, using Lemma 3.1.1 and 3.1.(1). Then by Lemma 1.3.1 there is unique arrow $v:H(\omega)\longrightarrow\tilde{\psi}$, s.t. $(\omega,v)^Y = v\circ H(m)$, i.e.

$$(\omega,v)^Y(t) = v(t)\circ mk(t) \qquad (2)$$

for all $t \in K$. Now defining $v_t(\bar{\varphi};\bar{a}) = v(t)$ we have (1). We shall write $v(t)$ for $v_t(\bar{\varphi};\bar{a})$ except in the special case when $(\bar{\varphi};\bar{a})$ is $(\omega k(\bar{x});\bar{w})$; in this case we shall write $v_\omega(t)$ for $v_t(\omega k(\bar{x});\bar{w})$ and otherwise the object $(\bar{\varphi};\bar{a})$ will be fixed.

3.5. Lemma.

For all $t,s \in K$ *s.t.* $(ts)^b \in K$ *we have*

$$v((ts)^b)\circ M(t,s) = \underline{b}(ts)\circ v(t)v(s) \qquad .$$

Proof. Fix $t \in K$ and define K' as in 3.2. First we shall construct for all $s \in K'$ an arrow

$$M'(s):\ \omega k(t)(\omega k(s)) \longrightarrow (ts)^{b}*(\bar{\varphi})$$

by the same method as in that 3.2 and 3.4. Define a normal functor $H:\mathcal{F}\longrightarrow\mathcal{F}^{K'}$ and an object $\tilde{\psi} \in \mathcal{F}^{K'}$ by $H(\xi)(s) = \omega k(t)(\xi k(s))$ and $\tilde{\psi}(s) = (ts)^{b}*(\bar{\varphi})$ respectively. Then there is an endofunctor Z in $(H\downarrow\tilde{\psi})$ s.t.

$$Z(\xi,x) = (U(\xi),(\xi,x)^{Z})$$

where

$$(\xi,x)^{Z}(s) = \begin{cases} v(t)\gamma\circ\omega k(t)\underline{n}_{s}(\xi)\ , & s = c\ ; \\ x(p)\gamma\circ\bar{\alpha}\circ\omega k(t)\underline{n}_{s}(\xi)\ , & s = pc\ ; \\ (x(q),x(r))\circ\underline{i}\circ\omega k(t)\underline{i}\circ\omega k(t)\underline{n}_{s}(\xi)\ , & s = (q,r)\ ; \\ t*(\bar{\varphi})a_{i}\circ\bar{b}(ts_{i})\circ x(s_{i})\circ\omega k(t)\underline{n}_{s}(\xi)\ , & s = \alpha_{i}\ ,\ i<n\ ; \\ (tp)^{b*}(\bar{\varphi})a_{i}\circ\bar{b}((tp)^{b}s_{i})\circ x((ps_{i})^{b})\circ\omega k(t)\underline{n}_{s}(\xi)\ , & s = p\alpha_{i}\ ,\ i<n\ ; \end{cases}$$

We leave to the reader to check that Z is indeed an endofunctor (the main necessary tools are Lemma 1.3.1 and Lemma 3.1.1). Then by Lemma 1.3.1 there is unique arrow $V:\ H(\omega) \longrightarrow \tilde{\psi}$ s.t.

$$(\omega,V)^{Z} = V\circ H(m) \qquad (1).$$

Define two arrows $V_1:\ H(\omega) \longrightarrow \tilde{\psi}$ and $V_2:\ H(\omega) \longrightarrow \tilde{\psi}$ by

$$V_1(s) = \underline{b}(ts)\circ v(t)v(s) \quad\text{and}\quad V_2(s) = v((ts)^{b})\circ M(t,s)$$

respectively. We shall show that V_1 and V_2 satisfy the equation (1) w.r.t. V whence it will follow that $V_1 = V_2$ and this will complete the proof of the Lemma. To prove that for all $s \in K'$

$$(\omega,V_1)^{Z}(s) = V_1(s)\circ\omega k(t)(mk(s)) \qquad (2)$$

consider cases for s as follows:

<u>Case 1.</u> $s = c$ is a constant with value γ . Then

$(\omega,V_1)^{Z}(s) = v(t)\gamma\circ\omega k(t)\underline{n}_{s}(\omega) = v(t)\underline{n}_{s}(\omega)$	(by 1.2.(1))
$= v(t)(\omega,v)^{Y}(s)$	(by definition of Y in 3.4)
$= v(t)(v(s)\circ mk(s))$	(by 3.4.(2))
$= v(t)v(s)\circ\omega k(t)(mk(s))$	(since $\mathbb{M}$ is a functor)
$= V_1(s)\circ\omega k(t)(mk(s))$	(by 1.6.(b1)) .

<u>Case 2.</u> s is of the form pc . Then

$(\omega,V_1)^Z(s) = V_1(p)\gamma\circ\bar{\alpha}\circ\omega k(t)\underline{n}_s(\omega) = \underline{b}(tp)\gamma\circ v(t)v(p)\gamma\circ\bar{\alpha}\circ\omega k(t)\underline{n}_s(\omega)$

$= \underline{b}(tp)\gamma\circ\bar{\alpha}\circ v(t)(v(p)\gamma\circ\underline{n}_s(\omega))$ (by 1.2.(2) and since $\mathbb{M}$ is a functor)

$= \underline{b}(ts)\circ v(t)(\omega,v)^Y(s)$ (by 1.6.(b3) and definition of Y)

$= \underline{b}(ts)\circ v(t)v(s)\circ\omega k(t)(mk(s))$ (as in case 1)

$= V_1(s)\circ\omega k(t)(mk(s))$.

Case 3. s is of the form (q,r) . Then

$(\omega,V_1)^Z(s) = (V_1(q),V_1(r))\circ\underline{i}\circ\omega k(t)\underline{i}\circ\omega k(t)\underline{n}_s(\omega)$

$= (\underline{b}(tq),\underline{b}(tr))\circ(v(t)v(q),v(t)v(r))\circ\underline{i}\circ\omega k(t)\underline{i}\circ\omega k(t)\underline{n}_s(\omega)$

$= \underline{b}(ts)\circ v(t)(v(q),v(r))\circ\omega k(t)\underline{i}\circ\omega k(t)\underline{n}_s(\omega)$ (by 1.2.(7), 1.6.(b4))

$= \underline{b}(ts)\circ v(t)((v(q),v(r))\circ\underline{i}\circ\underline{n}_s(\omega))$

$= \underline{b}(ts)\circ v(t)(\omega,v)^Y(s) = V_1(s)\circ\omega k(t)(mk(s))$ (as in case 2) .

Case 4. $s = x_i$, $i<n$. Then

$(\omega,V_1)^Z(s) = t*(\bar{\varphi})a_i\circ\bar{b}(ts_i)\circ V_1(s_i)\circ\omega k(t)\underline{n}_s(\omega)$

$= t*(\bar{\varphi})a_i\circ\bar{b}(ts_i)\circ\underline{b}(ts_i)\circ v(t)v(s_i)\circ\omega k(t)\underline{n}_s(\omega)$

$= v(t)(a_i\circ v(s_i)\circ\underline{n}_s(\omega)) = v(t)(\omega,v)^Y(s)$

$= V_1(s)\circ\omega k(t)(mk(s))$ (as in case 1) .

Case 5. $s = px_i$, $i<n$. Then

$(\omega,V_1)^Z(s) = (tp)^b*(\bar{\varphi})a_i\circ\bar{b}((tp)^b s_i)\circ V_1((ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$

$= (tp)^b*(\bar{\varphi})a_i\circ\bar{b}((tp)^b s_i)\circ\underline{b}(t(ps_i)^b)\circ v(t)v((ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$.

But

$(tp)^b*(\bar{\varphi})a_i\circ\bar{b}((tp)^b s_i)\circ\underline{b}(t(ps_i)^b)$

$= (tp)^b*(\bar{\varphi})a_i\circ\underline{b}(tp)s_i*(\bar{\varphi})\circ\bar{b}(tps_i)\circ\underline{b}(t(ps_i)^b)$ (by 1.6.(6))

$= (tp)^b*(\bar{\varphi})a_i\circ\underline{b}(tp)s_i*(\bar{\varphi})\circ\bar{\alpha}\circ\bar{b}(t(ps_i))\circ\underline{b}(t(ps_i)^b)$ (by 1.6.(7))

$= \underline{b}(tp)\varphi_i\circ(tp)*(\bar{\varphi})a_i\circ\bar{\alpha}\circ t*(\bar{\varphi})\bar{b}(ps_i)$ (by 1.2.(1) and 1.6.(b5))

$= \underline{b}(tp)\varphi_i\circ\bar{\alpha}\circ t*(\bar{\varphi})(p*(\bar{\varphi})a_i)\circ t*(\bar{\varphi})\bar{b}(ps_i)$ (by 1.2.(2))

$= \underline{b}(ts)\circ t*(\bar{\varphi})(p*(\bar{\varphi})a_i)\circ t*(\bar{\varphi})\bar{b}(ps_i)$ (by 1.6.(b3)) .

