Institute for Logic, Language and Computation
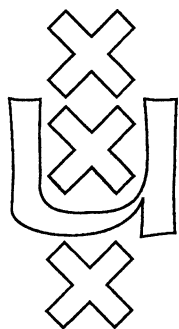
REMARKS ON THE THEORY OF SEMI-FUNCTORS

Raymond Hoofman
Ieke Moerdijk

University of Amsterdam

The ILLC Prepublication Series

1990

Logic, Semantics and Philosophy of Language

LP-90-01 Jaap van der Does   A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof   Dynamic Montague Grammar
LP-90-03 Renate Bartsch   Concept Formation and Concept Composition
LP-90-04 Aarne Ranta   Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn   Nominal Tense Logic
LP-90-06 Gennaro Chierchia   The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia   Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks   Flexible Montague Grammar
LP-90-09 Paul Dekker   The Scope of Negation in Discourse, towards a Flexible Dynamic Montague grammar
LP-90-10 Theo M.V. Janssen   Models for Discourse Markers
LP-90-11 Johan van Benthem   General Dynamics
LP-90-12 Serge Lapierre   A Functional Partial Semantics for Intensional Logic
LP-90-13 Zhisheng Huang   Logics for Belief Dependence
LP-90-14 Jeroen Groenendijk, Martin Stokhof   Two Theories of Dynamic Semantics
LP-90-15 Maarten de Rijke   The Modal Logic of Inequality
LP-90-16 Zhisheng Huang, Karen Kwast   Awareness, Negation and Logical Omniscience
LP-90-17 Paul Dekker   Existential Disclosure, Implicit Arguments in Dynamic Semantics

Mathematical Logic and Foundations

ML-90-01 Harold Schellinx   Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten   A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema   Relational Games
ML-90-04 Maarten de Rijke   Unary Interpretability Logic
ML-90-05 Domenico Zambella   Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten   Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke   A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx   Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani   Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen   Randomness in Set Theory
ML-90-11 Paul C. Gilmore   The Consistency of an Extended NaDSet

Computation and Complexity Theory

CT-90-01 John Tromp, Peter van Emde Boas   Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette   A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar   Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Edith Spaan, Leen Torenvliet   Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast   Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas   Dynamic Data Structures on Multiple Storage Media, a Tutorial
CT-90-07 Kees Doets   Greatest Fixed Points of Logic Programs
CT-90-08 Fred de Geus, Ernest Rotterdam, Sieger van Denneheuvel, Peter van Emde Boas   Physiological Modelling using RL
CT-90-09 Roel de Vrijer   Unique Normal Forms for Combinatory Logic with Parallel Conditional, a case study in conditional rewriting

Other Prepublications

X-90-01 A.S. Troelstra   Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke   Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev   On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04   Annual Report 1989
X-90-05 Valentin Shehtman   Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy   Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov   The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev   Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov   On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas   An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone   Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke   Bi-Unary Interpretability Logic
X-90-13 K.N. Ignatiev   Dzhaparidze's Polymodal Logic: Arithmetical Completeness, Fixed Point Property, Craig's Property
X-90-14 L.A. Chagrova   Undecidable Problems in Correspondence Theory
X-90-15 A.S. Troelstra   Lectures on Linear Logic

1991

Logic, Semantics and Philosophy of Langauge

LP-91-01 Wiebe van der Hoek, Maarten de Rijke   Generalized Quantifiers and Modal Logic
LP-91-02 Frank Veltman   Defaults in Update Semantics
LP-91-03 Willem Groeneveld   Dynamic Semantics and Circular Propositions
LP-91-04 Makoto Kanazawa   The Lambek Calculus enriched with Additional Connectives
LP-91-05 Zhisheng Huang, Peter van Emde Boas   The Schoenmakers Paradox: Its Solution in a Belief Dependence Framework
LP-91-06 Zhisheng Huang, Peter van Emde Boas   Belief Dependence, Revision and Persistence
LP-91-07 Henk Verkuyl, Jaap van der Does   The Semantics of Plural Noun Phrases
LP-91-08 Víctor Sánchez Valencia   Categorial Grammar and Natural Reasoning
LP-91-09 Arthur Nieuwendijk   Semantics and Comparative Logic
LP-91-10 Johan van Benthem   Logic and the Flow of Information

Mathematical Logic and Foundations

ML-91-01 Yde Venema   Cylindric Modal Logic
ML-91-02 Alessandro Berarducci, Rineke Verbrugge   On the Metamathematics of Weak Theories
ML-91-03 Domenico Zambella   On the Proofs of Arithmetical Completeness for Interpretability Logic
ML-91-04 Raymond Hoofman, Harold Schellinx   Collapsing Graph Models by Preorders
ML-91-05 A.S. Troelstra   History of Constructivism in the Twentieth Century
ML-91-06 Inge Bethke   Finite Type Structures within Combinatory Algebras
ML-91-07 Yde Venema   Modal Derivation Rules
ML-91-08 Inge Bethke   Going Stable in Graph Models
ML-91-09 V.Yu. Shavrukov   A Note on the Diagonalizable Algebras of PA and ZF
ML-91-10 Maarten de Rijke, Yde Venema   Sahlqvist's Theorem for Boolean Algebras with Operators
ML-91-11 Rineke Verbrugge   Feasible Interpretability
ML-91-12 Johan van Benthem   Modal Frame Classes, revisited

