Institute for Logic, Language and Computation

THE STRUCTURE OF EXPONENTIALS:

UNCOVERING THE DYNAMICS OF LINEAR LOGIC PROOFS

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The Structure of Exponentials: Uncovering the dynamics of Linear Logic proofs*

by

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Abstract

We construct the *exponential graph* of a proof π in (second order) linear logic, an artefact that displays the interdependencies of exponentials. Within this graph superfluous exponentials are defined, the removal of which is shown to yield a correct proof $\pi^{\triangleright}$ with essentially the same set of reductions.

Applications to intuitionistic and classical logic are obtained by means of reduction-preserving embeddings: a given proof is embedded into linear logic, then the removal-procedure is applied to it, resulting in a least (i.e. *optimally*) exponentiated linearization of the original proof.

The last part of the paper puts things the other way round, and defines families of linear logics in which exponential dependencies are ruled by a given graph. We sketch some work in progress and possible applications.

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1 Introduction

An exponential "!", "?" in a linear proof is superfluous if we can remove it and obtain a proof that (1) is still correct, and (2) has the same dynamics as the original one. If we can get rid of an exponential in a linear proof, we know that the subproof introducing it (by a L? or a R! rule) will endure no non-linear handling (erasure or duplication) during normalization: *"the fewer the exponentials, the more the information"*.

We characterize the superfluous exponentials in a given linear derivation π and show that removing them determines a lattice of linear derivations with top π and as bottom a unique normal form $\pi^{\triangleright}$. So there is a sole *best* result with respect to this removal method. Moreover our lattice has the property that all its elements have the same behaviour under reduction. Even better, in the 'mono' fragment of linear logic where modalities (i.e. exponentials occurring in a row) are required to be in $\{!, ?, !?, ?!\}$, no significant further improvement is possible: any 'subexponentiation' of π will have an associated 'exponential graph' containing that of $\pi^{\triangleright}$. (Because each exponential in $\pi^{\triangleright}$ is imperative one might think that for each subproof determined by an R! or a L? rule (a 'box' in terms of proofnets) at least one normalisation strategy exists, in the course of which it will be duplicated or erased. This is not the case: even when 'logically necessary', exponentials can be 'computationally superfluous' (see Danos et al.(1993)). In other words, our logical linearity analysis is but an approximation of the real linearity of proofs, which is likely to be revealed only by the tautological process of normalizing the proof).

In order to apply these results to intuitionistic and classical logic (formulated as suitable sequent calculi, as in Girard(1993) and Joinet(1993)), we need translations into linear logic such that reductions can be simulated by reductions of the image. A necessary condition for this to hold, is that the 'skeleton' of the original proof is preserved by the translation. Such translations we will call *decorations.*

Because of their plethoric use of exponentials *uniform* translations are bound to give only 'universal linearity information' about proofs. So we apply the internal machinery constructed to get proof-by-proof embeddings, displaying the hidden structure of 'specific linearity information' in a given derivation.

As was pointed out to us by J.-Y. Girard, the exponential graph suggests the study of linear logic in an extended language, containing a set of distinct exponentials $\underset{a}{!}, \underset{a}{?}$ whose logical interaction is determined by a binary relation R. In the last section we briefly indicate some of the joint work in progress on these extended ("multicolored") systems of linear logic.

Finally, let us observe that the notions and techniques introduced are not typical of linear logic, but might be set to work within the framework of Gentzen-style proof theory for modal logics in general.

2 Strips: an 'exponential removal' theory

Terminological conventions

Our object of study will be the full system of second-order classical linear logic, as introduced in Girard(1987). More precisely, we will consider derivations in the two-sided sequent calculus $\mathbf{CLL_2}$ which can be found in the appendix.

We will use the following terminology in order to distinguish between the occurrences of formulas in a given rule, e.g. L$\multimap$:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Delta_1, \Delta_2}$$

The formula $A \multimap B$ is called the *main* formula of the rule with main connective $\multimap$; the occurrences A and B in the premisses will be referred to as the *active* formulas; all other occurrences are said to be *passive*, and we distinguish in the obvious way between an *up* and a *down* occurrence of a given passive formula. In the case of second-order rules, e.g. L$\forall_2$ and R$\forall_2$

$$\frac{A[T], \Gamma \Rightarrow \Delta}{\forall X A[X], \Gamma \Rightarrow \Delta} \quad \text{and} \quad \frac{\Gamma \Rightarrow \Delta, A[Y]}{\Gamma \Rightarrow \Delta, \forall X A[X]}$$

the active occurrences are $A[T], A[Y]$. We refer to T, Y as the *abstracted* formulas.

We will encounter derivations that contain *repetitions* of sequents. We will in such cases speak of an application of the *repetition* rule, where all occurrences of formulas are said to be *passive.*

In the sequel the rules for the exponentials will have our special interest. We recall the *exponential contextual* or *promotion rules* L? and R! (in analogy with the proofnet formulation of linear logic also to be referred to as the *box* rules):

$$\frac{!\Gamma, C \Rightarrow ?\Delta}{!\Gamma, ?C \Rightarrow ?\Delta} \quad \text{and} \quad \frac{!\Gamma \Rightarrow C, ?\Delta}{!\Gamma \Rightarrow !C, ?\Delta}.$$

Observe that in these rules the formulas in the context *do* play an important role, in the sense that applicability of the rules depends crucially on their being 'exponentiated'. We therefore call them *side-active.*

Besides the two constraints originating in the modal condition imposed on formulas in the structural and contextual exponential rules of the linear sequent calculus (in the sequel we will refer to the *structural* and the *contextual* constraint), there is, as in any sequent calculus, another, fairly obvious, one: in writing down rules and derivations we implicitly demand the identity of some of the (sub)formulas occurring in the sequents appearing in it. In the sequel we will refer to the *identity* constraint. E.g. the occurrences of the contextual formulas $!\Gamma$ and $?\Delta$ in the premiss and conclusion of a promotion rule are occurrences of *identical* formulas. This implicit identity relation is made explicit in the following

2.1. DEFINITION. We call occurrences of (sub)formulas in a proof *identified* whenever they are the corresponding occurrences of the same[1] (sub)formula in

- the two formulas in an axiom;
- the cut formulas in a cut;
- the abstracted formulas in a second-order rule;
- an active formula and the corresponding subformula of the main formula in a logical or exponential rule (in the case of $L\forall_2$ and $R\exists_2$ rules a strict subformula of an abstracted occurrence, has no "correspondent" in the conclusion sequent of the rule);
- the up and down occurrences of passive or side-active formulas in a rule (this includes the implicit contextual contraction in additive binary rules). □

Let us denote by "$\sim$" the reflexive, symmetric and transitive closure of the identification relation. Note once more that all elements of a $\sim$-equivalence class are occurrences of the *same* (sub)formula (up to substitution).[2] In the sequel we will only deal with classes containing at least one formula whose main connective is an *exponential.* We denote the set of such classes in a proof π by $\mathcal{E}(\pi)$.