Therefore

$(\omega,V_1)^Z(s) = \underline{b}(ts)\circ t*(\bar{\varphi})(p*(\bar{\varphi})a_i)\circ t*(\bar{\varphi})\bar{b}(ps_i)\circ v(t)v((ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$

$= \underline{b}(ts)\circ v(t)(p*(\bar{\varphi})a_i\circ\bar{b}(ps_i)\circ v((ps_i)^b)\circ\underline{n}_s(\omega))$

$= \underline{b}(ts)\circ v(t)(\omega,v)^Y(s) = V_1(s)\circ\omega k(t)(mk(s))$ (as in case 2) .

This finishes the proof of (2). Thence $(\omega,V_1)^Z = V_1\circ H(m)$. To prove

$$(\omega,V_2)^Z = V_2\circ H(m)$$

we have to prove that for all $s \in K'$

$$(\omega,V_2)^Z(s) = V_2(s)\circ\omega k(t)(mk(s)) \tag{3}$$

which is done by considering cases for s as in the proof of (2) . The proof of (3) is simpler then that of (2) since coherence properties are not used in it. We shall illustrate it by treating one of the cases leaving the rest to the reader.

Case 5. $s = px_i$, $i<n$. Then

$(\omega,V_2)^Z(s) = (tp)^b*(\bar{\varphi})a_i\circ\bar{b}((tp)^b s_i)\circ V_2((ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$

$= (tp)^b*(\bar{\varphi})a_i\circ\bar{b}((tp)^b s_i)\circ v((tps_i)^b)\circ M(t,(ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$

$= (\omega,v)^Y((ts)^b)\circ\bar{n}_{(ts)^b}\circ M(t,(ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$ (by definition of Y)

$= v((ts)^b)\circ mk((ts)^b)\circ\bar{n}_{(ts)^b}\circ M(t,(ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$ (by 3.4.(2))

$= v((ts)^b)\circ\underline{m}((ts)^b)\circ M(t,(ps_i)^b)\circ\omega k(t)\underline{n}_s(\omega)$ (by 3.1.(2))

$= v((ts)^b)\circ(\omega,M)^X(t,s)$ (by definition of the endofunctor X in 3.2)

$= v((ts)^b)\circ M(t,s)\circ\omega k(t)(mk(s))$ (by 3.2.(2))

$= V_2(s)\circ\omega k(t)(k(s))$.

This finishes the proof of (3) and of Lemma 3.5. As a corollary we have: if $tx_i \in K$ where $i < n$, then

$$v(tx_i)\circ M(t,x_i) = v(t)v(x_i) \tag{4}.$$

We shall write $v(\bar{x})$ for the arrow $(v(x_0),\dots,v(x_{n-1})) : \omega k(\bar{x}) \longrightarrow \bar{\xi}$ in $\mathcal{F}^n$.

Corollary 1. *For all* $t \in K$ *we have* $v(t)\circ w(t) = t*(v(\bar{x}))$.

Proof. Induction on t. Consider cases for t as in the definition of w(t). All cases are easy but the last one $t = px_i$ in which Lemma 3.5 is used through (4):

$v(t)\circ w(t) = v(px_i)\circ M(p,x_i)\circ w(p)(\omega k(x_i))$ (by definition of w(t))

$= v(p)v(\mathfrak{x}_i)\circ w(p)(\omega k(\mathfrak{x}_i))$ (by (4))

$= (v(p)\circ w(p))v(\mathfrak{x}_i) = p*(v(\bar{\mathfrak{x}}))v(\mathfrak{x}_i)$ (by the induction hypothesis)

$= t*(v(\bar{\mathfrak{x}}))$. □

Corollary 2. $v(\bar{\mathfrak{x}})$ *is an arrow* $v(\bar{\mathfrak{x}}): (\omega k(\bar{\mathfrak{x}});\bar{w}) \longrightarrow (\bar{\varphi};\bar{a})$ *in the category* $(S* \Rightarrow \mathcal{F}^n)$ *, i.e. for all* $i < n$ *:*

$$v(\mathfrak{x}_i)\circ w_i = a_i\circ s_i*(v(\bar{\mathfrak{x}})) \qquad (5).$$

Indeed, by definition of w_i, 3.4.(2), 3.1.(2) and Corollary 1 :

$$v(\mathfrak{x}_i)\circ w_i = a_i\circ v(s_i)\circ \underline{n}_t(\omega)\circ m^{-1}k(t)\circ \underline{m}(\mathfrak{x}_i)\circ w(s_i) = a_i\circ v(s_i)\circ w(s_i)$$

$$= a_i\circ s_i*(v(\bar{\mathfrak{x}})) \quad .$$

It remains to show that $v(\bar{\mathfrak{x}})$ is the unique arrow $(\omega k(\bar{\mathfrak{x}});\bar{w}) \longrightarrow (\bar{\varphi};\bar{a})$ in $(S* \Rightarrow \mathcal{F}^n)$. For that suppose that $\bar{v}: (\omega k(\bar{\mathfrak{x}});\bar{w}) \longrightarrow (\bar{\varphi};\bar{a})$ is an arbitrary arrow in the last category, i.e. $\bar{v} = (v_0,\dots,v_{n-1})$ and

$$v_i\circ w_i = a_i\circ s_i*(\bar{v}) \qquad (6)$$

for all $i < n$.

3.6. Lemma.

For all $t \in K$ *we have* $v(t) = t*(\bar{v})\circ v_\omega(t)$.

Proof. Let $v' = \lambda t{\in}K.t*(\bar{v})\circ v_\omega(t)$. We shall show that v' satisfies 3.4.(2) i.e.

$$(\omega,v')^Y(t) = v'\circ mk(t) \qquad (1)$$

for all $t \in K$. Thence by the uniquity of the arrow v satisfying 3.4.(2) it will follow that $v = v'$. Consider cases for t as in the definition of the functor Y in 3.4. We shall treat one of the cases only, the other cases being similar and simpler. This is the case $t = p\mathfrak{x}_i$ where $i < n$; it is the only case of using the supposition 3.5.(6) . Let $t = p\mathfrak{x}_i$. Then

$(\omega, v')^Y(t) = p*(\bar{\varphi})a_i\circ \bar{b}(ps_i)\circ v'((ps_i)^b)\circ \underline{n}_t(\omega)$

$= p*(\bar{\varphi})a_i\circ \bar{b}(ps_i)\circ (ps_i)^b*(\bar{v})\circ v_\omega((ps_i)^b)\circ \underline{n}_t(\omega)$

$= p*(\bar{\varphi})a_i \circ (ps_i)*(\bar{v}) \circ \bar{b}(ps_i) \circ v_\omega((ps_i)^b) \circ \underline{n}_t(\omega)$ (since $\underline{b}$ is natural)

$= p*(\bar{v})(a_i \circ s_i*(\bar{v})) \circ \bar{b}(ps_i) \circ v_\omega((ps_i)^b) \circ \underline{n}_t(\omega)$

$= p*(\bar{v})(v_i \circ w_i) \circ \bar{b}(ps_i) \circ v_\omega((ps_i)^b) \circ \underline{n}_t(\omega)$ (by 3.5.(6))

$= p*(\bar{v})v_i \circ p*(\omega k(\bar{x}))w_i \circ \bar{b}(ps_i) \circ v_\omega((ps_i)^b) \circ \underline{n}_t(\omega)$

$= p*(\bar{v})v_i \circ (\omega, v_\omega)^Y(t)$ (by definition of Y in 3.4)

$= t*(\bar{v}) \circ v_\omega(t) \circ mk(t)$ (by 3.4.(2)).

This completes the proof of Lemma 3.6.

