Institute for Logic, Language and Computation

INVENTORY OF FRAGMENTS AND EXACT MODELS IN INTUITIONISTIC PROPOSITIONAL LOGIC

Lex Hendriks

ILLC Prepublication Series
for Mathematical Logic and Foundations ML-93-11

University of Amsterdam

The ILLC Prepublication Series

CL-91-03 Hub Prüst, Remko Scha, Martin van den Berg A Formal Discourse Grammar tackling Verb Phrase Anaphora

Other Prepublications

X-91-01 Alexander Chagrov, Michael Zakharyaschev The Disjunction Property of Intermediate Propositional Logics
X-91-02 Alexander Chagrov, Michael Zakharyaschev On the Undecidability of the Disjunction Property of Intermediate Propositional Logics
X-91-03 V. Yu. Shavrukov Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic
X-91-04 K.N. Ignatiev Partial Conservativity and Modal Logics
X-91-05 Johan van Benthem Temporal Logic
X-91-06 Annual Report 1990
X-91-07 A.S. Troelstra Lectures on Linear Logic, Errata and Supplement
X-91-08 Giorgie Dzhaparidze Logic of Tolerance
X-91-09 L.D. Beklemishev On Bimodal Provability Logics for Π_1-axiomatized Extensions of Arithmetical Theories
X-91-10 Michiel van Lambalgen Independence, Randomness and the Axiom of Choice
X-91-11 Michael Zakharyaschev Canonical Formulas for K4. Part I: Basic Results
X-91-12 Herman Hendriks Flexibele Categoriale Syntaxis en Semantiek: de proefschriften van Frans Zwarts en Michael Moortgat
X-91-13 Max I. Kanovich The Multiplicative Fragment of Linear Logic is NP-Complete
X-91-14 Max I. Kanovich The Horn Fragment of Linear Logic is NP-Complete
X-91-15 V. Yu. Shavrukov Subalgebras of Diagonalizable Algebras of Theories containing Arithmetic, revised version
X-91-16 V.G. Kanovei Undecidable Hypotheses in Edward Nelson's Internal Set Theory
X-91-17 Michiel van Lambalgen Independence, Randomness and the Axiom of Choice, Revised Version
X-91-18 Giovanna Cepparello New Semantics for Predicate Modal Logic: an Analysis from a standard point of view
X-91-19 Papers presented at the Provability Interpretability Arithmetic Conference, 24-31 Aug. 1991, Dept. of Phil., Utrecht University
Annual Report 1991

1992 *Logic, Semantics and Philosophy of Langauge*

LP-92-01 Víctor Sánchez Valencia Lambek Grammar: an Information-based Categorial Grammar
LP-92-02 Patrick Blackburn Modal Logic and Attribute Value Structures
LP-92-03 Szabolcs Mikulás The Completeness of the Lambek Calculus with respect to Relational Semantics
LP-92-04 Paul Dekker An Update Semantics for Dynamic Predicate Logic
LP-92-05 David I. Beaver The Kinematics of Presupposition
LP-92-06 Patrick Blackburn, Edith Spaan A Modal Perspective on the Computational Complexity of Attribute Value Grammar
LP-92-07 Jeroen Groenendijk, Martin Stokhof A Note on Interrogatives and Adverbs of Quantification
LP-92-08 Maarten de Rijke A System of Dynamic Modal Logic
LP-92-09 Johan van Benthem Quantifiers in the world of Types
LP-92-10 Maarten de Rijke Meeting Some Neighbours (a dynamic modal logic meets theories of change and knowledge representation)
LP-92-11 Johan van Benthem A note on Dynamic Arrow Logic
LP-92-12 Heinrich Wansing Sequent Caluli for Normal Modal Propositional Logics
LP-92-13 Dag Westerståhl Iterated Quantifiers
LP-92-14 Jeroen Groenendijk, Martin Stokhof Interrogatives and Adverbs of Quantification

Mathematical Logic and Foundations

ML-92-01 A.S. Troelstra Comparing the theory of Representations and Constructive Mathematics
ML-92-02 Dmitrij P. Skvortsov, Valentin B. Shehtman Maximal Kripke-type Semantics for Modal and Superintuitionistic Predicate Logics
ML-92-03 Zoran Marković On the Structure of Kripke Models of Heyting Arithmetic
ML-92-04 Dimiter Vakarelov A Modal Theory of Arrows, Arrow Logics I
ML-92-05 Domenico Zambella Shavrukov's Theorem on the Subalgebras of Diagonalizable Algebras for Theories containing $I\Delta_0$ + EXP
ML-92-06 D.M. Gabbay, Valentin B. Shehtman Undecidability of Modal and Intermediate First-Order Logics with Two Individual Variables
ML-92-07 Harold Schellinx How to Broaden your Horizon
ML-92-08 Raymond Hoofman Information Systems as Coalgebras
ML-92-09 A.S. Troelstra Realizability
ML-92-10 V.Yu. Shavrukov A Smart Child of Peano's

Compution and Complexity Theory

CT-92-01 Erik de Haas, Peter van Emde Boas Object Oriented Application Flow Graphs and their Semantics
CT-92-02 Karen L. Kwast, Sieger van Denneheuvel Weak Equivalence: Theory and Applications
CT-92-03 Krzysztof R. Apt, Kees Doets A new Definition of SLDNF-resolution

Other Prepublications

X-92-01 Heinrich Wansing The Logic of Information Structures
X-92-02 Konstantin N. Ignatiev The Closed Fragment of Dzhaparidze's Polymodal Logic and the Logic of Σ_1 conservativity
X-92-03 Willem Groeneveld Dynamic Semantics and Circular Propositions, revised version
X-92-04 Johan van Benthem Modeling the Kinematics of Meaning
X-92-05 Erik de Haas, Peter van Emde Boas Object Oriented Application Flow Graphs and their Semantics, revised version

1993 *Logic, Semantics and Philosophy of Langauge*

LP-93-01 Martijn Spaan Parallel Quantification
LP-93-02 Makoto Kanazawa Dynamic Generalized Quantifiers and Monotonicity
LP-93-03 Nikolai Pankrat'ev Completeness of the Lambek Calculus with respect to Relativized Relational Semantics
LP-93-04 Jacques van Leeuwen Identity, Quarrelling with an Unproblematic Notion
LP-93-05 Jaap van der Does Sums and Quantifiers
LP-93-06 Paul Dekker Updates in Dynamic Semantics
LP-93-07 Wojciech Buszkowski On the Equivalence of Lambek Categorial Grammars and Basic Categorial Grammars

Mathematical Logic and Foundations

ML-93-01 Maciej Kandulski Commutative Lambek Categorial Grammars
ML-93-02 Johan van Benthem, Natasha Alechina Modal Quantification over Structured Domains
ML-93-03 Mati Pentus The Conjoinablity Relation in Lambek Calculus and Linear Logic
ML-93-04 Andreja Prijatelj Bounded Contraction and Many-Valued Semantics
ML-93-05 Raymond Hoofman, Harold Schellinx Models of the Untyped λ-calculus in Semi Cartesian Closed Categories
ML-93-06 J. Zashev Categorial Generalization of Algebraic Recursion Theory
ML-93-07 A.V. Chagrov, L.A. Chagrova Algorithmic Problems Concerning First-Order Definability of Modal Formulas on the Class of All Finite Frames
ML-93-08 Raymond Hoofman, Ieke Moerdijk Remarks on the Theory of Semi-Functors
ML-93-09 A.S. Troelstra Natural Deduction for Intuitionistic Linear Logic
ML-93-10 Vincent Danos, Jean-Baptiste Joinet, Harold Schellinx The Structure of Exponentials: Uncovering the Dynamics of Linear Logic Proofs
ML-93-11 Lex Hendriks Inventory of Fragments and Exact Models in Intuitionistic Propositional Logic

Compution and Complexity Theory

CT-93-01 Marianne Kalsbeek The Vanilla Meta-Interpreter for Definite Logic Programs and Ambivalent Syntax
CT-93-02 Sophie Fischer A Note on the Complexity of Local Search Problems
CT-93-03 Johan van Benthem, Jan Bergstra Logic of Transition Systems
CT-93-04 Karen L. Kwast, Sieger van Denneheuvel The Meaning of Duplicates in the Relational Database Model
CT-93-05 Erik Aarts Proving Theorems of the Lambek Calculus of Order 2 in Polynomial Time

Other Prepublications

X-93-01 Paul Dekker Existential Disclosure, revised version
X-93-02 Maarten de Rijke What is Modal Logic?
X-93-03 Michiel Leezenberg Gorani Influence on Central Kurdish: Substratum or Prestige Borrowing
X-93-04 A.S. Troelstra (editor) Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, Corrections to the First Edition
X-93-05 A.S. Troelstra (editor) Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, Second, corrected Edition
X-93-06 Michael Zakharyashev Canonical Formulas for K4. Part II: Cofinal Subframe Logics

Institute for Logic, Language and Computation
Plantage Muidergracht 24
1018TV Amsterdam
Telephone 020-525.6051, Fax: 020-525.5101

INVENTORY OF FRAGMENTS AND EXACT MODELS IN INTUITIONISTIC PROPOSITIONAL LOGIC

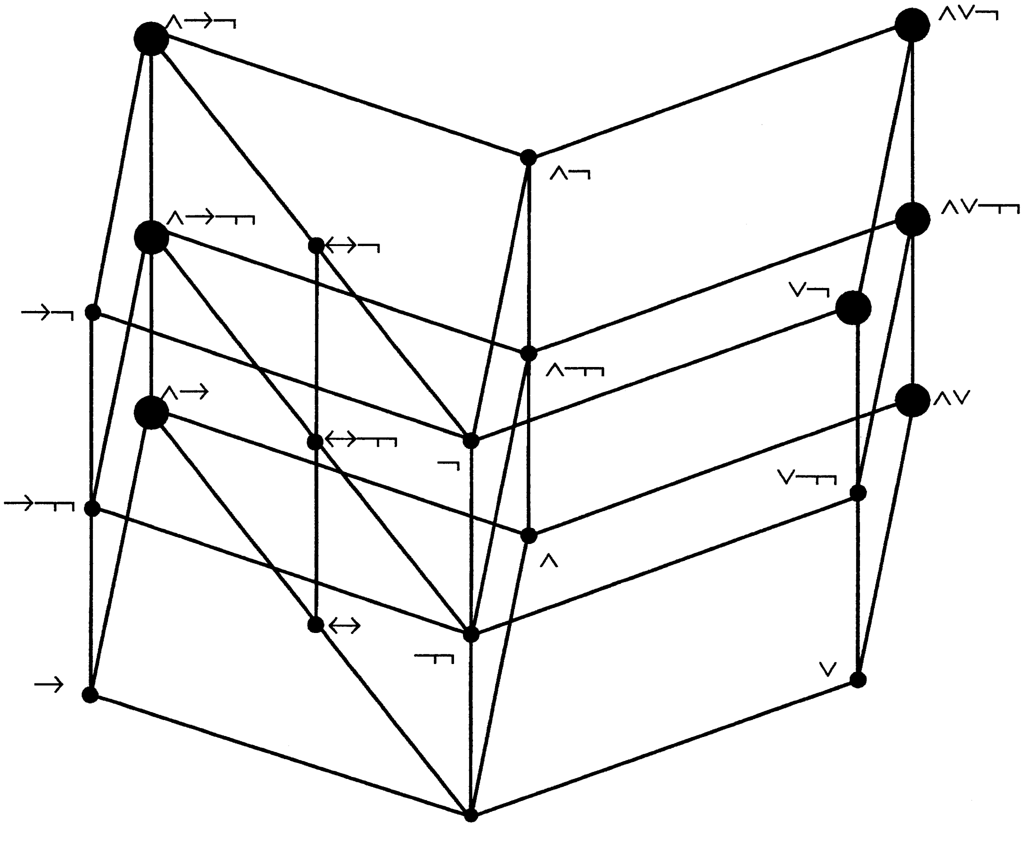
Lex Hendriks
Department of Mathematics and Computer Science
University of Amsterdam

ILLC Prepublications
for Mathematical Logic and Foundations
ISSN 0928-3315
Coordinating editor: Dick de Jongh

Research supported by the
Netherlands Organization for Scientific Research (NWO)

received July 1993

Inventory of Fragments and Exact Models in Intuitionistic Propositional Logic



Lex Hendriks

Department of Mathematics and Computer Science
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
email: lhendrik@fwi.uva.nl

Version 07.03
02/07/93

1. Introduction

The main subject of this report is the semantic structure of fragments of intuitionistic propositional logic (IpL). It is demonstrated that in some of the fragments there is a subordering of the Lindenbaum algebra of the fragment $\mathbb{F}$ which not only reflects its structural properties but also is -or can be extended to- a Kripke model, $\mathcal{K}$, which is complete for the fragment; that is:

$$\forall \varphi,\psi \in \mathbb{F}.\ \varphi \vdash \psi \Leftrightarrow \mathcal{K} \Vdash \varphi \rightarrow \psi$$

These concise representations of fragments were first introduced in 1975 by De Bruijn as *exact models* ([B75a]).

De Bruijn was aided by a computer in constructing the exact model of $[\wedge, \rightarrow]_3$ (the fragment with conjunction and implication over three atoms). The exact model, with 61 elements, was used to write a computer program testing validity in $[\wedge, \rightarrow]_3$([B75b]).
The interplay between logic, mathematics and computer programs has proved to be typical of this kind of 'computer aided logic' research on the semantic structure of IpL-fragments.

In the late 70's and early 80's attempts were made to calculate *diagrams*, the partial ordering of the Lindenbaum algebras, of IpL fragments using tableaux based theorem testers ([H80], [R85]). These attempts were hampered however by lack of computer time and memory (or, stated differently, by lack of sufficiently efficient formula testers).
The approach of De Bruijn proved to be a successful alternative ([JHR91]) although it needed knowledge of the structure of exact models (and hence of the semantic structure of the fragments) to be applicable. This inventory shows how the structure of most types of fragments has been unravelled now. The tableaux tester mentioned above still played its part from time to time to find an exact model or to check a candidate for an exact model.

This inventory mainly reports on the results of the search by D. de Jongh, G. Renardel de Lavalette and the author, for exact models in fragments of IpL and their use in calculating the diagrams of fragments.
This report would never have existed, if not for the generous support and valuable criticism of D. de Jongh and G. Renardel de Lavalette.

In the preliminary section, after the table of contents, the reader may find those notations, notions and conventions used in the text that may not be common knowledge in the logic community.
The structure of the main body of this treatise is explained in section 4: Fragments and exact models.

The spell of experimental logic however, the thrill of computer programs spitting out thousands of formulas, the excitement of a new drawing of an exact model, the fun of using one exact model to generate another one, it is all not included here.
The computer aided logic toolbox (that is the bunch of computer programs used to get most of the results described here) will be documented in a separate report.

2 Table of Contents

3 Preliminaries

For a background on intuitionistic propositional logic IpL the reader is referred to [TD88] (in which IpL has the name IPC).
Most of the tools and terminology from lattice theory used in this inventory can be found in [DP90].
For some application of lattice theory to IpL see [C77].

Definition 3.1 *Fragments* of IpL are sublanguages of $[\wedge, \vee, \rightarrow, \neg]$, obtained by restriction of the set of atoms or the application of connectives (or both).

Examples of restrictions of the application of connectives are *exclusion* (for example in the formulas of the fragment $[\wedge, \neg]_n$ there are no applications of $\vee$ or $\rightarrow$) and exclusion in combination with the use of *defined connectives* (like $\neg\neg$ and $\leftrightarrow$ in $[\neg\neg, \leftrightarrow]_n$).

Notation 3.2 In our notation of fragments there are six main rules:

$[\wedge, \vee]_n$	is the fragment with n atoms and $\wedge$, $\vee$ as its only connectives (likewise $[\wedge, \rightarrow, \neg]_n$ is the fragment over n atoms with $\wedge$, $\rightarrow$ and $\neg$ as its connectives, etc.).
$[\wedge, \vee]$	is the IpL fragment with $\wedge$, $\vee$ as its connectives and some countably infinite set of atoms. Likewise for $[\rightarrow, \neg\neg]$ etc..
$\Lambda[\rightarrow]_n$	is the fragment of conjunctions of formulas of $[\rightarrow]_n$. (Likewise $\mathrm{V}[\wedge, \neg\neg]_n$ is the fragment of disjunctions of formulas of $[\wedge, \neg\neg]_n$ etc.).
$P_n = \{p_1, \dots, p_n\}$	is assumed to be the set of atoms if a fragment has n atoms. In general we will use metavariables p, q, r etc. to range over atoms p_i.
$[\wedge, \vee, P_n]$	is used as an alternative notation for $[\wedge, \vee]_n$.
$\mathbb{F}(P_n)$	denotes that P_n is the set of atoms in the fragment $\mathbb{F}$.

Notation 3.3 Let $G \subseteq$ IpL, G finite:

ΛG	is the conjunction of all formulas in G
$\mathrm{V} G$	is the disjunction of all formulas in G.

If $\varphi_1, \dots, \varphi_n$ is a finite set of formulas and $1 \leq n$ then:

$\bigwedge_1^n \varphi_i$	is the conjunction of $\varphi_1, \dots, \varphi_n$.