The exponential graph

Let E be a subset of $\mathcal{E}(\pi)$. The *domain* of E is the union of the classes it contains. By a *strip* we mean the operation of simultaneous deletion in π of all external exponentials in the domain of E. The resulting pseudo-proof (which need in general not be a proof) is denoted by $\pi - E$. The corresponding instance of a rule r in $\pi - E$ is written as $r - E$.

Each formula B in π 're-appears' in $\pi - E$, though maybe slightly modified. To be precise it is modified if and only if some formula $!A$ or $?A$ in one of the classes in E is a subformula of B in π. If we want to specify the changes we will write $B - E$, though mostly we will continue to denote this, possibly modified, formula by B.

Take some box rule r in π, with its main occurrence in some class e and a side-active occurrence in some class e': we say that e *binds* e' (*via* r) and write this as $e \curvearrowright_1 e'$. The transitive closure of the relation $\curvearrowright_1$ will also be called *binding*, and is denoted by $\curvearrowright$. (Par abus de langage we will sometimes write $s \curvearrowright_1 s'$ and $s \curvearrowright s'$ also for proper *sub*sets of classes.)

[1] Exactly the same or, in case of quantifier rules, the same up to substitution.

[2] In general the converse does not hold, of course.

This defines a directed graph, the *exponential graph* $\mathcal{G}(\pi)$ of π, with as vertices the classes in $\mathcal{E}(\pi)$, and an arrow from e to e' if and only if $e \curvearrowright_1 e'$. If an occurrence of an element of a class e is main formula in a structural rule in π, then we label the corresponding vertex of the exponential graph by "s" (for 'source').

2.2. DEFINITION. A set $E \subset \mathcal{E}(\pi)$ is called *saturated* (or said to satisfy the *saturation condition*), in case for all $e' \in E$, if $e \curvearrowright e'$ for some $e \in \mathcal{E}(\pi)$, then also $e \in E$. If no class in E is labeled "s" then we will say that E verifies the *no sources condition.* If E satisfies both the saturation and the no sources condition, we say that it is *not relevantly exponentiated* (abbreviated by *nre*) in π. A *redex* is any non-empty set E that is *nre* and *minimal*, i.e. no proper subset of E is nre. □

2.3. THEOREM. (Stripping preserves correctness) *Let π be a proof, r a rule in π, and E nre in π; $r - E$ is still a correct rule, and hence $\pi - E$ is a proof. More precisely, either $r - E$ and r are instances of the same rule, or $r - E$ is a repetition rule.*

PROOF: First observe that, whatever rule r, because only classes are stripped, all identity constraints are obviously still satisfied by $r - E$.[3] Now, if r is a box rule, by the saturation condition, the (eventual) contextual constraint for $r - E$ will also be satisfied. And finally, if r is a stuctural rule, by the no sources condition, so is the structural constraint for $r - E$. (Clearly $r - E$ is a repetition rule only when r introduces an exponential that is stripped, i.e. when r is an exponential rule whose main formula is in the domain of E.) □

2.4. REMARK. We will in the sequel adopt the convention that all occurrences of the repetition rule in $\pi - E$ are eliminated. So possible repetitions of sequents are identified.

2.5. LEMMA. *Let E_1, E_2 be nre in π. Then so are $E_1 \cap E_2, E_1 \cup E_2$.* □

So $\mathcal{E}(\pi)$ contains a *largest* nre subset, which we denote by $E_{\mathsf{max}}(\pi)$. It is the largest saturated subset of $\mathcal{E}(\pi)$ that contains no vertices labeled "s".

2.6. LEMMA. *$\mathcal{E}(\pi - E) = \mathcal{E}(\pi)\backslash E$, and the exponential graph of $\pi - E$ is a full subgraph of that of π.*

PROOF: For the first claim, observe that any class not in E remains a class in $\pi - E$, while all classes in $\pi - E$ are classes in π. For the second claim, note

[3] Of course $\sim$-classes are defined precisely for that purpose!

that for e', e in $\mathcal{E}(\pi)\backslash E$ we have that $e' \curvearrowright_1 e$ in $\pi - E$ if and only if $e' \curvearrowright_1 e$ in π. $\square$

2.7. LEMMA. *If E, E' are nre in π, and E' is a subset of E, then $E\backslash E'$ is nre in $\pi - E'$.*

PROOF: As no class in E is labeled "s", the same holds for $E\backslash E'$. As E is nre in π and $E' \subseteq E$, the only possible elements of $\mathcal{E}(\pi)$ that bind elements of $E\backslash E'$ are in E'. So $E\backslash E'$ is saturated in $\mathcal{E}(\pi - E')$. $\square$

2.8. LEMMA. *Suppose E_1 is nre in π. Then E_2 is nre in $\pi - E_1$ if and only if $E_1 \cup E_2$ is nre in π.*

PROOF: ($\Rightarrow$) As E_1 and E_2 are nre, none of their elements is labeled by "s". Let $e' \in \mathcal{E}(\pi)$ bind an element of E_1. Then e' in E_1 by saturation. If it binds an element of E_2, and it is not an element of E_1, then $e' \in \mathcal{E}(\pi - E_1)$, so $e' \in E_2$, by saturation of E_2.