3.7. Lemma.

For all $t,s,r \in K$ *such that* $(ts)^b \in K$, $(sr)^b \in K$, *and* $(tsr)^b \in K$ *we have:*

$$M(t,(sr)^b) \circ \omega k(t)M(s,r) \circ \underline{\alpha} = M((ts)^b,r) \circ M(t,s)(\omega k(r)) \quad .$$

Proof. By the same method as in the proof of Lemma 3.5. Fix $t \in K$ and $s \in K$ s.t. $(ts)^b \in K$ and denote by K'' the set of all $r \in K$ s.t. $(sr)^b \in K$ and $(tsr)^b \in K$, and let $\vartheta = \omega k(t)(\omega k(s))$. Then define a normal functor $H: \mathcal{F} \longrightarrow \mathcal{F}^{K''}$ and an object $\tilde{\psi} \in \mathcal{F}^{K''}$ by

$$H(\xi)(r) = \vartheta(\xi k(r))$$

and

$$\tilde{\psi}(r) = \omega k((tsr)^b)$$

respectively. Consider an endofunctor A in $(H\downarrow\tilde{\psi})$ defined by

$$A(\xi,x) = (U(\xi),(\xi,x)^A)$$

where

$$(\xi,x)^A(r) = \begin{cases} \underline{m}((tsr)^b) \circ M(t,s)\gamma \circ \vartheta \underline{n}_r(\xi) & \text{if } r = c \ ; \\ \underline{m}((tsr)^b) \circ x(p)\gamma \circ \bar{\alpha} \circ \vartheta \underline{n}_r(\xi) & \text{if } r = pc \ ; \\ \underline{m}((tsr)^b) \circ \bar{i} \circ (x(p),x(q)) \circ \underline{i} \circ \vartheta \underline{i} \circ \vartheta \underline{n}_r(\xi) & \text{if } r = (p,q) \ ; \\ \underline{m}((tsr)^b) \circ x(s_i) \circ \vartheta \underline{n}_r(\xi) & \text{if } r = x_i \ ; \\ \underline{m}((tsr)^b) \circ x((ps_i)^b) \circ \vartheta \underline{n}_r(\xi) & \text{if } r = px_i \ . \end{cases}$$

By Lemma 1.3.1 there is unique arrow $u: H(\omega) \longrightarrow \tilde{\psi}$ s.t. $(\omega,u)^A = u \circ H(m)$.

Therefore to prove the Lemma it is enough to show that both arrows M_1 and M_2 satisfy the last equation w.r.t. u , where

$$M_1 = \lambda r{\in}K''.M(t,(sr)^b)\circ\omega k(t)M(s,r)\circ\underline{\alpha}$$

and

$$M_2 = \lambda r{\in}K''.M((ts)^b,r)\circ M(t,s)(\omega k(r)) \quad .$$

That means to show that for all $r \in K''$

$$(\omega,M_1)^A(r) = M_1(r)\circ\vartheta(mk(r)) \qquad (1)$$

and

$$(\omega,M_2)^A(r) = M_2(r)\circ\vartheta(mk(r)) \qquad (2).$$

To prove (1) consider cases for r as in the definition of A .

Case 1. $r = c$. Then

$(\omega,M_1)^A(r) = \underline{m}((tsr)^b)\circ M(t,s)\gamma\circ\vartheta\underline{n}_{\Gamma}(\omega)$

$= (\omega,M)^X(t,sr)\circ\omega k(t)\bar{n}_{sr}(\omega)\circ\underline{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$ (by definition of X in 3.2)

$= M(t,sr)\circ\omega k(t)(mk(sr))\circ\omega k(t)\bar{n}_{sr}(\omega)\circ\underline{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$ (by 3.2.(2))

$= M(t,sr)\circ\omega k(t)\underline{m}(sr)\circ\underline{\alpha}\circ\vartheta\bar{m}(r)\circ\vartheta(mk(r))$ (by 3.1.(2))

$= M(t,sr)\circ\omega k(t)\underline{m}(sr)\circ\omega k(t)(\omega k(s)\bar{m}(r))\circ\underline{\alpha}\circ\vartheta(mk(r))$ (by 1.2.(2))

$= M(t,sr)\circ\omega k(t)(\omega,M)^X(s,r)\circ\omega k(t)(\omega k(s)(m^{-1}k(r)))\circ\underline{\alpha}\circ\vartheta(mk(r))$

(by definition of X and 3.1.(2))

$= M(t,sr)\circ\omega k(t)M(s,r)\circ\underline{\alpha}\circ\vartheta(mk(r))$ (by 3.2.(2))

$= M_1(r)\circ\vartheta(mk(r))$.

Case 2. $r = pc$. Then

$(\omega,M_1)^A(r) = \underline{m}((tsr)^b)\circ M_1(p)\gamma\circ\bar{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$

$= \underline{m}((tsr)^b)\circ M(t,(sp)^b)\gamma\circ\omega k(t)M(s,p)\gamma\circ\underline{\alpha}\gamma\circ\bar{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$

$= (\omega,M)^X(t,(sp)^b c)\circ\omega k(t)\bar{n}_{(sp)^b c}(\omega)\circ\underline{\alpha}\circ\omega k(t)M(s,p)\gamma\circ\underline{\alpha}\gamma\circ\bar{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$

$= M(t,(sr)^b)\circ\omega k(t)\underline{m}((sr)^b)\circ\underline{\alpha}\circ\omega k(t)M(s,p)\gamma\circ\underline{\alpha}\gamma\circ\bar{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$ (by 3.2.(2))

$= M(t,(sr)^b)\circ\omega k(t)\underline{m}((sr)^b)\circ\omega k(t)(M(s,p)\gamma)\circ\underline{\alpha}\circ\underline{\alpha}\gamma\circ\bar{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$ (by 1.2.(2))

$= M(t,(sr)^b)\circ\omega k(t)(\omega,M)^X(s,r)\circ\omega k(t)(\omega k(s)\bar{n}_{r}(\omega))\circ\omega k(t)\underline{\alpha}\circ\underline{\alpha}\circ\underline{\alpha}\gamma\circ\bar{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$

$= M(t,(sr)^b)\circ\omega k(t)M(s,r)\circ\omega k(t)(\omega k(s)\underline{m}(r))\circ\underline{\alpha}\circ\vartheta\underline{n}_{\Gamma}(\omega)$ (by 3.2.(2), 1.2.(8))

$$= M_1(r)\circ\vartheta\underline{m}(r)\circ\vartheta\underline{n}_T(\omega) \qquad \text{(by 1.2.(2))}$$

$$= M_1(r)\circ\vartheta(mk(r)) \qquad \text{(by 3.1.(2)).}$$

<u>Case</u> <u>3.</u> $r = (p,q)$. Then

$$(\omega,M_1)^A(r) = \underline{m}((tsr)^b)\circ\bar{i}\circ(M_1(p),M_1(q))\circ\underline{i}\circ\vartheta\underline{i}\circ\vartheta\underline{n}_T(\omega) \quad .$$

But

$$\underline{m}((tsr)^b)\circ\bar{i}\circ(M_1(p),M_1(q))$$

$$= \underline{m}((tsr)^b)\circ\bar{i}\circ(M(t,(sp)^b),M(t,(sq)^b))\circ(\omega k(t)M(s,p),\omega k(t)M(s,q))\circ(\underline{\alpha},\underline{\alpha})$$

$$= (\omega,M)^X(t,(sr)^b)\circ\omega k(t)\bar{n}_{(sr)^b}\circ\omega k(t)\bar{i}\circ\bar{i}\circ(\omega k(t)M(s,p),\omega k(t)M(s,q))\circ(\underline{\alpha},\underline{\alpha})$$

$$= M(t,(sr)^b)\circ\omega k(t)\underline{m}((sr)^b)\circ\omega k(t)\bar{i}\circ\bar{i}\circ(\omega k(t)M(s,p),\omega k(t)M(s,q))\circ(\underline{\alpha},\underline{\alpha})$$

$$= M(t,(sr)^b)\circ\omega k(t)\underline{m}((sr)^b)\circ\omega k(t)\bar{i}\circ\omega k(t)(M(s,p),M(s,q))\circ\bar{i}\circ(\underline{\alpha},\underline{\alpha})$$

$$= M(t,(sr)^b)\circ\omega k(t)(\omega,M)^X(s,r)\circ\omega k(t)(\omega k(s)\bar{n}_r(\omega)\circ\omega k(s)\bar{i}\circ\bar{i})\circ\bar{i}\circ(\underline{\alpha},\underline{\alpha})$$

$$= M(t,(sr)^b)\circ\omega k(t)M(s,r)\circ\omega k(t)(\omega k(s)\underline{m}(r))\circ\omega k(t)(\omega k(s)\bar{i}\circ\bar{i})\circ\bar{i}\circ(\underline{\alpha},\underline{\alpha})$$

$$= M_1(r)\circ\bar{\alpha}\circ\omega k(t)(\omega k(s)\underline{m}(r))\circ\omega k(t)(\omega k(s)\bar{i}\circ\bar{i})\circ\bar{i}\circ(\underline{\alpha},\underline{\alpha})$$

$$= M_1(r)\circ\vartheta\underline{m}(r)\circ\vartheta\bar{i}\circ\bar{\alpha}\circ\omega k(t)\bar{i}\circ\bar{i}\circ(\underline{\alpha},\underline{\alpha}) \quad .$$

Therefore

$$(\omega,M_1)^A(r) = M_1(r)\circ\vartheta\underline{m}(r)\circ\vartheta\bar{i}\circ\bar{\alpha}\circ\omega k(t)\bar{i}\circ\bar{i}\circ(\underline{\alpha},\underline{\alpha})\circ\underline{i}\circ\vartheta\underline{i}\circ\vartheta\underline{n}_T(\omega)$$

$$= M_1(r)\circ\vartheta\underline{m}(r)\circ\vartheta\bar{i}\circ\bar{\alpha}\circ\underline{\alpha}\circ\bar{i}\circ\underline{i}\circ\vartheta\underline{i}\circ\vartheta\underline{n}_T(\omega) \qquad \text{(by 1.2.(9))}$$

$$= M_1(r)\circ\vartheta\underline{m}(r)\circ\vartheta\underline{n}_T(\omega) = M_1(r)\circ\vartheta(mk(r)) \quad .$$

