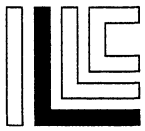

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NATURAL DEDUCTION FOR INTUITIONISTIC LINEAR LOGIC

A.S. Troelstra

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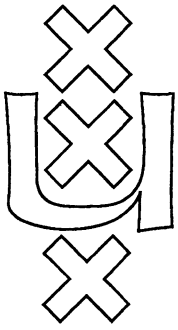

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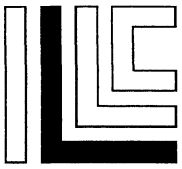

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Natural deduction for intuitionistic linear logic

A.S.Troelstra

Dedicated to Dirk van Dalen on occasion of his 60th birthday*

April 14, 1993

Abstract

The paper deals with two versions of the fragment with unit, tensor, linear implication and storage operator (the exponential !) of intuitionistic linear logic. The first version, **ILL**, appears in a paper by Benton, Bierman, Hyland and de Paiva; the second one, $\mathbf{ILL}^+$, is described in this paper. **ILL** has a contraction rule and an introduction rule !I for the exponential; in $\mathbf{ILL}^+$, instead of a contraction rule, multiple occurrences of labels for assumptions are permitted under certain conditions; moreover, there is a different introduction rule for the exponential, $!\mathrm{I}^+$, which is closer in spirit to the necessitation rule for the normalizable version of **S4** discussed by Prawitz in his monograph "Natural Deduction".

It is relatively easy to adapt Prawitz's treatment of natural deduction for intuitionistic logic to $\mathbf{ILL}^+$; in particular one can formulate a notion of strong validity (as in Prawitz's "Ideas and Results in Proof Theory") permitting a proof of strong normalization.

The conversion rules for **ILL** explicitly mentioned in the paper by Benton et. al. do not suffice for normal forms with subformula property, but we can show that this can be remedied by addition of a single conversion rule.

$\mathbf{ILL}^+$ also suggests the study of a class of categorical models, more special than the class introduced by Benton et. al.

1 Introduction

In this paper we shall assume familiarity with the proof-theoretic treatment of intuitionistic logic **IL** as presented e.g. in [P1, T1, T2].

We discuss natural deduction versions of the *multiplicative-exponential* fragment of intuitionistic linear logic, $\mathbf{ILL}_{\mathrm{me}}$ (usually shortened to **ILL** below, since we shall not deal with the full system **ILL** here). The operators and constants of $\mathbf{ILL}_{\mathrm{me}}$ are $\star$ (tensor), **1** (unit), $\multimap$ (linear implication), and ! (storage operator, exponential).

! behaves more or less like the modal necessity operator in the well-known sytem **S4** of modal logic; in particular, the first natural deduction formulations proposed for **ILL** (e.g. in [A]) had the following introduction rule for !

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}$$

*We gratefully acknowledge the use of D. Roorda's stylefile "exptrees.sty" for the typesetting of prooftrees. We are indebted to H. Schellinx for discussions and critical comments on a first draft of this paper.

(we use $\Gamma, \Gamma', \ldots, \Delta, \Delta', \ldots$ for multisets of assumptions) or in tree form

$$\begin{array}{c} [\![!\Gamma]\!] \\ \mathcal{D} \\ \dfrac{A}{!A} \end{array}$$

where the brackets $[\![$ and $]\!]$ in $[\![!\Gamma]\!]$ serve to indicate that $[\![!\Gamma]\!]$ is a *complete* multiset of open assumptions in $\mathcal{D}$, discharged at the application of !-introduction. This version has the disadvantage, as noted by several researchers, that the proof trees are not closed under substitution of deductions for open assumptions (substituting deductions for the assumptions $!\Gamma$ in an application of !-introduction leads to a deduction which ends in general not with a correct application of !-introduction). In [BBHP] it was proposed to generalize the !I-rule to

$$\frac{\Delta_1 \vdash !A_1, \ldots, \Delta_n \vdash !A_n \qquad !A_1, \ldots, !A_n \vdash B}{\Delta_1, \ldots, \Delta_n \vdash !B}$$

In the sequel we shall reserve the designation **ILL** for this version from [BBHP]. Closure under substitution is now taken care of, but for a proof-theoretic treatment the new version of the !I-rule turns out to be somewhat awkward; in a sense, the rule both introduces and eliminates !-formulas, and there is no direct relation in complexity between $!B$ and the formulas $!A_i$; the latter may be much more complex than the conclusion. Prawitz's treatment of **S4** in [P1] suggests another possibility, which we shall call $!\mathrm{I}^+$: a correct application of $!\mathrm{I}^+$ has the form

$$\begin{array}{c} \mathcal{D}_1 \qquad \mathcal{D}_n \\ [\![!A_1, \ldots, !A_n]\!] \\ \mathcal{D} \\ \dfrac{B}{!B} \end{array}$$

(no assumptions open in $\mathcal{D}_i$ become bound in $\mathcal{D}$; $[\![!A_1, \ldots, !A_n]\!]$ is a complete list of the open assumptions in $\mathcal{D}$). However, this does not combine very well with the contraction rule for the exponential. Therefore we study another version of $\mathbf{ILL}_{\mathrm{me}}$, called $\mathbf{ILL}^+$, in which !I is repaced by $!\mathrm{I}^+$, and contraction is eliminated by considering prooftrees where multiple labels of variables are permitted, if they arise by substituting isomorphic copies of a deduction $\mathcal{D}'$ for a collection of open assumptions of the form $!A$ in another deduction $\mathcal{D}$. Thus we suppress the *dynamic* aspect of contraction (i.e. the separate operation of replacing two distinctly labelled occurrences of a formula $!A$ by a single occurrence); a precise statement of the conditions permitting multiple occurrences of the same label will be given later on.

It appears that $\mathbf{ILL}^+$ permits a proof-theoretic treatment closely parallel to Prawitz's treatment of intuitionistic logic in [P1, P2]. In particular, we can formulate a notion of strong validity giving rise to a proof of strong normalization for $\mathbf{ILL}^+$; normal forms of deductions in $\mathbf{ILL}^+$ have the subformula property and can be analyzed in terms of the structure of tracks (track = path in [P1]), which in normal deductions always consist

of an elimination part, followed by a minimal part, followed by an introduction part. Applications of the kind given in [P1] follow.

Returning to **ILL** itself, the obvious "direct conversions" contracting an E-rule application with the conclusion of an I-rule as main premise, and the "permutative conversions" permitting to permute E-rule applications upward past minor premises of certain E-rules, do not suffice to give a normal form with subformula property. But one extra conversion, corresponding to one of the equalities in the notion of categorical model of **ILL** described in [BBHP], suffices for this; it is consideration of $\mathbf{ILL}^+$ which suggests a suitable normalization strategy for **ILL** relative to this set of conversion rules.

Finally, one may ask what notion of categorical model corresponds to $\mathbf{ILL}^+$? For $\mathbf{ILL}^+$ as such, the question does not make immediate sense, since the restrictions one has to impose on conversions in $\mathbf{ILL}^+$ are non-standard for a term-calculus. But the question does suggest the possible interest of a notion of categorical model obtained by imposing one extra equation on the set of equations listed for the models of BBHP, to the effect that any map from $!\Gamma$ to $!A$ can be obtained as the result of an !I-introduction to a map from $!\Gamma$ to A. This identity is true in algebraic models (trivially), but we do not know of a non-trivial type-theoretic or categorical model where it holds.

2 Notational representation of natural deductions

We recall that deductions in the system **IL** of natural deduction for intuitionistic propositional logic can be presented, in a highly redundant way, as trees where the nodes are labelled by sequents of the form

$$(*) \qquad x_1 : A_1, \ldots, x_n : A_n \vdash t : B$$

where t is a rigidly typed term of type B, and the free variables of t occur among $x_1, \ldots, x_n$. Such a representation obviously contains redundancies, since if t is rigidly typed, the variables x_i in t occur also typed as $x_i : A_i$; moreover, t reflects in its construction the complete prooftree up to this node, so the conclusion label at the bottom of the tree contains in fact all relevant information concerning the tree.

Several isomorphic forms of presentation of deductions in **IL** are obtained by stripping certain types of redundant information from the tree. Thus, for example, we obtain the usual *formula-tree presentation* by (1) stripping the terms and the context $x_1 : A_1, \ldots, x_n : A_n$ of each label, retaining only B in $(*)$ above, (2) retaining the variable labels of assumptions appearing at the top nodes (the leaves) of the tree; (3) indicating the rules used (when needed to avoid ambiguity) and (4) indicating, by repeating the labels, where assumptions are discharged.

The *term-presentation* is obtained by retaining only the rigidly typed term at the root of the tree, etc.

Each of these styles has its own merits; the formula-tree style has a certain "geometric flavour" and permits an appealing formulation of the structure of normalized proofs (as built from tracks with an elimination part, minimal part, and introduction part, cf. [P1]) from which we can neatly derive a number of corollaries (the subformula property, a generalized form of the disjunction rule etc.) It is true that for **IL** the $\vee$E-rule and the corresponding conversions (normalizing steps) are nastier than the other rules (a fact strongly emphasized in [GLT])— but really not too nasty, I think — it is still manageable.

The term presentation is very compact and precise, and makes the isomorphism between typed-term calculi and deduction systems fully explicit. It also suggests further normalization steps, which serve as a stepping stone towards a category-theoretic formulation of the logic.

The preceding remarks apply, mutatis mutandis, also to natural deduction formulations of intuitionistic linear logic.

In exhibiting deductions as formula trees, we use some standard conventions. We use calligraphic $\mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$, possibly sub-or superscripted, for formula prooftrees.

$$\begin{array}{c} \overset{x}{[A]} \\ \mathcal{D} \\ B \end{array}$$

is a prooftree $\mathcal{D}$ with $[A]$ the set of all open asumptions of the form A with the label x. The label is often dropped. Several assumption classes may appear as:

$$[\overset{x}{A}, \overset{y}{B}] \text{ or } [\overset{x}{A}][\overset{y}{B}]$$

Whenever an open hypothesis $\overset{x}{A}$ is discharged by a rule application, *all* occurrences of A with label x above the application of the rule are discharged (closed) simultaneously. It is usually convenient to assume that any label x discharged by rule application α occurs *only* above α; this can be achieved by relabelling closed assumptions if necessary (in term-notation this is just renaming bound variables).