($\Leftarrow$) By lemma 2.7. $\square$

2.9. PROPOSITION. *Let R_1, R_2 be distinct redexes in π. Then R_2 is a redex in $\pi - R_1$.*

PROOF: Observe that, by lemma 2.5, $R_1 \cap R_2 = \emptyset$, from which the claim easily follows, using lemma 2.6. $\square$

2.10. COROLLARY. *Let R_1, R_2 be distinct redexes in π. Then $(\pi - R_1) - R_2$ is a correct linear derivation. Moreover it is equal to $(\pi - R_2) - R_1$.* $\square$

Now define a reduction $\triangleright$ on linear derivations by $\pi \triangleright \pi - R$, for R a redex in π. Given some derivation π, clearly the number of potential redexes in π is finite. So all $\triangleright$-reduction-sequences are finite, ending in a $\triangleright$-normal form. As by the above $\triangleright$ is locally (1-1) confluent, in fact for each π we obtain a *unique* $\triangleright$-normal form, which we will denote by $\pi^{\triangleright}$.

Thus $\triangleright$ defines a complete lattice of linear derivations with top π, bottom $\pi^{\triangleright}$, and $\pi_i \triangleright \pi_j$ if and only if there is a (possibly empty) $\triangleright$-reduction-sequence leading from π_i to π_j.

We will refer to the lattice obtained as the "*$\triangleright$-lattice of π*".

2.11. LEMMA. *If E is nre in π, then $\pi \triangleright \pi - E$ and $E_{\mathsf{max}}(\pi - E) = E_{\mathsf{max}}(\pi)\backslash E$.*

PROOF: The first claim is shown by induction on the size of E, the second claim using lemma's 2.7, 2.8. $\square$

2.12. THEOREM. $\pi^{\triangleright} = \pi - E_{\mathsf{max}}(\pi)$, *and the exponential graph of* $\pi^{\triangleright}$ *is precisely the union of all directed paths in the graph of* π *starting from a vertex labeled "*s*".*

PROOF: By lemma 2.11, $\pi \triangleright \pi - E_{\mathsf{max}}(\pi)$, so $\pi - E_{\mathsf{max}}(\pi) \triangleright \pi^{\triangleright}$. But as $E_{\mathsf{max}}(\pi - E_{\mathsf{max}}(\pi)) = \emptyset$ in fact $\pi - E_{\mathsf{max}}(\pi) = \pi^{\triangleright}$. The second claim is immediate by 2.6 and the fact that we obtain the exponential graph of $\pi^{\triangleright}$ by removing all saturated subgraphs of the graph of π that do *not* contain a vertex labeled "s". □

Consequently we have shown:

> *"A class e remains in* $\pi^{\triangleright}$ *if and only if the corresponding class in* π *has a structural cause."*

The mono-stable fragment of $\mathbf{CLL}_2$

Let us call derivations π in linear logic 'mono' if the only modalities prefixing the skeleton of each formula appearing in π are among '!', '?', '!?' and '?!'. Observe that the collection of all first-order 'mono'-derivations is closed under cut-elimination. To get the same property in the second-order case, abstraction on externally modalized formulas should be prohibited. This defines a *proper* fragment of second-order linear logic: the *mono-stable* fragment. For 'mono'-derivations we are able to strengthen theorem 2.3, in the sense that we now also have the converse:

2.13. THEOREM. *Let* π *be 'mono', and* $E \subseteq \mathcal{E}(\pi)$. *Then* $\pi - E$ *is a correct linear derivation if* and only *if* E *is nre.*

PROOF: ($\Rightarrow$) If E is *not* nre and π is 'mono', then the strip defined by E will result in a derivation $\pi - E$ in which there is either an application of a structural rule on a non-exponentiated (not properly exponentiated) formula, or an application of an exponential contextual rule where the context contains (a) non-exponentiated (not properly exponentiated) formula(s). So $\pi - E$ can not possibly be correct. □

In general we can not be sure of the left-to-right direction: 'good' exponentials may be hidden (more or less directly) behind the 'stripped' ones, e.g. in case we strip in '!!' or '!??????!'.

Theorem 2.13 tells us that the minimum $\pi^{\triangleright}$ of the $\triangleright$-lattice of a 'mono'-derivation π is a minimum in a very strong sense: for *no* $E \subseteq \mathcal{E}(\pi^{\triangleright})$ the strip defined by E can possibly result in a derivation that is linearly correct.

This does *not* mean that for a 'mono'-derivation π, it is impossible to remove any more exponentials in $\pi^{\triangleright}$: what can't be done is remove one or more entire *classes*, but one still has the possibility to lower as much as possible the L! and R? rules that are left, in order to introduce them just before they are needed. If we apply this lowering of dereliction rules to $\pi^{\triangleright}$ we obtain derivations $(\pi^{\triangleright})'$. Clearly all of them have the same exponential graph. They also will be identified in their proofnet representation. In other words, the difference between them is negligible.

3 Strips and normalization

3.1. DEFINITION. Let c be a cut rule in a proof π. We will denote by $[c]$ the particular kind of *elementary normalization step* to be performed in order to eliminate the cut (following the standard normalization procedure for linear sequent calculus, see e.g. Troelstra(1992) for an exhaustive treatment of the first order case).

3.2. REMARK. The nature of $[c]$ depends on:
- the rules r_g and r_d surmounting c (in the left, respectively the right premiss);
- the status in r_g and r_d (main, passive, side-active) of the cutformula.

Accordingly we distinguish four kinds of elementary normalization steps: permutation steps, logical steps, structural steps, axiom steps.

We recall steps that in the sequel ask for a non trivial treatment, namely those where r_g or r_d is a box rule whose main formula is the cutformula; also we display the configuration where r_g and r_d are second-order rules introducing the cutformula. (For each we will show only one among the possible cases.)