<u>Case</u> <u>4.</u> $r = x_i$, $i < n$. Then

$$(\omega,M_1)^A(r) = \underline{m}((tsr)^b)\circ M_1(s_i)\circ\vartheta\underline{n}_T(\omega)$$

$$= \underline{m}((tsr)^b)\circ M(t,(ss_i)^b)\circ\omega k(t)M(s,s_i)\circ\underline{\alpha}\circ\vartheta\underline{n}_T(\omega)$$

$$= (\omega,M)^X(t,sr)\circ\omega k(t)\bar{n}_{sr}(\omega)\circ\omega k(t)M(s,s_i)\circ\underline{\alpha}\circ\vartheta\underline{n}_T(\omega)$$

$$= M(t,sr)\circ\omega k(t)\underline{m}(sr)\circ\omega k(t)M(s,s_i)\circ\underline{\alpha}\circ\vartheta\underline{n}_T(\omega)$$

$$= M(t,sr)\circ\omega k(t)(\omega,M)^X(s,r)\circ\omega k(t)(\omega k(s)\bar{n}_r(\omega))\circ\underline{\alpha}\circ\vartheta\underline{n}_T(\omega)$$

$$= M(t,sr)\circ\omega k(t)M(s,r)\circ\omega k(t)(\omega k(s)\underline{m}(r))\circ\underline{\alpha}\circ\vartheta\underline{n}_T(\omega)$$

$$= M_1(r)\circ\vartheta\underline{m}(r)\circ\vartheta\underline{n}_T(\omega) = M_1(r)\circ\vartheta(mk(r)) \quad .$$

<u>Case</u> <u>5.</u> $r = px_i$, $i < n$. This case is treated similarly to case 4. We leave it to the reader.

This completes the proof of (1). To prove (2) consider cases for r as in the proof of (1). We shall treat one of the cases only, the rest ones being similar but simpler. Let $r = (p,q)$. Then

$$(\omega,M_2)^A(r) = \underline{m}((tsr)^b)\circ\bar{i}\circ(M_2(p),M_2(q))\circ\underline{i}\circ\vartheta\underline{i}\circ\vartheta\underline{n}_r(\omega) \quad ,$$

and defining for short

$$\Phi = \underline{m}((tsr)^b)\circ\bar{i}\circ(M((ts)^b,p),M((ts)^b,q))$$

we have by definition of the functor A :

$(\omega,M_2)^A(r) = \Phi\circ(M(t,s)(\omega k(p)),M(t,s)(\omega k(q))\circ\underline{i}\circ\vartheta\underline{i}\circ\vartheta\underline{n}_r(\omega)$

$= \Phi\circ\underline{i}\circ M(t,s)(\omega k(p),\omega k(q))\circ\vartheta\underline{i}\circ\vartheta\underline{n}_r(\omega)$ (by 1.2.(7))

$= \Phi\circ\underline{i}\circ M(t,s)(\underline{i}\circ\omega(k(p),k(q))\circ\vartheta\underline{n}_r(\omega)$ (by 1.2.(7))

$= \Phi\circ\underline{i}\circ\omega k((ts)^b)\underline{i}\circ M(t,s)G_r(\omega)\circ\vartheta\underline{n}_r(\omega)$ (by 3.1.(1))

$= \Phi\circ\underline{i}\circ\omega k((ts)^b)\underline{i}\circ\omega k((ts)^b)\underline{n}_r(\omega)\circ M(t,s)(U(\omega)k(r))$ (by 1.2.(1))

$= (\omega,M_2)^X((ts)^b,r)\circ M(t,s)(U(\omega)k(r))$ (by definition of X in 3.2)

$= M((ts)^b,r)\circ\omega k((ts)^b)(mk(r))\circ M(t,s)(U(\omega)k(r))$ (by 3.2.(2))

$= M((ts)^b,r)\circ M(t,s)(\omega k(r))\circ\vartheta(mk(r))$ (by 1.2.(1))

$= M_2(r)\circ\vartheta(mk(r))$.

This completes the proof of Lemma 3.7.

We shall write $\underline{b}_\omega(t)$ for $\underline{b}_t(\omega k(\bar{x}))$ (see definition of $\underline{b}_t(\bar{\xi})$ in 1.6) and $\bar{b}_\omega(t)$ for $\underline{b}_\omega^{-1}(t)$.

3.8. Lemma.

For all $t,s \in K$ *, s.t.* $(ts)^b \in K$ *we have*

$$M(t,s)\circ w(t)w(s) = w((ts)^b)\circ\underline{b}_\omega(ts) \quad .$$

<u>Proof.</u> Induction on s . Consider cases for s as in the definition of w(s) . We shall treat only two of the cases: $s = pc$ and $s = px_i$. The rest ones can be considered similarly to the former.

Let $s = pc$. Then by 3.2.(2), 3.1.(2) and the definition of w(s)

$M(t,s)\circ w(t)w(s) = (\omega,M)^X(t,s)\circ\omega k(t)(m^{-1}k(s))\circ w(t)w(s)$

$= \underline{m}((ts)^b)\circ M(t,p)\gamma\circ\bar{\alpha}\circ\omega k(t)\bar{m}(s)\circ w(t)(\underline{m}(s)\circ w(p)\gamma)$

$= \underline{m}((ts)^b)\circ M(t,p)\gamma\circ\bar{\alpha}\circ w(t)(w(p)\gamma) = \underline{m}((ts)^b)\circ M(t,p)\gamma\circ w(t)w(p)\gamma\circ\bar{\alpha}$

$= \underline{m}((ts)^b)\circ w((tp)^b)\gamma\circ\underline{b}_\omega(tp)\gamma\circ\bar{\alpha}$ (by the induction hypothesis)

$= w((ts)^b)\circ\underline{b}_\omega(ts)$ (by definition of $w((ts)^b)$ and 1.6.(b3)).

Let $s = p\alpha_i$, $i < n$. Then by definition of $w(s)$

$M(t,s)\circ w(t)w(s) = M(t,s)\circ w(t)(M(p,\alpha_i)\circ w(p)(\omega k(\alpha_i)))$

$= M(t,s)\circ\omega k(t)M(p,\alpha_i)\circ\omega k(t)(w(p)(\omega k(\alpha_i)))\circ w(t)s*(\omega k(\bar{\alpha}))$ (by 1.2.(1))

$= M((tp)^b,\alpha_i)\circ M(t,p)(\omega k(\alpha_i))\circ\bar{\alpha}\circ\omega k(t)(w(p)(\omega k(\alpha_i)))\circ w(t)s*(\omega k(\bar{\alpha}))$

(by Lemma 3.7)

$= M((tp)^b,\alpha_i)\circ M(t,p)(\omega k(\alpha_i))\circ w(t)w(p)(\omega k(\alpha_i))\circ\bar{\alpha}$

$= M((tp)^b,\alpha_i)\circ w((tp)^b)(\omega k(\alpha_i))\circ\underline{b}_\omega(ts)(\omega k(\alpha_i))\circ\bar{\alpha}$

(by the hypothesis of the induction)

$= w((ts)^b)\circ\underline{b}_\omega(ts)$.

This completes the proof of Lemma 3.8.

3.9. Lemma.

For all $t \in K$ *we have* $w(t)\circ v_\omega(t) = \omega k(t) = 1_{\omega k(t)}$.