3 Intuitionistic linear logic

In presenting intuitionistic linear logic **ILL** care has to be taken in handling assumptions. For the purely multiplicative fragment with $\star, \multimap, \mathbf{1}$, this is simple: in the formula-tree style, the assumptions are treated as a multiset, or more precisely, as a set of occurrences, each occurrence with a distinct label; each $\multimap$I-application discharges precisely one occurrence, each $\star$E-application precisely two occurrences.

If we add the exponential !, however, we must build into the rules that multiple use is equivalent to single use of the assumption $!A$.

We can stick to the convention that distinct occurrences of assumptions always have distinct labels by having a contraction rule. The effect of this rule is to replace two distinct labels (x, y say) of a formula occurrence $!A$ with a new single occurrence with a new label (z say). In the formula-tree style an application of the contraction rule looks like

$$\dfrac{\begin{array}{c} \overset{x}{[!A}, \overset{y}{!A] } \\ \mathcal{D}' \\ \overset{z}{!A} \quad B \end{array}}{B}\, x, y \qquad \text{or more generally} \qquad \dfrac{\begin{array}{c} \overset{x}{[!A}, \overset{y}{!A] } \\ \mathcal{D} \quad \mathcal{D}' \\ !A \quad B \end{array}}{B}\, x, y$$

Similarly with weakening; the possibility of "vacuously" depending on assumption of the form $!A$ (labelled x) is expressed by a weakening rule:

$$\frac{\overset{x}{!A} \quad \begin{matrix}\mathcal{D}' \\ B\end{matrix}}{B} \quad \text{, more generally} \quad \frac{\begin{matrix}\mathcal{D} \\ !A\end{matrix} \quad \begin{matrix}\mathcal{D}' \\ B\end{matrix}}{B}$$

3.1. DEFINITION. In an application of the *promotion rule* !I

$$\frac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n & & \overset{x_1}{[\![A_1} \ldots, \overset{x_n}{A_n]\!]} \\ !A_1 & \ldots & !A_n & & \mathcal{E} \\ & & & & B\end{matrix}}{!B} \; x_1, \ldots, x_n$$

the conclusions $!A_i$ of the $\mathcal{D}_i$ are the *side premises* of the !I-application, and B is the *main premise*. □

3.2. NOTATION. If Γ, Δ are used for collections of formulas in versions of **ILL**, the Γ, Δ are treated as multisets; for sequences of formulas and derivations we use vectornotation $\vec{B}, \vec{\mathcal{D}}$ etc. □

3.3. DEFINITION. (*The system* **ILL**) For reference, we give a version of a natural deduction calculus for **ILL** (restricted to $!, \multimap, \star, \mathbf{1}$), presented as a termcalculus.

Axiom $\quad x : A \Rightarrow x : A$

$$\star\mathrm{I}\ \frac{\Gamma \Rightarrow s : A \quad \Delta \Rightarrow t : B}{\Gamma, \Delta \Rightarrow s \star t : A \star B} \qquad \star\,\mathrm{E}\frac{\Gamma \Rightarrow s : A \star B \quad \Delta, x : A, y : B \Rightarrow t : C}{\Gamma, \Delta \Rightarrow \mathrm{E}^{\star}_{x,y}(s,t) : C}$$

$$\multimap\mathrm{I}\ \frac{\Gamma, x : A \Rightarrow t : B}{\Gamma \Rightarrow \lambda x.t : A \multimap B} \qquad \multimap\mathrm{E}\ \frac{\Gamma \Rightarrow s : A \multimap B \quad \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow st : B}$$

$$\mathbf{1}\mathrm{I} \quad \Rightarrow * : \mathbf{1} \qquad \mathbf{1}\mathrm{E}\ \frac{\Gamma \Rightarrow s : \mathbf{1} \quad \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow \mathrm{E}^{\mathbf{1}}(s,t) : A}$$

$$!\mathrm{I}\ \frac{\Gamma_1 \vdash t_1 : !A_1, \ldots, \Gamma_n \vdash t_n : !A_n \qquad x_1 : !A_1, \ldots, x_n : !A_n \vdash s : B}{\Gamma_1, \ldots, \Gamma_n \vdash !_{x_1,\ldots,x_n}(t_1, \ldots, t_n; s)}$$

$$\mathrm{W}\ \frac{\Gamma \Rightarrow s : !B \quad \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow \mathrm{E}^{\mathrm{w}}(s,t) : A} \qquad \mathrm{D}\ \frac{\Gamma \Rightarrow s : !B}{\Gamma \Rightarrow \mathrm{E}^{\mathrm{d}}(s) : B}$$

$$\mathrm{C}\ \frac{\Gamma \Rightarrow s : !B \quad x : !B, y : !B, \Delta \Rightarrow t : A}{\Gamma, \Delta \Rightarrow \mathrm{E}^{\mathrm{c}}_{x,y}(s;t) : A}$$

Γ, Δ are sets of statements $x_i : A_i$ with the x_i all distinct; Γ, Δ disjoint. In $!_{\vec{x}}(\vec{t}; s)$ the operator $!_{\vec{x}}$ binds $\vec{x}$ in s; in $\mathrm{E}^{\mathrm{c}}_{x,y}(s,t)$ $\mathrm{E}^{\mathrm{c}}_{x,y}$ binds x, y in t. W = weakening, D = dereliction, C = contraction. □

In discussing **ILL** it is often advantageous to generalize both weakening and contraction. Weakening is generalized to:

$$\dfrac{\begin{array}{cccc}\mathcal{D}_1 & & \mathcal{D}_n & \mathcal{D}' \\ !A_1 & \ldots & !A_n & B\end{array}}{B}$$

and contraction to

$$\dfrac{\begin{array}{cccc} & & & [(!A_1)^{k_1}, \ldots, (!A_n)^{k_n}] \\ \mathcal{D}_1 & & \mathcal{D}_n & \mathcal{D}' \\ !A_1 & \ldots & !A_n & B\end{array}}{B}$$

where $(!A_i)^{k_i}$ refers to k_i ($k_i > 1$) assumptions of the form $!A_i$ in $\mathcal{D}'$. This form of contraction is a combination of n applications of

$$\dfrac{\begin{array}{cc} & [(!A_i)^{k_i}] \\ \mathcal{D}_i & \mathcal{D}' \\ !A_i & B\end{array}}{B}$$

which in turn is a mild generalization of the original contraction rule.

3.4. DEFINITION. In the applications of the general forms of W and C, the $!A_i$ appearing as conclusions of the $\mathcal{D}_i$ are called the *major premises* (plural!) of the application, and B the *minor premise.* □

3.5. In [BBHP] normalization for natural deduction is not discussed, but some conversions are listed, in particular

(1) "detour-conversions", i.e. the removal of a formula occurrence introduced by an I-rule, only to be immediately eliminated as major premise of an E-rule.

(2) permutation conversions of the following general form: a subdeduction of the form

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_{00} & \mathcal{D}_{01} \\ A & B\end{array}}{B} \quad \begin{array}{c}\mathcal{D}_1 \\ C\end{array}}{C} \qquad \text{converts to} \qquad \dfrac{\begin{array}{cc} & \mathcal{D}_{01} \\ \mathcal{D}_{00} & B\end{array} \quad \begin{array}{c}\mathcal{D}_1 \\ C\end{array}}{\dfrac{A \qquad C}{C}}$$

where the final rule is an E-rule with B as major premise (and similarly with more premises in the rule).

Normalization becomes rather complicated in **ILL**, due to the complicated form of the promotion rule, as illustrated by the conversion of an !-introduction followed by a

contraction. The dotted line in the second prooftree serves to make it visually clear that both formulas above it enter as assumptions in the deduction $\mathcal{F}$.

$$\dfrac{\dfrac{\mathcal{D}_1 \quad \mathcal{D}_n \quad \overset{x_1 \quad x_n}{[\![A_1 \ldots, A_n]\!]} \quad \mathcal{E} \quad [\overset{u \quad v}{!B\ !B}]}{!A_1 \quad \cdots \quad !A_n \quad B \quad \mathcal{F}}\, x_1, \ldots, x_n \quad C}{!B \quad C}\, u, v$$
$$C$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1 \quad \mathcal{D}_n \quad \overset{x_1 \quad x_n}{!A_1 \ \ldots \ !A_n} \quad \overset{[\![!A_1, \ldots, !A_n]\!]}{\mathcal{E}} \quad B}{[!B} \quad \overset{y_1 \quad y_n}{!A_1 \ \ldots \ !A_n} \quad \overset{[\![!A_1, \ldots, !A_n]\!]}{\mathcal{E}} \quad B}{!B]} \quad \mathcal{F}}{!A_1 \quad \cdots \quad !A_n \quad C}}{C}\ \text{(contractions)}\ x_1, y_1, \ldots, x_n, y_n$$

Detour conversions and permutative conversions are not sufficient to guarantee the subformula property for normal proofs, as we shall see.

3.6. EXAMPLE. Here is an example of a deduction in the fragment $-\circ, !$ which is normal w.r.t. detour- and permutation conversions, but which does not have the subformula property. (In particular, $!(B_1 -\circ B_2)$ does not occur as subformula in the conclusion).

$$\dfrac{\dfrac{\dfrac{!(!C-\circ(B_1-\circ B_2))^{(1)} \quad !C^{(2)}}{!(B_1-\circ B_2)}\, 3,4 \quad \dfrac{\dfrac{!(!C-\circ(B_1-\circ B_2))^{(3)}}{!C-\circ(B_1-\circ B_2)} \quad !C^{(4)}}{B_1-\circ B_2}\, 3,4 \quad !B_1^{(5)} \quad \dfrac{\dfrac{!(B_1-\circ B_2)^{(6)}}{B_1-\circ B_2} \quad \dfrac{!B_1^{(7)}}{B_1}}{B_2}\, 6,7}{!B_2}\, 5$$
$$\dfrac{(!B_1-\circ !B_2)}{!C-\circ(!B_1-\circ !B_2)}\, 2$$
$$\dfrac{}{!(!C-\circ(B_1-\circ B_2))-\circ(!C-\circ(!B_1-\circ !B_2))}\, 1$$

Formulating an analogue of the notion of path (*track* in our terminology) as used by Prawitz, it seems natural to let in an application of !I the occurrences of $!A_i$ as conclusion of $\mathcal{D}$ be followed by the assumption $!A_i$ in $\mathcal{E}$.