- If the cutformula is side-active in an exponential contextual rule surmounting c, we denote the associated reduction step by $[cc]$ ('commutative cut') being of the following form[4]:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array}}{!\Gamma \Rightarrow ?\Gamma', !A} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ !A, !\Delta \Rightarrow B, ?\Delta'\end{array}}{!A, !\Delta \Rightarrow !B, ?\Delta'}}{!\Gamma, !\Delta \Rightarrow !B, ?\Gamma', ?\Delta'} \quad \overset{[cc]}{\leadsto} \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array}}{!\Gamma \Rightarrow ?\Gamma', !A} \quad \begin{array}{c}\pi_2\\ \vdots\\ !A, !\Delta \Rightarrow B, ?\Delta'\end{array}}{!\Gamma, !\Delta \Rightarrow B, ?\Gamma', ?\Delta'}}{!\Gamma, !\Delta \Rightarrow !B, ?\Gamma', ?\Delta'}$$

- If the cutformula is main in a dereliction rule, we denote the associated reduction step by $[de]$ which is of the following form:

[4]**Thanks!** We gladly acknowledge the use of Dirk Roorda's 'exptrees'-macros for the typesetting of proof trees.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array}}{!\Gamma \Rightarrow ?\Gamma', !A} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ A, \Delta \Rightarrow \Delta'\end{array}}{!A, \Delta \Rightarrow \Delta'}}{!\Gamma, \Delta \Rightarrow ?\Gamma', \Delta'} \quad \overset{[de]}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ A, \Delta \Rightarrow \Delta'\end{array}}{!\Gamma, \Delta \Rightarrow ?\Gamma', \Delta'}$$

- If the cutformula is main in an instance of a contraction rule, we denote the associated reduction step by $[co]$ which is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array}}{!\Gamma \Rightarrow ?\Gamma', !A} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ !A, !A, \Delta \Rightarrow \Delta'\end{array}}{!A, \Delta \Rightarrow \Delta'}}{!\Gamma, \Delta \Rightarrow ?\Gamma', \Delta'} \quad \overset{[co]}{\rightsquigarrow} \quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array}}{!\Gamma \Rightarrow ?\Gamma', !A} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1'\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array}}{!\Gamma \Rightarrow ?\Gamma', !A} \quad \begin{array}{c}\pi_2\\ \vdots\\ !A, !A, \Delta \Rightarrow \Delta'\end{array}}{!A, !\Gamma, \Delta \Rightarrow ?\Gamma', \Delta'}}{!\Gamma, !\Gamma, \Delta \Rightarrow ?\Gamma', ?\Gamma', \Delta'}\\ \vdots\\ \hline !\Gamma, \Delta \Rightarrow ?\Gamma', \Delta'\end{array}$$

- If the cutformula is main in an instance of a weakening rule, we denote the associated reduction step by $[w]$ which is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma \Rightarrow ?\Gamma', A\end{array}}{!\Gamma \Rightarrow ?\Gamma', !A} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Delta \Rightarrow \Delta'\end{array}}{!A, \Delta \Rightarrow \Delta'}}{!\Gamma, \Delta \Rightarrow ?\Gamma', \Delta'} \quad \overset{[w]}{\rightsquigarrow} \quad \begin{array}{c}\pi_2\\ \vdots\\ \hline \Delta \Rightarrow \Delta'\\ \vdots\\ \hline !\Gamma, \Delta \Rightarrow ?\Gamma', \Delta'\end{array}$$

- If the cutformulas are main in $\forall_2$-rules, we denote the associated reduction step by $[\forall_2]$, which is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \Rightarrow \Gamma', A[X]\end{array}}{\Gamma \Rightarrow \Gamma', \forall X A[X]} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ A[T/X], \Delta \Rightarrow \Delta'\end{array}}{\forall X A[X], \Delta \Rightarrow \Delta'}}{\Gamma, \Delta \Rightarrow \Gamma', \Delta'} \quad \overset{[\forall_2]}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\pi_1[T/X]\\ \vdots\\ \Gamma \Rightarrow \Gamma', A[T/X]\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ A[T/X], \Delta \Rightarrow \Delta'\end{array}}{\Gamma, \Delta \Rightarrow \Gamma', \Delta'}$$

Let μ be an elementary normalization step. Any occurrence of a (sub)formula F (resp. any instance of a rule r) in $\mu(\pi)$ comes, in the obvious way, from a unique occurrence of a (sub)formula (resp. a rule) in π. Let us denote by $\mu_\star$ this lifting application.

3.3. LEMMA. (Lifting of classes) *For any elementary normalization step μ in a proof π, $\mu_\star$ respects classes. I.e., if F, G are occurrences of subformulas in $\mu(\pi)$ and $F \sim_{\mu(\pi)} G$, then $\mu_\star(F) \sim_\pi \mu_\star(G)$.* □

Hence each class e in $\mu(\pi)$ is mapped by $\mu_\star$ to a class e' of π (so $\mu_\star(e) \subset e'$). Note however, that this mapping is neither one-to-one, nor onto, in general.

3.4. LEMMA. (Lifting of binding) *For any elementary normalization step μ in a proof π, $\mu_\star$ respects binding. I.e., if e, e' are classes in $\mu(\pi)$ and $e \curvearrowright e'$ in $\mu(\pi)$, then $\mu_\star(e) \curvearrowright \mu_\star(e')$.*

PROOF: (Recall that $\curvearrowright$ is but the transitive closure of $\curvearrowright_1$.) Suppose μ is $[cc]$, and $e \curvearrowright_1 e'$ via the box rule permuted by μ with the cut rule. Either $\mu_\star(e) \curvearrowright_1 \mu_\star(e')$ or there is in π a class e'' (namely the class of the cutformulas) such that $\mu_\star(e) \curvearrowright_1 e''$ and $e'' \curvearrowright_1 \mu_\star(e')$. In all other cases $\mu_\star(e) \curvearrowright_1 \mu_\star(e')$ (in particular, note that for $T \equiv !T'$ or $?T'$ in $[\forall_2]$, there will be no binding involving T in $\pi_1[T/X]$). □

Let E be a set of classes in a proof π, and suppose μ is an elementary normalization step of π. Let us denote by $\mu(E)$ the set of classes in $\mu(\pi)$ mapped by $\mu_\star$ to a class in E. This makes sense, precisely because $\mu_\star$ respects classes.