Proof. By the method of the proofs of Lemma 3.5 and Lemma 3.7 . For each pair (ξ,x) where $x(t)\colon \xi k(t) \longrightarrow \omega k(t)$ or in other words x is an arrow

$$\lambda t{\in}K.\xi k(t) \longrightarrow \lambda t{\in}K.\omega k(t)$$

in $\mathcal{F}^K$, define another arrow

$$(\xi,x)^E\colon \lambda t{\in}K.U(\xi)k(t) \longrightarrow \lambda t{\in}K.\omega k(t)$$

in $\mathcal{F}^K$ by

$$(\xi,x)^E(t) = \begin{cases} \underline{m}(t)\circ\underline{n}_t(\xi) & \text{if } t = c \\ \underline{m}(t)\circ x(p)\gamma\circ\underline{n}_t(\xi) & \text{if } t = pc \\ \underline{m}(t)\circ\bar{i}\circ(x(p),x(q))\circ\underline{i}\circ\underline{n}_t(\xi) & \text{if } t = (p,q) \\ \underline{m}(t)\circ x(s_i)\circ\underline{n}_t(\xi) & \text{if } t = \alpha_i\ ,\ i < n \\ \underline{m}(t)\circ x((ps_i)^b)\circ\underline{n}_t(\xi) & \text{if } t = p\alpha_i\ ,\ i < n\ . \end{cases}$$

As in the proofs of Lemma 3.5 and Lemma 3.7 we see that there is unique arrow $e\colon \lambda t{\in}K.\omega k(t) \longrightarrow \lambda t{\in}K.\omega k(t)$ in $\mathcal{F}^K$ s.t.

$$(\omega,e)^E(t) = e(t)\circ m\mathit{k}(t) \tag{1}$$

for all $t \in K$. Therefore to prove the Lemma it is enough to show that (1) holds for both $e = e_1 = \lambda t{\in}K.w(t)\circ v_\omega(t)$ and $e = e_2 = \lambda t{\in}K.\omega\mathit{k}(t)$. We shall do this for e_1 only, the case with e_2 being easy. Again, in the proof of

$$(\omega,e_1)^E(t) = e_1(t)\circ m\mathit{k}(t)$$

we shall consider one of the cases for t only, namely the most interesting case in which Lemma 3.8 is used. This is the case $t = p\mathit{x}_i$. The rest of the cases we leave to the reader. Let $t = p\mathit{x}_i$, $i < n$. Then

$$\begin{aligned}
(\omega,e_1)^E(t) &= \underline{m}(t)\circ e_1((ps_i)^b)\circ\underline{n}_t(\omega) = \underline{m}(t)\circ w((ps_i)^b)\circ v_\omega((ps_i)^b)\circ\underline{n}_t(\omega) \\
&= \underline{m}(t)\circ w((ps_i)^b)\circ v_\omega((ps_i)^b)\circ\underline{n}_t(\omega) \\
&= \underline{m}(t)\circ M(p,s_i)\circ w(p)w(s_i)\circ\bar{b}_\omega(ps_i)\circ v_\omega((ps_i)^b)\circ\underline{n}_t(\omega) && \text{(by Lemma 3.8)} \\
&= (\omega,M)^X(p,\mathit{x}_i)\circ\omega\mathit{k}(t)\bar{n}_{\mathit{x}_i}(\omega)\circ w(p)w(s_i)\circ\bar{b}_\omega(ps_i)\circ v_\omega((ps_i)^b)\circ\underline{n}_t(\omega) \\
&= M(p,\mathit{x}_i)\circ\omega\mathit{k}(p)\underline{m}(\mathit{x}_i)\circ w(p)w(s_i)\circ\bar{b}_\omega(ps_i)\circ v_\omega((ps_i)^b)\circ\underline{n}_t(\omega) && \text{(by 3.2.(2))} \\
&= M(p,\mathit{x}_i)\circ w(p)w_i\circ\bar{b}_\omega(ps_i)\circ v_\omega((ps_i)^b)\circ\underline{n}_t(\omega) && \text{(by definition of } w_i) \\
&= M(p,\mathit{x}_i)\circ w(p)\omega\mathit{k}(\mathit{x}_i)\circ p{*}(\omega\mathit{k}(\bar{\mathit{x}}))w_i\circ\bar{b}_\omega(ps_i)\circ v_\omega((ps_i)^b)\circ\underline{n}_t(\omega) && \text{(by 1.2.(1))} \\
&= w(t)\circ(\omega,v_\omega)^Y(t) && \text{(by definitions of w and Y in 3.3 and 3.4)} \\
&= w(t)\circ v_\omega(t)\circ m\mathit{k}(t) && \text{(by 3.4.(2))} \\
&= e_1(t)\circ m\mathit{k}(t) .
\end{aligned}$$

This completes the proof of Lemma 3.9.

3.10. Final of the proof of the Theorem.

By Lemma 3.9 $\omega\mathit{k}(\mathit{x}_i) = w(\mathit{x}_i)\circ v_\omega(\mathit{x}_i) = \omega\mathit{k}(\mathit{x}_i)\circ v_\omega(\mathit{x}_i) = v_\omega(\mathit{x}_i)$, whence by Lemma 3.6 $v(\mathit{x}_i) = v_i\circ v_\omega(\mathit{x}_i) = v_i$. □

4.EXAMPLES.

4.1. Functorial DM-categories.

Denote by $\mathbf{C}_0$ the category of all categories in which exist initial objects and direct limits of ω-sequences $X_0 \longrightarrow X_1 \longrightarrow \dots$; morphisms in $\mathbf{C}_0$ are the functors which preserve such objects and limits. In the category $\mathbf{C}_0$ there exist binary sums $\mathcal{C}+\mathcal{D}$ for all objects $\mathcal{C},\mathcal{D} \in \mathbf{C}_0$; moreover all ω-sums $\Sigma_{i<\omega}\mathcal{C}_i$ exist in it. The construction of these sums is straightforward and is similar to that in the category of sets . Leaving details of it to the reader, we shall mention only following property of this construction: every object of the category $\Sigma_{i<\alpha}\mathcal{C}_i$ $(\alpha=\omega$ or $\alpha=2)$ can be uniquely represented in the form $I_i(X)$ for certain $i<\alpha$ and $X \in \mathcal{C}_i$, where $I_i\colon \mathcal{C}_i \longrightarrow \Sigma_{i<\alpha}\mathcal{C}_i$ are the canonical injections of the sum. Existence of ω-sums implies existence of objects $\mathcal{C} \in \mathbf{C}_0$ s.t. $\mathcal{C} \cong \mathcal{C}+\mathcal{C}$ in $\mathbf{C}_0$. Now suppose that we are given an object $\mathcal{C}$, two morphisms $G\colon \mathcal{C} \longrightarrow \mathcal{C}+\mathcal{C}$ and $H\colon \mathcal{C}+\mathcal{C} \longrightarrow \mathcal{C}$ in $\mathbf{C}_0$ and a natural isomorphism $\underline{n}\colon G\circ H \cong 1_{\mathcal{C}+\mathcal{C}}$. Let $\mathcal{F}_1$ be the category with objects the arrows $\varphi\colon \mathcal{C} \longrightarrow \mathcal{C}$ in $\mathbf{C}_0$, i.e. the endofunctors in $\mathcal{C}$ preserving initial objects and direct limits of ω-sequences, and morphisms the natural transformations. Then define multiplication $\mathbb{M}$ in $\mathcal{F}_1$ as composition, I as the identity functor $1_{\mathcal{C}}$, and define $L = H\circ I_0$ and $R = H\circ I_1$ where $I_0\colon \mathcal{C} \longrightarrow \mathcal{C}+\mathcal{C}$ and $I_1\colon \mathcal{C} \longrightarrow \mathcal{C}+\mathcal{C}$ are the canonical injections of the sum $\mathcal{C}+\mathcal{C}$. Next define cartesian functor in $\mathcal{F}_1$ by $\mathbb{D}(\varphi,\psi) = [\varphi,\psi]\circ G$ where $[\varphi,\psi]$ is the unique arrow $\mathcal{C}+\mathcal{C} \longrightarrow \mathcal{C}$ s.t. $[\varphi,\psi]\circ I_0 = \varphi$ and $[\varphi,\psi]\circ I_1 = \psi$. For arrows $f\colon \varphi \longrightarrow \varphi'$ and $g\colon \psi \longrightarrow \psi'$ define $\mathbb{D}(f,g) = [f,g]G$ where $[f,g]\colon [\varphi,\psi] \longrightarrow [\varphi',\psi']$ is the natural transformation uniquely determined by

$$[f,g](I_0(X)) = f(X) \text{ and } [f,g](I_1(X)) = g(X)$$

for $X \in \mathcal{C}$, and $[f,g]G$ is the natural transformation $\lambda X{\in}\mathcal{C}.[f,g](G(X))$.

Theorem 1. *The category* $\mathcal{F}_1$ *is a* DM-*category w.r.t.* $\mathbb{M},\mathbb{D},I,L,R$ *as defined above and suitable natural isomorphisms* $\underline{\alpha},\underline{\lambda},\underline{\rho},\underline{l},\underline{r},\underline{i}$.

Indeed, define $\underline{\alpha},\underline{\lambda},\underline{\rho},\underline{i}$ as corresponding units, and define

$$\underline{l}(\varphi,\psi) = \lambda X\in\mathfrak{C}.[\varphi,\psi](\underline{n}(I_0(X))) \quad \text{and} \quad \underline{r}(\varphi,\psi) = \lambda X\in\mathfrak{C}.[\varphi,\psi](\underline{n}(I_1(X))) \quad .$$

Then conditions (DM1) – (DM8) hold trivially except (DM4) and (DM5) which are but easy. □

If the natural transformation $\underline{n}$ is an unit, i.e. $\mathfrak{C}+\mathfrak{C}$ is a retract of $\mathfrak{C}$ in $\mathbf{C}_0$, then all isomorphisms $\underline{\alpha},\underline{\lambda},\underline{\rho},\underline{l},\underline{r},\underline{i}$ are units and the category $\mathcal{F}$ is what we can call a <u>strict</u> DM-category.