We then see that in our counterexample the introduction of $!(B_1 -\circ B_2)$ on the left is followed by a dereliction from the assumption $!(B_1 -\circ B_2)$ on the right; but the detour-conversions and permutation conversions mentioned before do not permit contracting the !I followed by !E (dereliction) in the path.

The following type of conversion permits us to contract the promotion/dereliction in our example.

$$\dfrac{\vec{\mathcal{G}}\quad \dfrac{\begin{array}{c}\vec{\mathcal{F}}\\ !\vec{B}\end{array}\quad \begin{array}{c}[\![!\vec{B}]\!]\\ \mathcal{D}\\ A_i\end{array}}{!A_i}\quad \vec{\mathcal{G}}'\quad \begin{array}{c}[\![\ldots !A_i \ldots]\!]\\ \mathcal{D}'\\ C\end{array}}{!C}$$

$\vec{\mathcal{F}}$ $\qquad\qquad$ $\mathcal{F}_i$

($!\vec{B}$ is a sequence of deductions of the form $!B_i$) is replaced by

$$\dfrac{\vec{\mathcal{G}}\quad !\vec{B}\quad \vec{\mathcal{G}}'\quad \begin{array}{c}\vec{\mathcal{F}}\\ \dfrac{\begin{array}{c}[\![!\vec{B}]\!]\\ \mathcal{D}\\ !\vec{B}\end{array}\quad \begin{array}{c}\\ \\ A_i\end{array}}{[\![\ldots !A_i \ldots]\!]}\\ \mathcal{D}'\\ C\end{array}}{!C}$$

This additional conversion permits normalization with subformula property for normal deductions, as we shall see later; an appropriate normalization strategy will be suggested by the system $\mathbf{ILL}^+$, to be discussed next.

For the reasons given above it seems worthwhile to explore the possibility of an alternative formula-tree presentation which is geometrically more manageable, at the expense of a slightly more complicated treatment of labelling of assumptions. Our solution (system $\mathbf{ILL}^+$) is closer in spirit to Prawitz's treatment of natural deduction for **S4** (cf. [P1]) and permits a satisfactory normalization theorem, with the subformula property for normal proofs, and a structure of paths in deductions similar to the case of intuitionistic logic.

4 The system $\mathbf{ILL}^+$

In comparing the two systems we shall stick to the convention that in a proof tree $[A]$ always refers to a single assumption occurrence of the form A. The principal features in which $\mathbf{ILL}^+$ differs from **ILL** are the following.

4.1. *The promotion rule*

In $\mathbf{ILL}^+$ the rule, now called $!\mathrm{I}^+$, takes the form

$$\dfrac{\begin{array}{c}\mathcal{E}_1 \quad \mathcal{E}_n\\ [\![!A_1 \cdots !A_n]\!]\\ \mathcal{D}\\ B\end{array}}{!B}$$

that is to say, in deduction $\mathcal{D}$ with conclusion B and complete set of open assumptions $!A_1, \ldots, !A_n$, deductions $\mathcal{E}_1, \ldots, \mathcal{E}_n$ have been substituted; this premise permits deriving $!B$ from B. In term style this becomes

$$\begin{array}{l}\text{If } t[x_1, \ldots, x_n/s_1, \ldots, s_n] : B, \quad \mathrm{FV}(t) = \{x_1 : !A_1, \ldots, x_n : !A_n\} \\ s_1 : !A_1, \ldots, s_n : !A_n, \text{ then } !t[\vec{x}/\vec{s}] : !B.\end{array}$$

So in $\mathbf{ILL}^+$ the operator ! does not bind variables; we may assume $(\mathrm{FV}(s_1) \cup \ldots \cup \mathrm{FV}(s_n)) \cap \mathrm{FV}(t) = \emptyset$.

DEFINITION. An application of $!\mathrm{I}^+$ as exhibited is said to be *based* on $\mathcal{E}_1, \ldots, \mathcal{E}_n$. □

4.2. *The contraction rule and multiple label occurrences*

The weakening rule is not changed. The contraction rule does not appear explicitly, but is built into the system by permitting multiple occurrences of the same free variable for assumptions.

Let us formulate the condition for multiple label occurrences more precisely. Whenever a label x for an open assumption A in deduction $\mathcal{D}$ is used precisely k times ($k > 1$), then there are k isomorphic copies $\mathcal{E}_1, \ldots, \mathcal{E}_k$ of the same deduction $\mathcal{F}$ with conclusion of the form $!B$, such that in each $\mathcal{E}_i$ there is a single occurrence of x, and $\mathcal{D}$ is of the form

$$\begin{array}{c}\mathcal{E}_1 \quad \mathcal{E}_k \\ [!B \ldots !B] \\ \mathcal{E}\end{array} \quad \equiv \quad \begin{array}{c}\mathcal{F} \quad \mathcal{F} \\ [!B \ldots !B] \\ \mathcal{E}\end{array} \quad \text{or} \quad t[x_1 : !B, \ldots, x_k : !B/s, \ldots, s]$$

A special case is where the $\mathcal{E}_i$ consist of $!B$ alone. Labels $y_1, \ldots, y_k$ in $\mathcal{E}_1, \ldots, \mathcal{E}_k$ respectively corresponding to a label y *bound* in $\mathcal{F}$, are all distinct. (This is necessary to guarantee that identical labels are always discharged simultaneously.) Intuitively we may think of the multiple occurrence as representing a generalized contraction rule application. The set of occurrences of $!B$ is called a *substitution location.*

The weakening rule is generalized as already indicated for **ILL**.

4.3. DEFINITION. (*The termsystem for* $\mathbf{ILL}^+$)

$$\star\mathrm{I}\ \frac{t : A \quad s : B}{t \star s : A \star B} \qquad \star\mathrm{E}\ \frac{t : A \star B \quad s[x : A, y : B] : C}{\mathrm{E}^{\star}_{x,y}(t, s) : C}$$

$$\multimap\mathrm{I}\ \frac{t[x : A] : B}{\lambda y.t[x/y] : A \multimap B} \qquad \multimap\mathrm{E}\ \frac{t : A \multimap B \quad s : A}{ts : B}$$

$$\mathbf{1}\mathrm{I}\ \ * : \mathbf{1} \qquad \mathbf{1}\mathrm{E}\ \frac{t[x : \mathbf{1}] : A}{\mathrm{E}^{\mathbf{1}}_{x}(t) : A}$$

$$!\mathrm{I}^+\ \frac{t[\vec{x} : !\vec{A}/\vec{s} : !\vec{A}] : B}{!t[\vec{x}/\vec{s}] : !B} \qquad \mathrm{D}\ \frac{t : !B}{\mathrm{E}^{\mathrm{d}}(t) : B} \qquad \mathrm{C}\ \frac{\vec{t} : !\vec{A} \quad s : B}{\mathrm{E}^{\mathrm{w}}(\vec{t}; s) : B}$$

with restrictions on variables as indicated above. □

4.4. PROPOSITION. *There is a map* $^\circ$ *from the deductions in* $\mathbf{ILL}^+$ *to the deductions in* **ILL**, *and a map* $^+$ *in the opposite direction, such that if* $\mathcal{D}$ *in* $\mathbf{ILL}^+$ *(*$\mathcal{E}$ *in* **ILL***) proves* $\Gamma \vdash A$, *then* $\mathcal{D}^\circ$ *in* **ILL** *(*$\mathcal{E}^+$ *in* $\mathbf{ILL}^+$*) proves* $\Gamma \vdash A$.

PROOF. We introduce an auxiliary system $\mathbf{ILL}^{++}$, containing all the rules of $\mathbf{ILL}^+$, having the same conditions on labels, and in addition has the (derivable) rules !I and the generalized contraction rule of **ILL**:

$$\dfrac{\begin{array}{cc} & [!A,\ldots,!A]^{x_1,\ldots,x_n} \\ \mathcal{D} & \mathcal{E} \\ !A & C \end{array}}{C}\ x_1,\ldots,x_n$$

The map $^+$ may be defined on the deductions of **ILL**, inductively on the length of derivations:

(1) replace any application of C as above by

$$\begin{array}{c} \mathcal{D} \quad \mathcal{D} \\ [!A,\ldots,!A] \\ \mathcal{E} \\ C \end{array}$$

(Nothing in any copy of $\mathcal{D}$ is bound in $\mathcal{E}$.)

(2) replace an !I-application as on the left by the $!\mathrm{I}^+$-application on the right.

$$\dfrac{\begin{array}{ccc} \mathcal{D}_1 & \mathcal{D}_n & [\![!A_1,\ldots,!A_n]\!]^{x_1,\ldots,x_n} \\ !A_1 & \ldots\ !A_n & \mathcal{E} \\ & & B \end{array}}{!B}\ x_1,\ldots,x_n \qquad \begin{array}{c} \begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_n \end{array} \\ [\![!A_1,\ldots,!A_n]\!] \\ \mathcal{E} \\ \dfrac{B}{!B} \end{array}$$

(Nothing in any $\mathcal{D}_i$ becomes bound in $\mathcal{E}$.) For the converse map $^\circ$, we proceed as follows. Let $\mathcal{D}$ be a deduction in $\mathbf{ILL}^{++}$; suppose a k-fold occurrence of a label results from the substitution of copies of $\mathcal{D}'$ at k $(k > 1)$ assumptions of the form $!A$ in $\mathcal{D}''$, i.e.

$$\mathcal{D} \quad \equiv \quad \begin{array}{c} \mathcal{D}' \quad \mathcal{D}' \\ [!A,\ldots,!A] \\ \mathcal{D}'' \end{array}$$

Then the k assumptions $!A$ form a substitution location ("subl") and the $\mathcal{D}'$ is the corresponding substitution deduction ("sded"). Define the *multiplicity degree* md of a deduction $\mathcal{D}$ as the sum of the lengths of its sded's.

N.B. If we encounter *nested* subdeductions

$$\mathcal{D}''' \quad \equiv \quad \begin{array}{c} \mathcal{D}' \\ [!B] \\ \mathcal{D}'' \\ !A \end{array}$$

with both $!B$ and $!A$ elements of substitution sets, then this contributes *at least* length($\mathcal{D}'$) + length($\mathcal{D}'''$) (i.e. the elements of $\mathcal{D}'$ count at least twice!).