3.5. LEMMA. *Let π be a proof, μ an elementary normalization step in π, and E a subset of $\mathcal{E}(\pi)$. If E is saturated, then so is $\mu(E)$.*

PROOF: Take a class e in $\mu(E)$ such that $e' \curvearrowright e$ for some class e' in $\mathcal{E}(\mu(\pi))$. By lemma 3.4, $\mu_\star(e') \curvearrowright \mu_\star(e)$. Because $e \in \mu(E)$, by definition $\mu_\star(e)$ is contained in a class of E. Hence, by saturation of E, the same holds for $\mu_\star(e')$, and, again by definition, $e' \in \mu(E)$. □

3.6. LEMMA. *Let π be a proof, μ an elementary normalization step in π, and E a subset of $\mathcal{E}(\pi)$. If E is nre, then so is $\mu(E)$.*

PROOF: By lemma 3.5, $\mu(E)$ is saturated. Now suppose there is a class e in $\mu(E)$ labeled "s". If an occurrence of a formula F is main in a contraction (resp. weakening) rule in $\mu(\pi)$, observe that either this already is the case for $\mu_\star(F)$ in π, or μ is $[co]$ (resp. $[w]$), and $\mu_\star(F)$ is side-active in the box rule to be duplicated (resp. erased). So either $\mu_\star(e)$ is also labeled "s", or there is in $\mathcal{E}(\pi)$ a class e' labeled "s" such that $e' \curvearrowright_1 e$, contradicting the hypothesis that E satisfies the no sources condition. □

3.7. DEFINITION. Let π be a proof, E an nre set of classes in π, μ an elementary normalization step performable in π. The equivalent of μ in $\pi - E$, denoted by $\hat{\mu}$, is defined as follows:

- $\hat{\mu} = [id]$ (the empty operation) if μ is either $[de]$ with active formulas in the domain of E, or a permutation step where the cut is permuted upwards from the conclusion to the premiss of an exponential rule with main formula in the domain of E.

- $\hat{\mu} = \mu$ in all other cases. □

Let r be a rule in a proof π, and μ an elementary normalization step of π. We denote by $\mu(r)$ the set of instances of rules in $\mu(\pi)$ mapped by $\mu_\star$ to r.

3.8. Remark. Let π be a proof, c a cut in π. If r is a rule in π that is neither c, nor r_g, r_d, then any rule $r' \in [c](r)$ is an instance (in $[c](\pi)$) of the same rule as r.

3.9. Lemma. *Let π be a proof, E nre in π, μ an elementary normalization step in π. If G is an occurrence of a (sub)formula in $\mu(\pi)$, then $G - \mu(E) = \mu_\star(G) - E$.*

Proof: For any subformula $!F$ of G we have that $!F$ is stripped in $G - \mu(E)$ if and only if $!F \in \mu(E)$ if and only if $\mu_\star(!F) \in E$ if and only if $\mu_\star(!F)$ is stripped in $\mu_\star(G) - E$. □

3.10. Theorem. (Stripping preserves normalization). *Let μ be an elementary normalization step in a proof π, and E nre in π. Then μ can be applied to π if and only if $\hat{\mu}$ can be applied to $\pi - E$, and $\hat{\mu}(\pi - E) = \mu(\pi) - \mu(E)$.*

Proof: (Sketch[5]) For the first half of the claim we have to check that whenever $\hat{\mu} \not\equiv [id]$, stripping does not change the *nature* of a given cut in π, i.e. $[c - E] = [c]$, which is immediate by remark 3.2 and theorem 2.3.

For the second half, we have to verify that, whatever μ, for any rule r in the proof tree π, it holds that $\mu(r) - \mu(E) = \hat{\mu}(r - E)$

Let c be the cut to which μ is applied.

- If r is neither c, nor r_g, r_d, then for any rule $r' \in \mu(r)$, by lemma 3.9 and remark 3.8, $r' - \mu(E)$ and $\mu_\star(r') - E$ (i.e. $r - E$) are instances of the same rule in respectively $\mu(\pi) - \mu(E)$ and $\pi - E$. Now, again by remark 3.8, rules in $\hat{\mu}(r - E)$ remain instances of the same rule.

- Let r be c or one of r_g, r_d. We consider just the case that μ is $[de]$ with cutformula in E. Then $\mu(E)$ is E minus the class consisting of the two active formulas of c. Neither of r_g, r_d have an equivalent in either $\pi - E$ (by remark 2.4) or $\mu(\pi)$. Remains the case that r in fact is c. Let c' be the (unique) rule in $\mu(c)$. Then $c' - \mu(E) = c' - (E \backslash \{A_1, A_2\}) = c - E$, so we are done. □

[5] A detailed proof would consist in a case-by-case inspection of all possible appearances of an instance of the cut rule in π. We refer the reader to section 5 for an alternative, intuitive, argument supporting the claim.

3.11. COROLLARY. *Let π be a proof, E nre in π, and μ an elementary normalization step in π. Then $\mu(\pi) \rhd \hat{\mu}(\pi - E)$.*

PROOF: By theorem 3.10 and lemma 2.11 $\square$

3.12. REMARK. Note that the converse of lemma 3.6 does *not* hold: $F \subseteq \mathcal{E}(\mu(\pi))$ might very well be nre, while $\mu_\star(F) \subseteq \mathcal{E}(\pi)$ is not.

A typical example is the class of a main formula in a box rule to be duplicated by $[co]$, which might become nre after duplication.