Definition 3.4 If $\mathbb{F}$ and $\mathbb{G}$ are fragments in logics with derivability relationships $\vdash_{\mathbb{F}}$ and $\vdash_{\mathbb{G}}$, then $\mathbb{F}$ is a *conservative extension* of $\mathbb{G}$ if

i. each formula of $\mathbb{G}$ is a formula of $\mathbb{F}$ ($\mathbb{G} \subseteq \mathbb{F}$)
ii. for all $\varphi, \psi \in \mathbb{G}$, $\varphi \vdash_{\mathbb{G}} \psi \Leftrightarrow \varphi \vdash_{\mathbb{F}} \psi$

As most of the fragments in this report are in IpL, often the two derivability relationships mentioned in definition 3.4 are identical. Hence in this report it is almost always trivial that an extension is conservative (an important exception is in section 7.2).

Definition 3.5 Let S be an ordered set (p.o. set, see [DP90]), $U \subseteq S$ then:

$\downarrow U ::= \{x \in S \mid \exists y \in U . x \leq y\}$	$\downarrow U$ is called the *down set* of U.
$\uparrow U ::= \{x \in S \mid \exists y \in U . x \geq y\}$	$\uparrow U$ is called the *up set* of U.
$U^\circ ::= \{x \in U \mid \forall y \leq x . y \in U\}$	U° is called the *interior* of U.

In case of down sets of singletons, instead of $\downarrow\{x\}$ we usualy write $\downarrow x$ (likewise for up sets).

Notation 3.6 The *complement* of a set U will be denoted by U^c.

Fact 3.7 $U^{\circ} = (\uparrow(U^{c}))^{c}$

Fact 3.7 is a straightforward consequence of definition 3.5.

Definition 3.8 If S is an ordered set, then:

a. S is a *lattice* (see [DP90]) if there are operations $\cap$ (*join*) and $\cup$ (*meet*) on S such that:

a.1 $\forall x, y, z \in S.\ x \leq y \wedge x \leq z \leftrightarrow x \leq y \cap z$

a.2 $\forall x, y, z \in S.\ x \leq z \wedge y \leq z \leftrightarrow x \cup y \leq z$

b. A lattice S is *distributive* iff

$\forall x, y, z \in S.\ (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$

c. If S has a bottom (a minimum), then this element is denoted by $\perp$

d. If S is a lattice, an element $x \in S$ is called *join-irreducible* (or simply *irreducible*) if $x \neq \perp$ and

$\forall y, z \in S.\ x = y \cup z \rightarrow (x = y) \vee (x = z)$.

Definition 3.9 Let $\mathbf{K} = < K, \leq >$ be a finite ordered set and $k \in K$. The *depth* of k, $\delta(k)$, is defined as:

$\delta(k) = 0$ if k a minimal element (so $\forall l \in K.\ l \leq k \rightarrow l = k$),

$\delta(k) = n + 1$ if $\max\{\delta(l) \mid l < k\} = n$.

Definition 3.10 Let $\mathbf{K} = <K, \leq>$ be an ordered set and for each atomic formula p, $\omega(p)$ a down set of $\mathbf{K}$ then $\mathcal{K} = <\mathbf{K}, \omega>$ is a *Kripke model*, $\mathbf{K}$ is sometimes called a Kripke frame. For each $k \in K$ and each (IpL-)formula φ the (forcing) relation $k \Vdash \varphi$ is defined inductively:

$k \Vdash p$	for atomic p, if $k \in \omega(p)$
$k \Vdash \varphi \wedge \psi$	iff $k \Vdash \varphi$ and $k \Vdash \psi$
$k \Vdash \varphi \vee \psi$	iff $k \Vdash \varphi$ or $k \Vdash \psi$
$k \Vdash \varphi \rightarrow \psi$	iff $\forall l \leq k (l \Vdash \varphi \Rightarrow l \Vdash \psi)$
$k \Vdash \neg\varphi$	iff $\forall l \leq k (l \nVdash \varphi)$

As usual we define:

$\mathcal{K} \Vdash \varphi$	for: $\forall k \in K.\ k \Vdash \varphi$
$\mathbf{K} \Vdash \varphi$	if for all ω, $<\mathbf{K}, \omega> \Vdash \varphi$.
$\Vdash \varphi$	for: $\forall \mathcal{K}.\ \mathcal{K} \Vdash \varphi$ (or $\forall \mathbf{K}.\ \mathbf{K} \Vdash \varphi$)
$\varphi \Vdash \psi$	for: $\forall \mathcal{K}.\ \mathcal{K} \Vdash \varphi \Rightarrow \mathcal{K} \Vdash \psi$

If $k \in K$ and $l \leq k$ for all $l \in K$ then k is called the *root* of $\mathbf{K}$.
If $k \in K$ and $l < k$ for no $l \in K$ then k is called a *terminal node* of $\mathbf{K}$.
If $k \in K$ then *atom* (k) is defined as: *atom* $(k) = \{p \text{ atomic} \mid k \Vdash p\}$.

The most important reason to repeat the definition of the Kripke semantics here is the reversal of the usual order. This is convenient in case our Kripke frame is a set of equivalence classes of formulas ordered by $\vdash$, in the definition above $\vdash$ will correspond naturally with $\leq$. In this way $\wedge$ will behave as a meet and $\vee$ as a join in a lattice of equivalence classes.
Most of the notions and terminology used to describe the semantics of IpL stem from Kripke semantics in modal logic [B85].

Definition 3.11 A Kripke model $\mathcal{L} = \langle L, \nu \rangle$ is a *submodel* of model $\mathcal{K} = \langle K, \omega \rangle$ if **L** is a down set in **K** (with inherited order) and for all atoms p: $\nu(p) = \omega(p) \cap L$.

In the literature on semantics for modal logic, **L** is called a *generated subframe* of **K** [B85] and the kind of submodel defined above is also known as a *generated submodel.*

Definition 3.12 Two Kripke models $\mathcal{K}$ and $\mathcal{L}$ are called *equivalent* ($\mathcal{K} \equiv \mathcal{L}$), if for every IpL-formula φ: $\mathcal{K} \Vdash \varphi \Leftrightarrow \mathcal{L} \Vdash \varphi$.

Definition 3.13 Let **K** and **L** be Kripke-frames. A function ρ: **K** $\rightarrow$ **L** is a *reduction* from **K** to **L** if:

i. for all $k, l \in$ **K**: $l \leq k \Rightarrow \rho(l) \leq \rho(k)$
ii. for all $k \in$ **K** and $l \in$ **L**: $l \leq \rho(k) \Rightarrow \exists m \in \rho^{-1}(l).\ m \leq k$
iii. ρ is surjective.

If $\mathcal{K} = \langle K, \omega \rangle$ and $\mathcal{L} = \langle L, \nu \rangle$ Kripke models and ρ: **K** $\rightarrow$ **L** a reduction such that for all $k \in$ **K** and all atomic p: $k \in \omega(p) \Leftrightarrow \rho(k) \in \nu(p)$, then ρ is a reduction of the Kripke model $\mathcal{K}$ to $\mathcal{L}$.

Functions for which the first two conditions of definition 3.13 hold are sometimes called p-*morphisms* (originally *strongly isotone*)[JT66, B85, R86, TD88a]. The first condition will be recognised as a homorphism (or monotonicy) condition. The second condition is known as the p-morphism condition.

Fact 3.14

a. If $\mathcal{K}$ and $\mathcal{L}$ are Kripke models and $\mathcal{L}$ is a reduction from $\mathcal{K}$, then $\mathcal{K} \equiv \mathcal{L}$
b. A Kripke model $\mathcal{K}$ is irreducible iff every subframe (submodel) of $\mathcal{K}$ is irreducible.

Definition 3.15 A Kripke frame **K** is *rooted* if **K** has one maximal element (there is a $m \in$ **K** such that m is the greatest, **K** $= \downarrow m$). A Kripke model is called rooted if its frame is rooted.

Definition 3.16 Let **K** be a Kripke frame, P a finite set of atoms, then ω is a P-*valuation* (on **K**) if: $\omega(p) \neq \varnothing \Rightarrow p \in P$.
$\mathcal{K} = \langle K, \omega \rangle$ is a P-*model* (on **K**) if ω is a P-valuation.

Fact 3.17 (Kripke semantics completeness for IpL)
For any IpL formula φ:

a. $\vdash \varphi \Leftrightarrow \Vdash \varphi$
b. if $\nvdash \varphi$ then there is a finite rooted Kripke model $\mathcal{K}$ such that $\mathcal{K} \nVdash \varphi$
c. if $Val(\varphi) \subseteq P$ and $\nvdash \varphi$ then there is a (rooted) P-model $\mathcal{K}$ such that $\mathcal{K} \nVdash \varphi$

The completeness theorem for finite rooted Kripke models, which results from 3.17.a, is just a weaker case of the completeness theorem for finite trees [TD88] and c. is a refinement of b. which will be clear from a careful inspection of the proof of b.

Notation 3.18 Equivalence of formulas φ and ψ is denoted by $\varphi \equiv \psi$.
So $\varphi \equiv \psi$ iff $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

Definition 3.19 A formula φ of IpL is called *∨-irreducible* or simply *irreducible*, if $\varphi \not\equiv \bot$ and for all ψ and χ: $\varphi \vdash \psi \vee \chi \quad \Leftrightarrow \quad \varphi \vdash \psi$ or $\varphi \vdash \chi$

Note that in IpL (ordered by $\vdash$) the $\wedge$ and $\vee$ are lattice operators on equivalence classes Hence in definitions 3.8.d and 3.19 the concept of irreducibility is essentially the same.

To verify facts about irreducibility we will often use the notion of the *Aczel slash*.

Definition 3.20 (Aczel slash) Let Γ be a set of formulas, and φ a formula, then $\Gamma \mid \varphi$ is defined inductively:

$$\begin{array}{lcl} \Gamma \mid p & \Leftrightarrow & \Gamma \vdash p \text{ for } p \text{ atomic or } p = \bot \\ \Gamma \mid \varphi \wedge \psi & \Leftrightarrow & \Gamma \mid \varphi \text{ and } \Gamma \mid \psi \\ \Gamma \mid \varphi \vee \psi & \Leftrightarrow & \Gamma \mid \varphi \text{ or } \Gamma \mid \psi \\ \Gamma \mid \varphi \rightarrow \psi & \Leftrightarrow & \Gamma \vdash \varphi \rightarrow \psi \text{ and } (\Gamma \mid \varphi \Rightarrow \Gamma \mid \psi) \end{array}$$

Fact 3.21

a. (Kleene, [K62]) If $\varphi \not\equiv \bot$ then is φ irreducible iff $\varphi \mid \varphi$

b. $\Gamma \mid \varphi \Rightarrow \Gamma \vdash \varphi$

c. If $\Gamma \vdash \varphi \rightarrow \psi$ and $\Gamma \nvdash \varphi$ then $\Gamma \mid \varphi \rightarrow \psi$

d. All formulas in $[\wedge, \rightarrow, \neg] \backslash \bot$ are irreducible

In the sequel we will need some elementary facts (see for example TD88) about classical propositional logic, CpL, and its relation with IpL.

Definition 3.22 If P_n is a set of atomic formulas and $Q \subseteq P_n$, then define:

$$\varphi_Q := \bigwedge\{p \mid p \in Q\} \wedge \bigwedge\{\neg p \mid p \in P_n \backslash Q\}$$

Fact 3.23 Let $\vdash_c$ be the derivability relation in CpL. Then

a. The set $\{\varphi_Q \mid Q \subseteq P_n\}$ is the set of irreducibles in the diagram of the classical fragment $[\wedge, \neg, P_n]^c$

b. If $Var(\psi) \subseteq P_n$ and $\psi \not\equiv \bot$ then for some $Q \subseteq P_n$: $\varphi_Q \vdash \psi$

c. $\neg\varphi \vdash \neg\psi \quad \Leftrightarrow \quad \neg\varphi \vdash_c \neg\psi$

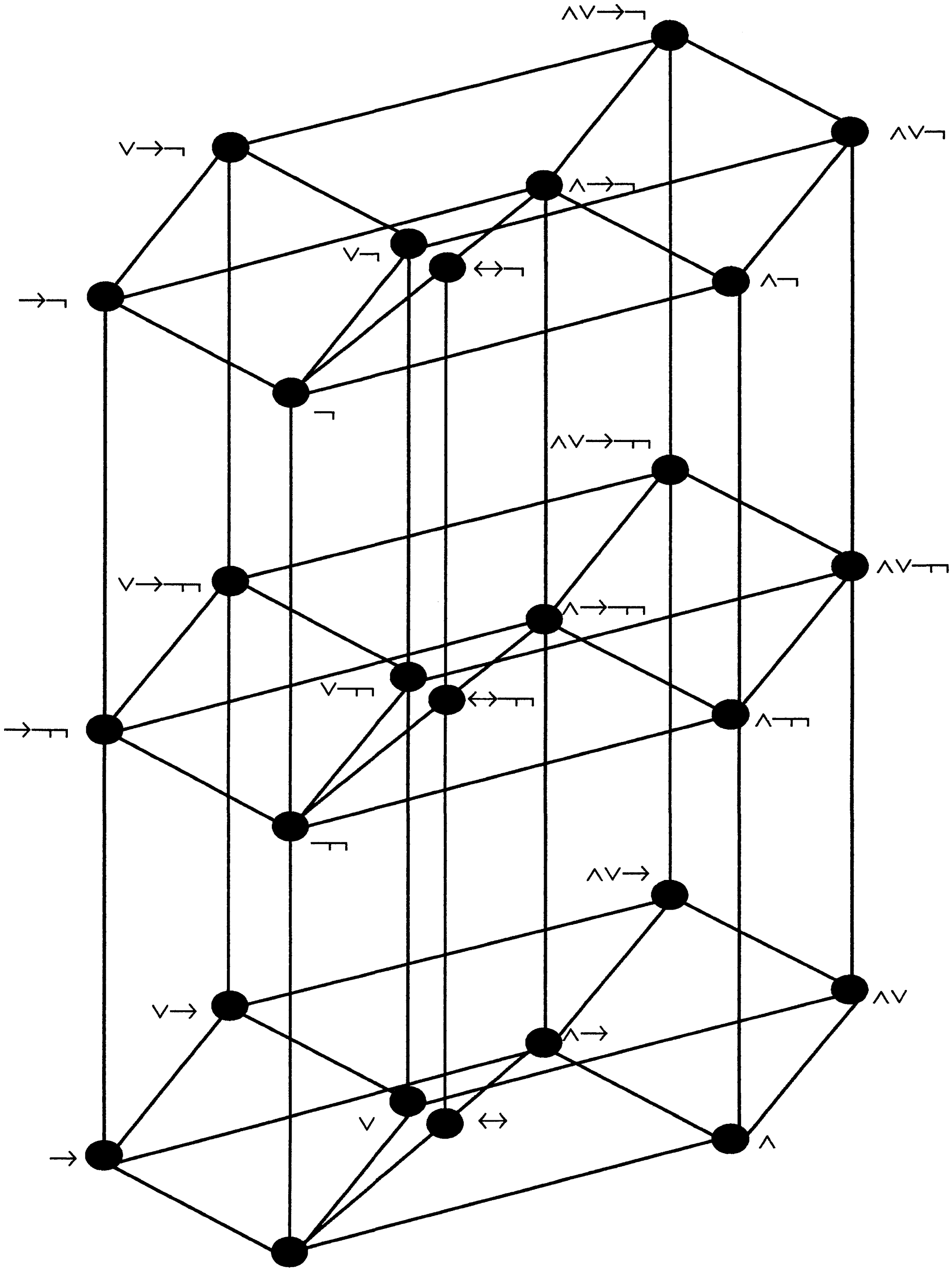
d. $\neg\varphi \equiv \neg\psi \quad \Leftrightarrow \quad \varphi \equiv_c \psi$

e. $\varphi \vdash \neg\neg\psi \quad \Leftrightarrow \quad \varphi \vdash_c \psi$

4 Fragments and exact models

This inventory describes almost all finite fragments of intuitionistic propositional logic with connectives in the set {$\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\neg$, $\neg\neg$} and a finite set of atoms. Not included are the [$\leftrightarrow$] fragments and some very trivial ones, like $[\neg]_n$.

Modulo logical equivalence (for example $[\wedge, \leftrightarrow]_n \equiv [\wedge, \rightarrow]_n$), for any finite number of atoms n, the fragments of IpL can be ordered by conservative extension into a lattice:



Definition 4.1 The diagram of a fragment $\mathbb{F}$, *Diag* ($\mathbb{F}$), is the set of equivalence classes of formulas of $\mathbb{F}$, ordered by the derivability relationship $\vdash$.

In talking about 'finite' fragments we identified the fragment with its diagram (taking it 'modulo equivalence').

From the 27 fragments above, 21 are finite for each finite number of atoms n. The 6 extensions of $[\vee, \rightarrow]_n$ are infinite for n >1 and already $[\vee, \rightarrow, \neg]_1$ (and hence $[\wedge, \vee, \rightarrow, \neg]_1$) is infinite ([R49], [N60]).

The diagrams of 7 of the finite fragments are distributive lattices (as will be proved in consecutive sections). For finite distributive lattices Birkhoff's representation theorem is applicable:

Theorem 4.2 (Birkhoff) Any finite distributive lattice is isomorphic to the lattice of the down sets of its irreducible elements.

Pf. A proof can be found in [DP90]. □

Definition 4.3 If the diagram *Diag* ($\mathbb{F}$) of fragment $\mathbb{F}$ is a finite distributive lattice and *Irr* (*Diag* ($\mathbb{F}$)) is the subset of irreducible elements of *Diag* ($\mathbb{F}$) (with inherited order), then *Irr* (*Diag* ($\mathbb{F}$)) is the *exact model* of $\mathbb{F}$: *Exm* ($\mathbb{F}$) = *Irr* (*Diag* ($\mathbb{F}$)).