Theorem 2. *The category* $\mathcal{F}_1$ *is iterative and there is a standard translation functor in* $\mathcal{F}_1$.

This follows from Theorem 1.4.2 and Corollary 2.2.1. We leave to the reader to check that conditions of Theorem 1.4.2 hold.□

4.2. Category of abstract programs with correctness proofs.

We shall construct a category $\mathcal{F}_2$ whose objects are ternary relations $\varphi \subseteq \mathbb{F}\times M\times M$ where M is a set treated as data domain and elements of $\mathbb{F}$ are conceived as proofs. Such a relation φ will be considered as an abstract representation of a data processing device Φ which given an input from M gives a set of outputs from M and s.t. $\varphi(\underline{u},x,y)$ is equivalent to '$\underline{u}$ is a proof that given the input x to Φ y will belong to the set of outputs'. For the set M we shall suppose that there are two nonempty disjoint subsets $M_0,M_1 \subseteq M$ and three mappings d_0,d_1,d: $M \longrightarrow M$, s.t. $d_i(x) \in M_i$, and $d(d_i(x)) = x$ for all $x \in M$ and $i < 2$. For the set $\mathbb{F}$ we shall take the typed structure of hereditary partial functionals over M. We should note that this construction can be carried on with other structures for $\mathbb{F}$, for instance it is enough to suppose that $\mathbb{F}$ has the structure of a linear combinatory algebra with surjective pairing. But we prefer to exhibit an example in which using the notion of iteratively closed DM-category seems to be essential.

Define <u>types</u> inductively as follows:

1) 0 is a type;

2) if a and b are types, then $a \longrightarrow b$ and $a \times b$ are types;

3) if $a_0, a_1, \ldots$ is an infinite sequence of types, then $a_0 \times a_1 \times a_2 \ldots$ (or shortly $\prod i.a_i$) is a type.

The set $\mathbb{F}_a$ of hereditary partial functionals of type a over M is defined by induction on a as follows: $\mathbb{F}_0$ is M ; $\mathbb{F}_{a \longrightarrow b}$ is the set of all partial functions from $\mathbb{F}_a$ to $\mathbb{F}_b$; $\mathbb{F}_{a \times b}$ is $\mathbb{F}_a \times \mathbb{F}_b$; and $\mathbb{F}_{\prod i.a_i}$ is the product $\prod_{i=0}^{\omega} \mathbb{F}_{a_i}$.

Now the category $\mathcal{F}_2$ is defined as follows: objects are all relations $\varphi \subseteq \mathbb{F}_a \times M \times M$ for all types a ; $\varphi \in \mathcal{F}_2$ will be called to be of type a iff $\varphi \subseteq \mathbb{F}_a \times M \times M$. Arrows in $\mathcal{F}_2$ from $\varphi \in \mathcal{F}_2$ of type a to $\psi \in \mathcal{F}_2$ of type b are functionals $f \in \mathbb{F}_{a \longrightarrow (0 \longrightarrow (0 \longrightarrow b)))}$, s.t. $f\underline{u}xy$ is defined iff $\varphi(\underline{u},x,y)$ (we are writing $f\underline{u}xy$ for $f(\underline{u})(x)(y)$ etc.), and

$$\forall\ \underline{u},x,y\ (\ \varphi(\underline{u},x,y) \Rightarrow \psi(f\underline{u}xy,x,y)\ ) .$$

Composition $g \circ f$ of arrows f from φ to ψ and g from ψ to χ is defined by $(g \circ f)\underline{u}xy = g(f\underline{u}xy)xy$, and for every $\varphi \in \mathcal{F}_2$ let $1_{\varphi}\underline{u}xy = \underline{u}$ if $\varphi(\underline{u},x,y)$ and let $1_{\varphi}\underline{u}xy$ be not defined otherwise.

Next we define functors $\mathbb{M}$ and $\mathbb{D}$; the idea is that they will correspond to composition and branching of programs respectively. The definition is as follows:

$$(\varphi\psi)(\underline{w},x,y) \iff \exists z,\underline{u},\underline{v}(\ \underline{w} = \langle z,\langle \underline{u},\underline{v}\rangle\rangle\ \&\ \varphi(\underline{u},z,y)\ \&\ \psi(\underline{v},x,z)\ )$$

the type of $\varphi\psi$ is $0 \times (a \times b)$ if a and b are the types of φ and ψ respectively;

$$(fg)\langle z,\langle \underline{u},\underline{v}\rangle\rangle xy = \langle z,\langle f\underline{u}zy, g\underline{v}xz\rangle\rangle ,$$

where $f \in \mathcal{F}_2(\varphi,\varphi')$, $g \in \mathcal{F}_2(\psi,\psi')$ and $(\varphi\psi)(\langle z,\langle \underline{u},\underline{v}\rangle\rangle,x,y)$, and

$(fg)\langle z,\langle \underline{u},\underline{v}\rangle\rangle xy$ is undefined otherwise;

$$\mathbb{D}(\varphi,\psi)(\underline{w},x,y) \Leftrightarrow \exists \underline{u},\underline{v}(\ \underline{w} = \langle \underline{u},\underline{v}\rangle \ \&\ ((x \in M_0 \ \&\ \underline{v} = \underline{o}_b \ \&\ \varphi(\underline{u},d(x),y)) \vee (x \in M_1 \ \&\ \underline{u} = \underline{o}_a \ \&\ \psi(\underline{v},d(x),y)))\ ),$$

where φ and ψ are objects of types a and b respectively, and $\underline{o}_c \in \mathbb{F}_c$ is any fixed functional of type c ; the type of $\langle\varphi,\psi\rangle$ is $a \times b$;

$$\mathbb{D}(f,g)\langle \underline{u},\underline{v}\rangle xy = \begin{cases} \langle f\underline{u}d(x)y,\underline{o}_b\rangle & \text{if } x \in M_0 \ \&\ \underline{v} = \underline{o}_b \ \&\ \varphi(\underline{u},d(x),y) \\ \langle \underline{o}_a,g\underline{v}d(x)y\rangle & \text{if } x \in M_1 \ \&\ \underline{u} = \underline{o}_a \ \&\ \psi(\underline{v},d(x),y)\ , \\ \text{undefined otherwise} & . \end{cases}$$

Let I be the object of $\mathcal{F}_2$ of type 0 defined by $I(u,x,y) \Leftrightarrow u=\underline{o}_0 \ \&\ x=y$, and define $L \in \mathcal{F}_2$ and $R \in \mathcal{F}_2$ respectively by $L(u,x,y) \Leftrightarrow I(u,d_0(x),y)$ and $R(u,x,y) \Leftrightarrow I(u,d_1(x),y)$.

Theorem 1. *The category* $\mathcal{F}_2$ *is a* DM-*category w.r.t.* $\mathbb{M},\mathbb{D},I,L,R$ *as defined above and suitable natural isomorphisms* $\underline{\alpha},\underline{\lambda},\underline{\rho},\underline{l},\underline{r},\underline{i}$.

The proof this theorem is long but straightforward.□

Theorem 2. *The category* $\mathcal{F}_2$ *is iteratively closed and there is a standard translation functor in it.*

Proof (a sketch). Denote by $(\varphi)_L$ and $(\varphi)_R$ the multiplication functor with fixed left and right argument φ respectively:

$$(\varphi)_L(\xi) = \varphi\xi \quad \text{and} \quad (\varphi)_R(\xi) = \xi\varphi \quad , \ \xi \in \mathcal{F} .$$

These functors have right adjoints $(\varphi)_L^*$ and $(\varphi)_R^*$ respectively: for $\psi \in \mathcal{F}$ $(\varphi)_L^*(\psi)$ is defined by

$$(\varphi)_L^*(\psi)(\underline{w},x,z) \Leftrightarrow \forall \underline{u},y(\ \varphi(\underline{u},z,y) \Rightarrow \psi(\underline{wuy},x,y)\ ) ,$$

and the arrow $h\colon \varphi((\varphi)_L^*(\psi)) \longrightarrow \psi$ defined by $h\langle z,\langle \underline{u},\underline{w}\rangle\rangle xy = \underline{wuy}$ is universal: for every arrow $f\colon \varphi\xi \longrightarrow \psi$ the arrow $f'\colon \xi \longrightarrow (\varphi)_L^*(\psi)$ defined by $f'\underline{v}xz\underline{u}y = f\langle z,\langle \underline{u},\underline{v}\rangle\rangle xy$ is the unique one satisfying $f = h\circ\varphi f'$; the construction of right adjoint to $(\varphi)_R$ is similar. For every sequence $\lambda i.\varphi_i$ of objects of $\mathcal{F}_2$ of type a_i respectively the object φ of type $\Pi i.a_i$ defined by

$$\varphi(\lambda i.\underline{u}_i,x,y) \Leftrightarrow \forall i \varphi_i(\underline{u}_i,x,y)$$