3.13. THEOREM. *Let $\mu_k \ldots \mu_1$ be a reduction-sequence in π. Then*

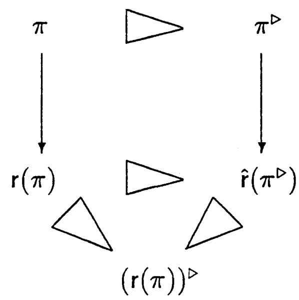
$$\hat{\mu}_k \ldots \hat{\mu}_1(\pi^{\rhd}) \rhd (\mu_k \ldots \mu_1(\pi))^{\rhd}.$$

PROOF: By iteration of corollary 3.11 we find $\mu_k \ldots \mu_1(\pi) \rhd \hat{\mu}_k \ldots \hat{\mu}_1(\pi^{\rhd})$. We conclude that $\hat{\mu}_k \ldots \hat{\mu}_1(\pi^{\rhd})$ is in the $\rhd$-lattice of $\mu_k \ldots \mu_1(\pi)$, where $(\mu_k \ldots \mu_1(\pi))^{\rhd}$ is the bottom-element. $\square$

We established an important property of the $\rhd$-lattice of a derivation π that intuitively can be expressed as follows:

> *"Derivations in the $\rhd$-lattice of π have, essentially, the same set of reductions."*

Writing r to denote a reduction sequence in π, the content of the above can be visualised in the following diagram:



In remark 2.13 we observed that if π is *not* 'mono', it may be possible to strip sets of classes in π that are not nre, and still get a derivation that is linearly correct. However, the result of such a strip is likely to have a behaviour under reduction quite different from that of π, and in general theorem 3.10 will no longer hold.

4 Application to intuitionistic and classical logic

In Danos et al.(1993) we introduced the concept of *linear decoration* of a given proof in one of the standard sequent formulations for intuitionistic or classical logic, being a derivation in linear logic having the same *skeleton* as the proof of departure. I.e., if we delete all exponentials, replace the linear connectives by their non-linear counterparts and eliminate possible repetitions of sequents, then 'what we get is what we got'.

There exist uniform translations of intuitionistic as well as of classical logic into linear logic, which are also decorations for the standard sequent-calculi, though not *all* uniform translations automatically define a decoration.

Here we will strengthen the concept of decoration as follows:

4.1. DEFINITION. Let L be a sequent calculus and σ a procedure for cut elimination in L. A decoration δ for L is said to be a *strong decoration* (with respect to σ) if and only if any elementary normalisation step in σ, transforming a derivation π in L into π', can be simulated by one or more elementary steps in the standard procedure for linear sequent calculus, leading from $\delta(\pi)$ to $\delta(\pi')$. □

In other words, δ is a strong decoration if and only if the following diagram commutes:

$$\begin{array}{ccc} \pi & \xrightarrow{\quad\delta\quad} & \delta(\pi) \\ {\scriptstyle \mu_L}\downarrow & & \downarrow{\scriptstyle \overline{\mu}_{LL}} \\ \pi' & \xrightarrow[\quad\delta\quad]{} & \delta(\pi') \end{array}$$

4.2. DEFINITION. We will say that δ is a 'mono' decoration if $\delta(\pi)$ is a 'mono' derivation for any proof π in L. □

Now if π is an L-derivation, we apply the decoration in order to obtain a linear derivation $\delta(\pi)$. We strip $\delta(\pi)$, and get the minimum $\delta(\pi)^{\triangleright}$ in the $\triangleright$-lattice.

Using the results of the previous sections, we have the following:

4.3. THEOREM. *If δ is a strong 'mono' decoration for L, then $\delta(\pi)^{\triangleright}$ is an optimal linear version of π, with essentially the same set of reductions.* □

Note that for any decoration δ of intuitionistic or classical sequent calculus there exists a normalization strategy, say σ_l, such that δ is strong with respect to σ_l: it suffices to define σ_l as the reflection of the procedure in linear logic. But e.g. in the case of standard versions of classical sequent calculus and uniform translations as the one defined (for the propositional fragment) by

for atoms take $p^{\diamond} := !p$; then put

$$\begin{array}{rcl}(A \wedge B)^{\diamond} & := & !(!?A^{\diamond} \otimes !?B^{\diamond}) \\ (A \vee B)^{\diamond} & := & !(?A^{\diamond} \wp ?B^{\diamond}) \\ (A \rightarrow B)^{\diamond} & := & !(?A^{\diamond} \multimap ?B^{\diamond}),\end{array}$$

it is not obvious how one should formulate the corresponding strategy σ_l *directly*, and independent of specific derivations π. Moreover, in general, the decoration of a proof will not be 'mono'.

In Joinet(1993) linear translations (comparable to the one above) are defined that *do* take us into the 'mono-stable' realm of linear logic. Though these do not define decorations of the *standard* sequent calculus formulations of classical logic, they are easily seen to impose certain restrictions on the structure of classical derivations, that can be built into *non-standard* sequent formulations, for which these translations then *are* decorations. Also, by construction, we find σ_l as a 'natural' procedure of cut-elimination, so that in fact we obtain *strong* decorations. To these calculi, complete for classical second-order logic (and baptized **LKT**, **LKQ** in Joinet(1993)), consequently our 'optimal linearization' analysis can be applied.

The description in Danos et al.(1993) of the construction of a non-standard sequent calculus (**ILU**) for *intuitionistic* implicational logic may serve as an illustration of the principle also behind the calculi **LKT** and **LKQ**: the linear translation $(\cdot)^{\circledast}$ which is the identity on atoms and maps an implicational formula $A \supset B$ to $!A^{\circledast} \multimap !B^{\circledast}$ defines a decoration of the usual sequent calculus for intuitionistic implicational logic, while Girard's well-known translation $(\cdot)^*$ (mapping $A \supset B$ to $!A^* \multimap B^*$) does *not*. This is related to the fact that in the standard formulations application of the rule $L \supset$ is allowed also when the active formula B in the right premiss has been main formula in a structural rule. Note that these instances of $L \supset$ have no equivalent in the natural deduction formulation of intuitionistic logic, which suggests that the collection of derivations that do *not* use it in such cases is complete. This indeed is so and a non-standard sequent calculus formulation (which consequently is closer to natural deduction and the simply typed lambda calculus than the standard one) is obtained by a straightforward abstraction of the structure of linear derivations of sequents of the form $!\Gamma^{\circledast} \Rightarrow A^{\circledast}$.

The systems **LKQ** and **LKT** are similarly based upon (dual) decompositions of classical implication, resp. as $!A \multimap ?!B$ and $!?A \multimap ?B$.[6] As expected, our theorem 4.3 applies to each of these systems.