Fact 4.4 If *Diag* ($\mathbb{F}$) is the diagram of fragment $\mathbb{F}$, *Exm* ($\mathbb{F}$) its exact model and $\cup$ is the join of formulas (representing equivalence classes) in *Diag* ($\mathbb{F}$), then for $\varphi, \psi \in \mathbb{F}$ and $\chi \in$ *Exm* ($\mathbb{F}$):

a. $\varphi \vee \psi \vdash \varphi \cup \psi$
b. $\chi \vdash \varphi \cup \psi \Leftrightarrow \chi \vdash \varphi$ or $\chi \vdash \psi$

If *Exm* ($\mathbb{F}$) is the exact model of $\mathbb{F}$ there is a 1-1 mapping between formulas of $\mathbb{F}$ (modulo equivalence) and the down sets of *Exm* ($\mathbb{F}$): $\omega(\varphi) := \{\psi \in Exm(\mathbb{F}) \mid \psi \vdash \varphi\}$
Note that ω here is not (yet) a valuation of a Kripke frame.

Notation 4.5 If E is a set of formulas, $<E, \vdash>$ is the Kripke model with the set E ordered by $\vdash$ as its frame and valuation: $\omega(p) := \{\varphi \in E \mid \varphi \vdash p\}$. If $\varphi \in E$, the node in $<E, \vdash>$ corresponding to φ will be denoted by k_φ.

The introduction of this notation helps us to distinguish $k_\varphi \Vdash \psi$ (in $<E, \vdash>$ the node of φ forces ψ) from $\varphi \Vdash \psi$ (every Kripke model forcing φ forces ψ).

The exact model *Exm* ($\mathbb{F}$) can be taken as a Kripke model $<Exm(\mathbb{F}), \vdash>$, but in this report we will meet exact models which are not faithful as Kripke models (i.e. its valuation ω do not correspond to the 1-1 mapping ω defined above).

Definition 4.6 The exact model *Exm* ($\mathbb{F}$) of $\mathbb{F}$ is called an *exact Kripke model* if in the Kripke model $<Exm(\mathbb{F}), \vdash>$ and for any $\varphi \in \mathbb{F}$:

$\{\psi \in Exm(\mathbb{F}) \mid \psi \vdash \varphi\} = \{\psi \in Exm(\mathbb{F}) \mid k_\psi \Vdash \varphi\}$

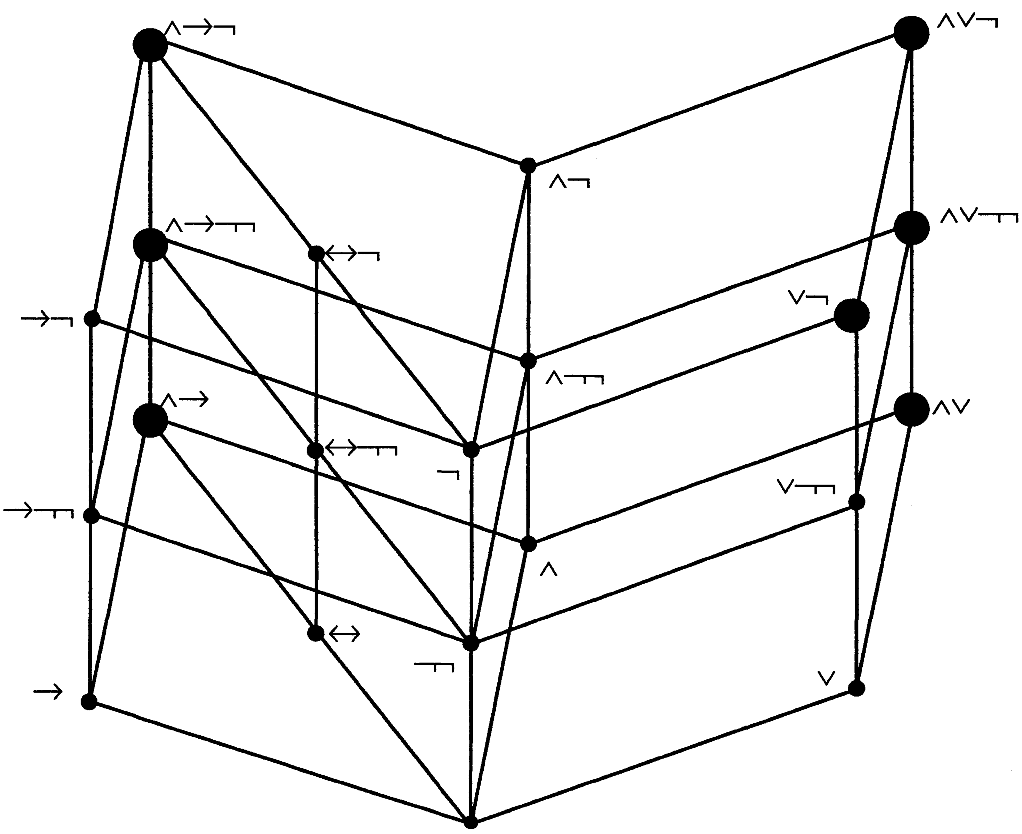
If $\mathbb{F}$ has an exact Kripke model, it will be denoted by *ExKm* ($\mathbb{F}$).

In case of an exact Kripke model *ExKm* ($\mathbb{F}$) of $\mathbb{F}$ it is straightforwardly proved that ω is a topological valuation of formulas of $\mathbb{F}$ on *Exm* ($\mathbb{F}$), in the sense of [MT48].
Fact 4.7 is a recapitulation of the details of this topological valuation.

Fact 4.7 If *ExKm* ($\mathbb{F}$) an exact Kripke model of $\mathbb{F}$ and ω as above, then (as far as the connectives are applicable in $\mathbb{F}$):

$$
\begin{aligned}
\omega(\varphi\wedge\psi) &= \omega(\varphi)\cap\omega(\psi)\\
\omega(\varphi\vee\psi) &= \omega(\varphi)\cup\omega(\psi)\\
\omega(\varphi\rightarrow\psi) &= \omega(\varphi)\Rightarrow\omega(\psi) \qquad = (\omega(\varphi)^{c}\cup\omega(\psi))^{\circ}\\
\omega(\bot) &= \varnothing\\
\omega(\neg\varphi) &= \omega(\varphi)^{c\circ}\\
\omega(\neg\neg\varphi) &= \omega(\varphi)^{c\circ c\circ}
\end{aligned}
$$

The 21 types of finite fragments mentioned before form a sublattice of the lattice of fragments above. In the diagram of this lattice below, the 7 types of fragments having exact models are indicated by a bold dot:



For each n the exact models of $[\wedge,\vee]_n$, $[\wedge,\vee,\neg]_n$, $[\wedge,\rightarrow]_n$ and $[\wedge,\rightarrow,\neg]_n$ are exact Kripke models, as will be proved in the sequel of this treatise (see 6.1, 7.1, 10.1 and 11). For $n>1$ the exact models of $[\vee,\neg]_n$, $[\wedge,\vee,\neg\neg]_n$ and $[\wedge,\rightarrow,\neg\neg]_n$ are not exact Kripke models (see 8, 9.1 and 12).

Definition 4.8 A fragment $\mathbb{F}$ is an *exact hull* (*exact Kripke hull*) of a fragment $\mathbb{G}$ if $\mathbb{F}$ has an exact model (exact Kripke model) and is a (conservative) extension of $\mathbb{G}$.

Note that the condition that $\mathbb{G}$ is a conservative is only important if the logic of $\mathbb{G}$ differs from that of $\mathbb{F}$.

From the diagram above we derive a first fact about exact Kripke models.

Fact 4.9 Each finite fragment of IpL with connectives in the set $\{\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \neg\neg\}$ has a finite fragment of IpL as an exact Kripke hull.

So the type of efficient theorem provers based on exact models [B75b, JHR91] can in principle be used to calculate the structure of all diagrams of finite fragments of IpL (with connectives in the set mentioned before).

As we are particularly interested in the structure of diagrams of IpL as reflected by the structure of exact models, the fragments of IpL have been grouped in this inventory around the fragments with an exact model.

section	main fragment	subfragments
6	$[\wedge,\vee]$	$[\wedge],[\vee]$
7	$[\wedge,\vee,\neg]$	$[\wedge,\neg]$
8	$[\vee,\neg]$	
9	$[\wedge,\vee,\neg\neg]$	$[\wedge,\neg\neg],[\vee,\neg\neg]$
10	$[\wedge,\rightarrow]$	$[\rightarrow]$
11	$[\wedge,\rightarrow,\neg]$	$[\rightarrow,\neg]$
12	$[\wedge,\rightarrow,\neg\neg]$	$[\rightarrow,\neg\neg]$

A stronger version of fact 4.9 can be obtained using the notion of restricted depth of left nesting of implication.

Definition 4.10 The *left nesting* level of an IpL formula φ, $\lambda(\varphi)$, is defined inductively by:

$$\lambda(p) = 0 \text{ for atomic } p \text{ or } p = \bot$$
$$\lambda(\varphi\wedge\psi) = \lambda(\varphi\vee\psi) = \max(\lambda(\varphi), \lambda(\psi))$$
$$\lambda(\varphi\rightarrow\psi) = \max(\lambda(\varphi)+1, \lambda(\psi))$$

If $|\varphi|$ is the class of formulas equivalent with φ, the level of left nesting of $|\varphi|$ is the minimum of $\lambda(\psi)$ for $\psi\in|\varphi|$.

Notation 4.11 A fragment $\mathbb{F}\subseteq[\wedge, \vee, \rightarrow]_n$ with left nesting restricted to some finite number k will be denoted as a subfragment of $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ (hence as $[\wedge, {}^k\!\!\rightarrow]_n$ or $[{}^k\!\!\rightarrow]_n$ etc.). As negation is definable if $\bot$ is in the language of $\mathbb{F}$, fragments with negation and a restricted level of left nesting will be denoted as subfragments of $[\wedge, \vee, {}^k\!\!\rightarrow, \bot]_n$.

Note that we could have introduced a negation rule for λ, $\lambda(\neg\varphi) = \lambda(\varphi) + 1$, but the notation of a fragment with negation and restricted level of left nesting k as $[\wedge, \vee, {}^k\!\!\rightarrow, \neg]_n$ would not express the restriction on the negation.
By the way, not restricting the application of negation would yield an alternative level of left nesting λ' by stipulating that apart from $\lambda'(\neg\varphi) = \lambda'(\varphi)$ all rules for λ and λ' are the same, for which one easily proves that for every φ there is an equivalent ψ such that $\lambda(\varphi) \leq \lambda'(\psi) + 2$.

The structure of fragments with restricted left nesting of implications is not yet fully unraveled. But we do know that all fragments of type $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ and $[\wedge, \vee, {}^k\!\!\rightarrow, \bot]_n$ are finite and have an exact Kripke model.

Lemma 4.12 (A. Visser) The diagram of $[\wedge, \vee, {}^1\!\!\rightarrow]_n$ is finite.

Pf. By induction on *n*.
$\underline{n = 1}$: The diagram of $[\wedge, \vee, {}^1\!\!\rightarrow]_1$ has clearly only two classes: p and $p \rightarrow p$.
$\underline{n+1}$: In $[\wedge, \vee, {}^1\!\!\rightarrow]_{n+1}$ all formulas can (up to equivalence) be built with $\wedge$ and $\vee$ from atomic formulas and formulas of the form $p \rightarrow \varphi$, where $\varphi \in [\wedge, \vee, {}^1\!\!\rightarrow]_{n+1}$ and p atomic.
We prove that there are only finitely many $p \rightarrow \varphi$ in $[\wedge, \vee, {}^1\!\!\rightarrow]_{n+1}$ which proves $[\wedge, \vee, {}^1\!\!\rightarrow]_{n+1}$ to be a finitely generated distributive lattice and hence finite.
Without loss of generality assume $p \notin [\wedge, \vee, {}^1\!\!\rightarrow]_n$. Let $\varphi \in [\wedge, \vee, {}^1\!\!\rightarrow]_{n+1}$ and $\psi = \varphi[p := \top]$ (where $\top \equiv q \rightarrow q \in [\wedge, \vee, {}^1\!\!\rightarrow]_n$). One simply proves $p \rightarrow \varphi \equiv p \rightarrow \psi$ and as $\psi \in [\wedge, \vee, {}^1\!\!\rightarrow]_n$ there are only finitely many $p \rightarrow \varphi$. □

Theorem 4.13 The diagram of $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ is finite.

Pf. By induction on *k*.
$\underline{k = 1}$: By lemma 4.12.
$\underline{k + 1}$: The diagram of $[\wedge, \vee, {}^{k+1}\!\!\rightarrow]_n$ can be generated using the operators $\wedge, \vee, {}^1\!\!\rightarrow$ and using the equivalence classes of $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ as a set of generators. By induction hypothesis the diagram of $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ is finite. Hence the diagram of $[\wedge, \vee, {}^{k+1}\!\!\rightarrow]_n$ is the homomorphic image of the diagram of $[\wedge, \vee, {}^1\!\!\rightarrow]_m$ for some m, and by lemma 4.12 is finite. □

Corollary 4.14 $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ has an exact model, the set of $\vee$-irreducibles in $[\wedge, \vee, {}^k\!\!\rightarrow]_n$.

Pf. The diagram of $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ is a finite distributive lattice the join of which corresponds to the disjunction ($\vee$). So by definition 4.3 the set of irreducible elements of $[\wedge, \vee, {}^k\!\!\rightarrow]_n$ is an exact model. □

The diagram of $[\wedge, \vee, {}^k\!\!\rightarrow, \bot]_n$ is a homomorphic image of the diagram of $[\wedge, \vee, {}^k\!\!\rightarrow]_{n+1}$ by $F(\varphi) := \varphi[z := \bot]$. The proof of the next fact is essentially the same as that of corollary 11.7 in section 11.

Fact 4.15 Let P be the set of atomic formulas in $[\wedge, \vee, \rightarrow, \neg]_n$ and $P \cup \{z\}$ be the set of atomic formulas in $[\wedge, \vee, \rightarrow]_{n+1}$. Then the submodel corresponding to $\omega(z \rightarrow \Lambda P)$ in the exact model of $[\wedge, \vee, {}^k\!\!\rightarrow]_{n+1}$ is the exact Kripke model of $[\wedge, \vee, {}^k\!\!\rightarrow, \bot]_n$.

Theorem 4.16 Each finite subset of IpL has an exact hull which is a finite fragment of IpL with an exact Kripke model.

Pf. By definition all connectives in an IpL-fragment are definable by a $[\wedge, \vee, \rightarrow, \bot]$ formula (possibly with extra restrictions, like reduced nesting). If $\mathbb{F}$ is a finite fragment, there is a maximum number n of atoms in $\mathbb{F}$ and a maximum k of the left nesting of formulas of $\mathbb{F}$ written as $[\wedge, \vee, \rightarrow, \bot]$ formulas. Hence $\mathbb{F}$ is (modulo equivalence, can be conservatively embedded into) a part of $[\wedge, \vee, {}^k\!\!\rightarrow, \bot]_n$. □

Corollary 4.17 In IpL every formula is a finite disjunction of irreducible formulas.

Pf. In a finite fragment containing disjunction and with an exact model, each formula is a finite disjunction of irreducibles. By theorem 4.16 it is sufficient to prove each IpL formula φ to be formula of a finite fragment (containing $\vee$).
As φ has a finite number, say n, of atoms as subformulas and a finite level of left nesting of implication, say k, $\varphi \in [\wedge, \vee, \overset{k}{\rightarrow}, \bot]_n$ which is finite (and even has itself an exact Kripke model). □

Corollary 4.17 can also be proved in a more direct fashion, using the properties of the Aczel slash (fact 3.21):

Pf. By induction on the length of (i.e. the number of symbols in) the IpL formula φ.
If $\varphi = \bot$ then $\varphi = \bigvee\varnothing$.
If φ atomic then trivially $\varphi \mid \varphi$.
Note that if $\varphi \neq \bot$ then φ is a finite conjunction of atoms, disjunctions and implications.
If one of the conjuncts of φ is a disjunction, $\varphi \equiv \psi\vee\chi$ for some ψ, χ which are shorter than φ. Now apply the induction hypothesis to ψ and χ, to prove φ is a finite disjunction of irreducible formulas.
Otherwise, $\varphi = \bigwedge_1 p_i \wedge \bigwedge_m (\psi_j \rightarrow \chi_j)$
If $\varphi \vdash \psi_k$ for some k, $1 \leq k \leq m$, then $\varphi \equiv \chi_k \wedge \bigwedge_1 p_i \wedge \bigwedge_{\substack{m \\ j\neq k}} (\psi_j \rightarrow \chi_j)$ and hence reduces to a strictly shorter formula.
Else $\varphi \nvdash \psi_k$ for all k, $1 \leq k \leq m$. So $\varphi \mid (\psi_j \rightarrow \chi_j)$ for all $j \leq m$ and $\varphi \mid \bigwedge_m (\psi_j \rightarrow \chi_j)$.
As trivially $\varphi \mid \bigwedge_1 p_i$, we have $\varphi \mid \varphi$. □

Theorem 4.18 For every IpL formula φ there is an (up to equivalence) unique set of irreducible formulas $\psi_1, \ldots, \psi_n$ such that:

a. if $\psi_i \vdash \psi_j$ then $i = j$ (independency)
b. $\varphi \equiv \bigvee \psi_i$.