Any replacement of a subdeduction $\mathcal{D}'''$ of the form

$$\begin{array}{c} \mathcal{D}' \quad \mathcal{D}' \\ [!A,\ldots,!A] \\ \mathcal{D}'' \\ B \end{array} \quad \text{by} \quad \begin{array}{c} \overset{x_1}{[!A},\ldots,\overset{x_n}{!A}] \\ \mathcal{D}' \quad \mathcal{D}'' \\ !A \quad B \\ \hline B \end{array} x_1,\ldots,x_n$$

lowers the md of the deduction. We successively remove multiple occurrences of labels from a given deduction $\mathcal{D}$ as follows. Given a multiple label x, arising from substitution of deduction $\mathcal{D}'$ at k occurrences of $!A$ in $\mathcal{D}$, we distinguish two cases:

(1) x is open in the whole deduction $\mathcal{D}$, and we replace the multiple substitution of $\mathcal{D}'$ by a contraction applied after the last rule application.

(2) if x is bound, there must be a rule application where all occurrences of x become bound simultaneously, so $\mathcal{D}$ contains a subdeduction $\mathcal{D}^*$

$$\begin{array}{c} \mathcal{D}' \quad \mathcal{D}' \\ [!A,\ldots,!A] \\ \mathcal{D}'' \\ \dfrac{B \qquad \mathcal{E}}{B'}\, x,y,\ldots \end{array} \quad \text{or} \quad \begin{array}{c} \mathcal{D}' \quad \mathcal{D}' \\ [!A,\ldots,!A] \\ \mathcal{D}'' \\ \dfrac{\mathcal{E} \qquad B}{B'}\, x,y,\ldots \end{array}$$

In this case we introduce the contraction after the conclusion B of $\mathcal{D}''$.

We continue till we have found a deduction of md zero. The result is almost an **ILL**-proof, except for the possible occurrences of $!\mathrm{I}^+$-applications

$$\begin{array}{c} \mathcal{D}_1 \quad \mathcal{D}_n \\ [\![!A_1,\ldots,!A_n]\!] \\ \mathcal{E} \\ \dfrac{B}{!B} \end{array}$$

We may now assume that all assumption occurrences have distinct labels in the whole deduction. We replace such an $!\mathrm{I}^+$-application by an $!\mathrm{I}$-application

$$\begin{array}{c} \qquad\qquad\qquad\quad \overset{x_i}{[\![\ldots !A_i \ldots]\!]} \\ \mathcal{D}_1 \quad\quad \mathcal{D}_n \quad\quad \mathcal{E} \\ !A_1 \quad \ldots \quad !A_n \quad\quad B \\ \hline !B \end{array} x_1,\ldots,x_n$$

($x_1,\ldots,x_n$ fresh labels). In finitely many steps we reach an **ILL**-deduction. While there are several possibilities for transformation, as e.g. in the case of $!\mathrm{I}^+$-applications, a unique choice is easily stipulated (e.g. choose highest occurrences $!A_1,\ldots,!A_n$ which can serve as a basis for the $!\mathrm{I}$+-application). □

5 Conversions of ILL$^+$

5.1. *New conversions*

Permutative conversions are defined as usual, and involve $\star$E, **1**E, and (multiple) weakening followed by some elimination rule. Also standard are the $\star$-, $\multimap$- and **1**-conversions.

5.2. NOTATION. We write $\mathcal{D} \rhd \mathcal{D}'$ if $\mathcal{D}$ reduces to $\mathcal{D}'$, and $\mathcal{D} \rhd_1 \mathcal{D}'$ or $\mathcal{D}' \lhd_1 \mathcal{D}$, if $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a single conversion. □

!I$^+$ followed by dereliction contracts according to

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_i}{[\![\ldots, !A_i \ldots]\!]}\ \mathcal{D}}{B}}{\dfrac{!B}{B}} \quad \rhd_1 \quad \dfrac{\dfrac{\mathcal{D}_i}{[\![\ldots !A_i \ldots]\!]}\ \mathcal{D}}{B}$$

i.e.

D4$^+$ $\qquad \mathrm{E}^{\mathrm{d}}(!t[\vec{x}/\vec{s}]) = t[\vec{x}/\vec{s}]$

where $!t$ is based on the $\vec{s}$. !I$^+$ followed by weakening contracts as follows:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_i}{[\![\ldots !A_i \ldots]\!]}\ \mathcal{D}}{B} \quad \mathcal{E}}{\dfrac{!B \qquad C}{C}} \quad \rhd_1 \quad \dfrac{\mathcal{D}_i \qquad \mathcal{E}}{\dfrac{\ldots !A_i \ldots \quad C}{C}}$$

an instance of the generalized rule of weakening (which may be replaced by n successive weakenings). In term notation:

$$\mathrm{E}^{\mathrm{w}}(!t[\vec{x}/\vec{s}]; t') = \mathrm{E}^{\mathrm{w}}(\vec{s}, t')$$

With n-fold weakening as primitive, the !I$^+$–W contraction may be formulated accordingly:

W1$^+$ $\qquad \mathrm{E}^{\mathrm{w}}(\vec{t_1}, !t[\vec{x}/\vec{s}], \vec{t_2}; t') = \mathrm{E}^{\mathrm{w}}(\vec{t_1}, \vec{s}, \vec{t_2}; t')$

Here the $\vec{x}$ indicates the set of occurrences $[\![!A_1, \ldots, !A_n]\!]$ in a promotion application, i.e. t is based on $\vec{s}$.

It is to be noted that application of a single conversion in a subtree belonging to a set of isomorphic subtrees inserted at a substitution location, may fall outside our class of prooftrees for **ILL**$^+$; but finitely many "isomorphic" conversions will then bring us back into the class of **ILL**$^+$-prooftrees.

5.3. *The term equations for* **ILL**

We shall briefly compare the term-equivalences of [BBHP] with the equivalences generated by our conversions. For brevity, we state the term equivalences of [BBHP] in our notation; it is instructive to write them out as operations on prooftrees. We arrange the equations in groups.

1. Equalities corresponding to detour-conversions for $\mathbf{1}, \star, \multimap$ *and* !I–!E.

D1 $\quad \mathrm{E}^{\mathbf{1}}(*, s) = s,$

D2 $\quad \mathrm{E}^{\star}_{x,y}(t \star t', s) = s[x, y/t, t'],$

D3 $\quad (\lambda x.t)s = t[x/s],$

D4 $\quad \mathrm{E}^{\mathrm{d}}(!_{\vec{x}}(\vec{s}, t)) = t[\vec{x}/\vec{s}].$

In $\mathbf{ILL}^+$, D1–3 also hold as conversions, to D4 corresponds the conversion of dereliction following promotion in the form D4$^+$ mentioned before.

2. Extensionalities (analogues of η*-conversion).*

E1 $\quad \mathrm{E}^{\mathbf{1}}(t, f[z/*]) = f[z/t],$

E2 $\quad \mathrm{E}^{\star}_{x,y}(t, f[z/x \star y]) = f[z/t],$

E3 $\quad \lambda x.tx = t \ (x \notin \mathrm{FV}(t)),$

E4 $\quad !_x(t, \mathrm{E}^{\mathrm{d}}(x)) = t.$

To E1–4 correspond in $\mathbf{ILL}^+$ E1–3 and E4 in modified form

E4$^+$ $\quad !(\mathrm{E}^{\mathrm{d}}(t)) = t.$

3. Equalities involving weakening. $\mathrm{E}^{\mathrm{w}}(\vec{s}; t)$ is short for

$$\mathrm{E}^{\mathrm{w}}(s_1, \mathrm{E}^{\mathrm{w}}(s_2, \ldots \mathrm{E}^{\mathrm{w}}(s_n, t)\ldots))$$

The equations are

W1 $\quad \mathrm{E}^{\mathrm{w}}(!_{\vec{x}}(\vec{s}; t), t') = \mathrm{E}^{\mathrm{w}}(\vec{s}; t'),$

W2 $\quad !_{x,\vec{x}'}(s, \vec{s}'; \mathrm{E}^{\mathrm{w}}(x, t)) = \mathrm{E}^{\mathrm{w}}(s, !_{\vec{x}'}(\vec{s}', t)),$

W3 $\quad \mathrm{E}^{\mathrm{c}}_{x,y}(s, \mathrm{E}^{\mathrm{w}}(x, t)) = t[y/s], \ \ \mathrm{E}^{\mathrm{c}}_{x,y}(s, \mathrm{E}^{\mathrm{w}}(y, t)) = t[x/s],$

W4 $\quad f[\mathrm{E}^{\mathrm{w}}(z, s)] = \mathrm{E}^{\mathrm{w}}(z, f[s]).$

W1 corresponds to the conversion of promotion followed by weakening and corresponds in $\mathbf{ILL}^+$ to W1$^+$ mentioned above.

W2 expresses that for a promotion following a weakening the weakening may be pushed "downward" past the promotion. In $\mathbf{ILL}^+$ it corresponds to (in term notation)

W2$^+$ $\quad !\mathrm{E}^{\mathrm{w}}(x; t[\vec{x}/\vec{s}]) = \mathrm{E}^{\mathrm{w}}(x, !t[\vec{x}/\vec{s}])$.

W3 corresponds to

W3$^+$ $\quad \mathrm{E}^{\mathrm{w}}(t, s[x/t]) = s[x/t]$.

which is also not among our conversions. W4 permits us to push weakening up/down as long as no binding of hypotheses is involved, and contains our permutation conversions for weakening as a special case.