Computation and Complexity Theory

CT-91-01 Ming Li, Paul M.B. Vitányi   Kolmogorov Complexity Arguments in Combinatorics
CT-91-02 Ming Li, John Tromp, Paul M.B. Vitányi   How to Share Concurrent Wait-Free Variables
CT-91-03 Ming Li, Paul M.B. Vitányi   Average Case Complexity under the Universal Distribution Equals Worst Case Complexity
CT-91-04 Sieger van Denneheuvel, Karen Kwast   Weak Equivalence
CT-91-05 Sieger van Denneheuvel, Karen Kwast   Weak Equivalence for Constraint Sets
CT-91-06 Edith Spaan   Census Techniques on Relativized Space Classes
CT-91-07 Karen L. Kwast   The Incomplete Database
CT-91-08 Kees Doets   Levationis Laus
CT-91-09 Ming Li, Paul M.B. Vitányi   Combinatorial Properties of Finite Sequences with high Kolmogorov Complexity
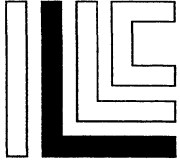
REMARKS ON THE THEORY OF SEMI-FUNCTORS

Raymond Hoofman
Department of Mathematics and Computer Science
University of Amsterdam

Ieke Moerdijk
Department of Mathematics and Computer Science
University of Utrecht

Remarks on
the Theory of Semi-Functors

R. Hoofman*   I. Moerdijk[†]

University of Amsterdam   University of Utrecht

Abstract

By establishing an appropriate equivalence, we show that the theory of semi-functors can be fully embedded in the theory of (ordinary) functors. As a result, standard properties and constructions on functors extend automatically to semi-functors.

1 Introduction

Categories and functors play a central role in the semantics of various formal languages. For example, it is well-known that extensional typed lambda calculi are essentially equivalent to cartesian closed categories (see e.g. [12]).

In order to develop a similar categorical semantics for *non*-extensional typed lambda calculi, S. Hayashi introduced in [3] the notion of a *semi-functor* between categories. He developed in loc. sit. the theory of semi-adjunctions and semi cartesian closed categories, and showed that the latter correspond to non-extensional typed lambda calculi. He also showed how the theory of semi-functors provides a natural approach to the "extensionalization" of models of the lambda calculus. Later on, semi-functors were studied in the context of linear logic and the second-order lambda calculus; see e.g. [4, 7, 9, 14, 15, 16]. A systematic exposition of the theory of semi-functors is given in [5].

---

*Email: raymond@fwi.uva.nl.

[†]Email: moerdijk@math.ruu.nl.

Although the introduction of semi-functors at first sight appears to be a proper extension of ordinary category theory, we show in this paper that the theory of semi-functors can be fully embedded into the theory of (ordinary) functors. More specifically, we establish an appropriate equivalence between, on the one hand, the theory of categories, semi-functors, and natural transformations between such semi-functors, and, on the other hand, the theory of Karoubi-complete categories, ordinary functors, and ordinary natural transformations. A precise statement of this equivalence occurs as our main Theorem 1 below.

As a consequence of this result, many standard properties and constructions of functors extend automatically to semi-functors (see section 3). In particular, many results of [5] are immediate consequences of our main theorem.

In a next section (section 4), we investigate the relation of our approach to the work of Koymans [10], while in the final section we will present an extension of our main theorem to the context of fibred (or indexed) categories. We are motivated by the fact that such fibred categories (with appropriate properties) provide models of second-order languages (see e.g. [9, 17]). Indeed, in [8] we will show that this extension to fibred categories provides a method for the extensionalization of second-order calculi.

2   Semi-Functors and the Karoubi Envelope

In this preliminary section we review some basic definitions. We assume that the reader is familiar with the notions of *2-category*, *2-functor*, etc. (a basic reference is [11]). For a 2-category C, the *category* obtained by forgetting the 2-cells of C will again be denoted by C.

Let C and D be categories. Recall from [3, 5] and the introduction that a *semi-functor* $F : C \to D$ maps objects, respectively arrows, in C to objects, respectively arrows, in D, preserving domain, codomain and composition. Hence a semi-functor is defined just as a functor, except that a semi-functor need not preserve identities. For semi-functors F and $G : C \to D$, a *natural transformation between semi-functors* $\alpha : F \to G$ is a family $(\alpha_C : FC \to GC \mid C \in C)$ of arrows in the category D, satisfying the two conditions

1. $\alpha_D \circ F(f) = G(f) \circ \alpha_C,$

2. $\alpha_C \circ F(id_C) = \alpha_C$.

Semi-functors and natural transformations between semi-functors can be composed in the obvious ways, thus yielding a 2-category denoted by Cat_s. The usual 2-category Cat of categories, functors and natural transformations is a subcategory of Cat_s, full on 2-cells.

Recall that an *idempotent* in a category C is an arrow $p : C \to C$ in C such that $p \circ p = p$. A *splitting* of p is a triple $\langle R, r, s \rangle$ where $r : C \to R$ and $s : R \to C$ are arrows satisfying $r \circ s = id_R$ and $s \circ r = p$. Such a splitting of a given idempotent is unique up to isomorphism. One says that a category C *has splittings* (or that *idempotents split in* C, or that C is *Karoubi complete*) iff a splitting exists for every idempotent arrow. (Observe that we do not require a "canonical" *choice* of splittings; see section 4 below.) We write Cat_i for the full 2-subcategory of Cat with as objects the categories that have splittings. Thus

$$\mathsf{Cat}_i \hookrightarrow \mathsf{Cat} \hookrightarrow \mathsf{Cat}_s.$$

The *Karoubi envelope* $\mathcal{K}\mathsf{C}$ of a category C is obtained by freely adding splittings for idempotents in C: objects of $\mathcal{K}\mathsf{C}$ are pairs $\langle C, p \rangle$ where $p : C \to C$ is an idempotent in C, while arrows $f : \langle C, p \rangle \to \langle D, q \rangle$ in $\mathcal{K}\mathsf{C}$ are arrows $f : C \to D$ in C with the property that $q \circ f \circ p = f$. (Note that the identity arrow on $\langle C, p \rangle$ is not the identity arrow id_C of C but $p : \langle C, p \rangle \to \langle C, p \rangle$.) It is easy to see that any idempotent $f : \langle C, p \rangle \to \langle C, p \rangle$ in $\mathcal{K}\mathsf{C}$ splits, via the object $R = \langle C, f \rangle$.