Pf. From the previous corollary it is clear that there is a finite independent set of irreducible ψ_i's such that $\varphi = \bigvee\psi_i$. To prove this set unique up to equivalence, assume $\chi_1, \ldots, \chi_m$ is an independent set of irreducibles (that is satisfying the independency condition a) and $\varphi \equiv \bigvee\chi_j$. We prove each χ_j to be equivalent to some ψ_i.
From $\chi_j \vdash \bigvee\psi_i$ and the irreducibility of χ_j, infer that for some i: $\chi_j \vdash \psi_i$.
On the other hand, from $\psi_i \vdash \bigvee\chi_j$ and the irreducibility of ψ_i, infer that for some k: $\psi_i \vdash \chi_k$. Hence $\chi_j \vdash \chi_k$ and by independence $j = k$, which proves $\chi_j \equiv \psi_i$. □

5 γ-Reductions, three-valued Heyting logic and Kripke completions

In this section some general notions and methods to describe and construct Kripke models are introduced and a detour is made into the intermediate propositional logic H_3, the three-valued Heyting logic. The reader may skip this section and return later to be informed about details of γ-reductions, exact frames, H_3 and Kripke completions which will be used after section 6.

5.1 Exact frames

Reductions (or surjective p -morphisms, see definition 3.13) can be regarded as built up from α- and β-reductions (see JT66), where α-reductions eliminate repetitions of 'equivalent' worlds and β-reductions eliminate repetitions of 'equivalent' submodels. We may (as will be proved in the sequel) reduce a finite Kripke model as much as possible, by combining all possible α- and β-reductions. This is essentially what is happening in what will be called a γ-reduction.

Recall the definition of *atom* (k) as the set of atoms forced in k (definition 3.10).
If $\mathbf{K} = < K, \leq >$ a Kripke frame and $\mathcal{K} = <\mathbf{K}, \omega>$ a Kripke model we will use $<K, \leq, \omega>$ as an alternative description for $\mathcal{K}$.

Definition 5.1.1 Let $\mathcal{K} = <K, \leq, \omega>$ be a finite Kripke model. The γ-reduction of $\mathcal{K}$ is defined as $\mathcal{K}^\gamma = <K^\gamma, \leq^\gamma, \omega^\gamma>$, where $K^\gamma = \{\gamma(k) \mid k \in K\}$ and $\gamma(k)$ is defined inductively over the depth of k, $\delta(k)$:

$\delta(k) = 0$: $\quad \gamma(k) = <atom(k), \varnothing>$

$\delta(k) = n+1$: $\quad \gamma(k) = \gamma(l)$ $\quad$ if $atom(l) = atom(k)$ and $\downarrow l = \{m \mid m < k\}$

$\quad\quad = <atom(k), \{\gamma(l) \mid l < k\}>$ $\quad$ otherwise.

The order $\leq^\gamma$ is induced by defining: $\gamma(l) < \gamma(k) \Leftrightarrow \gamma(k) = <Q, R>$ and $\gamma(l) \in R$.
The valuation ω^γ is defined as: $\gamma(k) \in \omega(p) \Leftrightarrow p \in atom(k)$.

Fact 5.1.2 Let $\mathcal{K}$ be a finite Kripke model, then $\mathcal{K}^\gamma$ is a finite Kripke model.

Note that the definition of γ identifies isomorphic submodels (applies β-reductions) and by its casewise definition, if δ(k) > 0, also applies α-reductions.

Lemma 5.1.3 Let $\mathcal{K}$ be a finite Kripke model, then $\mathcal{K}^\gamma$ is a reduction of $\mathcal{K}$.

Pf. $\gamma: \mathcal{K} \to \mathcal{K}^\gamma$ is the required reduction (see definition 3.13).
The proof that γ satisfies all conditions is straightforward. □

Corollary 5.1.4 If $\mathcal{K}$ is a finite Kripke model, then $\mathcal{K} \equiv \mathcal{K}^\gamma$.

Pf. Apply fact 3.14. □

Definition 5.1.5 A Kripke model $\mathcal{K} = <\mathbf{K}, \omega>$ is called *γ-irreducible* if:
$\forall k, k' \in \mathbf{K}\ (atom(k) = atom(k') \wedge \downarrow k \setminus \{k, k'\} = \downarrow k' \setminus \{k, k'\} \to k = k')$.

Definition 5.1.5 may read as an alternative definition of γ-reduction, as is stipulated in the following fact.

Fact 5.1.6 If $\mathcal{K}$ is γ-irreducible, then $\mathcal{K}$ is isomorphic to $\mathcal{K}^\gamma$.

Lemma 5.1.7 If $\mathcal{K}$ is a finite Kripke model, then $\mathcal{K}^{\gamma} = \mathcal{K}^{\gamma\gamma}$.

Pf. First, note that by definition 5.1.1 $atom(\gamma\gamma(k)) = atom(\gamma(k)) = atom(k)$.
By induction on the depth of k, $\delta(k)$, we prove $\gamma(k)=\gamma\gamma(k)$.
If $\delta(k) = 0$, then $\gamma\gamma(k) = \langle atom(\gamma(k)), \varnothing\rangle$. Hence $\gamma\gamma(k) = \gamma(k)$.
If $\delta(k) = n+1$ and $\gamma(k) = \gamma(l)$ for some $l < k$, then by induction hypothesis:
$\gamma\gamma(k) = \gamma\gamma(l) = \gamma(l) = \gamma(k)$.
If $\delta(k) = n+1$ and $\gamma(k) = \langle atom(k), \{\gamma(l) \mid l < k\}$, then by definition 5.1.1 we have:
$\gamma\gamma(k) = \langle atom(k), \{\gamma\gamma(l) \mid l < k\}$.
Hence, by induction hypothesis one infers $\gamma(k) = \gamma\gamma(k)$. □

Lemma 5.1.8 If $\mathcal{K}$ is a finite Kripke model and $\mathcal{K}^{\gamma}$ is its γ-reduction, then every reduction of $\mathcal{K}^{\gamma}$ is an isomorphism.

Pf. Let $\mathbf{L}$ be a reduction of $\mathcal{K}^{\gamma}$, f a reduction and:

$$\mathcal{K} \xrightarrow{\gamma} \mathcal{K}^{\gamma} \xrightarrow{f} \mathbf{L}$$

As f is a reduction, it is surjective.
To prove f to be injective, let $f(\langle L, S\rangle) = f(\langle L', S'\rangle)$.
By the definition of reduction we have:
$f(\langle L, S\rangle) \Vdash p \quad \Leftrightarrow \quad \langle L, S\rangle \Vdash p \quad \Leftrightarrow \quad p \in L$
Hence $L = L'$.
To proof $S = S'$ we proceed by induction over the depth of $\langle L, S\rangle$.
Let $S = \varnothing$. By the p-morphism condition, if $f(k) \leq f(\langle L, \varnothing\rangle)$ then $k = \langle L, \varnothing\rangle$.
So in this case we also have $S' = \varnothing$.
Let $k \in S$. As $k < \langle L, S\rangle$, also $f(k) \leq f(\langle L, S\rangle) = f(\langle L', S'\rangle)$.
By the p-morphism condition:
$\exists l \in f^{-1}(f(k))$. $l \leq \langle L', S'\rangle$.
For such an l, by definition of the γ-reduction, $l \in S'$.
By the induction hypotheses, from $f(k) = f(l)$ infer that $k = l$. Hence $S \subseteq S'$.
In the same way one proves $S' \subseteq S$, which proves $S = S'$. □

Notation 5.1.9 Recall the definition of P-model (definition 3.16)
If **K** is a finite Kripke frame, we will write $\Omega_P(\mathbf{K})$ for the set of all P-valuations on **K**.

Definition 5.1.10 Let **K** be a finite Kripke frame and P a finite set of atoms.
Let, for each valuation ω on **K**, $\langle \mathbf{K}, \omega\rangle^{\gamma}$ be the γ-reduction of $\langle \mathbf{K}, \omega\rangle$.
Define the *canonical P-model* on **K**, $\mathbf{Mod}_P(\mathbf{K})$, as:

$$\mathbf{Mod}_P(\mathbf{K}) = \bigcup\{\langle \mathbf{K}, \omega\rangle^{\gamma} \mid \omega \in \Omega_P(\mathbf{K})\}.$$

The canonical P-model (or canonical model for short) on **K** is an ordered set. Its order is defined by stipulating: $\langle L, S\rangle < \langle L', S'\rangle \Leftrightarrow \langle L, S\rangle \in S'$

Fact 5.1.11 $\mathbf{Mod}_P(\mathbf{K})$, defined above, is a well-defined Kripke model.

Lemma 5.1.12 Let **K** be a finite Kripke frame and P a finite set of atoms. Then every P-model on **K** is reducible to (hence equivalent with) a submodel of **Mod**$_P$(**K**), the canonical P-model of **K**.

Pf. Let $\mathcal{K}$ be a P-model, $\mathcal{K}^\gamma$ its γ-reduction. We prove $\mathcal{K}^\gamma = <\mathbf{K}^\gamma, \mu'>$ to be a submodel of **Mod**$_P$(**K**) $= <\mathbf{M_K}, \mu>$.
By the definition of **Mod**$_P$(**K**), $\mathbf{K}^\gamma \subseteq \mathbf{M_K}$. For the valuations we have:
$\mu'(p) = \{<L, S> \in \mathbf{K}^\gamma \mid p \in L\} = \{ <L, S> \in \mathbf{M_K} \mid p \in L\} \cap \mathbf{K}^\gamma = \mu(p) \cap \mathbf{K}^\gamma$.
To prove $\mathbf{K}^\gamma$ to be a down set:
Let $k \in \mathbf{K}^\gamma$, $k = <L, S>$ and $l \in \mathbf{M_K}$ such that $l < k$. From $l < k$ infer that $l \in S$.
But $S \subseteq \mathbf{K}^\gamma$, so $l \in \mathbf{K}^\gamma$. □

Lemma 5.1.12 stipulates that the canonical P-model on **K** contains only the most 'reduced' P-models on **K**. Combining lemma 5.1.8 and lemma 5.1.12 yields the following corollary.

Corollary 5.1.13 **Mod**$_P$(**K**) (the canonical P-model for some finite frame **K** and finite set of atoms P) is irreducible.

Lemma 5.1.14 Let **Mod**$_P$(**K**) $= <\mathbf{M_K}, \mu>$ be the canonical P-model of **K**, then every rooted submodel of **Mod**$_P$(**K**) is a reduction of some submodel of a P-model of **K**.

Pf. Let $k \in \mathbf{M_K}$, then by definition (5.1.10) for some P-model $\mathcal{K} = <\mathbf{K}, \omega>$ and $m \in \mathbf{K}$ and $k = <L, S>$ such that $\downarrow k$ is the γ-reduction of $\downarrow m$ (as in the proof of 5.1.12). □

In case the frame **K** in lemma 5.1.14 is a tree, one can prove a stronger version of this lemma: every rooted submodel of **Mod**$_P$(**K**) is a reduction of a P-model. To prove this use the fact that the application of a reduction to a reduction yields a reduction and the next lemma.

Lemma 5.1.15 Let **K** be a finite tree and P a finite set of atoms, then every rooted submodel of a P-model of **K** is a reduction of some P-model of **K**.

Pf. Let $\mathcal{K} = <\mathbf{K}, \omega>$ and $k \in \mathbf{K}$. Let m be some terminal node in $\downarrow k$ and define:

$$f(l) = l \quad \text{if } l \leq k$$
$$= k \quad \text{if } k \leq l$$
$$= m \quad \text{otherwise}$$

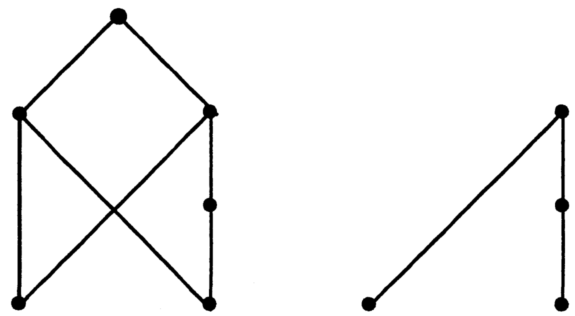
Let $\nu(p) = \{l \mid f(l) \in \omega(p)\}$.
Then ν clearly is the required valuation on **K** such that $<\mathbf{K}, \nu>$ is a P-model reduced by f to $\downarrow k$, if f indeed is a reduction.
To prove f a reduction, first note that f certainly is surjective.
Next, let $i \leq j$, we prove $f(i) \leq f(j)$.
Let $x <> y$ denote $\neg(x \leq y) \wedge \neg(x \geq y)$.
In general there are five cases:
- $j \leq k$ as $f(i) = i$ and $f(j) = j$, indeed $f(i) \leq f(j)$;
- $k \leq j$ as $f(j) = k$ and $f(i) \leq k$, indeed $f(i) \leq f(j)$;
- $j <> k$ there are three subcases:
  - $i \leq k$ impossible here, as **K** is a tree and hence $\uparrow i$ is lineairly ordered, so from $i \leq k$ and $i \leq j$ we know $j <> k$ is excluded;
  - $k \leq i$ impossible as it would by $k \leq i \leq j$ imply $k \leq j$;
  - $i <> k$ then $f(i) = m = f(j)$.

Remains to prove the p-morphism condition: $i \leq f(j) \Rightarrow \exists h \in f^{-1}(i).\ h \leq j$.

In general, there are three cases:

$j \leq k$	then $f(j) = j$ and hence $i \leq j$;
$k \leq j$	then $f(j) = k$ and $i \leq f(j) = k \leq j$;
$j <> k$	then $f(j) = m$.
	As m is a terminal node, $i = m$ and hence $j \in f^{-1}(i)$. □

Example 5.1.16
Here is an example showing that lemma 5.1.15 is not true in general for al Kripke frames. The frame on the left is not reducible to its subframes on the right.



Theorem 5.1.17 If **K** a finite frame, P a finite set of atoms and $\mathbf{Mod_P(K)}$ the canonical P-model of **K** then:

(i) every P-model on **K** is equivalent with a submodel of $\mathbf{Mod_P(K)}$;
(ii) every rooted submodel of $\mathbf{Mod_P(K)}$ is a reduction of a submodel of some P-model on **K**;
(iii) if **K** is a tree, then every rooted submodel of $\mathbf{Mod_P(K)}$ is a reduction of some P-model on **K**;
(iv) every reduction of a submodel of $\mathbf{Mod_P(K)}$ is an isomorphism.

Pf. By the lemmas above. □

Definition 5.1.18 If **K** is a finite frame and $\mathbb{F}$ a fragment such that: $\varphi \nvdash \psi$ iff there is a valuation ω such that for $\mathcal{K} = \langle \mathbf{K}, \omega \rangle$, $\mathcal{K} \nVdash \varphi \rightarrow \psi$, then **K** is a *canonical frame* for $\mathbb{F}$.

The notion of derivability here is not restricted to IpL, but applies to every propositional logic (having the same formulas as IpL) which has a completeness theorem for some class of Kripke models (like H_3 for example, see section 5.2).
If **K** is a canonical frame for $\mathbb{F}$ then every $\varphi \in \mathbb{F}$ corresponds uniquely with a down set on the canonical model of **K**. However, it may still be that $\mathbf{Mod_P(K)}$ is not the exact model of $\mathbb{F}$ because the mapping of formulas of $\mathbb{F}$ to down sets of $\mathbf{Mod_P(K)}$ may be non-surjective.

Definition 5.1.19 If **K** is a canonical frame for $\mathbb{F}$ and its canonical model, $\mathbf{Mod_P(K)}$, is the exact model of $\mathbb{F}$, **K** is called the *exact frame* of $\mathbb{F}$.

Example 5.1.20 The classical propositional logic, CpL, has an exact frame: $\mathbb{1}$, the frame with exactly one world.

5.2 The three-valued Heyting logic H_3

Defnition 5.2.1 H_3 is defined as the logic with as its axioms those of IpL (see for example [TD88]) plus the Gödel-formula (see [G32]) expressing that there are only three truth values:

(G3) $(p \leftrightarrow q) \vee (p \leftrightarrow r) \vee (p \leftrightarrow s) \vee (q \leftrightarrow r) \vee (q \leftrightarrow s) \vee (r \leftrightarrow s)$.

Derivability in H_3 is denoted by $\vdash_3$.

Fact 5.2.2 H_3 can alternatively be axiomatized as:

a.1 IpL + $p \vee (p \rightarrow q) \vee \neg q$

a.2 IpL + $((p \rightarrow (((q \rightarrow r) \rightarrow q) \rightarrow q)) \rightarrow p) \rightarrow p$ + $(p \rightarrow q) \vee (q \rightarrow p)$

a.3 IpL + $((p \rightarrow q) \rightarrow r) \rightarrow (((s \rightarrow p) \rightarrow r) \rightarrow r)$

The H_3 axiom in a.1 is a simplified version of Hosoi's $p \vee \neg p \vee (p \rightarrow q) \vee (q \rightarrow r)$ [H66]. For the Kripke-models of H_3 the axioms in a.2 correspond to linearity and maximal depth 1 (definition 3.9): $(p \rightarrow q) \vee (q \rightarrow p)$ is Dummett's axiom for LC and $((p \rightarrow (((q \rightarrow r) \rightarrow q) \rightarrow q)) \rightarrow p) \rightarrow p$ is the iterated Peirce axiom.
For the details see for example [T65].
The scheme a.3 stems from Ivo Thomas [T62].

Fact 5.2.3
The truth value of formulas in H_3 can be calculated by the matrices:

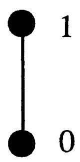
$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

	$\neg$
0	2
1	0
2	0

The behavior of the matrices above is modelled by the behavior of valuations on the frame 2 defined as the set $\{0, 1\}$ ordered by $0 \leq 1$ (and the order is of course reflexive).