4. Equalities with contraction. We use an abbreviation

$$\mathrm{E}^{\mathrm{c}}_{\vec{y},\vec{z}}(\vec{t}; s) := \mathrm{E}^{\mathrm{c}}_{y_1,z_1}(t_1, \mathrm{E}^{\mathrm{c}}_{y_2,z_2}(t_2, \ldots \mathrm{E}^{\mathrm{c}}_{y_n,z_n}(t_n, s)\ldots))$$

where $\vec{y} \equiv y_1, \ldots, y_n$, $\vec{z} \equiv z_1, \ldots, z_n$. The equalities are

C1 $\quad \mathrm{E}^{\mathrm{c}}_{y,z}(!_{\vec{x}}(\vec{s}; t); s') = \mathrm{E}^{\mathrm{c}}_{\vec{x}',\vec{x}''}(\vec{s}; s'[y, z/!_{\vec{x}}(\vec{x}'; t), !_{\vec{x}}(\vec{x}''; t)])$,

C2 $\quad !_{z,\vec{z}'}(s, \vec{s}'; \mathrm{E}^{\mathrm{c}}_{x,y}(z, t)) = \mathrm{E}^{\mathrm{c}}_{x',y'}(s, !_{x,y,\vec{z}'}(x', y', \vec{s}'; t))$,

C3 $\quad \mathrm{E}^{\mathrm{c}}_{x,y}(s, t) = \mathrm{E}^{\mathrm{c}}_{y,x}(s, t)$,

C4 $\quad \mathrm{E}^{\mathrm{c}}_{x,w}(s, \mathrm{E}^{\mathrm{c}}_{y,z}(w, t)) = \mathrm{E}^{\mathrm{c}}_{w,z}(s, \mathrm{E}^{\mathrm{c}}_{x,y}(w, t))$,

C5 $\quad f[z/\mathrm{E}^{\mathrm{c}}_{x,y}(s, t)] = \mathrm{E}^{\mathrm{c}}_{x,y}(s, f[z/t])$.

C1–5 disappear (i.e. left- and righthand side of the equation translate into identical terms) in $\mathbf{ILL}^+$.

The *generalized* form of contraction requires a much more involved term operator.

5. Other rules.

P1 $\quad f[w/\mathrm{E}^{\mathbf{1}}(s, t)] = \mathrm{E}^{\mathbf{1}}(s, f[w/t])$,

P2 $\quad f[w/\mathrm{E}^{\star}_{x,y}(s, t)] = \mathrm{E}^{\star}_{x,y}(t, f[w/s])$.

The same equations can be adopted in $\mathbf{ILL}^+$; these equalities contain the permutation conversions for $\mathrm{E}^{\star}$ and $\mathrm{E}^{\mathbf{1}}$ as special cases.

X1 $\quad !_{\vec{y}',y,\vec{y}''}(\vec{t}', !_{\vec{x}}(\vec{t}; f), \vec{t}''; g) = !_{\vec{y}',\vec{x}',\vec{y}''}(\vec{t}', \vec{t}, \vec{t}''; g[y/!_{\vec{x}}(\vec{x}'; f)])$.

In $\mathbf{ILL}^+$ X1 disappears.

REMARK. If there is a notion of categorical model corresponding to $\mathbf{ILL}^+$, which might be seen as a strengthening of the conversion rules for $\mathbf{ILL}^+$ as well as the categorical identities for **ILL** as stipulated in [BBHP], it should be based on the $\mathbf{ILL}^+$-conversion rules plus E1–3, $\mathrm{E4}^+$, $\mathrm{W2}^+$,$\mathrm{W3}^+$,W4, P1–2.

However, the term calculus of $\mathbf{ILL}^+$ does not behave in the standard way, as we already pointed out: isomorphic subterms of type $!A$ giving rise to multiple occurrences of the same variable (multiple labels) ought always to be converted simultaneously in order to stay within the same class of $\mathbf{ILL}^+$-terms.

It does make immediate sense however, to ask for the notion of categorical model corresponding to a system based on contraction as for **ILL**, but with the rule $!\mathrm{I}^+$. In this intermediate system a very natural conversion rule suggests itself, namely

$$\mathrm{E4}^* \qquad !_{\vec{x}}(\vec{y}; \mathrm{E}^{\mathrm{d}}(t)) = t[\vec{x}/\vec{y}],$$

from which it follows that

$$!_{\vec{x}}(\vec{s}; \mathrm{E}^{\mathrm{d}}(t)) = t[\vec{x}/\vec{s}].$$

This conversion has in **ILL** the effect that

$$\dfrac{\begin{array}{cc}[\![!\vec{B}]\!] & [\![!A_1, \ldots, !A_n]\!] \\ \mathcal{E}_i & \mathcal{D} \\ \ldots !A_{i-1}\, !A_i\, !A_{i+1} \ldots & C\end{array}}{!C}$$

is equivalent to

$$\dfrac{\begin{array}{c}[\![!\vec{B}]\!] \\ \mathcal{E}_i \\ [\![!A_1 \ldots, !A_i, \ldots, !A_n]\!] \\ \mathcal{D} \\ !A_1 \ldots !A_{i-1}\, !\vec{B}\, !A_{i+1} \ldots !A_n \qquad C\end{array}}{!C}$$

(Replace $\mathcal{E}_i$ by

$$\dfrac{\begin{array}{c}[\![!\vec{B}]\!] \\ \mathcal{E}_i \\ !A_i \\ !\vec{B} \quad A_i\end{array}}{!A_i}$$

and apply the rule X1, etc.)

The rule $\mathrm{E4}^*$ holds in algebraic models for linear logic (intuitionistic linear logic with storage, in the terminology of [T2]), but we do not know of a non-trivial type-theoretic

model where E4* is fulfilled. E4* is in fact *equivalent* to the requirement that a "change of basis" for the promotion rule leaves the proof term the same. Specifically,

$$\begin{array}{c} [\![!\vec{B}]\!] \\ \mathcal{D} \\ !A \end{array} \equiv \begin{array}{c} [\![!\vec{B}]\!] \\ \mathcal{D} \quad \underline{!A} \\ \underline{!A \quad A} \\ !A \end{array} \equiv \begin{array}{c} [\![!\vec{B}]\!] \\ \mathcal{D} \\ \underline{!A} \\ \underline{!\vec{B} \quad A} \\ !A \end{array}$$

where the first instance of !I has basis $!A$, the second the basis $[\![!\vec{B}]\!]$; the first equivalence is the usual E4, the second is the "change of basis" equivalence, the combination yields E4*.

5.4. *Strategies for normalizing*

5.5. DEFINITION. A *segment* in a deduction is a set of formula occurrences $A_1, \ldots, A_n$ of the same formula, such that A_{i+1} is immediately below A_i for $1 \leq i < n$, A_i for $i < n$ is minor premise of W,1E or $\star$E, A_1 is not conclusion of such a rule, and A_n is not minor premise of such a rule.

A segment is *maximal* if either $n = 1$ and $A_1 = A_n$ is conclusion of an I-rule and major premise of an E-rule, or $n > 1$ and A_n is major premise of an E-rule.

A *terminal* segment of $\mathcal{D}$ is a segment where A_n is the conclusion of $\mathcal{D}$. (N.B. In our fragment of $\mathbf{ILL}^+$ the terminal segment is unique.)

A segment is *critical* if it is a maximal segment of maximal degree (degree of a segment = complexity of the formula of the segment). □

5.6. PROPOSITION. *(Normalization for $\mathbf{ILL}^+$) Each deduction $\mathcal{D}$ in $\mathbf{ILL}^+$ can be brought into normal form by a finite sequence of reduction steps.*

PROOF. We may normalize deductions by making conversions at the leftmost-topmost critical segment. If this is done in the leftmost subdeduction $\mathcal{D}'$ of a finite set of copies of $\mathcal{D}'$ inserted for several occurrences of a formula $!A$, then the result might fall outside our class of deductions; but if we next successively make the same conversion in each of the copies, we are back at a deduction of $\mathbf{ILL}^+$. Each step in this procedure according to the strategy just described results in a diminishing of the total length of all maximal segments of maximal complexity.

One case requires attention: what if for a substituted deduction $\mathcal{D}'$, at two occurrences o_1 and o_2 of the substitution location consisting of occurrences of $!A$ say, at o_1 a conversion cutting out $!A$ is possible, and at o_2 not? But in this case Rule ($\mathcal{D}'$) is promotion, and it is easy to see that in this case we can take the basis of the promotion (a set of occurrences of a formula $!B$ say) in all copies of the original $\mathcal{D}'$ as new substitution locations.

The form of normal deductions may now be analyzed as done in [P1]; more details are given in section 6.

6 Strong normalization in $\mathbf{ILL}^+$

We may prove strong normalization for $\mathbf{ILL}^+$ using Prawitz's concept of strong validity, adapted to the present system (for an exposition of the intuitionistic case see e.g. [T1]).

Since single conversions do sometimes lead outside $\mathbf{ILL}^+$, we consider a wider class of proof trees, where multiple labels are permitted (as usual, open assumptions with the same label always have to be discharged simultaneously); for the rest the rules have the same form as for $\mathbf{ILL}^+$.

If we can prove strong normalization for this wider class of deductions, we have SN for $\mathbf{ILL}^+$, with respect to those normalization strategies where, if one of a series of copies of $\mathcal{D}$ substituted at a set of occurrences of $!A$ is converted, then all the others are converted in the same way at the next steps until the whole group has again become isomorphic.

6.1. DEFINITION. $\mathcal{Z}$ is a *conversion candidate* of $\mathcal{D}$ if the terminal segment of $\mathcal{D}$ is of the form $A \star B$ and begins with an $\star$-introduction with deductions $\mathcal{F}, \mathcal{F}'$ of the premises, and $\mathcal{Z} \equiv (\mathcal{F}, \mathcal{F}')$.

If $\mathcal{D} \rhd \mathcal{D}^*$ and $\mathcal{D}^*$ has a conversion candidate $\mathcal{Z}$, we say that $\mathcal{Z}$ is a *derivate* of $\mathcal{D}$. □

NOTATION. Below we shall (unless indicated otherwise) stick to the convention that given a derivation $\mathcal{D}$ not ending with W, the subdeductions of the premises from left to right are $\mathcal{D}_0, \mathcal{D}_1, \ldots$; in the case of weakening $\mathcal{D}_1$ is the minor premiss and $\mathcal{D}_0$ stands for one of the major premises. This may be iterated giving rise to notations $\mathcal{D}_{01}$ etc. Similarly for $\mathcal{D}'$.

We write $\mathcal{D} \rhd \mathcal{D}'$ or $\mathcal{D}' \lhd \mathcal{D}$ if $\mathcal{D}$ reduces to $\mathcal{D}'$, i.e. $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a series of conversion steps. We write $\mathcal{D} \rhd_1 \mathcal{D}'$ or $\mathcal{D}' \lhd_1 \mathcal{D}$ if $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a single conversion applied to some subdeduction of $\mathcal{D}$.

Rule($\mathcal{D}$) is the last rule applied in $\mathcal{D}$.