ILU is the neutral fragment of intuitionistic implicational logic in Girard's system of Unified Logic **LU** (Girard(1993)), and indeed our methods are not limited to merely this fragment. If we decorate negative atoms N as $?N$, positive atoms P as $!P$, and follow the linear definitions of the classical and intuitionistic connectives, we get a strong 'mono' decoration. As a result theorem 4.3 applies to *all* of non-linear **LU**, which besides e.g. **ILU** includes Girard's system of classical logic **LC** (Girard(1991)).[7]

(Note an important difference between **LC** and systems like **LKT**, **LKQ**, **ILU**, being that unlike the latter, it is *not* based upon an underlying *direct* linear translation of classical(intuitionistic) logic, but passes, through the concept of *polarity*, via an intermediate language.)

5 Multicolor linear logic

As is well known, the sequent calculus rules for the exponentials, unlike those for the other connectives, do not imply their uniqueness modulo linear equivalence: if we introduce a unary connective ¡, with the *same* rules as the exponential !, then neither $!A \Rightarrow ¡A$, nor $¡A \Rightarrow !A$ are derivable. Given a linear proof, this suggests the possibility, pointed out to us by J.-Y. Girard, to use a *distinct* exponential as main connective for each class in $\mathcal{E}(\pi)$; or, otherwise said, use a different *colour* for each vertex in the exponential graph of π. The binding relation then gives rise to a natural notion of interprovability between some (or all) of our colours, in the form of *promotional constraints*. The idea is easily formalized:

Let R be some binary relation (on some index-set); extend the language of linear logic by indexed exponentials $\underset{a}{!}, \underset{a}{?}$. We define the sequent calculus R-**CLL** by adding to the non-exponential fragment of **CLL**

- for all indexed exponentials $\underset{a}{!}, \underset{a}{?}$ the usual dereliction rules;

- structural permissions restricted to specified sets of indices $\mathcal{W}_{\mathsf{R}}$ and $\mathcal{C}_{\mathsf{R}}$, i.e. structural rules of weakening(contraction) for $\underset{a}{!}, \underset{a}{?}$ iff $a \in \mathcal{W}_{\mathsf{R}}(\mathcal{C}_{\mathsf{R}})$;

[6] We recover **ILU** by the usual intuitionistic restriction of **LKT** to single-conclusion sequents. **LKT** appears to be closely related to the natural deduction system for classical logic studied in Parigot(1992). A more detailed account will be given in Danos et al.(199).

[7] This, by the way, gives us an indirect proof of cut elimination for these systems.

- the following promotion rules (subject to the restriction that $z\mathsf{R}x_i, z\mathsf{R}y_j$ for all x_i, y_j) :

$$\dfrac{\underset{x_1}{!}G_1,\ldots,\underset{x_n}{!}G_n \Rightarrow A,\underset{y_1}{?}D_1,\ldots\underset{y_m}{?}D_m}{\underset{x_1}{!}G_1,\ldots,\underset{x_n}{!}G_n \Rightarrow \underset{z}{!}A,\underset{y_1}{?}D_1,\ldots\underset{y_m}{?}D_m}\ \mathrm{R!}$$

$$\dfrac{\underset{x_1}{!}G_1,\ldots,\underset{x_n}{!}G_n, A \Rightarrow \underset{y_1}{?}D_1,\ldots\underset{y_m}{?}D_m}{\underset{x_1}{!}G_1,\ldots,\underset{x_n}{!}G_n, \underset{z}{?}A \Rightarrow \underset{y_1}{?}D_1,\ldots\underset{y_m}{?}D_m}\ \mathrm{L?}$$

5.1. PROPOSITION. R *is reflexive iff* $\underset{a}{!}\phi \Rightarrow \underset{a}{!}\phi$ *and* $\underset{a}{?}\phi \Rightarrow \underset{a}{?}\phi$ *are cutfree derivable in* R-**CLL**, *for any formula* ϕ, *any index* a. □

5.2. PROPOSITION. R-**CLL** *enjoys cut-elimination iff (1)* R *is transitive and (2)* $\mathcal{W}_{\mathsf{R}}$ *and* $\mathcal{C}_{\mathsf{R}}$ *are upwardly closed (i.e. if* $i \in \mathcal{W}_{\mathsf{R}}(\mathcal{C}_{\mathsf{R}})$ *and* $i\mathsf{R}j$, *then* $j \in \mathcal{W}_{\mathsf{R}}(\mathcal{C}_{\mathsf{R}})$*).*
PROOF: ($\Leftarrow$) Observe that (1) implies correctness of $[cc]$, (2) correctness of $[w]$ and $[co]$.

($\Rightarrow$) Suppose aRb and bRc. Then the following is a derivation in R-**CLL**:

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\underset{c}{!}p \Rightarrow p}}{\underset{c}{!}p \Rightarrow \underset{b}{!}p} \qquad \dfrac{\dfrac{p \Rightarrow p}{\underset{b}{!}p \Rightarrow p}}{\underset{b}{!}p \Rightarrow \underset{a}{!}p}}{\underset{c}{!}p \Rightarrow \underset{a}{!}p}$$

But obviously there is no cut-free proof of $\underset{c}{!}p \Rightarrow \underset{a}{!}p$ in case $(a,c) \notin R$.

Similarly we obtain contradicitions in case $i \in \mathcal{W}_{\mathsf{R}}, iRj$ and $j \notin \mathcal{W}_{\mathsf{R}}$ or $i \in \mathcal{C}_{\mathsf{R}}, iRj$ and $j \notin \mathcal{C}_{\mathsf{R}}$, e.g. using the following derivations:

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\underset{j}{!}p \Rightarrow p}}{\underset{j}{!}p \Rightarrow \underset{i}{!}p} \qquad \dfrac{q \Rightarrow q}{\underset{i}{!}p, q \Rightarrow q}}{\underset{j}{!}p, q \Rightarrow q} \qquad\qquad \dfrac{\underset{j}{!}p \Rightarrow \underset{i}{!}p \qquad \dfrac{\dfrac{\underset{i}{!}p \Rightarrow p \quad \underset{i}{!}p \Rightarrow p}{\underset{i}{!}p, \underset{i}{!}p \Rightarrow p \otimes p}}{\underset{i}{!}p \Rightarrow p \otimes p}}{\underset{j}{!}p \Rightarrow p \otimes p}$$

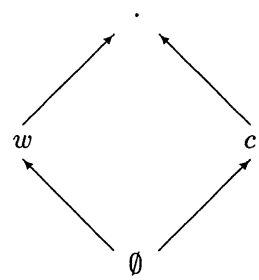
□

We get the standard calculus by taking the relation $\mathrm{I} = \{(\cdot,\cdot)\}$ on a one-point index set $\{\cdot\}$. So **CLL** = I-**CLL**. This reflects the fact that all modalized formulas in linear logic obtain *full* structural permissions.