The frame 2.

To prove 5.2.3, first observe that G3 is only true in rooted frames with less then three worlds. Next prove that the Dummett axiom is derivable in H_3:
$G3[r := p \vee q, s := p \wedge q] \vdash (p \rightarrow q) \vee (q \rightarrow p)$. Hence models of H_3 are (equivalent to) valuations on $\mathbb{2}$.

Fact 5.2.4 The logic H_3 is characterized by the Kripke frame $\mathbb{2}$, that is:

$$\vdash_3 \varphi \text{ iff, for all valuations } \omega \text{ on } \mathbb{2},\ \mathbb{2} \Vdash \varphi[\omega]$$

Lemma 5.2.5 Let H_{3n} be H_3 restricted to n atoms, then $\mathbb{2}$ is a canonical model of H_{3n}.

Pf. By fact 5.2.4.

Theorem 5.2.6 $\mathbb{2}$ is the exact frame of H_{3n}.

Pf. Let P be the set of atoms in H_{3n}, $\mathbf{Mod}_n(\mathbb{2}) = \langle \mathbf{M}_{2n}, \mu \rangle$ the canonical P-model on $\mathbb{2}$. As observed above, after lemma 5.1.12 we only have to prove that every down set in $\mathbf{Mod}_n(\mathbb{2})$ corresponds with a formula in H_{3n} in the mapping $\mu(\varphi) = \{k \in \mathbf{M}_{2n} \mid k \Vdash \varphi\}$.
One may check that $l \in \mathbf{M}_{2n}$ is of the form:

$$l = \langle L, S \rangle \quad \text{such that } L \subseteq P \text{ and } S = \emptyset \qquad (1)$$
$$\text{or } S = \{\langle L', \emptyset \rangle\} \text{ and } L \subset L' \qquad (2)$$

Define casewise:

$$\varphi_l = \bigwedge L \wedge \bigwedge\{\neg p \mid p \in P \setminus L\} \qquad \text{(in case 1)}$$
$$\bigwedge L \wedge \bigwedge\{\neg p \mid p \in P \setminus L'\} \wedge \bigwedge\{\neg\neg p \mid p \in L' \setminus L\} \wedge \bigwedge\{p \leftrightarrow q \mid p, q \in L \setminus L\} \text{ (if 2)}$$

It is readily checked that $\mu(\varphi_l) = \{k \in \mathbf{M}_{2n} \mid k \Vdash \varphi_l\} = \downarrow l$.
For any down set S in $\mathbf{M}_{2n}$ we may take the disjunction of the φ_l such that $l \in S$. □

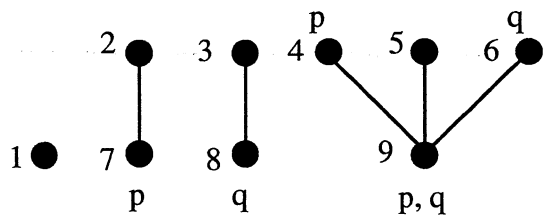
Of course in the proof above, the φ_l can be simplified (with the added equivalences only one $\neg\neg p$ such that $p \in L' \setminus L$ will do). Also the formula for down set S could be restricted to the φ_l such that l is a maximal element in S.

Corollary 5.2.7 From the proof of theorem 5.2.6 one may deduce the structure of $\mathbf{Mod}_n(\mathbb{2})$ as a Kripke model such that:

a. there are 2^n terminal elements each representing a subset of the set of atoms P;
b. above each terminal node, say corresponding to $Q \subseteq P$, the nodes (if any) correspond exactly to the proper subsets of Q;
c. the maximal depth of each node is 1.

Corollary 5.2.8 $\mathbf{Mod}_n(\mathbb{2})$ has $\Sigma_{k=0}^{n}\binom{n}{k}2^k$ elements and $\Pi_{k=0}^{n}(2^{2^k-1}+1)^{\binom{n}{k}}$ open subsets. Hence H_{3n} has $\Pi_{k=0}^{n}(2^{2^k-1}+1)^{\binom{n}{k}}$ equivalence classes.

Example 5.2.9
The exact model of H_{32} is the model $\mathbf{Mod_2}(\mathbb{2})$:



The corresponding formulas:

1. $\neg p \wedge \neg q$	4. $p \wedge \neg\neg q$	7. $p \wedge \neg q$
2. $\neg\neg p \wedge \neg q$	5. $\neg\neg p \wedge \neg\neg q$	8. $\neg p \wedge q$
3. $\neg p \wedge \neg\neg q$	6. $\neg\neg p \wedge q$	9. $p \wedge q$

5.3 Kripke completions of exact models

As announced in section 4 and as will be proved in section 8, some fragments $\mathbb{F}$ of IpL do posess an exact model, $<E, \vdash>$ which is not an exact Kripke model (hence not for all $\varphi, \psi \in \mathbb{F}: \varphi \vdash \psi \Leftrightarrow <E, \vdash> \Vdash \varphi \to \psi$).
In this section a technique is introduced to extend E to E^+ such that for all $\varphi, \psi \in \mathbb{F}: \varphi \vdash \psi \Leftrightarrow <E^+, \vdash> \Vdash \varphi \to \psi$.
In the proofs of this section we will need the fact that $\mathbb{F}$ satisfies some conditions, i.e. $\mathbb{F}$ is 'normal' in a sense to be made more precise in the following definition.

Definition 5.3.1 A fragment $\mathbb{F}$ is called a *normal fragment* if:
a. the connectives of $\mathbb{F}$ are among those in $\{\wedge, \vee, \to, \leftrightarrow, \neg, \neg\neg\}$
b. if $\to$ or $\leftrightarrow$ are connectives of $\mathbb{F}$, so is $\wedge$.

Notation 5.3.2 Let P_n be a finite set of atoms, E a set of formulas. Recall the definition of φ_Q from definition 3.22 and from notation 4.5, for $\varphi \in E$, that of $k_\varphi \in <E, \vdash>$.
If $\varphi = \varphi_Q$ then k_φ will be denoted as k_Q.

In the following lemma we need the notion of a P_n-model, defined in definition 3.16.

Lemma 5.3.3 Let $Var(\varphi) \subseteq P_n$, $\varphi \not\equiv \bot$ and $Q \subseteq P_n$. Then:
a. if k_Q is a terminal node in a P_n-model, then: $k_Q \Vdash \varphi \Leftrightarrow \varphi_Q \vdash \varphi$
b. $\varphi \vdash \varphi_Q \quad \Rightarrow \quad \varphi \equiv \varphi_Q$
c. $\varphi_Q \vdash \neg\neg\varphi \quad \Rightarrow \quad \varphi_Q \vdash \varphi$
d. there is an $R \subseteq P_n$ such that if k_R is a terminal node in a P_n-model, $k_R \Vdash \varphi$

Pf. a. Let k_Q be a terminal node in a P_n-model. Then $k_Q \nVdash \varphi \Rightarrow k_Q \Vdash \neg\varphi$.
Applying this to p gives $k_Q \Vdash \neg p$ for $p \in P_n \backslash Q$, which leads to $k_Q \Vdash \varphi_Q$.
As a consequence $\varphi_Q \vdash \varphi \Rightarrow k_Q \Vdash \varphi$.
Assume $k_Q \Vdash \varphi$. If $\mathbf{L}$ is a P_n-model such that $\mathbf{L} \Vdash \varphi_Q$, then clearly $\mathbf{L}$ is reducible to $<k_Q>$, the Kripke model with k_Q as its only node. As $<k_Q> \Vdash \varphi$, so does $\mathbf{L}$.
By the completeness theorem for P_n-models (fact 3.17.c) $\varphi_Q \vdash \varphi$.
b. As a simple consequence of 3.23.b: if $\varphi \not\equiv \bot$, then for some $R \subseteq P_n$ it is true that $\varphi_R \vdash \varphi$. Hence, if $\varphi \vdash \varphi_Q$ also $\varphi_R \vdash \varphi_Q$ and by the definition of φ_Q and φ_R it will be clear that $\varphi_R = \varphi_Q$. Which proves $\varphi \equiv \varphi_Q$.
c. Let $<k_Q>$ be the model defined in the proof of part a. If $\varphi_Q \vdash \neg\neg\varphi$, then as a consequence of a: $k_Q \Vdash \neg\neg\varphi$ and, as k_Q is a terminal node, $k_Q \Vdash \varphi$. Again using part a. of this lemma: $\varphi_Q \vdash \varphi$.
d. This is a combination of fact 3.23.b and part a. of this lemma. □

Lemma 5.3.3 justifies the name 'end extension' in the next definition.

Definition 5.3.4 Let $E \subseteq \mathbb{F}(P_n)$. Then E^+ is defined as: $E^+ := E \cup \{\varphi_Q \mid Q \subseteq P_n\}$
The Kripke model $<E^+, \vdash>$ is called the *end extension* of $<E, \vdash>$.
If γ is the reduction defined in 5.1.1, then $<E^+, \vdash>^\gamma$ is called the *Kripke completion* of $<E, \vdash>$.

Lemma 5.3.5 If $<E, \vdash>$ is the exact model of the normal fragment $\mathbb{F}(P_n)$ and $<E^+, \vdash>$ is its end extension, then for each $\psi \in E$ and each $\varphi \in E^+$: $k_\varphi \Vdash \psi \quad \Leftrightarrow \quad \varphi \vdash \psi$.

Pf. By induction on the length of ψ. Use the induction hypothesis that for all $\xi \in E$ strictly shorter then ψ we have:

(IH) $\quad \forall \varphi \in E^+ : k_\varphi \Vdash \xi \Leftrightarrow \varphi \vdash \xi$

$\psi \in P_n$: by definition if $p \in P_n$ then $k_\varphi \Vdash p \Leftrightarrow \varphi \vdash p$

$\psi = \chi \wedge \sigma$: $k_\varphi \Vdash \chi \wedge \sigma \Leftrightarrow k_\varphi \Vdash \chi$ and $k_\varphi \Vdash \sigma$
$\overset{\text{(IH)}}{\Leftrightarrow} \varphi \vdash \chi$ and $\varphi \vdash \sigma \Leftrightarrow \varphi \vdash \chi \wedge \sigma$

$\psi = \chi \vee \sigma$: as $\psi \in E$, ψ is irreducible in *Diag* ($\mathbb{F}$) and if $\vee$ is a connective of $\mathbb{F}$, ψ is $\vee$-irreducible (see the remark after definition 3.19). Hence $\psi \equiv \chi$ or $\psi \equiv \sigma$ and in both cases by IH: $k_\varphi \Vdash \psi \quad \Leftrightarrow \quad \varphi \vdash \psi$

$\psi = \chi \to \sigma$: a. Assume $\varphi \vdash \chi \to \sigma$. For all $\tau \in E^+ : \tau \vdash \varphi \Rightarrow (\tau \vdash \chi \Rightarrow \tau \vdash \sigma)$
If $k_\tau \leq k_\varphi$ then $\tau \vdash \varphi$ and by IH: $k_\tau \Vdash \chi \quad \Rightarrow \quad k_\tau \Vdash \sigma$.
Hence $k_\varphi \Vdash \chi \to \sigma$.
b. Assume $k_\varphi \Vdash \chi \to \sigma$. If $\tau \vdash \varphi$ then $k_\tau \leq k_\varphi$, so $k_\tau \Vdash \chi \to \sigma$, and by IH: $\tau \vdash \varphi \wedge \chi$ implies $\tau \vdash \sigma$.
As $<E, \vdash>$ is the exact model of $\mathbb{F}$, if ω the 1-1 correspondence between *Diag* ($\mathbb{F}$) and the downsets of $<E, \vdash>$, we have: $\omega(\varphi \wedge \chi) \subseteq \omega(\sigma)$. (Note that by definition of a normal fragment $\varphi \wedge \chi$ is in $\mathbb{F}$). Hence $\varphi \vdash \chi \to \sigma$.

$\psi = \chi \leftrightarrow \sigma$: in this case the proof closely resembles the proof in case of $\psi = \chi \to \sigma$.

$\psi = \neg\chi$: a. Assume $\varphi \vdash \neg\chi$. For all $\tau \in E^+ : \tau \vdash \varphi \Rightarrow \tau \nvdash \chi$ (as $\bot \notin E^+$).
If $k_\tau \leq k_\varphi$ then $\tau \vdash \varphi$ and by IH $k_\tau \nVdash \chi$. Hence $k_\varphi \Vdash \neg\chi$.
b. Assume $k_\varphi \Vdash \neg\chi$. Let $Q \subseteq P_n$ then k_Q a terminal node such that $k_Q \leq k_\varphi$ iff $\varphi_Q \vdash \varphi$. Hence for all $Q \subseteq P_n$ we have (using lemma 5.3.3) $\varphi_Q \vdash \neg\chi$. According to fact 3.23 in CpL we have $\varphi = \bigvee\{\varphi_Q \mid Q \subseteq P_n$ and $\varphi_Q \vdash \varphi\}$, which proves $\varphi \vdash_c \neg\chi$.
Again by fact 3.23 this implies $\varphi \vdash \neg\chi$.

$\psi = \neg\neg\chi$: a. Assume $\varphi \vdash \neg\neg\chi$. If τ is minimal in E^+ and $\tau \vdash \varphi$ then k_τ is a terminal node of $<E^+, \vdash>$ such that $k_\tau \leq k_\varphi$.
By lemma 5.3.3 $\tau = \varphi_Q$ for some $Q \subseteq P_n$ and, also by lemma 5.3.3 $k_Q \Vdash \chi$. Hence $k_\varphi \Vdash \neg\neg\chi$.
b. Assume $k_\varphi \Vdash \neg\neg\chi$. Let $Q \subseteq P_n$ then k_Q a terminal node such that $k_Q \leq k_\varphi$ iff $\varphi_Q \vdash \varphi$. Hence for all $Q \subseteq P_n$ we have (using lemma 5.3.3) $\varphi_Q \vdash \chi$. According to fact 3.23 in CpL we have $\varphi = \bigvee\{\varphi_Q \mid Q \subseteq P_n$ and $\varphi_Q \vdash \varphi\}$, which proves $\varphi \vdash_c \neg\neg\chi$.
Again by fact 3.23 this implies $\varphi \vdash \neg\neg\chi$. □

Corollary 5.3.6 If $<E, \vdash>$ is the exact model of the normal fragment $\mathbb{F}(P_n)$ and $<E^+, \vdash>$ is its end extension, then $\forall\varphi, \psi \in \mathbb{F}$: $\varphi \vdash \psi \Leftrightarrow <E^+, \vdash> \Vdash \varphi \to \psi$.

Pf. Assume $<E^+, \vdash> \Vdash \varphi \to \psi$. For every $\xi \in E$ such that $\xi \vdash \varphi$ it is true in $<E^+, \vdash>$, by lemma 5.3.5, that $k_\xi \Vdash \xi$ and hence $k_\xi \Vdash \varphi$. By assumption we have $k_\xi \Vdash \psi$ and, again by lemma 5.3.5, $\xi \vdash \psi$.
As $<E, \vdash>$ is the exact model of $\mathbb{F}$, this proves $\varphi \vdash \psi$. □

The following theorem shows that the Kripke completion of an exact model, $<E, \vdash>$, of a normal fragment is an end extension, modulo the identification of some of the k_Q with terminal nodes of $<E, \vdash>$.
So, the exact model of a normal fragment is isomorphic to a subset of its Kripke completion (with the order inherited from the Kripke frame).

Theorem 5.3.7 Let $<E, \vdash>$ be the exact model of the normal fragment $\mathbb{F}(P_n)$ and let $<E^+, \vdash>^\gamma$ be its Kripke completion. Then: $\forall\varphi, \psi \in \mathbb{F}$: $\varphi \vdash \psi \Leftrightarrow <E^+, \vdash>^\gamma \Vdash \varphi \to \psi$.
Moreover, if γ is the reduction of the end extension $<E^+, \vdash>$ to $<E^+, \vdash>^\gamma$, then γ restricted to E is injective (a monomorphism).

Pf. That $\forall\varphi, \psi \in \mathbb{F}$: $\varphi \vdash \psi \Leftrightarrow <E^+, \vdash>^\gamma \Vdash \varphi \to \psi$ is a direct consequence of corollary 5.3.6 and corollary 5.1.4.
Assume $\varphi, \psi \in E$ and $\gamma(k_\varphi) = \gamma(k_\psi)$. In $<E^+, \vdash>$ we have, by lemma 5.3.5, $k_\varphi \Vdash \varphi$.
As γ is a reduction, this is also true in $<E^+, \vdash>^\gamma$ and hence $k_\psi \Vdash \varphi$ in both $<E^+, \vdash>^\gamma$ and $<E^+, \vdash>$. Again by 5.3.5, this proves $\psi \vdash \varphi$. In the same way one proves $\varphi \vdash \psi$.
Which proves γ restricted to E to be an injection. □

Note that the restriction of theorem 5.3.7 to normal fragments is not very essential.
From the overview of fragments in section 4 it is clear that the only fragments in this report that have an exact model are normal.

According to lemma 5.1.8, the Kripke completion of an exact model is in a sense minimal.
Still it is sometimes possible to be more economical about extending the exact model, as is stipulated by the following lemma.

Lemma 5.3.8 Let $\mathcal{K}$ be a Kripke completion of the exact model of normal fragment $\mathbb{F}(P_n)$. If for some terminal node k of $\mathcal{K}$, $k \nVdash \varphi$ for all $\varphi \in \mathbb{F}$ then in theorem 5.3.7 $\mathcal{K}$ may be replaced by $\mathcal{K} \setminus \{k\}$.