"SN" abbbreviates "strongly normalizable". □

6.2. DEFINITION. A deduction $\mathcal{D}$ is said to be *strongly valid* (SV) if one of the following clauses applies:

1. $\mathcal{D}$ consists of an assumption, or the axiom **1**I.
2. Rule($\mathcal{D}$) $\in \{\star\text{I}, !\text{I}^+\}$ and the subdeductions of the premises are SV.
3. Rule($\mathcal{D}$) $= \multimap$I, i.e. $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}[A] \\ \mathcal{D}' \\ B\end{array}}{A \multimap B}$$

then $\mathcal{D} \in$ SV if for all $\mathcal{D}^* \in$ SV with conclusion A

$$\begin{array}{c}\mathcal{D}^* \\ [A] \\ \mathcal{D} \\ B\end{array}$$

is SV.

4. Rule($\mathcal{D}$) $\in \{-\circ\text{E}, !\text{E}\}$, $\mathcal{D}$ is normal or for all $\mathcal{D}' \lhd_1 \mathcal{D}'$, $\mathcal{D}'$ is SV.
5. Rule($\mathcal{D}$) = W, and $\mathcal{D}$ is normal or for all $\mathcal{D}' \lhd_1 \mathcal{D}$, $\mathcal{D}'$ is SV, $\vec{\mathcal{D}}_0 \in$ SN, $\mathcal{D}_1 \in$ SV.
6. Rule($\mathcal{D}$) = **1**E, and $\mathcal{D}$ is normal, or for all $\mathcal{D}' \lhd_1 \mathcal{D}$, $\mathcal{D}'$ is SV, SN($\mathcal{D}_0$) and SV($\mathcal{D}_1$).
7. Rule(D) = $\star$E, and $\mathcal{D}$ is normal or for all $\mathcal{D}' \lhd_1 \mathcal{D}$, $\mathcal{D}'$ is SV, and condition (*) holds, that is to say SN($\mathcal{D}_0$), SV($\mathcal{D}_1$), and whenever the deduction $\mathcal{D}_0$ of the main premise $A \star B$ has a derivate $(\mathcal{F}, \mathcal{F}')$, and the minor premise has deduction

$$\begin{array}{c}[A,B] \\ \mathcal{D}_1 \\ C\end{array} \quad \text{, then} \quad \begin{array}{c}\mathcal{F}\ \ \mathcal{F}' \\ [A,\ B] \\ \mathcal{D}_1 \\ C\end{array} \quad \text{is SV.}$$

□

6.3. LEMMA. *Let Rule($\mathcal{D}$) $\in \{\star\text{I}, !\text{I}^+, -\circ\text{I}\}$. Then, if $\mathcal{D} \rhd_1 \mathcal{D}^{(1)} \rhd_1 \mathcal{D}^{(2)} \rhd_1 \ldots$, where*

$$\mathcal{D}^{(n)} = \frac{\mathcal{D}_0^{(n)}}{A} \quad \text{or} \quad \frac{\mathcal{D}_0^{(n)} \quad \mathcal{D}_1^{(n)}}{A}$$

then $\ldots \rhd_1 \mathcal{D}_i^{(n)} \rhd_1 \mathcal{D}_i^{(n+1)} \rhd_1 \mathcal{D}_i^{(n+2)} \rhd_1 \ldots$ become reduction sequences after deletion of repetitions.

6.4. LEMMA. *If $\mathcal{D} \rhd \mathcal{D}'$ and* SV*($\mathcal{D}$) then* SV*($\mathcal{D}'$).*
PROOF. By induction over the inductively defined class of SV deductions. Obviously it suffices to show that if $\mathcal{D} \rhd_1 \mathcal{D}'$, and SV($\mathcal{D}$), then SV($\mathcal{D}'$). □

6.5. LEMMA. $\mathcal{D} \in$ SV $\Rightarrow \mathcal{D} \in$ SN.
PROOF. By induction over the class of SV deductions. The induction step is immediate if $\mathcal{D} \in$ SV by clauses 3–6, since then if $\mathcal{D}$ is not normal all $\mathcal{D}' \lhd_1 \mathcal{D}$ are SV, and so by the induction hypothesis SN. If $\mathcal{D} \in$ SV by clauses 1–2, strong normalizability is also immediate from the induction hypothesis. □

6.6. LEMMA. *Let Rule($\mathcal{D}$) $\in \{\star\mathrm{E}, \multimap\mathrm{E}, !\mathrm{E}, \mathrm{W}, \mathbf{1}\mathrm{E}\}$. Then $\mathcal{D} \in \mathrm{SV}$ if*

(i) $\mathrm{SN}(\mathcal{D}_i)$ *for all immediate subdeductions* $\mathcal{D}_i$.

(ii) If Rule $(\mathcal{D}) = \multimap\mathrm{E}$ *or* $!\mathrm{E}$, *then* $\mathrm{SV}(\mathcal{D}_i)$.

(iii) If Rule$(\mathcal{D}) = W$, *then* $\mathrm{SN}(\vec{\mathcal{D}}_0)$ *and* $\mathrm{SV}(\mathcal{D}_1)$.

(iv) If Rule$(\mathcal{D}) = \mathbf{1}\mathrm{E}$, *then* $\mathrm{SN}(\mathcal{D}_0)$ *and* $\mathrm{SV}(\mathcal{D}_1)$.

(v) If Rule$(\mathcal{D}) = \star\mathrm{E}$, *clause* $(*)$ *in the definition of* SV *applies.*

PROOF. In order to prove this lemma, we assign to each $\mathcal{D}$ satisfying the conditions of the lemma with conclusion of given complexity an induction value $\mathrm{IV}(\mathcal{D}) = (\beta, \gamma, \delta)$ where

- β = (sum of) length(s) of reduction tree(s) of $\mathcal{D}_0$ (resp. $\vec{\mathcal{D}}_0$);
- γ = (sum of) length(s) of $\mathcal{D}_0$ (resp. $\vec{\mathcal{D}}_0$);
- δ = sum of lengths of reduction trees of the deductions of the premises.

The ordering is lexicographic: $(\beta, \gamma, \delta) < (\beta', \gamma', \delta') := (\beta < \beta') \vee (\beta = \beta' \wedge \gamma < \gamma') \vee (\beta = \beta' \wedge \gamma = \gamma' \wedge \delta < \delta')$. We prove the lemma by induction on $\mathrm{IV}(\mathcal{D})$. It suffices to prove $\forall \mathcal{D}' \lhd_1 \mathcal{D}(\mathrm{SV}(\mathcal{D}'))$, since the other conditions imposed on SV by the definition hold automatically if the assumptions of the lemma are satisfied.
Case 1. $\mathcal{D}$ normal: we are done.

In all other cases, let $\mathcal{D}' \lhd_1 \mathcal{D}$; let $\mathrm{IV}(\mathcal{D}) = (\beta, \gamma, \delta)$ and $\mathrm{IV}(\mathcal{D}') = (\beta', \gamma', \delta')$ (if defined, which has to be shown).
Case 2. $\mathcal{D}'$ is obtained by a conversion step applied to the deduction of one of the primises of the last rule application in $\mathcal{D}$. Then $\mathcal{D}'$ falls under the conditions of the lemma, and has a well-defined lower IV.
Case 3. $\mathcal{D}' \lhd_1 \mathcal{D}$ by a detour-conversion involving the final rule-application. Then the major premiss of the last rule in $\mathcal{D}$ is obtained by an I-rule. For example,

$$\mathcal{D} \equiv \dfrac{\dfrac{\begin{matrix}[A]\\ \mathcal{D}_{00}\\ B\end{matrix}}{A \multimap B} \quad \begin{matrix}\mathcal{D}_1\\ A\end{matrix}}{B} \qquad \mathcal{D}' \equiv \begin{matrix}\mathcal{D}_1\\ [A]\\ \mathcal{D}_{00}\\ B\end{matrix}$$

By clause (ii), $\mathrm{SV}(\mathcal{D}_1)$, $\mathrm{SV}(\mathcal{D}_0)$, and hence by the definition of SV, it follows that $\mathrm{SV}(\mathcal{D}')$. if

$$\mathcal{D} \equiv \dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00}\\ B\end{matrix}}{!B}}{B} \qquad \mathcal{D}' \equiv \begin{matrix}\mathcal{D}_{00}\\ B\end{matrix}$$

then $\mathrm{SV}(\mathcal{D}_0)$, hence $\mathrm{SV}(\mathcal{D}_{00})$, where $\mathcal{D}_{00} \equiv \mathcal{D}'$.
Case 4. $\mathcal{D}' \lhd_1 \mathcal{D}$ by a permutative reduction involving the final rule application.

Subcase 4.1. $\multimap$E or !E is permuted over $\star$E. Let

$$\mathcal{D} \equiv \dfrac{\dfrac{\mathcal{D}_{00} \quad \overset{[A,B]}{\mathcal{D}_{0,1}}}{\dfrac{A \star B \quad C \multimap D}{C \multimap D}} \quad \mathcal{D}_1}{D} \qquad \mathcal{D}' \equiv \dfrac{\mathcal{D}_{00} \quad \dfrac{\overset{[A,B]}{\mathcal{D}_{0,1}} \quad C}{D}}{\dfrac{A \star B}{D}}$$

$\mathrm{SN}(\mathcal{D}_0)$, hence also $\mathrm{SN}(\mathcal{D}_0')$. The induction value of $\mathcal{D}'$, if defined, is clearly lower: $\beta < \beta'$ or $\beta = \beta' \wedge \gamma' < \gamma$. We must show that $\mathcal{D}'$ again satisfies the conditions of the lemma.

Note: $\mathrm{SV}(\mathcal{D}_0), \mathrm{SV}(\mathcal{D}_1), \mathrm{SV}(\mathcal{D}_{01})$, hence $\mathrm{SV}(\mathcal{D}_1')$ by the lemma, since $\mathrm{IV}(\mathcal{D}_1') = (\beta'', \gamma'', \delta'')$ with $\beta'' < \beta \vee (\beta'' = \beta \wedge \gamma'' < \gamma)$, hence $\mathrm{SV}(\mathcal{D}_1')$.

Also, if $\mathcal{D}_{00}$ has a derivative $(\mathcal{F}, \mathcal{F}')$, we must show that

$$\mathcal{F}'' \equiv \dfrac{\overset{\mathcal{F}\mathcal{F}'}{\overset{[A\,B]}{\mathcal{D}_{01}}} \quad \mathcal{D}_1}{\dfrac{C \multimap D \quad C}{D}}$$

is SV. For this we need that the left subdeduction of $\mathcal{F}''$ is SV. But $\mathrm{SV}(\mathcal{D}_0)$, hence this follows by condition $(*)$ in the definition of SV.