If $F : \mathsf{C} \to \mathsf{D}$ is a semi-functor, then one obtains an ordinary functor $\mathcal{K}(F) : \mathcal{K}\mathsf{C} \to \mathcal{K}\mathsf{D}$ defined on objects by $\mathcal{K}(F)\langle C, p \rangle = \langle FC, F(p) \rangle$, while on arrows $\mathcal{K}(F)(f) = F(f)$. Similarly, a semi natural transformation $\alpha : F \to G$ between semi-functors yields an ordinary natural transformation $\mathcal{K}(\alpha) : \mathcal{K}(F) \to \mathcal{K}(G)$, with components $\alpha_{\langle C, p \rangle} = \alpha_C \circ F(p)(= G(p) \circ \alpha_C)$. In this way one obtains a 2-functor

$$\mathcal{K} : \mathsf{Cat}_s \to \mathsf{Cat}_i.$$

In [2] it is observed that, when viewed as a functor $\mathcal{K} : \mathsf{Cat}_s \to \mathsf{Cat}$, the Karoubi envelope is right adjoint to the inclusion $\mathsf{Cat} \hookrightarrow \mathsf{Cat}_s$; see also [5].

3   The Main Theorem

Our main result is the following:

Theorem 1 *(Main Theorem) The Karoubi envelope $\mathcal{K}$ defines an equivalence of 2-categories*

$$\mathcal{K} : \mathsf{Cat}_s \stackrel{\simeq}{\Rightarrow} \mathsf{Cat}_i.$$

As we shall see, this theorem is not difficult to prove. Nonetheless, it is fundamental for the theory of semi-functors, since it allows us to extend properties of functors and natural transformations in a systematic way to semi-functors and natural transformations between semi-functors. Some instances are mentioned below.

For the proof of the theorem, first recall that a functor $F : \mathsf{C} \to \mathsf{D}$ is an (ordinary) equivalence of categories iff F induces an isomorphism from $\mathsf{C}(C, C')$ to $\mathsf{D}(FC, FC')$ and, moreover, F is surjective up to isomorphism (in the sense that for each $D \in \mathsf{D}$ there exists $C \in \mathsf{C}$ such that $FC \cong D$). Uniformly substituting "equivalent" for "isomorphic" in this definition yields the notion of an equivalence of 2-categories. In detail,

Definition 2 *Let C, D be 2-categories. A 2-functor $F : \mathsf{C} \to \mathsf{D}$ is an equivalence of 2-categories (and C and D are 2-equivalent) iff the following conditions hold:*

1. For any objects $C, C' \in \mathsf{C}$ the 2-functor F restricted to $\mathsf{C}(C, C') \to \mathsf{D}(FC, FC')$ is an ordinary equivalence of categories.

2. The 2-functor F is surjective up to equivalence: for each $D \in \mathsf{D}$ there exists $C \in \mathsf{C}$ such that $FC \simeq D$ (i.e., there are one-cells $f : FC \to D$, $g : D \to FC$, and invertible 2-cells $f \circ g \cong 1_D$ and $g \circ f \cong 1_{FC}$).

Now we give the proof of Theorem 1:

Proof: For any category C, there is an evident functor $\eta_{\mathsf{C}} : \mathsf{C} \to \mathcal{K}(\mathsf{C})$ given on objects by $\eta_{\mathsf{C}}(C) = \langle C, id_C \rangle$. This functor is full and faithful. Furthermore, it is well-known that η_{C} is an equivalence of categories when idempotents split in C. (Indeed, if $\langle R, r, s \rangle$ is a splitting for $\langle C, p \rangle$, then r and s define an isomorphism $\eta_{\mathsf{C}}(R) \cong \langle C, p \rangle$ in $\mathcal{K}\mathsf{C}$.) This shows that the 2-functor $\mathcal{K} : \mathsf{Cat}_s \to \mathsf{Cat}_i$ is surjective up to equivalence.

The "forgetful" semi-functor $\varepsilon_{\mathsf{C}} : \mathcal{K}(\mathsf{C}) \to \mathsf{C}$ runs into the opposite direction of η_{C}, and is defined on objects by $\varepsilon_{\mathsf{C}}\langle C, p \rangle = C$. Both η_{C} and ε_{C} are natural in C in the sense that for any functor F,

$$\eta_{\mathsf{D}} \circ F = \mathcal{K}(F) \circ \eta_{\mathsf{C}} : \mathsf{C} \to \mathcal{K}\mathsf{D}, \tag{1}$$

4

and for any semi-functor G,

$$\varepsilon_{\mathsf{D}} \circ \mathcal{K}(G) = G \circ \varepsilon_{\mathsf{C}} : \mathcal{K}\mathsf{C} \to \mathsf{D}. \tag{2}$$

Observe, furthermore, that for any category C we have the identity

$$\mathcal{K}(\varepsilon_{\mathsf{C}}) \circ \eta_{\mathcal{K}\mathsf{C}} = id : \mathcal{K}\mathsf{C} \to \mathcal{K}\mathsf{C}, \tag{3}$$

as well as a canonical natural isomorphism

$$\mathcal{K}(\eta_{\mathsf{C}}) \cong \eta_{\mathcal{K}\mathsf{C}} : \mathcal{K}\mathsf{C} \to \mathcal{K}^2\mathsf{C}. \tag{4}$$

To complete the proof of the theorem, we now show that for any categories C and D, the Karoubi envelope gives an (ordinary) equivalence of categories

$$\mathcal{K} : \mathsf{Cat}_s(\mathsf{C}, \mathsf{D}) \xrightarrow{\sim} \mathsf{Cat}_i(\mathcal{K}\mathsf{C}, \mathcal{K}\mathsf{D}).$$