Pf. Let $<E, \vdash>$ the exact model of $\mathbb{F}$ and $<E^+, \vdash>$ is its end extension.
For no $\xi \in E$ we have (by lemma 5.3.5) $k_\xi \Vdash \xi$.
So if $k \nVdash \varphi$ for all $\varphi \in \mathbb{F}$, then $k = k_Q$ for some $Q \subseteq P_n$.
Note that k_Q is an isolated terminal node, not below any of the other nodes in $\mathcal{K}$.
Hence $\mathcal{K} \setminus \{k_Q\}$ is a generated submodel of K and for $\varphi \in \mathbb{F}$:
$\{k \in \mathcal{K} \mid k \Vdash \varphi\} = \{k \in \mathcal{K} \setminus \{k_Q\} \mid k \Vdash \varphi\}$ which proves $\mathcal{K}$ and $\mathcal{K} \setminus \{k_Q\}$ to be equivalent for $\mathbb{F}$ formulas. □

6.1 The [∧, ∨]-fragments

The structure of the [∧, ∨]-fragments is relatively well known [C77, DP91, S13, S36].

Fact 6.1.1

a. For [∧, ∨] formulas classical propositional logic (CpL) and IpL coincide, i.e. for all $\varphi, \psi \in [\wedge, \vee]$: $\varphi \vdash \psi \Leftrightarrow \varphi \vdash_c \psi$

b. The Lindenbaum algebra of $[\wedge, \vee]_n$ is isomorphic to the free distributive lattice over n generators.

c. (Disjunctive normal form) Each $\varphi \in [\wedge, \vee]_n$ is equivalent to a finite disjunction of $[\wedge]_n$ formulas.

d. All $\varphi \in [\wedge]_n$ are ∨-irreducible.

e. [∧] is dual to [∨]. If $\varphi \in [\wedge]$ define $\varphi^* := \varphi[\vee/\wedge, \wedge/\vee]$ (replacing all ∧ by ∨ and vice versa). Then for all $\varphi, \psi \in [\wedge, \vee]$ we have:

$$\varphi \vdash \psi \;\Leftrightarrow\; \psi^* \vdash \varphi^*$$

f. If $\varphi \in [\wedge, \vee P_n]$ then: $\bigwedge P_n \vdash \varphi$.

Theorem 6.1.2 The diagram of $[\wedge]_n \backslash \bigwedge P_n$ is the exact Kripke model of $[\wedge, \vee, P_n]$.

Pf. Define $\omega: [\wedge, \vee]_n \rightarrow [\wedge]_n \backslash \bigwedge P_n$ as $\omega(\varphi) = \{\bigwedge Q \mid Q \subset P_n \text{ and } \bigwedge Q \vdash \varphi\}$. We prove ω is the required isomorphism.
It is easily verified that $\omega(\varphi)$ is a down set in the diagram of $[\wedge]_n \backslash \bigwedge P_n$. That ω is surjective is also obvious as for a down set U in the diagram of $[\wedge]_n \backslash \bigwedge P_n$, $\omega(\bigvee U) = U$ (if we define $\bigvee U = \bigwedge P_n$ if $U = \emptyset$).
To show ω is injective, let $\varphi, \psi \in [\wedge, \vee, P_n]$ be in disjunctive normal form (fact 6.1.c) and $\varphi = \bigvee U$, $\psi = \bigvee W$.
Now $\bigvee U \equiv \bigvee{\downarrow}U$ and for all $\chi \in [\wedge]_n \backslash \bigwedge P_n : \chi \vdash \varphi \Leftrightarrow \chi \in {\downarrow}U$.
So $\omega(\varphi) = {\downarrow}U$ and from $\omega(\varphi) = \omega(\psi)$ we infer $\bigvee{\downarrow}U = \bigvee{\downarrow}W$ and hence $\varphi \equiv \psi$.

To show $[\wedge]_n \backslash \bigwedge P_n$ is an exact Kripke model, we have to verify:

$$\omega(\varphi) = \{\chi \in [\wedge]_n \backslash \bigwedge P_n \mid k_\chi \Vdash \varphi\} \qquad \text{(for } k_\chi \text{ see notaition 4.5)}$$

This can be done by proving for all $\chi \in [\wedge]_n \backslash \bigwedge P_n$ by induction over the complexity of $\varphi \in [\wedge, \vee, P_n]$:

$$\text{(IH)} \quad \chi \in \omega(\varphi) \Leftrightarrow k_\chi \Vdash \varphi$$

<u>For atomic p</u>: by definition (definition 4.2) $\chi \Vdash p \Leftrightarrow \chi \vdash p \Leftrightarrow \chi \in \omega(p)$.

<u>$\varphi = \psi \wedge \phi$</u>: $\chi \in \omega(\psi \wedge \phi) \Leftrightarrow \chi \vdash \psi$ and $\chi \vdash \phi \overset{\text{(IH)}}{\Leftrightarrow} k_\chi \Vdash \psi$ and $k_\chi \Vdash \phi$
$\Leftrightarrow k_\chi \Vdash \psi \wedge \phi$

<u>$\varphi = \psi \vee \phi$</u>: $\chi \in \omega(\psi \vee \phi) \Leftrightarrow \chi \vdash \psi \vee \phi$
$\Leftrightarrow \chi \vdash \psi$ or $\chi \vdash \phi$ (χ is irreducible, fact 6.1.1.d)
$\overset{\text{(IH)}}{\Leftrightarrow} k_\chi \Vdash \psi$ or $k_\chi \Vdash \phi \Leftrightarrow k_\chi \Vdash \psi \vee \phi$ □

Corollary 6.1.3 The exact Kripke model of $[\wedge, \vee, P_n]$ is (isomorphic to) the p.o. set of proper nonempty subsets of the set P_n, ordered by ⊇.

Pf. Define $\upsilon: [\wedge]_n \backslash \bigwedge P_n \rightarrow \wp(P_n) \backslash P_n$ by:

$$\upsilon(\chi) = \emptyset \quad \text{if } \chi = \bigwedge P_n$$
$$= \{p \in P \mid \chi \vdash p\} \quad \text{otherwise.}$$

Obviously υ is the required isomorphism. □

As the characteristic functions of down sets are exactly the monotonic functions into {0, 1}, corollary 6.1.3 in fact establishes the correspondence between formulas of $[\wedge, \vee]_n$ and monotonic functions of $2^n \rightarrow 2$. The problem of determining the number $D(n)$ of these functions (for each n) goes back to Dedekind and is also known in a different, but equivalent, form as the Sperner problem. The number $D(n)$ corresponds to the number of elements in the diagram of $[\wedge, \vee]_n$ (see [S28], [K69] and [K88] for more details).

In [S73] there is a table (1439) for $D(n)$:

n	$D(n)$
1	1
2	4
3	18
4	166
5	7 579
6	7 828 352
7	2 414 682 040 996

6.2 The [∧]-fragments

The structure of the $[\wedge]_n$ diagram was already revealed in corollary 6.1.3: the $[\wedge, P_n]$-diagram is isomorphic to the p.o. set of non-empty subsets of the set P_n.
This is trivial, as each $\chi \in [\wedge, P_n]$ is equivalent to ΛU for some non-empty $U \subseteq P_n$.
From this observation the number of equivalence classes in $[\wedge]_n$ is easily calculated: 2^n-1.

6.3 The [∨]-fragments

The structure of the $[\vee]_n$ diagrams is dual to that of their $[\wedge]_n$ counterparts (that is, in the categorical way, using the correspondence of fact 6.1.1.e, the diagram of $[\vee]_n$ is obtained from that of $[\wedge]_n$ by reversing the direction of all arrows).

7.1 The $[\wedge, \vee, \neg]$-fragments

In this section we will prove that the diagram of $[\wedge, \neg]_n \backslash \bot$ is the exact Kripke model of the fragment $[\wedge, \vee, \neg]_n$. Also a simple family of frames $\mathbf{F_n}$ is presenented such that each $\mathbf{F_n}$ is a canonical frame for $[\wedge, \vee, \neg]_n$.

Lemma 7.1.1 (The $[\wedge, \neg]$ normal form) Each $\varphi \in [\wedge, \vee, \neg]_n$ is equivalent to a finite disjunction of $[\wedge, \neg]_n$ formulas ($[\wedge, \vee, \neg]_n = \bigvee[\wedge, \neg]_n$).

Pf. Straightforward, using formula induction. The most interesting step is negation, where we use $\neg(\psi_1 \vee ... \vee \psi_m) \equiv \neg\neg(\neg\psi_1 \wedge ... \wedge \neg\psi_m)$ to prove that:

$$\neg\varphi \in [\wedge, \vee, \neg]_n \Leftrightarrow \neg\varphi \in [\wedge, \neg]_n$$

$\neg(\psi_1 \vee ... \vee \psi_m) \equiv \neg\neg(\neg\psi_1 \wedge ... \wedge \neg\psi_m)$ is a simple consequence from fact 3.23 and one of the classical De Morgan laws. □

Theorem 7.1.2 The diagram of $[\wedge, \neg]_n \backslash \bot$ is the exact Kripke model of $[\wedge, \vee, \neg]_n$

Pf. Recall from fact 3.21.d that all formulas of $[\wedge, \neg]_n \backslash \bot$ are irreducible.
Hence by lemma 7.1.1 and definition 4.2 $[\wedge, \neg]_n \backslash \bot$ is the exact model of $[\wedge, \vee, \neg]_n$.
To prove this model to be an exact Kripke model, with valuation:

$$\omega(p) = \{\chi \in [\wedge, \neg]_n \backslash \bot \mid \chi \vdash p\}$$

we prove by formula induction, for all $\varphi \in [\wedge, \vee, \neg]_n$

(IH) for all $\chi \in [\wedge, \neg]_n \backslash \bot$ we have $\chi \vdash \varphi \Leftrightarrow k_\chi \Vdash \varphi$.

Recall that (notation 4.5) k_χ is the class of χ, as a node in the Kripke model.

<u>φ atomic</u>: by definition

<u>$\varphi \wedge \psi$</u>: $k_\chi \Vdash \varphi \wedge \psi \Leftrightarrow k_\chi \Vdash \varphi$ and $k_\chi \Vdash \psi \underset{(IH)}{\Leftrightarrow} \chi \vdash \varphi$ and $\chi \vdash \psi \Leftrightarrow \chi \vdash \varphi \wedge \psi$

<u>$\varphi \vee \psi$</u>: $k_\chi \Vdash \varphi \vee \psi \Leftrightarrow k_\chi \Vdash \varphi$ or $k_\chi \Vdash \psi \underset{(IH)}{\Leftrightarrow} \chi \vdash \varphi$ or $\chi \vdash \psi$
$\Leftrightarrow \chi \vdash \varphi \vee \psi$ (as χ is irreducible)

<u>$\neg\psi$</u>: by fact 3.23 and completeness of $[\wedge, \neg]$ for classical propositional logic, there is a $\sigma \in [\wedge, \neg]_n$ such that $\sigma \equiv_c \psi$ and hence $\neg\psi \equiv \neg\sigma$. If $\sigma = \bot$ then (IH) is trivially true. So assume $\sigma \in [\wedge, \neg]_n \backslash \bot$ and let θ in the sequel range over formulas of $[\wedge, \neg]_n \backslash \bot$.

$\chi \vdash \neg\sigma \Rightarrow \forall\theta \vdash \chi.\ \theta \nVdash \psi \underset{(IH)}{\Leftrightarrow} \forall\theta \vdash \chi.\ k_\theta \nVdash \psi \Leftrightarrow k_\chi \Vdash \neg\psi$

To show $\forall\theta \vdash \chi. \theta \nVdash \psi \Rightarrow \chi \vdash \neg\sigma$, observe that $\chi \wedge \sigma \in [\wedge, \neg]_n$ and both $\chi \wedge \sigma \vdash \chi$ and $\chi \wedge \sigma \vdash \sigma$.

Assuming $\forall\theta \vdash \chi.\ \theta \nvdash \sigma$ we find $\chi \wedge \sigma \equiv \bot$ and hence $\chi \vdash \neg\sigma$. □

The fragment $[\wedge, \vee, \neg]$ has the intermediate logic of all frames with maximal depth 1 as a conservative extension. To prove this we need the following lemma.

Lemma 7.1.3 Let $\mathcal{K}$ be a rooted Kripke model with k_0 as its root and L its set of terminal nodes. If $\mathcal{K}'$ is the Kripke model $\{k_0\} \cup L$ (with inherited order and valuation from $\mathcal{K}$) then for all $\varphi \in [\wedge, \vee, \neg]$: $\mathcal{K} \Vdash \varphi \Leftrightarrow \mathcal{K}' \Vdash \varphi$.

Pf. By induction on the complexity of φ.
If $\varphi = \neg\psi$ then use:

$$\mathcal{K} \Vdash \neg\psi \Leftrightarrow \forall l \in L\ (l \Vdash \neg\psi) \Leftrightarrow \forall l \in L\ (l \nVdash \psi) \Leftrightarrow \mathcal{K}' \Vdash \neg\psi$$

The other cases are even more trivial. □

There are simple counterexamples to prove that (the $\Leftarrow$-part of) lemma 7.1.3 fails if implications are involved.
For example take a Kripke model $\mathcal{K}$ with three worlds $\{k_a, k_b, k_c\}$ such that:
$k_a > k_b > k_c$ and $atom(k_a) = \emptyset$, $atom(k_b) = \{p\}$ and $atom(k_c) = \{p, q\}$.
Let $\mathcal{K}'$ be as defined above, then $\mathcal{K}' \Vdash p \rightarrow q$ and $\mathcal{K} \nVdash p \rightarrow q$.

Corollary 7.1.4 The rooted frame $\mathbf{F}_n$ of depth 1 and 2^n+1 elements is a canonical frame for the fragment $[\wedge, \vee, \neg]_n$.

Pf. Let $\varphi, \psi \in [\wedge, \vee, \neg]_n$ and $\varphi \nvdash \psi$.
According to definition 5.1.18 we have to prove that there is a valuation ω on $\mathbf{F}_n$ such that $<\mathbf{F}_n, \omega> \Vdash \varphi$ and $<\mathbf{F}_n, \omega> \nVdash \psi$.
By the completeness theorem (see fact 3.17) there is a rooted Kripke model $\mathcal{K}$, $\mathcal{K} \Vdash \varphi$ and $\mathcal{K} \nVdash \psi$. According to lemma 7.1.3 there is also a Kripke model $\mathcal{K}'$ which is rooted, has a maximal depth of 1 and is equivalent to $\mathcal{K}$ for $[\wedge, \vee, \neg]_n$-formulas.
As far as $[\wedge, \vee, \neg]_n$-formulas are concerned, each terminal node k in a Kripke model is characterized (up to equivalency) by *atom* $(k) \cap P_n$. That is, there is a maximum of 2^n terminal nodes in the γ-reduction of a P_n-model (see definitions 5.1.1 and 5.1.9). This proves that $\mathcal{K}'^{\gamma}$, the γ-reduction of the model $\mathcal{K}'$ above, is (isomorphic to) a P_n-model on $\mathbf{F}_n$. As $\mathcal{K}'^{\gamma}$ is equivalent to $\mathcal{K}'$ (corollary 5.1.4) this proves $\mathbf{F}_n$ to be canonical for $[\wedge, \vee, \neg]_n$. □

Theorem 7.1.5 The intermediate logic IpL + $((p\to(((q\to r)\to q)\to q))\to p)\to p$ is a conservative extension of the IpL fragment $[\wedge, \vee, \neg]$.

Pf. Recall (see 5.2.2, [G81]) that $((p\to(((q\to r)\to q)\to q))\to p)\to p$ is the iterated Peirce formula, which characterizes the frames of maximal depth 1. □

The intermediate logic IpL + $((p\to(((q\to r)\to q)\to q))\to p)\to p$ is a maximal conservative extension of $[\wedge, \vee, \neg]$. But it not unique, other intermediate logics exist that are also maximal conservative extensions of $[\wedge, \vee, \neg]$.

7.2 The [$\wedge$, $\neg$]-fragments

In this section we will prove the three-valued Heyting logic H_3 to be an exact hull of the fragment [$\wedge,\neg$].

Lemma 7.2.1 (Normal form of [$\wedge$, $\neg$]) For each $\varphi \in [\wedge,\neg]_n$ there is a $Q \subseteq P_n$ and a formula $\psi \in [\wedge,\neg]_n$ such that $\varphi \equiv \Lambda Q \wedge \neg\psi$.

Pf. By a simple induction on the complexity of φ. □

An immediate consequence of lemma 7.2.1 is the existence of a simple algorithm to decide for every $\varphi, \chi \in [\wedge,\neg]$ whether $\varphi \vdash \chi$. If $\varphi \equiv \Lambda Q \wedge \neg\psi$ and $\chi \equiv \Lambda R \wedge \neg\rho$ then:

$\varphi \vdash \chi \quad \Leftrightarrow \quad R \subseteq Q$ and $\rho \vdash_c \psi$ (ψ is a consequence of ρ in classical logic).

Definition 7.2.2 If l is a terminal node in the finite rooted Kripke model $\mathcal{K}$, with k_0 as its root, and ω as its valuation, the *terminal submodel* $\mathcal{K}^l$ is defined as the frame $\{k_0, l\}$, such that $l \leq k_0$, and a valuation ν such that $\forall k \in \mathcal{K}^l.\ k \in \nu(p) \leftrightarrow k \in \omega(p)$

Note that the terminal submodels are models with frame $\mathbb{2}$.