As a result, $\mathcal{D}_0', \mathcal{D}_1'$ are SV, hence SN, and $\mathrm{IV}(\mathcal{D}')$ is defined and the IH applies.

The treatment of !E over $\star$E is completely similar.

Subcase 4.2. $\multimap$E or !E over W or **1**E. The arguments are quite similar to, but slightly simpler than in the preceding case.

Subcase 4.3. $\star$E or W or **1**E over $\star$E or W or **1**E. Let us consider the most complicated case of $\star$E over $\star$E.

$$\mathcal{D} \equiv \dfrac{\dfrac{\mathcal{D}_{00} \quad \overset{[A,B]}{\mathcal{D}_{01}} \quad \overset{[C,D]}{\mathcal{D}_1}}{\dfrac{A \star B \quad C \star D}{C \star D}} \quad E}{E} \qquad \mathcal{D}' \equiv \dfrac{\mathcal{D}_{00} \quad \dfrac{\overset{[A,B]}{\mathcal{D}_{01}} \quad \overset{[C,D]}{\mathcal{D}_1}}{C \star D} \quad E}{\dfrac{A \star B}{E}}$$

$\mathrm{IV}(\mathcal{D}) = (\beta, \gamma, \delta)$. We have

(a) $\mathcal{D}_0 \in \mathrm{SN}$, $\mathcal{D}_1 \in \mathrm{SV}$;

(b) If $\mathcal{D}_0$ has a derivate $(\mathcal{F}, \mathcal{F}')$, then

$$\mathcal{H} \equiv \begin{array}{c} \overset{\mathcal{F}\mathcal{F}'}{\overset{[C\,D]}{\mathcal{D}_1}} \\ E \end{array}$$

is also SV.

We have to show that $\mathcal{D}'$ falls under the IH. In the first place $\mathrm{SN}(\mathcal{D}_{00})$ holds, since $\mathrm{SN}(\mathcal{D}_0)$. Secondly, we must check that $\mathcal{D}'_1$ is SV. This requires (1) $\mathrm{SN}(\mathcal{D}_{01})$, which follows from $\mathrm{SN}(\mathcal{D}_0)$, (2) $\mathrm{SV}(\mathcal{D}_1)$ which holds by (a), and (3) whenever $\mathcal{D}_{01}$ has a derivate $(\mathcal{F}, \mathcal{F}')$, then $\mathcal{H}$ as above is SV.

But if $\mathcal{D}_{01}$ has a derivate $(\mathcal{F}, \mathcal{F}')$ then $(\mathcal{F}, \mathcal{F}')$ is also a derivate of $\mathcal{D}_0$, so (3) follows from (b). Hence $\mathrm{SV}(\mathcal{D}'_1)$ follows by IH, since $\mathcal{D}'_1$ has a lower IV.

In order to get $\mathrm{SV}(\mathcal{D}')$ it remains to be shown that if $\mathcal{D}_{00}$ has a derivate $(\mathcal{G}, \mathcal{G}')$, then

$$\mathcal{G}'' \quad \equiv \quad \dfrac{\dfrac{\mathcal{G} \quad \mathcal{G}'}{[A,B] \quad [C,D]}}{\dfrac{\mathcal{D}_{01} \quad \mathcal{D}_1}{C \star D \quad E}}{E}$$

is also SV. This is similar to the preceding part of the argument; the crucial clause to be verified is now: if $\mathcal{G}''_0$ has a derivate $(\mathcal{F}, \mathcal{F}')$, then $\mathcal{H}$ is SV.

However, if $\mathcal{D}_{00}$ has a derivate $(\mathcal{G}, \mathcal{G}')$, this means that $\mathcal{D}_0$ reduces to something like

$$\dfrac{\mathcal{E}_0 \quad \dfrac{\dfrac{\mathcal{G} \quad \mathcal{G}'}{A \quad B}}{\mathcal{E}_n \quad A \star B}}{A \star B} \quad \dfrac{\vdots \quad [A,B]}{A \star B \quad \mathcal{D}_{01}}{C \star D}$$

and then $\mathcal{D}_0$ also reduces to

$$\dfrac{\mathcal{E}_0 \quad \dfrac{\mathcal{E}_n \quad \dfrac{\dfrac{\mathcal{G} \quad \mathcal{G}'}{A \quad B} \quad [A,B]}{A \star B \quad \mathcal{D}_{01}}{C \star D}}{C \star D}}{C \star D}$$

so $\mathcal{D}_0$ reduces to a deduction

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G}\ \ \mathcal{G}'}{[A,B]}}{\mathcal{D}_{01}}}{\mathcal{E}_n \quad C\star D}}{C\star D}}{\vdots}}{\dfrac{\mathcal{E}_0 \quad C\star D}{C\star D}}$$

and it appears that $(\mathcal{F},\mathcal{F}')$ is also a derivate of $\mathcal{D}_0$, hence $\mathcal{H}$ is indeed SV.
Subcase 4.4. Permutation of **1**E over $\star$E. This case is similar to earlier cases, but simpler. □

6.7. DEFINITION. $\mathcal{D}$ is SVS (*strongly valid under substitution*) if every substitution of SV deductions for open assumptions in $\mathcal{D}$ yields a SV deduction. □

6.8. PROPOSITION. *All deductions in* $\mathbf{ILL}^+$ *are* SVS.
PROOF. By induction on the lengths of deductions. We consider two typical cases.
Case 1. Let Rule $(\mathcal{D})$ = W. Then

$$\mathcal{D} \equiv \dfrac{\begin{matrix}\mathcal{D}_0 & \mathcal{D}_1\\ !A & B\end{matrix}}{B}$$

$\mathcal{D}_0, \mathcal{D}_1$ are SVS by induction hypothesis. Let $\mathcal{D}^*$ be a substitution instance of $\mathcal{D}$, so

$$\mathcal{D}^* \equiv \dfrac{\begin{matrix}\mathcal{D}_0^* & \mathcal{D}_1^*\\ !A & B\end{matrix}}{B}$$

then $\mathcal{D}_0^*, \mathcal{D}_1^*$ are SV, hence $\mathcal{D}_0^*$ is SN and so $\mathcal{D}^*$ is SV by the preceding lemma.
Case 2. Rule$(\mathcal{D})$ is $\star$E, so

$$\mathcal{D} \equiv \dfrac{\begin{matrix}\mathcal{D}_0 & [A,B]\\ A\star B & \\ & \mathcal{D}_1\\ & C\end{matrix}}{C}$$

and let $\mathcal{D}^*$ be a substitution instance

$$\mathcal{D}^* \equiv \dfrac{\begin{matrix}\mathcal{D}_0^* & [A,B]\\ A\star B & \\ & \mathcal{D}_1^*\\ & C\end{matrix}}{C}$$

By induction hypothesis $\mathcal{D}_0^*, \mathcal{D}_1^*$ are SV, so $\mathcal{D}_0^*$ is SN. Suppose $\mathcal{D}_0^*$ has a derivate $(\mathcal{F}, \mathcal{F}')$ occurring in a $\mathcal{D}^{**} \lhd \mathcal{D}^*$; $\mathcal{D}^{**}$ is SV, and it follows that $\mathcal{F}, \mathcal{F}'$ are SV. Then

$$\begin{array}{c} \mathcal{F} \quad \mathcal{F}' \\ [A, B] \\ \mathcal{D}_1^* \\ C \end{array}$$

is SV. etc. □

7 The structure of normal deductions in ILL$^+$

7.1. DEFINITION. A *track* in a normal $\mathcal{D}$ is a sequence of formula occurrences A_0, A_1, $A_2, \ldots, A_n$ such that

1. A_0 is an axiom, open assumption or assumption closed by $\multimap$I in $\mathcal{D}$;
2. A_{i+1} is immediately below A_i if A_{i+1} is conclusion of an I-rule, A_i a premise of the same rule;
3. A_{i+1} is immediately below A_i if A_i is major premiss of an application of !E = D, $\multimap$E, or minor premise of an application of $\star$E, W, or **1**E;
4. A_i is major premise of an application of $\star$E and A_{i+1} is an assumption discharged by that application;
5. A_n is either conclusion of $\mathcal{D}$, or major premise of **1**E, or a major premise of W.

□

We can divide a track into segments as in the case of intuitionistic logic; in a track of a normal deduction we can then distinguish the elimination part, followed by the minimal part, followed by the introduction part.

7.2. LEMMA. *Each formula occurrence in the proof tree of a normal deduction belongs to some track .*
PROOF. By induction on the depth of deductions. □

7.3. PROPOSITION. *(Subformula property) Let $\Gamma \vdash A$ by a normal deduction $\mathcal{D}$. Then all formulas in $\mathcal{D}$ are subformulas of $\Gamma \cup \{A\}$.*
PROOF. A *track of order 1* of a deduction $\mathcal{D}$ ends in the conclusion of $\mathcal{D}$ (i.e. is a terminal track). A *track of order $n+1$* terminates either in a major premise of **1**E, or in a major premise $!B$ of W, or in a minor premise of $\multimap$E, while the minor premise in the case of **1**E, W, and the major premise in the case of $\multimap$E, belong to a track of order n.

We prove by induction on the order of tracks that all formulas in a track are subformulas of $\Gamma \cup \{A\}$.

For the track of any order we have that all formulas occurring in it are subformulas of the open assumptions of the deduction or of the final formula of the track. Let π be a track of $\mathcal{D}$ (with conclusion A) of order $n+1$.

If π terminates in a minor premise B of $\multimap$E, then the major premise $B \multimap C$ belongs to a track of order n and so by induction hypothesis, $B \multimap C$ is subformula of $\Gamma \cup \{A\}$. Then B is also subformula of $\Gamma \cup \{A\}$, so π satisfies the subformula property.

If π terminates in a major premise of an **1**E- or W-application, the last rule must be an elimination, and the whole track consists of subformulas of the first formula in the track. The first formula is either an open assumption of the deduction, or is discharged below the end of the track. If discharged by $\multimap$I, or by $\star$E, this happens in a track of lower order, and the IH applies to this track. □

As an example of an application we give the next proposition.