First we show that this restriction of $\mathcal{K}$ to hom-sets is surjective up to isomorphism. To this end, let $F : \mathcal{K}\mathsf{C} \to \mathcal{K}\mathsf{D}$ be any functor, and consider the semi-functor $\overline{F} = \varepsilon_{\mathsf{D}} \circ F \circ \eta_{\mathsf{C}} : \mathsf{C} \to \mathsf{D}$. Then

$$
\begin{aligned}
\mathcal{K}\overline{F} &= \mathcal{K}(\varepsilon_{\mathsf{D}}) \circ \mathcal{K}(F) \circ \mathcal{K}(\eta_{\mathsf{C}}) \\
&\overset{(4)}{\cong} \mathcal{K}(\varepsilon_{\mathsf{D}}) \circ \mathcal{K}(F) \circ \eta_{\mathcal{K}\mathsf{C}} \\
&\overset{(1)}{=} \mathcal{K}(\varepsilon_{\mathsf{D}}) \circ \eta_{\mathcal{K}\mathsf{C}} \circ F \\
&\overset{(3)}{=} F.
\end{aligned}
$$

Next we observe that the Karoubi envelope $\mathcal{K}$ restricted to hom-sets is full and faithful. Indeed, if $F, G : \mathsf{C} \to \mathsf{D}$ are semi-functors, then the operation

$$\mathrm{Nat}(F, G) \to \mathrm{Nat}(\mathcal{K}(F), \mathcal{K}(G)) : \alpha \mapsto \mathcal{K}(\alpha),$$

from natural transformations $F \to G$ to natural transformations $\mathcal{K}(F) \to \mathcal{K}(G)$, is a bijection. Its inverse is

$$\mathrm{Nat}(\mathcal{K}(F), \mathcal{K}(G)) \to \mathrm{Nat}(F, G) : \beta \mapsto \beta \cdot \eta_{\mathsf{C}},$$

where the components of $\beta \cdot \eta_{\mathsf{C}}$ are given by $(\beta \cdot \eta_{\mathsf{C}})_C = \beta_{\langle C, id \rangle}$, for any object $C \in \mathsf{C}$. ∎

5

Note that the inclusion $\mathsf{Cat}_i \hookrightarrow \mathsf{Cat}_s$ is a quasi inverse for $\mathcal{K} : \mathsf{Cat}_s \to \mathsf{Cat}_i$. In other words, any category C is semi-equivalent to its Karoubi envelope $\mathcal{K}\mathsf{C}$ via $\eta_\mathsf{C} : \mathsf{C} \to \mathcal{K}\mathsf{C}$, $\varepsilon_\mathsf{C} : \mathcal{K}\mathsf{C} \to \mathsf{C}$.

A large part of the theory of semi-functors presented in [5] can be viewed as an immediate application of Theorem 1. For example, when one *defines* a semi-adjunction between semi-functors $F : \mathsf{C} \to \mathsf{D}$, $G : \mathsf{D} \to \mathsf{C}$ in the standard way in the language of the 2-category Cat_s (see [11]), then F is left semi-adjoint to G iff $\mathcal{K}(F) : \mathcal{K}\mathsf{C} \to \mathcal{K}\mathsf{D}$ is left adjoint to $\mathcal{K}(G)$. Furthermore, all well-known categorical properties of adjoint functors immediately transport along the equivalence $\mathsf{Cat}_i \simeq \mathsf{Cat}_s$ to give the same properties of semi-adjoint semi-functors. (For example, semi-adjoints are unique up to semi natural isomorphism, semi-adjunctions compose, etc., etc.)

The same principal applies to other properties which can be expressed in the language of 2-categories. Thus a category C has semi-products iff $\mathcal{K}\mathsf{C}$ has products. (Indeed, the existence of a right (semi-)adjoint to the diagonal $\mathsf{C} \to \mathsf{C} \times \mathsf{C}$ is a 2-categorical property of C as an object of Cat_s, and the equivalence between Cat_s and Cat_i preserves products and diagonals.) Similarly, a category is semi cartesian closed iff $\mathcal{K}\mathsf{C}$ is cartesian closed. As a last example, one can develop a theory of semi-monads and semi-algebras by transporting the usual theory of algebras and monads (formulated for objects in a 2-category in [11, 18]) along the equivalence of Theorem 1. The notion of a semi-algebra finds an application in [6].

4 The Karoubi Envelope as a Monad

The main purpose of this section is to point out some relations with the work of Koymans [10]. We assume that the reader is familiar with the notions of a *monad*, the *Kleisli category construction*, and the *Eilenberg-Moore category* (a basic reference is [13]).

To begin with, we observe that the Karoubi envelope yields a monad on the category Cat. Indeed, the functor $\mathcal{K} : \mathsf{Cat} \to \mathsf{Cat}$ (i.e., $\mathsf{Cat} \to \mathsf{Cat}_i \hookrightarrow \mathsf{Cat}$) comes equipped with natural transformations $\eta : Id \to \mathcal{K}$ and $\mu : \mathcal{K}^2 \to \mathcal{K}$ as follows: define η as in the proof of Theorem 1, and take $\mu_\mathsf{C}\langle\langle C, p\rangle, f\rangle = \langle C, f\rangle$. (Actually, $\mathcal{K}$ is a 2-comonad on the 2-category Cat.)

Next we consider the Kleisli category of the Karoubi monad. Recall that

the objects in the Kleisli category are the same as in Cat, viz. categories, whereas arrows $F : \mathsf{C} \to \mathsf{D}$ in the Kleisli category are functors $F : \mathsf{C} \to \mathcal{K}\mathsf{D}$ in Cat. We observe that each such functor F gives rise to a semi-functor $\eta_\mathsf{D} \circ F : \mathsf{C} \to \mathsf{D}$, while each semi-functor $G : \mathsf{C} \to \mathsf{D}$ yields a functor $\mathcal{K}(G) \circ \varepsilon_{\mathcal{K}\mathsf{D}} : \mathsf{C} \to \mathsf{D}$. As this defines a one-to-one correspondence between arrows in Cat_s and arrows in the Kleisli category of $\mathcal{K}$, it follows that the last category is actually isomorphic to Cat_s.