The characterization of superfluous exponentials in the exponential graph of a linear derivation π boils down to the identification of those modalized formulas in the proof for which there is purely *logical* evidence that at no point (during normalization) they will use the 'talents bestowed upon them'. Using the above, we then can reformulate their *removal* as the interpretation of the proof π as a II-**CLL**-proof, where II denotes the reflexive closure of the relation $\emptyset \longrightarrow \cdot$, distinguishing the usual expontials $!, ?$ from the 'no-permission' exponentials $\underset{\emptyset}{!}, \underset{\emptyset}{?}$, which correspond to the superfluous exponentials in π.

Observe that a linear derivation π and its interpretation II(π) as a derivation in II-**CLL** have *exactly* the same set of reductions. Now obviously an $\emptyset$-exponentiated formula during reduction will never be cutformula in a $[w]$ or $[co]$ reduction-step, which thus provides us with alternative evidence for theorem 3.10.

An even more refined analysis can be obtained by interpreting π in IV-**CLL**. Here IV denotes the reflexive, transitive closure of the relation



and we distinguish *four* types of exponentials corresponding to the four possible structural permissions occurring in a linear logic proof. An optimal four-colouring of a given linear derivation π is easily obtained from $\mathcal{G}(\pi)$ if we distinguish w(eakening) and c(ontraction) as source-labels. It is not difficult to see that the results of sections 2 and 3 extend in an obvious way to the rewriting of a **CLL**-derivation which removes superfluous exponentials and moreover replaces $\underset{w}{!}A$ by $\mathbf{1}\&A$ for all w-coloured vertices of $\mathcal{G}(\pi)$.

Derivations in R-**CLL** where R is an *order* (i.e. transitive and non-reflexive) correspond to linear derivations having an exponential graph that is *acyclic*. If $\mathcal{U}$ is a *universal* order[8], then any derivation with an acyclic exponential graph can be interpreted as a derivation in $\mathcal{U}$-**CLL**. Note that (1) it is easy to check whether or not a given derivation has an acyclic exponential graph (so is an $\mathcal{U}$-**CLL**-derivation) and (2) that acyclicity of $\mathcal{G}(\pi)$, intuitively, prohibits the occurrence of *auto-duplication effects* during normalization. This in turn implies the interesting fact that it is possible to generalize the so-called 'approximation theorem' for linear derivations in normal form (Girard(1987)) to the class of

[8]I.e. $\mathcal{U}$ is a countable order into which any finite order X can be embedded, and having the property that for each such embedding ϕ of X, and for each finite extension Y of X, there exists an embedding ψ of Y whose restriction to X equals ϕ.

all $\mathcal{U}$-**CLL**-derivations. Moreover, it seems not unlikely that (a suitable variant of) $\mathcal{U}$-**CLL** will precisely characterize the class of polynomial time computable functions, and thus provide an alternative for the system **BLL** of bounded linear logic studied in Girard et al.(1992). We hope to return to this question in the near future.

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Appendix: $\mathbf{CLL_2}$

Identity axiom and cut rule:

$$\text{Ax}\quad A \Rightarrow A \qquad\qquad \text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Rules and axioms for the constants:

$$(\text{no L}\top) \qquad\qquad \text{R}\top \quad \Gamma \Rightarrow \top, \Delta$$

$$\text{L}\bot \quad \bot \Rightarrow \qquad\qquad \text{R}\bot\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \bot}$$

Logical rules:

$$\text{L}{-\!\circ}\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma', A \multimap B \Rightarrow \Delta, \Delta'} \qquad\qquad \text{R}{-\!\circ}\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta}$$

$$\text{L}\wp\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma', B \Rightarrow \Delta'}{\Gamma, \Gamma', A\wp B \Rightarrow \Delta, \Delta'} \qquad\qquad \text{R}\wp\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A\wp B, \Delta}$$

$$\text{L\&}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A\&B \Rightarrow \Delta}\ \ \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A\&B \Rightarrow \Delta} \qquad\qquad \text{R\&}\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A\&B, \Delta}$$

Rules for the first order quantifiers (y not free in Γ, Δ):

$$\text{L}\forall\ \frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta} \qquad\qquad \text{R}\forall\ \frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}$$

Rules for the second order quantifiers (Y not free in Γ, Δ):

$$\text{L}\forall_2\ \frac{\Gamma, A[T/X] \Rightarrow \Delta}{\Gamma, \forall X\, A \Rightarrow \Delta} \qquad\qquad \text{R}\forall_2\ \frac{\Gamma \Rightarrow \Delta, A[Y/X]}{\Gamma \Rightarrow \Delta, \forall X\, A}$$

Exponential structural rules:

$$\text{W!}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad \text{W?}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow ?A, \Delta} \qquad \text{C!}\ \frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad \text{C?}\ \frac{\Gamma \Rightarrow ?A, ?A, \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

Exponential contextual rules:

$$\text{L?}\ \frac{!\Gamma, A \Rightarrow ?\Delta}{!\Gamma, ?A \Rightarrow ?\Delta} \qquad\qquad \text{R!}\ \frac{!\Gamma \Rightarrow A, ?\Delta}{!\Gamma \Rightarrow !A, ?\Delta}$$

Exponential dereliction rules:

$$\text{R?}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow ?A, \Delta} \qquad\qquad \text{L!}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$

Linear negation is defined by $A^{\bot} = A \multimap \bot$; the rules and axioms for $\mathbf{1}, \mathbf{0}, \otimes, \oplus, \exists, \exists_2$ are obtainable in the obvious ('De Morgan') way from those for $\bot, \top, \wp, \&, \forall, \forall_2$.

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