7.4. DEFINITION. A formula (-occurrence) in a formula A is said to be a *strictly positive part (s.p.p.)* of A according to one of the following clauses:

1. A is s.p.p. of A;
2. if $B \star C$ is s.p.p of A, then B, C are s.p.p. of A;
3. if $!B$ is s.p.p. of A, so is B;
4. if $B \multimap C$ is s.p.p of A, so is C.

□

7.5. PROPOSITION. *If* $\Gamma \vdash A \star B$ *in* **ILL**, *and* Γ *does not contain* $\star$ *in a strictly positive position* ($\star$ *not main operator of a s.p.p. subformula of* Γ), *then* $\Gamma_1 \vdash A$ *and* $\Gamma_2 \vdash B$ *with* Γ_1 *and* Γ_2 *sub-multisets of* Γ.

PROOF. Let $\mathcal{D}$ be a normal deduction of $\Gamma \vdash A \star B$. If $\mathcal{D}$ ends with an I-rule we are done. If the terminal segment starts with an introduction, the deduction takes the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{F} \quad \mathcal{F}'}{A \quad B}}{\mathcal{E}_n \quad A \star B}}{A \star B}}{\vdots}}{\mathcal{E}_2 \quad A \star B}}{\mathcal{E}_1 \quad A \star B}}{A \star B}$$

where the final segment passes through a number of $\star$E, **1**E, and W-applications. However, $\star$E-applications are in fact excluded, since no strictly positive occurrence of $\star$ appears in

Γ. But this means that in $\mathcal{F}$ and $\mathcal{F}'$ no assumptions are discharged, and

$$\dfrac{\dfrac{\dfrac{\mathcal{E}_1 \quad \dfrac{\mathcal{E}_2 \quad \dfrac{\mathcal{E}_n \quad A}{A}}{\vdots}}{A}}{A}}{A} \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\mathcal{E}_1 \quad \dfrac{\mathcal{E}_2 \quad \dfrac{\mathcal{E}_n \quad B}{B}}{\vdots}}{B}}{B}}{B}$$

are both correct deductions. □

REMARK. The statement of the proposition may be considerable refined, e.g. by noting that assumptions common to Γ_1, Γ_2 must permit to derive exponential formulas, etc.

Another application may be (almost) copied from [P1, page 57].

7.6. PROPOSITION. *Let C be without $\multimap$, $\Gamma \equiv \{A_i \multimap B_i : 1 \leq i \leq n\}$, and assume $\Gamma \vdash C$. then $\Gamma' \vdash A_i$ for some $i \leq n$, Γ' a sub-multiset of Γ.*

8 Normalization in ILL

We shall now show how the normalization strategy in $\mathbf{ILL}^+$ suggests a corresponding strategy in **ILL**.

8.1. DEFINITION. A *segment* in **ILL** is a sequence of occurrences $A_1, \ldots, A_n$ of the same formula such that

1. A_1 not conclusion of W, **1**E, $\star$E, nor assumption discharged by C or !I;
2. A_n not minor premise of W, **1**E, $\star$E, or side premise of !I, or major premise of C;
3. for $1 \leq i < n$, either A_i is minor premise of an application α of W, **1**E or $\star$E, and A_i is the conclusion of α; or A_i is major premise of C and A_{i+1} is one of the assumptions discharged by α; or A_i is side premise of an instance α of !I, and A_{i+1} is an assumption discharged by α.

A segment is *maximal* if $\sigma \equiv A_1, \ldots, A_n$, A_1 conclusion of I-rule, A_n major premise of E-rule. As before, we define a *critical* segment as a maximal segment of maximal degree. □

8.2. DEFINITION. A *track* of $\mathcal{D}$ in **ILL** is a sequence of formula occurrences $A_1, \ldots, A_n$ in $\mathcal{D}$ such that

1. A_1 is an open assumption or an axiom or an assumption discharged by $\multimap$I;
2. A_i is major premise of $\multimap$E or D, or minor premise of $\star$E, C, **1**E, W, or premise of $\multimap$I, $\star$I, and A_{i+1} is the conclusion;

3. $A_i \equiv C \star D$ is major premise of $\star$E, and A_{i+1} is one of the assumptions discharged;

4. A_i is major premise of a contraction, and A_{i+1} one of the assumptions discharged;

5. A_i is a side premise of a promotion, and A_{i+1} is an occurrence discharged by the promotion.

6. A_n is major premise of a weakening or **1**E, or minor premise of $-\circ$E;

□

8.3. *Description of a strategy for normalization*

We look for an analogue of the strategy which works well in the intuitionistic case and for $\mathbf{ILL}^+$, namely: look for the rightmost branch in the formula tree containing a critical segment; apply a conversion in the topmost critical segment in this branch.

This strategy works for $\mathbf{ILL}^+$, because segments (in contrast to tracks) always belong to a unique branch of the tree. But this is not any longer the case for **ILL**. So in order to determine the proper place for a conversion, we construct, inspired by $\mathbf{ILL}^+$, an auxiliary partially ordered system with nodes labeled by formulas as follows. Given $\mathcal{D}$, the auxiliary structure $[\mathcal{D}]$ is obtained by systematically replacing

$$\dfrac{\mathcal{D}_1 \quad \mathcal{D}_n \quad \overset{[\![!A_1,\ldots,!A_n]\!]}{\mathcal{D}'}}{!B} \qquad \text{by} \qquad \dfrac{\dfrac{[\mathcal{D}_1] \quad \cdots \quad [\mathcal{D}_n]}{!A_1 \qquad\quad !A_n}}{\dfrac{[!A_1 \quad \cdots \quad !A_n]}{[\mathcal{D}']}\;\dfrac{B}{!B}}$$

and

$$\dfrac{\mathcal{D} \quad \overset{x_1}{[!A},\ldots,\overset{x_n}{!A]}}{B} \qquad \text{by} \qquad \dfrac{\dfrac{[\mathcal{D}]}{!A}}{\dfrac{/\cdots\backslash}{[!A\ldots!A]}\;\dfrac{[\mathcal{D}']}{\dfrac{B}{B}}}$$

letting the map [ ] act as a homomorphism for all other rules. There is a bijective correspondence between the formula occurrences in $\mathcal{D}$ and in $[\mathcal{D}]$.

A more formal description of the partial order of $[\mathcal{D}]$ is as follows: $[\mathcal{D}] = (\mathcal{D}, <)$ is a partial ordered set of the formula occurrences of $\mathcal{D}$, extending the tree order $(\mathcal{D}, \prec)$ of $\mathcal{D}$. Occurrence A is below occurrence B in $[\mathcal{D}]$ if

(a) $A \prec B$ in $\mathcal{D}$, or

(b) there is an !I-application where B is in the deduction of the main premise below assumption $!A_i$, and A is in the deduction of the side premise $!A_i$, or

(c) there is a C-application with A in the deduction of the major premise $!C$, and B in the deduction of the minor premise below one of the occurrences of $!C$ discharged by the C-application, or

(d) A below B by an application of transitivity for the ordering $<$.

.

For an instance of !I, all occurrences in the derivations of the side premises are above the occurrences in the derivation of the main premise in $[\mathcal{D}]$; and for an instance of contraction, all occurrences in the derivation of the major premise are above all occurrences in the derivation of the minor premise.

The strategy is now described as follows. Select in $[\mathcal{D}]$ a rightmost branch containing a topmost critical segment; take the topmost critical segment in this branch and apply the conversion to this segment.

A crucial instance of conversion may serve to show that the strategy has the desired effect.

$$\dfrac{\dfrac{\vec{\mathcal{E}} \quad \mathcal{D}' \quad \vec{\mathcal{E}}' \quad \overset{[\![!\vec{E}, !A, !\vec{E}']\!]}{\mathcal{D}} \quad \overset{[!B, !B]}{\mathcal{D}''}}{\dfrac{!\vec{E} \quad !A \quad !\vec{E}' \quad B}{!B} \quad C}}{C}$$

converted to

$$\dfrac{\vec{\mathcal{E}} \quad \mathcal{D}' \quad \vec{\mathcal{E}}' \quad \dfrac{\dfrac{!\vec{E} \quad !A \quad !\vec{E}' \quad \overset{[\![!\vec{E}, !A, !\vec{E}']\!]}{\mathcal{D}} \quad B}{[!B} \quad \mathcal{D}''}{C}}{!\vec{E} \quad !A \quad !\vec{E}' \quad \dfrac{\dfrac{!\vec{E} \quad !A \quad !\vec{E}' \quad \overset{[\![!\vec{E}, !A, !\vec{E}']\!]}{\mathcal{D}} \quad B}{!B]} \quad \mathcal{D}''}{C}}{C}$$

has as effect that indeed an occurrence of $!B$ is removed, but on the other hand, there is on the right hand side an extra occurrence of $!A$; but if we have chosen the critical segment according to our strategy, the occurrences of $!A$ cannot belong to a critical segment.

9 Concluding remarks

The simpler version of promotion plus the *removal* of the "dynamical" aspect of contraction (by which we mean that identifying the labels of two distinct assumptions of a formula of the form $!A$ is made into a separate *operation*) result in a variant $\mathbf{ILL}^+$ with a relatively simple proof theory. In addition, it suggests the consideration of a special class of categorical models for **ILL**.

There is a price to pay: the condition on the occurrence of multiple labels for $\mathbf{ILL}^+$ is not difficult to manage, but if we want to extend $\mathbf{ILL}^+$ by additive operators and constants, it becomes rather unwieldy.

On the other hand, the study of $\mathbf{ILL}^+_{\mathrm{me}}$ suggested a suitable normalization strategy for $\mathbf{ILL}_{\mathrm{me}}$ as well; this strategy seems also suitable for a complete system **ILL**. Although there seems no reason to doubt strong normalization for **ILL** (presumably a variant of the method in [G] would do the job), it is not clear how to extend strong validity to **ILL**.

We have not troubled ourselves with the Church-Rosser property (confluence), which holds for **ILL** and $\mathbf{ILL}^+$. The significance of confluence for these systems seems to be limited, as it is not likely that the conversion rules identify all intuitively equal deductions.

It would be interesting to extend the treatment of $\mathbf{ILL}^+$ to a multiple conclusion sequent calculus for a correspondence fragment for classical linear logic (cf. [C] for classical logic) and compare normal forms for this case with proofnets.

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