Proposition 3 *The category* Cat_s *of categories and semi-functors is isomorphic to the Kleisli category of the Karoubi monad* $\langle \mathcal{K}, \eta, \mu \rangle$ *on* Cat.

We remark that this isomorphism can also be extended to an isomorphism of 2-categories. As an easy consequence of the proposition, standard monad-theory tells us that there exists a (2-)adjunction between Cat and Cat_s, which is of course the adjunction mentioned in section 2.

We consider the Eilenberg-Moore category (the category of algebras) of the Karoubi monad. Recall that the objects of this category are functors of the form $F : \mathcal{K}\mathsf{C} \to \mathsf{C}$ that satisfy $F \circ \eta_\mathsf{C} = Id_\mathsf{C}$ and $F \circ \mu_\mathsf{C} = F \circ \mathcal{K}(F)$. An arrow $H : (F : \mathcal{K}\mathsf{C} \to \mathsf{C}) \to (G : \mathcal{K}\mathsf{D} \to \mathsf{D})$ consists of a functor $H : \mathsf{C} \to \mathsf{D}$ that satisfies the requirement $G \circ \mathcal{K}(H) = H \circ F$. The Eilenberg-Moore category can be described as the category Cat_{ic} of categories with *chosen* splittings and splitting-preserving functors, defined as follows:

Definition 4 *A category* C *has chosen splittings if there is given a family* $(R_p, r_p, s_p \mid p$ *idempotent in* $\mathsf{C})$, *where each triple* $\langle R_p, r_p, s_p \rangle$ *is a splitting of* p, *satisfying the following requirements:*

1. $R_{id_A} = A$,

2. $r_{id_A} = id_A$,

3. $s_{id_A} = id_A$.

Moreover, for each pair of idempotents p, q *satisfying* $p \circ q \circ p = p$,

4. $R_{r_p \circ q \circ s_p} = R_q$,

5. $r_{r_p \circ q \circ s_p} \circ r_p \circ q = r_q$,

6. $q \circ s_f \circ s_{r_p \circ q \circ s_p} = s_q$.

7

If also D *has chosen splittings denoted by* $\langle R'_p, r'_p, s'_p \rangle$, *then the functor* $H :$ C → D *is said to* preserve chosen splittings *iff* $H(R_p) = R'_{H(p)}$, $H(r_p) = r'_{H(p)}$, *and* $H(s_p) = s'_{H(p)}$. *For splitting preserving functors* $H, K :$ C → D, *a natural transformation* $\alpha : H \to K$ *is said to* preserve chosen splittings *iff*

$$r'_{K(p)} \circ \alpha_A \circ H(p) \circ s'_{H(p)} = \alpha_{R_p}.$$

The notion from [10] of a category with *explicit* splittings differs slightly from the notion introduced here. Specifically, in the definition of a category with explicit splittings, requirement (5) above is replaced by the somewhat stronger requirement

$$r_{r_p \circ q \circ s_p} \circ r_p = r_q$$

and analogous for requirement (6).

Let Cat_{ic} denote the (2-)category of categories with chosen splittings and splitting preserving functors. Similarly, Cat_{ie} denotes the full (2-)subcategory of Cat_{ic} of categories with explicit splittings.

Proposition 5 *The Eilenberg-Moore category of the Karoubi monad on* Cat *is isomorphic to the category* Cat_{ic} *of categories with chosen splittings.*

Proof: Each chosen splitting on C determines an algebra $F : \mathcal{K}C \to C$ as follows. Let $(R_p, r_p, s_p \mid p$ idempotent in C$)$ be a family of chosen splittings on C, then take $F\langle C, p \rangle = R_p$ and $F(f : \langle C, p \rangle \to \langle D, q \rangle) = r_q \circ f \circ s_p$. The other way round, algebras $F : \mathcal{K}C \to C$ give rise to chosen splittings: take $R_p = F\langle C, p \rangle$, $r_p = F(p)$, and $s_p = F(p)$. It is easy to check that this yields a one-to-one correspondence between algebras and chosen splittings. Furthermore, splitting preserving functors correspond to morphisms between algebras. ∎

As in the case of proposition 3, the isomorphism of proposition 5 can be extended to an isomorphism of 2-categories. Furthermore, by standard monad-theory we find an adjunction between Cat and Cat_{ic}, which restricts to the (2-)adjunction described in [10] between Cat and Cat_{ie}.

5 Extension to Fibred Categories

Motivated by the fact that the (categorical) semantics of second-order languages require fibred categories, we indicate in this section how our main

Theorem 1 can be extended to this context. These results will be applied in [8]. We assume that the reader is familiar with the notions of *fibred category* and *cartesian functor* (see e.g. [1]).

Let B be a fixed category, and let Fib(B) be the 2-category of fibered categories over B. So its objects are (Grothendieck) fibrations $P : \mathsf{E} \to \mathsf{B}$, its arrows are cartesian functors over B, and its 2-cells are vertical natural transformations, all as usual. Say a fibered category $P : \mathsf{E} \to \mathsf{B}$ *has fibrewise splittings* iff each category $\mathsf{E}_B = P^{-1}(B)$ has splittings. Denote by $\mathsf{Fib}_{vi}(\mathsf{B})$ the full 2-subcategory of Fib(B) whose objects have fibrewise splittings (where *vi* stands for *vertical idempotent*).