In the lemma below we will need a simple fact about Kripke models:

Fact 7.2.3 Let $\mathcal{K}$ be a finite rooted Kripke model and $\mathcal{K}^l$ a terminal submodel of $\mathcal{K}$ and $l \Vdash_{\mathcal{K}} \varphi$ then $l \Vdash_{\mathcal{K}^l} \varphi$.

Lemma 7.2.4 Let $\mathcal{K}^l$ be a terminal submodel of (finite rooted) $\mathcal{K}$, $\varphi \in [\wedge, \vee, \neg]$ and $\mathcal{K} \Vdash \varphi$, then $\mathcal{K}^l \Vdash \varphi$.

Pf. By formula induction: φ atomic, a conjunction or a disjunction: trivial
In case $\varphi = \neg\psi$: if $\mathcal{K} \Vdash \neg\psi$ then, by 7.2.3, $l \Vdash \neg\psi$. Hence $\mathcal{K}^l \Vdash \neg\psi$. □

Lemma 7.2.5 Let $\mathcal{K}$ be a finite rooted Kripke model, $\varphi \in [\wedge,\neg]$ and $\mathcal{K} \nVdash \varphi$, then there is a terminal submodel $\mathcal{K}^l$ such that $\mathcal{K}^l \nVdash \varphi$.

Pf. By formula induction:

<u>φ atomic:</u> trivial

<u>$\varphi = \psi\wedge\chi$:</u> $\mathcal{K} \nVdash \psi\wedge\chi \Rightarrow \mathcal{K} \nVdash \psi$ or $\mathcal{K} \nVdash \chi$
$\overset{\Rightarrow}{(IH)} \exists l.\ \mathcal{K}^l \nVdash \psi$ or $\exists l.\ \mathcal{K}^l \nVdash \chi \Leftrightarrow \exists l.\ \mathcal{K}^l \nVdash \psi\wedge\chi$

<u>$\varphi = \neg\psi$:</u> $\mathcal{K} \nVdash \neg\psi \Rightarrow$ for some $k \in \mathcal{K}$, $m \Vdash \psi$ for all $m \in \downarrow k$ and hence for terminal nodes l of $\downarrow k$: $l \nVdash \neg\psi$ and $\mathcal{K}^l \nVdash \neg\psi$ □

Theorem 7.2.6 H_3 is a conservative extension of [$\wedge$, $\neg$]

Pf. If $\varphi, \psi \in$ IpL then $\varphi \vdash \psi$ implies $\varphi \vdash_3 \psi$ (by the completeness theorems; the $\mathbb{2}$-models of H_3 are a special kind of Kripke models).
Let $\varphi \nvdash \psi$. By the completeness theorem (fact 3.17) there is a finite rooted $\mathcal{K}$ such that $\mathcal{K} \Vdash \varphi$ and $\mathcal{K} \nVdash \psi$. By lemma 7.2.5 there is a terminal submodel $\mathcal{K}^l$ such that $\mathcal{K}^l \nVdash \psi$ and by lemma 7.1.3 $\mathcal{K}^l \Vdash \varphi$. This $\mathcal{K}^l$ is a $\mathbb{2}$-model disproving $\varphi \vdash_3 \psi$. □

Let again H_{3n} be H_3 restricted to n atoms. In section 5.2 we proved H_{3n} to have an exact frame, $\mathbb{2}$. Hence has an exact Kripke model, $\mathbf{Mod_n}(\mathbb{2})$.

Corollary 7.2.6 H_{3n} is an exact Kripke hull for [$\wedge$, $\neg$].

8 The [∨, ¬]-fragments

The finite [∨, ¬]-fragments in IpL do have exact models which, for $n > 1$, are not exact Kripke models. In this section the structure of the exact model of $[\vee, \neg]_n$ will be described and we will use the technique of Kripke completions from section 5.3 to obtain Kripke models that are 'almost' exact Kripke models.

Lemma 8.1 (Normal form of [∨, ¬]) For every $\varphi \in [\vee, \neg]$, there is a finite set of atoms P and a finite set of $\psi_i \in [\vee, \neg]$ $(1 \leq i \leq m$ for some m) such that $\varphi = \bigvee P \vee \bigvee \neg\psi_i$.

Pf. By a simple induction on the complexity of φ. □

Theorem 8.2 The fragment $[\vee, \neg]_n$ has an exact model: EN_n, the set of atoms and negations in $[\vee, \neg]_n$, ordered by ⊢.

Pf. From the previous lemma we know every formula in $[\vee, \neg]_n$ is a disjunction of formulas in EN_n. According to fact 3.21.d both atoms and negations are always irreducible. □

Corollary 8.3 The exact model of $[\vee, \neg]_n$ is a copy of the classical fragment $[\vee, \neg]_n \backslash \bot$ (where all formulas are preceded by a double negation) and the n atoms added.

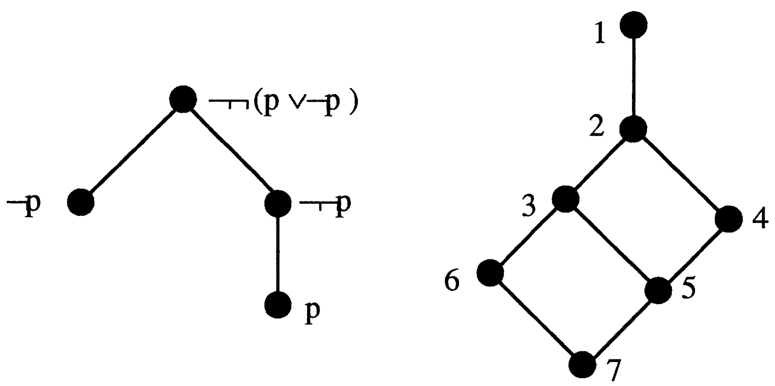
The normal form of lemma 8.1 can be restated in terms of EN_n. For every $\varphi \in [\wedge, \neg]_n$ there is a subset $\psi_1, \ldots, \psi_m \in EN_n$, such that $\varphi \equiv \bigvee \psi_i$. Note that if all $\psi_i, \chi_j \in EN_n$, then

$$\bigvee \psi_i \vdash \bigvee \chi_j \quad \Leftrightarrow \quad \forall i \exists j.\ \psi_i \vdash \chi_j$$

Note that the CpL fragment $[\vee, \neg]_n$ is equivalent with the CpL fragment $[\wedge, \neg]_n$. Hence the irreducible formulas of $[\vee, \neg]_n$ (see fact 3.23) are equivalent with (irreducible) formulas of $[\wedge, \neg]_n$.

Fact 8.4 The exact model of $[\vee, \neg]_n$ consists of those formulas in $[\wedge, \neg]_n$ (the exact model of $[\wedge, \vee, \neg]_n$) that are (equivalent with a formula) in $[\vee, \neg]_n$.

Example 8.5
The exact model of $[\vee, \neg, p]$ and its diagram:



1. $\neg\neg(p \vee \neg p)$	2. $\neg\neg p \vee \neg p$	3. $p \vee \neg p$	4. $\neg\neg p$
5. p	6. $\neg p$	7. $\neg(p \vee \neg p)$	

Lemma 8.6 If $p \in P_n$ then for each $[\vee, \neg]_n$-formula $\varphi \in EN_n$:

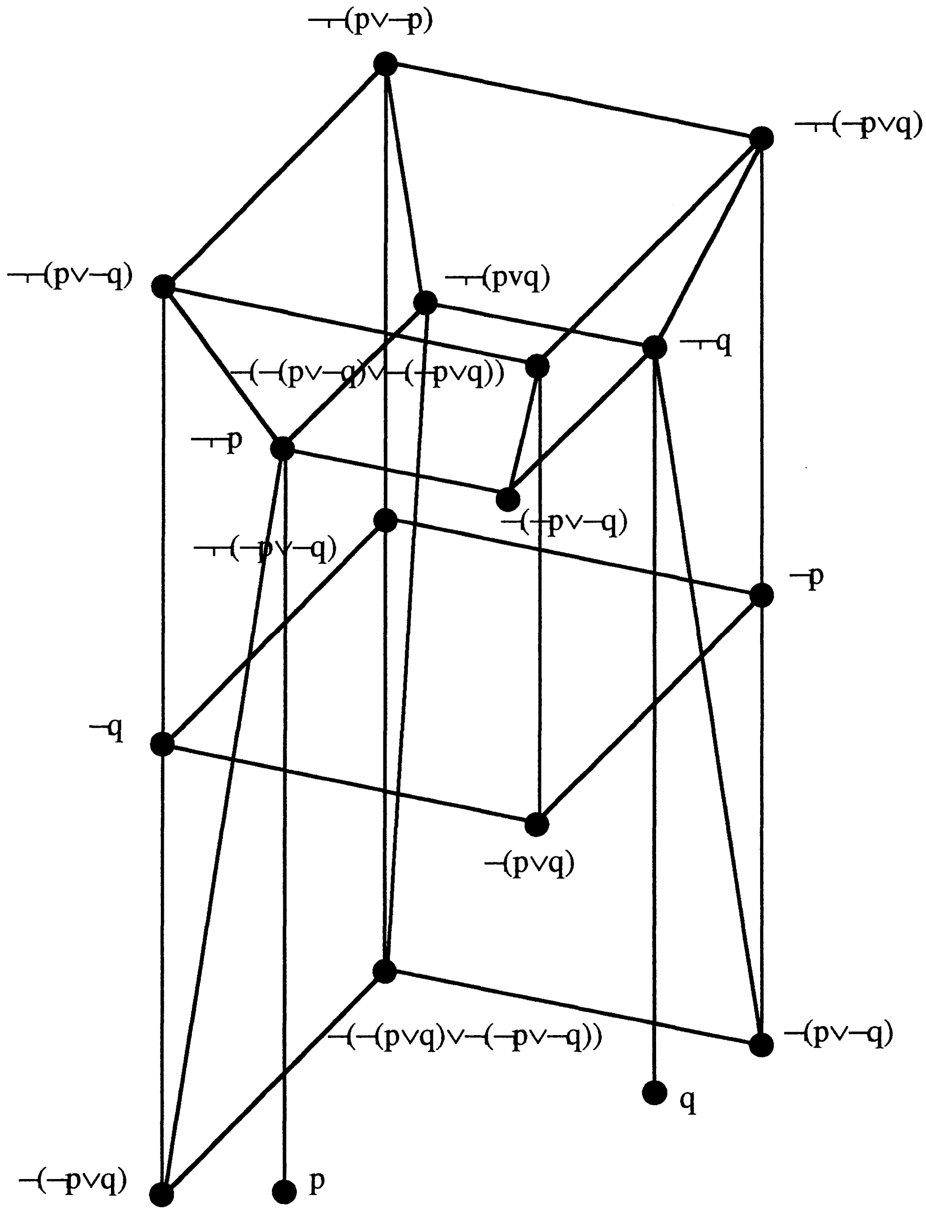
$$p \vdash \varphi \quad \Leftrightarrow \quad p = \varphi \text{ or } \neg\neg p \vdash \varphi$$

Pf. If φ atomic, then trivially $p = \varphi$. So let $\varphi = \neg\psi$. Then:

$$p \vdash \neg\psi \Rightarrow \neg\neg\psi \vdash \neg p \Rightarrow \neg\neg p \vdash \neg\psi \Rightarrow \neg\neg p \vdash \varphi$$ □

By lemma 8.6 the atomic p in the exact model of $[\vee, \neg]_n$ is a direct predecessor of $\neg\neg p$.

Example 8.7
The exact model of $[\vee, \neg, p, q]$, EN_2 :



There are 385 down sets in this exact model, so the fragment $[\vee, \neg]_2$ has 385 equivalence classes.

The example above shows that for $n \geq 2$, $[\vee, \neg]_n$ does not have an exact Kripke model. Equipped with the valuation $\omega(p) = \{ \psi \in EN_2 \mid \psi \vdash p \}$ in the model EN_2 the node corresponding to q would force $\neg p$ for example.

Let $\mathfrak{E}_n = \langle EN_n, \vdash \rangle$ be the exact model of $[\vee, \neg, P_n]$. From the structure of the exact model (and that of the diagram of the corresponding classical fragment) the minimal elements of EN_n are known. The classical fragment $[\vee, \neg]_n$ is a Boolean algebra which has as its atoms the formulas $\varphi_Q = \bigwedge Q \wedge \bigwedge\{\neg p \mid p \in P_n \backslash Q\}$, where $Q \subseteq P_n$.

Fact 8.8 The minimal elements of $\mathfrak{E}_n$ are the atomic formulas in P_n and (in case $n > 1$) the formulas of the form $\neg\neg\varphi_Q$.

This fact is a consequence of lemma 8.6. Recall that for these atoms φ_Q of the Boolean algebra of $[\vee, \neg]_n$ and for any $\psi \in [\vee, \neg]_n$ in classical propositional logic it is true that:

$$\varphi_Q \nvdash_c \psi \Leftrightarrow \varphi_Q \vdash_c \neg\psi \Leftrightarrow \varphi_Q \vdash \neg\psi.$$

The φ_Q above are the irreducible elements in the Boolean algebra and act like an exact model (for the classical fragment).

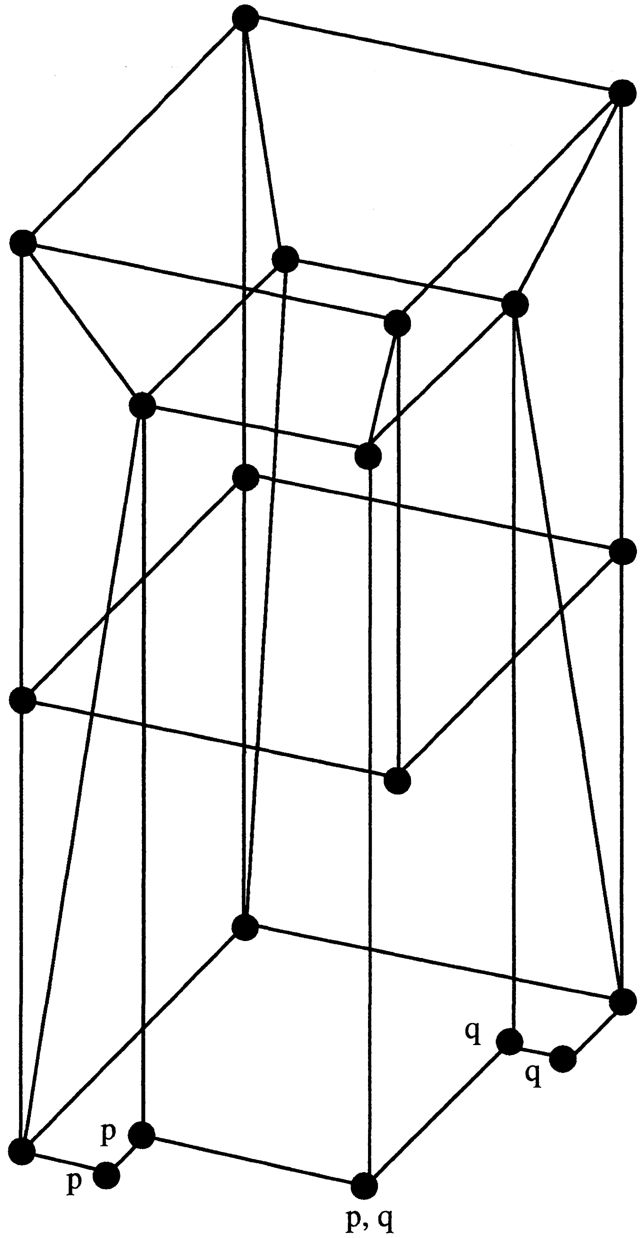
Theorem 8.9 Let $\mathfrak{E}_n = \langle EN_n, \vdash \rangle$ be the exact model of $[\vee, \neg, P_n]$, n > 1.
Extend $\mathfrak{E}_n$ to $\mathfrak{K}_n = \langle \mathbf{K}_n, \omega \rangle$ by adding, for every nonempty $Q \subseteq P_n$, elements k_Q such that:

(i) k_Q forces exactly the atoms in Q;
(ii) for all $\chi \in \mathbf{K}_n$:
$k_Q < \chi \quad \Leftrightarrow \chi \in EN_n$ and $\varphi_Q \vdash \chi$ (φ_Q as defined above).
(iii) for $p \in P_n$ and $\varphi \in EN_n$: $k_\varphi \Vdash p \Leftrightarrow \varphi \vdash p$

Then $\mathfrak{K}_n$ is the Kripke completion of $\mathfrak{E}_n$.

Pf. Note that for $n > 1$ and every nonempty $Q \subseteq P_n$, for some $p \in P_n$ k_Q forces either $p \wedge q$ or $p \wedge \neg q$, for every $q \in P_n$.
Suppose for some nonempty $Q \subseteq P_n$ that $p \in Q$ and (after application of γ) k_Q reduces to a terminal node k_ψ of EN_n in the end extension of $\langle EN_n, \vdash \rangle$.
If $\psi \vdash p$, then by theorem 8.2 $\psi \equiv p$. From $\gamma(k_Q) = \gamma(p)$ and $k_Q \Vdash \neg q$ one could (using theorem 5.3.7) infer that $p \vdash \neg q$, which is not the case.
On the other hand, if $\psi \nvdash p$ then $\gamma(k_Q) = \gamma(k_\psi)$ would imply $\psi \vdash p$, a contradiction.
Hence for $Q \neq \varnothing$ k_Q will not be identified with an element of EN_n by reduction γ.
As $\varphi_\varnothing \equiv \bigvee P_n$ is in EN_n, it is not necessary to add $k_\varnothing$. □

Example 8.10
The Kripke completion of $\mathfrak{E}_2 = <EN_2, \vdash>$ the exact model of $[\vee, \neg, p, q]$:



Note that in case of $[\vee, \neg, p]$ the Kripke completion is isomorphic to the exact model itself.