is a product of $\lambda i.\varphi_i$ in $\mathcal{F}_2$; projections $p_i\colon \varphi \longrightarrow \varphi_i$ are defined by $p_i(\lambda i.\underline{u}_i)xy = \underline{u}_i$. Existence of ω-products and right adjoints to $(\varphi)_L$ and $(\varphi)_R$ implies existence of right adjoint to every normal functor $H{:}\mathcal{F}_2 \longrightarrow \mathcal{F}_2^{\ \omega}$. Let $\varphi \in \mathcal{F}_2$ be an object of type a . Then the standard endofunctor $\Gamma(\xi)=\mathbb{D}(I,\xi)\varphi$ has l.f.p. $(\gamma,\underline{m})$ which can be constructed directly as follows. Let b be the type $b_0 \times b_1 \times \ldots$ where $b_0 = 0$ and $b_{n+1} = 0\times((0\times b_n)\times a)$ for all natural n. The set M is infinite; therefore we may identify natural numbers with certain elements of M. Denote by $\mathbb{F}_{b,n}$ the set of all functionals of type b of the form $\langle n,\underline{w}_0,\ldots,\underline{w}_n,\underline{o},\underline{o},\ldots\rangle$ where we write $\langle \underline{v}_0,\ldots\rangle$ for $\lambda i.\underline{v}_i$ and omit subscripts in $\underline{o}_c$ etc. and n is a natural number as element of M. Then define a functional $\underline{m}$ of suitable type by

$$\underline{m}\langle z,\langle\langle \underline{o}_0,\underline{w}\rangle,\underline{v}\rangle\rangle = \begin{cases} \langle 0,\langle z,\langle\langle \underline{o},\underline{o}\rangle,\underline{v}\rangle\rangle,\underline{o},\underline{o},\ldots\rangle & \text{if } z \in M_0 \\ \langle n+1,\underline{w}_1,\ldots,\underline{w}_{n+1},\underline{o},\underline{o},\ldots\rangle & \text{if } z \in M_1 \text{ and } \underline{w} = \lambda i.\underline{w}_i \in \mathbb{F}_{b,n} \\ \text{undefined otherwise} & , \end{cases}$$

where $z,\underline{w},\underline{v}$ are of types 0,b,a respectively. Next define an object γ of type b as the least relation satisfying following two conditions:

a) $z \in M_0$ & $d(z) = y$ & $\varphi(\underline{v},x,z)$ & $\underline{w} = \underline{m}\langle z,\langle\langle \underline{o},\underline{o}\rangle,\underline{v}\rangle\rangle xy \Rightarrow \gamma(\underline{w},x,y)$

b) $z \in M_1$ & $\varphi(\underline{v},x,z)$ & $\gamma(\underline{w}',d(z),y)$ & $\underline{w} = \underline{m}\langle z,\langle\langle \underline{o},\underline{w}'\rangle,\underline{v}\rangle\rangle xy \Rightarrow \gamma(\underline{w},x,y)$.

Then $\underline{m}\colon \Gamma(\gamma) \longrightarrow \gamma$ is an arrow in $\mathcal{F}_2$. Given an arrow $f\colon \Gamma(\xi) \longrightarrow \xi$ in $\mathcal{F}_2$ the equality $h\circ\underline{m} = f\circ\Gamma(h)$ for $h\colon \gamma \longrightarrow \xi$ is equivalent to :

$$h\underline{w}xy = \begin{cases} f\langle z,\langle\langle \underline{o},\underline{o}\rangle,\underline{v}\rangle\rangle xy & \text{if the hypothesis of a) holds} \\ f\langle z,\langle\langle \underline{o},h\underline{w}'d(z)\,y\rangle,\underline{v}\rangle\rangle xy & \text{if the hypothesis of b) holds} \\ \text{undefined otherwise} \end{cases}$$

The last equality is satisfied by unique functional h defined by corresponding recursion. Therefore $(\gamma,\underline{m})$ is l.f.p. of Γ . In a similar way can be shown that there is l.f.p. of the functor $\lambda\xi.\mathbb{D}(L\varphi,R\xi)$ whence there is a standard translation functor in $\mathcal{F}_2$.□

4.3. Category of logical programs with correctness proofs.

We shall construct an example similar to the previous one in which elements of $\mathbb{F}$ will be real proofs and objects of the category will correspond to logical programs. For the set M we shall suppose that a structure is given on it and by Δ we shall denote the positive diagram of that structure, i.e. the set of all true atomic formulas in the language $\mathcal{L}$ of the structure enriched with constants for all elements of M (which we shall identify with corresponding constants). By logical program we shall mean as usual a finite set of first order predicate formulas in the language $\mathcal{L}$ which are universal closures of Horn clauses written by conjunction & and implication $\supset$, i.e. formulas of the form $\forall x_0 \ldots \forall x_{n-1}(P_0 \& \ldots \& P_m \supset P)$ or $\forall x_0 \ldots \forall x_{n-1} P$ where $P_0, \ldots, P_m$, P are atomic. We shall suppose for the language $\mathcal{L}$ that it contains for each arity a countable list of predicate variables and by X, X_1, X_2 we shall denote the first three predicate variables of arity two respectively. For any formal object (formula, proof etc.) or finite set of such objects Q, by Q^1 (respectively Q^2) we shall denote the result of replacement in Q of every predicate variable of each arity with number i with that with number $2i+1$ (respectively $2i+2$). The set of all <u>normal</u> <u>natural</u> <u>derivations</u> (in the sense of Pravitz [13]) in the language $\mathcal{L}$ will be denoted by $\mathbb{N}$. We shall write $d\colon \Gamma \longrightarrow A$ for 'd is a natural derivation with conclusion formula A and all uneliminated hypotheses of d belong to Γ '. It will be convenient to use some termal notations for natural derivations, namely given two derivations $d_0\colon \Gamma \longrightarrow A_0$ and $d_1\colon \Gamma \longrightarrow A_1$ by $\langle d_0, d_1 \rangle$ we shall denote the the obvious derivation $d\colon \Gamma \longrightarrow A_0 \& A_1$ obtained from d_0 and d_1 by applying &-introduction to their conclusions; similarly, given $d\colon \Gamma \longrightarrow A \supset B$ and $e\colon \Gamma \longrightarrow A$ let $de\colon \Gamma \longrightarrow B$ be the derivation obtained from d and e by applying $\supset$-elimination to their conclusions. A ternary relation $\varphi \subseteq \mathbb{N} \times M \times M$ will be called to correspond to a logical program Φ iff

$$\varphi(\underline{u}, x, y) \iff \underline{u}\colon \Delta, \Phi \longrightarrow X(x,y)$$

for all triples $(\underline{u},x,y) \in \mathbb{N}\times M\times M$. (We are writing Δ,Φ for $\Delta\cup\Phi$ etc.)

Now define a category $\mathcal{F}_3$ with objects relations $\varphi \subseteq \mathbb{N}\times M\times M$ corresponding to logical programs and arrows $f\colon \varphi \longrightarrow \psi$ the functions $f\colon \varphi \longrightarrow \mathbb{N}$ s.t.

$$\varphi(\underline{u},x,y) \Rightarrow \psi(f(\underline{u},x,y),x,y)$$

for all $(\underline{u},x,y) \in \mathbb{N}\times M\times M$. Composition of arrows and units in $\mathcal{F}_3$ are defined as in section 4.2.

Denote by C the formula $\forall \mathfrak{x}\mathfrak{y}\mathfrak{z}(X_1(\mathfrak{z},\mathfrak{y})\&X_2(\mathfrak{x},\mathfrak{z})\supset X(\mathfrak{x},\mathfrak{y}))$ and let $d(x,y,z)$ be the obvious normal derivation $d(x,y,z)\colon\ C \longrightarrow X_1(z,y)\&X_2(x,z)\supset X(x,y))$ where $x,y,z \in M$. For any two relations $\varphi,\psi \subseteq \mathbb{N}\times M\times M$ define a relation $\varphi\psi \subseteq \mathbb{N}\times M\times M$ as in 4.2:

$$(\varphi\psi)(\underline{w},x,y) \Longleftrightarrow \exists z,\underline{u},\underline{v}(\ \underline{w} = d(x,y,z)\langle\underline{u}^1,\underline{v}^2\rangle\ \&\ \varphi(\underline{u},z,y)\ \&\ \psi(\underline{v},x,z)\ )\quad .$$

Lemma 1. *If* $\varphi,\psi \in \mathcal{F}_3$ *then* $\varphi\psi \in \mathcal{F}_3$.