For a fibred category $P : \mathsf{E} \to \mathsf{B}$, we construct a new fibred category $\mathcal{K}_v\mathsf{E} \to \mathsf{B}$ as follows. The objects in $\mathcal{K}_v\mathsf{E}$ are pairs $\langle E, p \rangle$ where E is an object in E and $p : E \to E$ is a vertical idempotent: $P(p) = id_{PE}$ and $p^2 = p$. The arrows $\langle E, p \rangle \to \langle E', p' \rangle$ in $\mathcal{K}_v\mathsf{E}$ are arrows $f : E \to E'$ in E satisfying $p' \circ f \circ p = f$. The composition and identities for $\mathcal{K}_v\mathsf{E}$ are obvious (just as for the ordinary $\mathcal{K}$). Finally, we extend the functor $P : \mathsf{E} \to \mathsf{B}$ to a functor $\mathcal{K}_v\mathsf{E} \to \mathsf{B}$ by $\langle E, p \rangle \mapsto PE$ on objects and in the obvious way on arrows. We call $\mathcal{K}_v(P)$ the *vertical Karoubi envelope* of P.

Proposition 6 *The functor* $\mathcal{K}_v\mathsf{E} \to \mathsf{B}$ *defines a fibred category with fibrewise splittings.*

Proof: Let $u : B' \to B$ be an arrow in B and $\langle E, p \rangle$ an object in $\mathcal{K}_v\mathsf{E}$ with $PE = B$. We indicate how to define a cartesian lifting $u^\sharp : \langle E', p' \rangle \to \langle E, p \rangle$ of u, and leave further details to the reader. First choose a cartesian lifting $u^\flat : E' \to E$ of u in E. Then since E is fibred over B there is a unique $p' : E' \to E'$ with $u^\flat \circ p' = u^\flat$. By uniqueness, p' is idempotent. Finally, take $u^\sharp = p \circ u^\flat : \langle E', p' \rangle \to \langle E, p \rangle$ as the desired cartesian lifting of u in $\mathcal{K}_v\mathsf{E}$. ∎

It is easy to see that if the fibred category $\mathsf{E} \to \mathsf{B}$ is defined from an indexed category $\mathsf{B}^{op} \to \mathsf{Cat}$, then $\mathcal{K}_v\mathsf{E} \to \mathsf{B}$ is induced from the composite indexed category $\mathsf{B}^{op} \to \mathsf{Cat} \xrightarrow{\mathcal{K}} \mathsf{Cat}$.

Many properties of fibred categories are preserved by the vertical Karoubi envelope. For models of second order languages the most relevant ones are: (1) If $\mathsf{E} \to \mathsf{B}$ is fibrewise cartesian closed, then so is $\mathcal{K}_v\mathsf{E} \to \mathsf{B}$. (2) If $\mathsf{E} \to \mathsf{B}$ is complete (cocomplete), then so is $\mathcal{K}_v\mathsf{E} \to \mathsf{B}$. [For this last preservation

fibred category is a semi-fibred category. We leave it to the reader to check the following proposition: the functor $\mathcal{K}_v\mathsf{E} \to \mathsf{B}$ defines a fibred category iff $\mathsf{E} \to \mathsf{B}$ is a semi-fibred category.

References

[1] Bénabou, J., *Fibred Categories and the Foundations of Naive Category Theory*, J. Symb. Log., vol. 50, pp. 10-37.

[2] Freyd, P. A., A. Scedrov, *Categories, Allegories*, North-Holland Publ. Comp., Amsterdam, 1989.

[3] Hayashi, S., *Adjunction of Semifunctors: Categorical Structures in Non-Extensional Lambda-Calculus*, Theor. Comput. Sci. 41 (1985), pp. 95-104.

[4] Hoofman, R., *Non-Stable Models of Linear Logic*, in: A. Nerode, M. Taitslin (Eds.), Logical Foundations of Computer Science, Proceedings, Lect. Notes in Comp. Sci., vol. 620, Springer-Verlag, Berlin, 1992, pp. 209-220.

[5] Hoofman, R., *The Theory of Semi-Functors*, Mathematical Structures in Computer Science, vol. 3, pp. 93-128.

[6] Hoofman, R., *Information Systems as Coalgebras*, ILLC Prepublication Series, ML-92-08, University of Amsterdam.

[7] Hoofman, R., H. Schellinx, *Collapsing Graph Models by Preorders*, in: D.H. Pitt, P.-L. Curien, S. Abramsky, A.M. Pitts, A. Poigné and D.E. Rydeheard (Eds.), Category Theory and Computer Science, Proceedings, Lect. Notes in Comp. Sci., vol. 530, Springer-Verlag, Berlin, 1991, pp. 53-73.

[8] Hoofman, R., I. Moerdijk, *On the Extensionalization of Second-Order Models*, manuscript.

[9] Jacobs, B., *Semantics of Second Order Lambda Calculus*, Mathematical Structures in Computer Science, vol. 2, 1991.

property, recall from [1] that $P : \mathsf{E} \to \mathsf{B}$ is said to be complete (cocomplete) iff B has pullbacks and if, for each morphism u in B, the functor $u^* = P(u)$ has a right adjoint (left adjoint) satisfying the Beck-Chevalley condition.]

Let $P : \mathsf{E} \to \mathsf{B}$ and $P' : \mathsf{E}' \to \mathsf{B}$ be fibred categories over B. A semi-functor $F : \mathsf{E} \to \mathsf{E}'$ over B (i.e., $P' \circ F = P$) induces an ordinary functor $\mathcal{K}_v(F) : \mathcal{K}_v\mathsf{E} \to \mathcal{K}_v\mathsf{E}'$, exactly as in section 2. Furthermore, a natural transformation $\alpha : F \to G$ between two such semi-functors induces an ordinary natural transformation $\mathcal{K}_v(\alpha) : \mathcal{K}_v(F) \to \mathcal{K}_v(G)$. Say that F is *semi-cartesian* iff $\mathcal{K}_v(F)$ is cartesian, and say that α is *vertical* iff $P(\alpha_E) = id_{FE}$ for all $E \in \mathsf{E}$. This defines a 2-category

$$\mathsf{Fib}_s(\mathsf{B})$$

with fibred categories as objects, cartesian semi-functors as 1-cells, and vertical natural transformations between such semi-functors as 2-cells. The following result is now a straightforward extension of Theorem 1.

Theorem 7 *The vertical Karoubi envelope defines an equivalence of 2-categories*

$$\mathcal{K}_v : \mathsf{Fib}_s(\mathsf{B}) \to \mathsf{Fib}_{vi}(\mathsf{B}).$$

Remark: For a semi-functor $F : \mathsf{E} \to \mathsf{E}'$ over B as above, the property of being semi-cartesian can be described explicitly as follows. The semi-functor F is semi-cartesian iff for every cartesian arrow $f : D \to E$ in E its image $F(f)$ has the following property: $F(f)$ can be factorized uniquely (up to isomorphism) as $g \circ h$, where $h : FD \to K$ vertical and $g : K \to FE$ cartesian; for this factorization there exists a vertical $h' : K \to FD$ so that $h' \circ h = F(id_D)$ and $g \circ h \circ h' = F(id_E) \circ g$.

Remark: Various of the results in this section can be further extended. For example, proposition 6 above can be strengthened as follows. Let $u : B' \to B$ be an arrow in B. A *semi-cartesian arrow over* u is an arrow $f : E_1 \to E$ over u (i.e., $P(f) = u$) equipped with an operation assigning to each arrow $g : E_2 \to E$ and factorization $P(g) = u \circ v$ an arrow $\bar{g} : E_2 \to E_1$ over v satisfying $f \circ \bar{g} = g$. Moreover, we require the operation to be natural in the sense that $\bar{g} \circ h = \overline{g \circ h}$, for each $h : D \to E_2$. A functor $P : \mathsf{E} \to \mathsf{B}$ is a *semi-fibred category* iff for each $u : B' \to B$ in B and $E \in \mathsf{E}_B$ there exists a semi-cartesian morphism over u with codomain E. Note that each

10

[10] Koymans, C. P. J., *Models of the Lambda Calculus*, Inform. and Control 52 (1982), pp. 306-322.

[11] Kelly, G. M., R. Street, *Review of the Elements of 2-Categories*, in: Category Seminar, Lect. Notes in Mathematics, vol. 420, Springer-Verlag, Berlin, 1974, pp. 75-103.

[12] Lambek, J., P. J. Scott, *Introduction to Higher Order Categorical Logic*, Studies in Advanced Mathematics 7, Cambridge University Press, 1986.

[13] Maclane, S., *Categories for the Working Mathematician*, Springer-Verlag, New-York, 1971.

[14] Martini, S., *An Interval Model for Second Order Lambda Calculus*, in: D. H. Pitt, A. Poigné and D. Rydeheard (Eds.), Category Theory and Computer Science, Proceedings, Lect. Notes in Comp. Sci., vol. 283, Springer-Verlag, Berlin, 1987, pp. 219-237.

[15] Martini, S., *Categorical Models for Non-Extensional λ-Calculi and Combinatory Logic*, Math. Struct. in Comp. Sci., vol. 2, 1992, pp. 327-357.

[16] Román, L., *On Partial Cartesian Closed Categories*, in: J.W. Gray and A. Scedrov (Eds.), Categories in Computer Science and Logic, Proceedings, Contemporary Mathematics, vol. 92, American Mathematical Society, Providence-Rhode Island, 1989, pp. 349-356.

[17] Seely, R. A. G., *Categorical Semantics for Higher Order Polymorphic Lambda Calculus*, J. Symb. Logic., No. 52, 1987, pp. 145-156.

[18] Street, R., *The Formal Theory of Monads*, Journal of Pure and Applied Algebra, 2 (1972), pp. 149-168.