Indeed, if φ and ψ correspond to logical programs Φ and Ψ respectively, then $\varphi\psi$ corresponds to the logical program Φ^1,Ψ^2,C . The proof uses a standard analysis of normal derivations $\underline{w}\colon\ \Delta,\Phi^1,\Psi^2,C \longrightarrow X(x,y)$ which should be of the form $d(x,y,z)\langle\underline{u}^1,\underline{v}^2\rangle$.□

The last Lemma enables us to define multiplication functor $\mathbb{M}$ as in the previous section: $\mathbb{M}(\varphi,\psi) = \varphi\psi$, and for arrows $f\colon\ \varphi \longrightarrow \varphi'$ and $g\colon\ \psi \longrightarrow \psi'$ define

$$(fg)(d(x,y,z)\langle\underline{u}^1,\underline{v}^2\rangle,x,y) = d(x,y,z)\langle(f(\underline{u},z,y))^1,(g(\underline{v},x,z))^2\rangle\ .$$

To define cartesian functor $\mathbb{D}$ in $\mathcal{F}_3$ we need some additional suppositions. We shall suppose that a logical program Σ is given together with predicate variables T,F,D,D_0,D_1 occurring in it of arity 1,1,2,2,2 respectively s.t. following conditions are fulfilled:

1) the program Σ have no predicate variables among other ones i.e. those who are subject to substitutions denoted as Q^1 etc. (in other words predicate variables in Σ are treated as constants however different from basic constants of the structure M and the program itself as set of axioms for those

constants);

2) for every $x \in \mathrm{M}$ there is at most one normal derivation

$$\underline{\mathrm{d}}_T(x):\ \Delta,\Sigma \longrightarrow T(x)\ ;$$

we shall write $\mathrm{M}_T(x)$ for 'the derivation $\underline{\mathrm{d}}_T(x)$ exists';

3) for every $x \in \mathrm{M}$ there is at most one normal derivation

$$\underline{\mathrm{d}}_F(x):\ \Delta,\Sigma \longrightarrow F(x)\ ;$$

we shall write $\mathrm{M}_F(x)$ for 'the derivation $\underline{\mathrm{d}}_F(x)$ exists';

4) for no $x \in \mathrm{M}$ both $\mathrm{M}_T(x)$ and $\mathrm{M}_F(x)$ hold;

5) for every $x \in \mathrm{M}$ there is unique $\mathrm{d}(x) \in \mathrm{M}$ and unique normal

$$\underline{\mathrm{d}}(x):\ \Delta,\Sigma \longrightarrow D(x,\mathrm{d}(x)) \qquad ;$$

6) for every $x \in \mathrm{M}$ there are unique $\mathrm{d}_0(x) \in \mathrm{M}$ and unique $\mathrm{d}_1(x) \in \mathrm{M}$ and unique normal derivations

$$\underline{\mathrm{d}}_i(x):\ \Delta,\Sigma \longrightarrow D_i(x,\mathrm{d}_i(x)) \quad , \quad i = 0,1\ ;$$

7) for every $x \in \mathrm{M}$ and both $i = 0,1$ we have $\mathrm{d}(\mathrm{d}_i(x)) = x$, $\mathrm{M}_T(\mathrm{d}_0(x))$, and $\mathrm{M}_F(\mathrm{d}_1(x))$.

The suppositions 1) - 7) are fulfilled when the program Σ consists of trivial formulas of the form $\forall x(P(x) \supset T(x))$ where P is a basic predicate of the structure M etc.; they are fulfilled also in following natural case: the structure M is that of natural numbers with basic relations the equality to zero $x = 0$ and the successor relation $x = y+1$; T,F,D,D_0,D_1 are respectively following relations: 'x is even', 'x is odd', $y = [x/2]$, $y = 2x$, and $y = 2x+1$; and Σ is the obvious inductive definition of those relations.

Now let $B_T(x,u,y)$ be the formula $T(x)\ \&\ D(x,u)\ \&\ X_1(u,y) \supset X(x,y)$ and let $\forall B_T$ be the universal closure $\forall x \forall u \forall y B_T(x,u,y)$. Similarly define $B_F(x,u,y)$ as $F(x)\ \&\ D(x,u)\ \&\ X_2(u,y) \supset X(x,y)$ and $\forall B_F$ as the universal closure of B_F . For all $x,y \in \mathrm{M}$ denote by $\underline{\mathrm{e}}_T(x,y)$ and $\underline{\mathrm{e}}_F(x,y)$ the obvious normal derivations

$$\underline{\mathrm{e}}_T:\ \forall B_T \longrightarrow B_T(x,\mathrm{d}(x),y) \quad \text{and} \quad \underline{\mathrm{e}}_F:\ \forall B_F \longrightarrow B_F(x,\mathrm{d}(x),y)$$

respectively. Given normal derivations

$$\underline{u}: \Gamma \longrightarrow X(\underline{d}(x),y) \quad \text{and} \quad \underline{v}: \Theta \longrightarrow X(\underline{d}(x),y)$$

we may define two normal derivations

$$T^*(\underline{u},x,y): \Delta,\Sigma,\Gamma,\forall B_T \longrightarrow X(x,y) \quad \text{and} \quad F^*(\underline{v},x,y): \Delta,\Sigma,\Theta,\forall B_F \longrightarrow X(x,y)$$

by

$$T^*(\underline{u},x,y) = \underline{e}_T(x,y)\langle\langle \underline{d}_T(x),\underline{d}(x)\rangle,\underline{u}^1\rangle$$

and

$$F^*(\underline{v},x,y) = \underline{e}_F(x,y)\langle\langle \underline{d}_F(x),\underline{d}(x)\rangle,\underline{v}^2\rangle$$

respectively. Then for any relations $\varphi,\psi \subseteq \mathbb{N}\times\mathbb{M}\times\mathbb{M}$ define a relation $\mathbb{D}(\varphi,\psi) \subseteq \mathbb{N}\times\mathbb{M}\times\mathbb{M}$ as follows:

$$\mathbb{D}(\varphi,\psi)(\underline{w},x,y) \Longleftrightarrow \exists \underline{u},\underline{v}((M_T(x) \ \&\ \varphi(\underline{u},d(x),y) \ \&\ \underline{w} = T^*(\underline{u},x,y)) \vee$$
$$(M_F(x) \ \&\ \psi(\underline{v},d(x),y) \ \&\ \underline{w} = F^*(\underline{v},x,y))) \quad .$$

Lemma 2. *If* $\varphi,\psi \in \mathcal{F}_3$ *then* $\mathbb{D}(\varphi,\psi) \in \mathcal{F}_3$.

If φ and ψ correspond to logical programs Φ and Ψ respectively, then $\mathbb{D}(\varphi,\psi)$ corresponds to the logical program $\mathbb{D}(\Phi,\Psi) = \Sigma,\Phi^1,\Psi^2,\forall B_T,\forall B_F$. As in the proof of Lemma 1 a standard analysis of normal derivations $\underline{w}: \Delta,\mathbb{D}(\Phi,\Psi) \longrightarrow X(x,y)$ shows that $\underline{w}$ should be of the form $T^*(\underline{u},x,y)$ or $F^*(\underline{u},x,y)$ for suitable $\underline{u}$ s.t. $\varphi(\underline{u},d(x),y)$ or $\psi(\underline{u},d(x),y)$ respectively.□

As before Lemma 2 enables us to define cartesian functor $\mathbb{D}$ in $\mathcal{F}_3$: for arrows $f: \varphi \longrightarrow \varphi'$ and $g: \psi \longrightarrow \psi'$ we define

$$\mathbb{D}(f,g)(\underline{w},x,y) = \begin{cases} T^*(f(\underline{u},d(x),y),x,y) & \text{if} \quad M_T(x) \ \&\ \underline{w} = T^*(\underline{u},x,y) \\ F^*(g(\underline{u},d(x),y),x,y) & \text{if} \quad M_F(x) \ \&\ \underline{w} = T^*(\underline{u},x,y) \end{cases} .$$

Next define objects I,L,R of $\mathcal{F}_3$ as corresponding to the logical programs $\forall x X(x,x)$; $\Sigma,\forall x\forall y(D_0(x,y) \supset X(x,y))$; $\Sigma,\forall x\forall y(D_1(x,y) \supset X(x,y))$ respectively.

Theorem 1. *The category* $\mathcal{F}_3$ *is a* DM*-category w.r.t.* $\mathbb{M},\mathbb{D}$,I,L,R *as defined above and suitable natural isomorphisms* $\underline{\alpha},\underline{\lambda},\underline{\rho},\underline{l},\underline{r},\underline{i}$.

The proof of this theorem is similar to that of Theorem 4.2.1.□

Theorem 2. *The category* $\mathcal{F}_3$ *is iterative and there is a standard translation functor in it.*

The proof of this theorem follows that of Theorem 1.4.2. We can not apply immediately the last theorem because it is not seen how to construct limit to arbitrary ω-sequence in $\mathcal{F}_3$. But we can construct limits to the concrete ω-sequences arising as in the proof in Theorem 1.4.2. This is done by a direct construction similar to that in the proof of Theorem 4.2.2 and using an obvious logical program for the iteration. For the translation functor the proof is similar but easier.□

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