The ILLC Prepublication Series

CT-91-10 John Tromp, Paul Vitányi   A Randomized Algorithm for Two-Process Wait-Free Test-and-Set
CT-91-11 Lane A. Hemachandra, Edith Spaan Quasi-Injective Reductions
CT-91-12 Krzysztof R. Apt, Dino Pedreschi Reasoning about Termination of Prolog Programs
Computational Linguistics
CL-91-01 J.C. Scholtes   Kohonen Feature Maps in Natural Language Processing
CL-91-02 J.C. Scholtes   Neural Nets and their Relevance for Information Retrieval
CL-91-03 Hub Prüst, Remko Scha, Martin van den Berg   A Formal Discourse Grammar tackling Verb Phrase Anaphora
Other Prepublications
X-91-01 Alexander Chagrov, Michael Zakharyaschev The Disjunction Property of Intermediate Propositional Logics
X-91-02 Alexander Chagrov, Michael Zakharyaschev On the Undecidability of the Disjunction Property of Intermediate Propositional
                                            Logics
X-91-03 V. Yu. Shavrukov   Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic
X-91-04 K.N. Ignatiev   Partial Conservativity and Modal Logics
X-91-05 Johan van Benthem   Temporal Logic
X-91-06   Annual Report 1990
X-91-07 A.S. Troelstra   Lectures on Linear Logic, Errata and Supplement
X-91-08 Giorgie Dzhaparidze   Logic of Tolerance
X-91-09 L.D. Beklemishev   On Bimodal Provability Logics for Π_1-axiomatized Extensions of Arithmetical
                                            Theories
X-91-10 Michiel van Lambalgen   Independence, Randomness and the Axiom of Choice
X-91-11 Michael Zakharyaschev   Canonical Formulas for K4. Part I: Basic Results
X-91-12 Herman Hendriks   Flexibele Categoriale Syntaxis en Semantiek: de proefschriften van Frans Zwarts en
                                            Michael Moortgat
X-91-13 Max I. Kanovich   The Multiplicative Fragment of Linear Logic is NP-Complete
X-91-14 Max I. Kanovich   The Horn Fragment of Linear Logic is NP-Complete
X-91-15 V. Yu. Shavrukov   Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic, revised
                                            version
X-91-16 V.G. Kanovei   Undecidable Hypotheses in Edward Nelson's Internal Set Theory
X-91-17 Michiel van Lambalgen   Independence, Randomness and the Axiom of Choice, Revised Version
X-91-18 Giovanna Cepparello   New Semantics for Predicate Modal Logic: an Analysis from a standard point of view
X-91-19 Papers presented at the Provability Interpretability Arithmetic Conference, 24-31 Aug. 1991, Dept. of Phil., Utrecht University
1992   Annual Report 1991
Logic, Semantics and Philosophy of Langauge
LP-92-01 Víctor Sánchez Valencia   Lambek Grammar: an Information-based Categorial Grammar
LP-92-02 Patrick Blackburn   Modal Logic and Attribute Value Structures
LP-92-03 Szabolcs Mikulás   The Completeness of the Lambek Calculus with respect to Relational Semantics
LP-92-04 Paul Dekker   An Update Semantics for Dynamic Predicate Logic
LP-92-05 David I. Beaver   The Kinematics of Presupposition
LP-92-06 Patrick Blackburn, Edith Spaan   A Modal Perspective on the Computational Complexity of Attribute Value Grammar
LP-92-07 Jeroen Groenendijk, Martin Stokhof A Note on Interrogatives and Adverbs of Quantification
LP-92-08 Maarten de Rijke   A System of Dynamic Modal Logic
LP-92-09 Johan van Benthem   Quantifiers in the world of Types
LP-92-10 Maarten de Rijke   Meeting Some Neighbours (a dynamic modal logic meets theories of change and
                                            knowledge representation)
LP-92-11 Johan van Benthem   A note on Dynamic Arrow Logic
LP-92-12 Heinrich Wansing   Sequent Caluli for Normal Modal Propositional Logics
LP-92-13 Dag Westerståhl   Iterated Quantifiers
LP-92-14 Jeroen Groenendijk, Martin Stokhof Interrogatives and Adverbs of Quantification
Mathematical Logic and Foundations
ML-92-01 A.S. Troelstra   Comparing the theory of Representations and Constructive Mathematics
ML-92-02 Dmitrij P. Skvortsov, Valentin B. Shehtman Maximal Kripke-type Semantics for Modal and Superintuitionistic
                                            Predicate Logics
ML-92-03 Zoran Marković   On the Structure of Kripke Models of Heyting Arithmetic
ML-92-04 Dimiter Vakarelov   A Modal Theory of Arrows, Arrow Logics I
ML-92-05 Domenico Zambella   Shavrukov's Theorem on the Subalgebras of Diagonalizable Algebras for Theories
                                            containing $I\Delta_0 + EXP$
ML-92-06 D.M. Gabbay, Valentin B. Shehtman   Undecidability of Modal and Intermediate First-Order Logics with Two Individual
                                            Variables
ML-92-07 Harold Schellinx   How to Broaden your Horizon
ML-92-08 Raymond Hoofman   Information Systems as Coalgebras
ML-92-09 A.S. Troelstra   Realizability
ML-92-10 V.Yu. Shavrukov   A Smart Child of Peano's
Compution and Complexity Theory
CT-92-01 Erik de Haas, Peter van Emde Boas   Object Oriented Application Flow Graphs and their Semantics
CT-92-02 Karen L. Kwast, Sieger van Denneheuvel Weak Equivalence: Theory and Applications
CT-92-03 Krzysztof R. Apt, Kees Doets   A new Definition of SLDNF-resolution
Other Prepublications
X-92-01 Heinrich Wansing   The Logic of Information Structures
X-92-02 Konstantin N. Ignatiev   The Closed Fragment of Dzhaparidze's Polymodal Logic and the Logic of Σ_1
                                            conservativity
X-92-03 Willem Groeneveld   Dynamic Semantics and Circular Propositions, revised version
X-92-04 Johan van Benthem   Modeling the Kinematics of Meaning
X-92-05 Erik de Haas, Peter van Emde Boas   Object Oriented Application Flow Graphs and their Semantics, revised version
1993
Logic, Semantics and Philosophy of Langauge
LP-93-01 Martijn Spaan   Parallel Quantification
LP-93-02 Makoto Kanazawa   Dynamic Generalized Quantifiers and Monotonicity
LP-93-03 Nikolai Pankrat'ev   Completeness of the Lambek Calculus with respect to Relativized Relational Semantics
LP-93-04 Jacques van Leeuwen   Identity, Quarrelling with an Unproblematic Notion
Mathematical Logic and Foundations
ML-93-01 Maciej Kandulski   Commutative Lambek Categorial Grammars
ML-93-02 Johan van Benthem, Natasha Alechina Modal Quantification over Structured Domains
ML-93-03 Mati Pentus   The Conjoinablity Relation in Lambek Calculus and Linear Logic
ML-93-04 Andreja Prijatelj   Bounded Contraction and Many-Valued Semantics
ML-93-05 Raymond Hoofman, Harold Schellinx   Models of the Untyped λ-calculus in Semi Cartesian Closed Categories
ML-93-06 J. Zashev   Categorial Generalization of Algebraic Recursion Theory
ML-93-07 A.V. Chagrov, L.A. Chagrova   Algorithmic Problems Concerning First-Order Definability of Modal Formulas on the
                                            Class of All Finite Frames
ML-93-08 Raymond Hoofman, Ieke Moerdijk   Remarks on the Theory of Semi-Functors
Compution and Complexity Theory
CT-93-01 Marianne Kalsbeek   The Vanilla Meta-Interpreter for Definite Logic Programs and Ambivalent Syntax
CT-93-02 Sophie Fischer   A Note on the Complexity of Local Search Problems
CT-93-03 Johan van Benthem, Jan Bergstra   Logic of Transition Systems
Other Prepublications
X-93-01 Paul Dekker   Existential Disclosure, revised version
X-93-02 Maarten de Rijke   What is Modal Logic?