9.1 The $[\wedge, \vee, \neg\neg]$-fragments

The finite $[\wedge, \vee, \neg\neg]$-fragments have exact models which are not exact Kripke models. In this section we will prove that their Kripke completions can be replaced by slightly smaller models, equivalent with the Kripke completions for $[\wedge, \vee, \neg\neg]$-formulas.

Lemma 9.1.1 (Normal form of $[\wedge, \vee, \neg\neg]$) For every $\varphi \in [\wedge, \vee, \neg\neg]_n$ there is a finite set of formulas ψ_i such that
(i) $\psi_i = \Lambda Q \wedge \neg\neg\chi_i$ for Q a set of atoms, $\chi_i \in [\wedge, \vee]$
(ii) $\varphi \equiv \mathrm{V}\psi_i$

Pf. By formula induction. For the $\wedge$-case observe that:

$$((\Lambda Q \wedge \neg\neg\alpha) \vee (\Lambda R \wedge \neg\neg\beta)) \wedge ((\Lambda S \wedge \neg\neg\gamma) \vee (\Lambda T \wedge \neg\neg\delta)) \equiv$$
$$((\Lambda Q \wedge \neg\neg\alpha) \wedge (\Lambda S \wedge \neg\neg\gamma)) \vee ((\Lambda Q \wedge \neg\neg\alpha) \wedge (\Lambda T \wedge \neg\neg\delta)) \vee$$
$$((\Lambda R \wedge \neg\neg\beta) \wedge (\Lambda S \wedge \neg\neg\gamma)) \vee ((\Lambda R \wedge \neg\neg\beta) \wedge (\Lambda T \wedge \neg\neg\delta)) \equiv$$
$$(\Lambda(Q \cup S) \wedge \neg\neg(\alpha \wedge \gamma)) \vee (\Lambda(Q \cup T) \wedge \neg\neg(\alpha \wedge \delta)) \vee$$
$$(\Lambda(R \cup S) \wedge \neg\neg(\beta \wedge \gamma)) \vee (\Lambda(R \cup T) \wedge \neg\neg(\beta \wedge \delta))$$

and the last formula is of the required form. □

Theorem 9.1.2 The fragment $[\wedge, \vee, \neg\neg, P_n]$ has an exact model: the set ED_n of formulas of the form $\Lambda Q \wedge \neg\neg\varphi$ where $Q \subseteq P_n$ and $\varphi \in [\wedge, \vee]_n$.

Pf. Apart from ΛP_n, every formula of $[\wedge, \vee, \neg\neg, P_n]$ is a disjunction of formulas of the set ED_n by lemma 9.1.1.
Let ΛP_n correspond to the empty set, then every formula φ in $[\wedge, \vee, \neg\neg, P_n]$ corresponds to a down set $\omega(\varphi)$ of ED_n (in the order of $\vdash$) and vice versa via

$$\omega(\varphi) = \{\psi \in ED_n \mid \psi \vdash \varphi\}$$

Clearly this ω is the required isomorphism. □

The formulas in ED_n of the form ΛQ play an important role in the construction of the Kripke completion of the exact model of $[\wedge, \vee, \neg\neg, P_n]$.

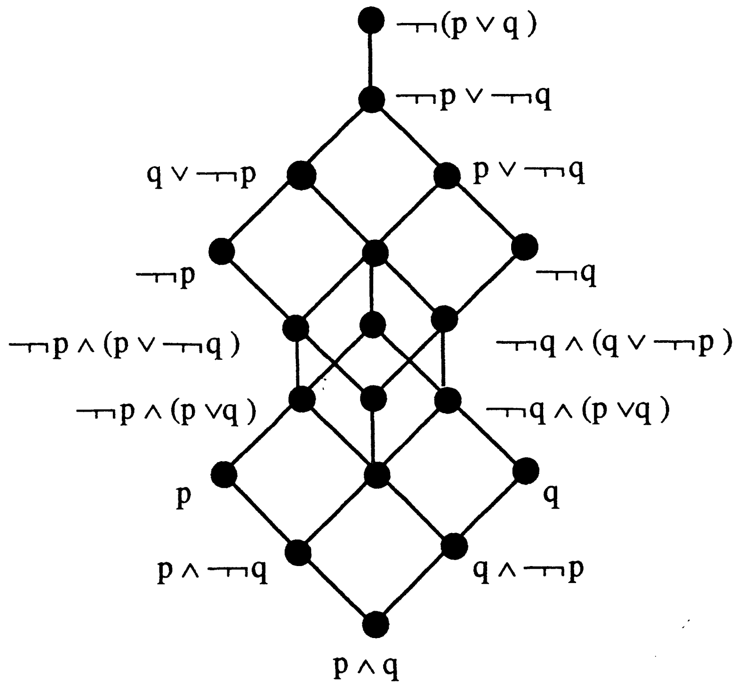
Fact 9.1.3 The minima of the exact model ED_n of $[\wedge, \vee, \neg\neg, P_n]$ are formulas of the form $\Lambda(P_n \backslash \{p\}) \wedge \neg\neg p$ where $p \in P_n$.

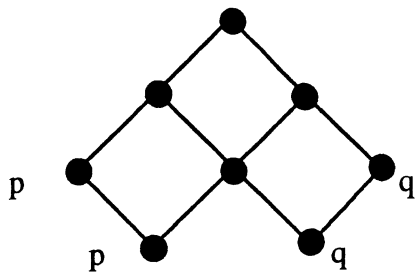
From fact 9.1.3 it is clear that in the Kripke completion of the exact model of $[\wedge, \vee, \neg\neg, P_n]$ none of the k_Q's will reduce to a minimal element in the exact model. Moreover, as all minima force at least all $\neg\neg p$ where $p \in P_n$, the node $k_\varnothing$ will be an isolated element as described in lemma 5.3.8.

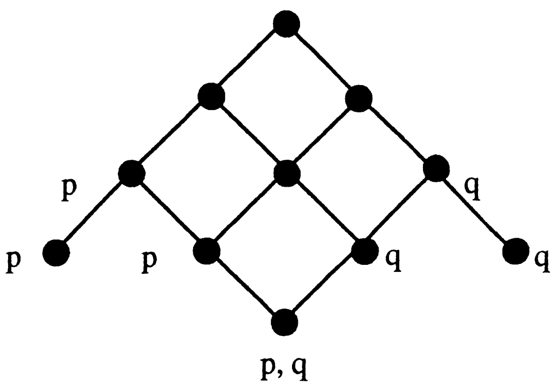
Theorem 9.1.4 If $<ED_n, \vdash>$ is the exact model of $[\wedge, \vee, \neg\neg, P_n]$ as defined in theorem 9.1.2, then for $[\wedge, \vee, \neg\neg, P_n]$-formulas its Kripke completion is equivalent to the end extension of $<ED_n, \vdash>$ without the node $k_\varnothing$.

Pf. By the remark above and lemma 5.3.8. □

Example 9.1.5
The diagram of [$\wedge, \vee, \neg\neg$, p, q], its exact model and the Kripke completion of its exact model (without $k_{\varnothing}$):







9.2 The $[\wedge, \neg\neg]$-fragments

In this section we will show that each $[\wedge, \neg\neg]_n$ fragment has as a simple exact hull the fragment $[\wedge, \neg\neg, \top]_n$ that is: the fragment $[\wedge, \neg\neg]_n$ where there is a top element $\top$ added.

Lemma 9.2.1 (Normal form for $[\wedge, \neg\neg]$) Each formula $\varphi \in [\wedge, \neg\neg]_n$ is equivalent with a formula of the form:

$$\Lambda A \wedge \neg\neg \Lambda B$$

where A , B are sets of atoms of the fragment such that $A \cap B = \varnothing$ and $A \cup B \neq \varnothing$.

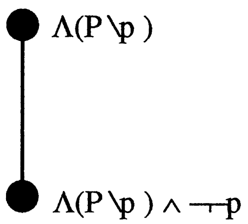
Pf. Trivial. □

Corollary 9.2.2 The number of equivalence classes in $[\wedge, \neg\neg]_n$ is 3^n-1.

Pf. It is easily verified that if $\Lambda A \wedge \neg\neg \Lambda B$ and $\Lambda C \wedge \neg\neg D$ are normal forms then they are equivalent only if $A = C$ and $B = D$.
For every formula $\Lambda A \wedge \neg\neg \Lambda B$ and every atom p there are three possibilities: $p \in A$, $p \in B$ or $p \notin A \cup B$. Note that this is in fact a three-valued valuation (compare H_3 in section 3.23). Every valuation of the n atoms will correspond to a formula in normal form in the same way, but for $\Lambda\varnothing$, the conjunction of an empty set of formulas. Hence the number of equivalence classes in $[\wedge, \neg\neg]_n$ is 3^n-1. □

The proof of corollary 9.2.2 in fact already points out how, by adding a top $\top$ (corresponding to $\Lambda\varnothing$) the diagram of the $[\wedge, \neg\neg]_n$ fragment becomes isomorphic to the lattice of down sets in the structure of n copies of **2**.

Theorem 9.2.3 The fragment $[\wedge, \neg\neg, \top]_n$ has an exact model, $\pounds_n$, the n copies of **2** such that:



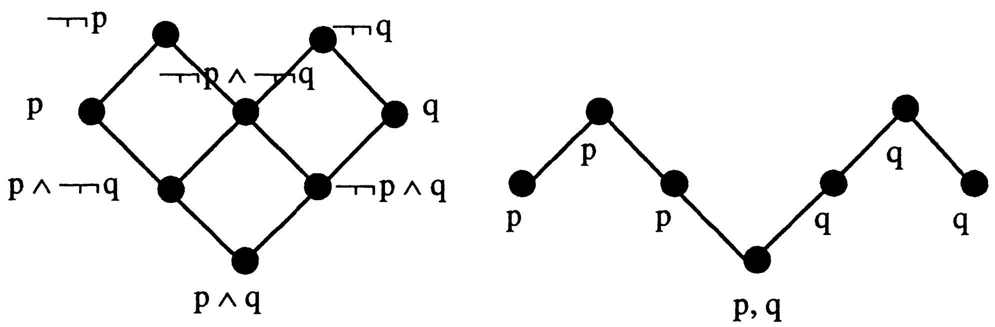
where $p \in P_n$.

Pf. We prove that every down set in $\pounds_n$ corresponds to a normal form and vice versa. As in the proof of corollary 9.2.2 for each normal form φ and each atom p there is a copy of **2** with exactly three possibilities (either $\varphi \vdash p$, $\varphi \vdash \neg\neg p$ or $\varphi \nvdash \neg\neg p$), corresponding with φ beeing forced in the top, at the bottom or nowhere in that copy. Which gives an exact correspondence between formulas of $[\wedge, \neg\neg, \top]_n$ and down sets in the specified structure provided that we have a formula for the whole set. □

Theorem 9.2.4 If $\mathcal{E}_n = < E_n, \vdash >$ is the exact model of $[\wedge, \neg\neg, \mathrm{T}, P_n]$ and $\mathcal{K}_n = < W_n, \vdash >$ is an extension of $\mathcal{E}_n$ where $W_n = E_n \cup \{\Lambda P_n \backslash Q \mid |Q| \leq 1\}$, then $\mathcal{K}_n$ is equivalent to the Kripke completion of $\mathcal{E}$ for $[\wedge, \neg\neg, \mathrm{T}, P_n]$-formulas

Pf. Clearly $\mathcal{K}_n$ is a part of the end extension of $\mathcal{E}_n$ and if $Q = P_n$ or $Q = P_n \backslash \{p\}$ then no element of $\mathcal{E}_n$ will reduce to k_Q in the Kripke completion.
If $Q = P_n \backslash \{p, q\}$ for some $p \neq q$ then φ_Q does not imply any of the formulas in E_n. For assume that $\varphi_Q \vdash \Lambda P_n \backslash \{p\}$ then $\varphi_Q \vdash q$ and also $\varphi_Q \vdash \neg q$, but of course φ_Q is consistent.
By lemma 5.3.8 the Kripke completion without such k_Q's is equivalent to the Kripke completion itself, as far as the $[\wedge, \neg\neg, \mathrm{T}, P_n]$-formulas are concerned. □

Example 9.2.5 The diagram of $[\wedge, \neg\neg, p, q]$ and a Kripke completion for $[\wedge, \neg\neg, \mathrm{T}, p, q]$



9.3 The [∨, ¬¬]-fragments

The diagram of a $[\vee, \neg\neg]_n$ fragment is almost a (distributive) lattice, with the bottom element missing. In this section we will demonstrate that the extension of the fragment with a bottom element $\bot$, to $[\vee, \neg\neg, \bot]_n$, is an exact Kripke hull for $[\vee, \neg\neg]_n$.

Lemma 9.3.1 (Normal form for $[\vee, \neg\neg]$) For all $\varphi \in [\vee, \neg\neg, P_n]$ there is an equivalent formula of the form:

$$\bigvee A \vee \bigvee\{\neg\neg\bigvee B \mid B \in \mathcal{B}\}$$

where $A \subseteq P_n$ $\mathcal{B} \subseteq \wp(P_n)$ and $A \cup \bigcup\mathcal{B} \neq \varnothing$.

Pf. By formula induction on φ. □

Corollary 9.3.2 Every formula in $[\vee, \neg\neg]_n$ is a finite disjunction of atoms and formulas of the form $\neg\neg\varphi$.

Definition 9.3.3 Let $\mathbf{D}_n$ be the diagram of $[\vee]_n$, then $\mathbf{D}_n^{\neg\neg}$ is the set $\{\neg\neg\varphi \mid \varphi \in \mathbf{D}_n\}$ ordered by $\vdash$.

Fact 9.3.4

a. $\mathbf{D}_n^{\neg\neg}$ is isomorphic to $\mathbf{D}_n$.
b. $\mathbf{D}_n^{\neg\neg}$ is isomorphic to the p.o. set of non empty subsets of P_n (or to the the n-dimensional hypercube minus bottom).
c. the minimal elements in $\mathbf{D}_n^{\neg\neg}$ are the formulas of the form $\neg\neg p$ for atomic p.

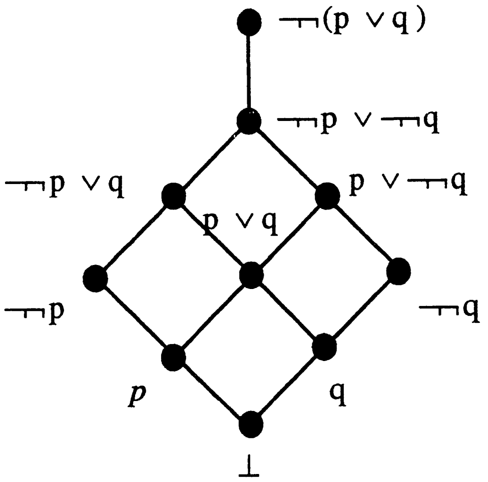
Theorem 9.3.5 The exact Kripke model of $[\vee, \neg\neg, \bot]_n$ is a copy of $\mathbf{D}_n^{\neg\neg}$ where the atomic formulas are added (such that $p <_1 \chi \Leftrightarrow \chi \equiv \neg\neg p$).

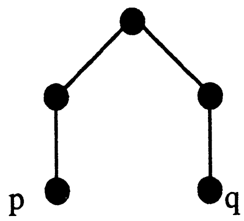
Pf. Let $\mathcal{K}$ be the supposed exact Kripke model of $[\vee, \neg\neg, \bot]_n$ as specified.
From corollary 9.3.2 infer that $\mathcal{K}$ is the exact model of $[\vee, \neg\neg, \bot]_n$.
Inspection of the structure of $\mathcal{K}$, using fact 9.3.4 shows $\mathcal{K}$ is exact. □

Corollary 9.3.6 The number of equivalence classes in $[\vee, \neg\neg, \bot]_n$ is $\Sigma_{k=0}^{n}\binom{n}{k}(\mathsf{D}(k)+1)$, where $\mathsf{D}(k)$ is the k-th Dedekind number (see section 6.1).

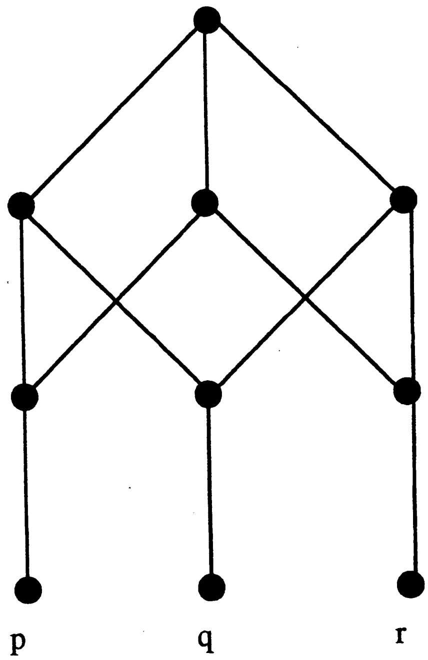
Pf. The number of equivalence classes in $[\vee, \neg\neg, \bot]_n$ can be calculated from the structure of the exact model (theorem 9.3.5). But it is also possible to use the normal form for $[\vee, \neg\neg, \bot]_n$ formulas.
For this calculation, we need a more economical version of the normal form of lemma 9.3.1. First observe that we can take all the B in this normal form to be disjunct from the set A. Moreover, one can take $\mathcal{B}$ to be either $\{\varnothing\}$ or a so called *Spernersystem* (an "independent" set of subsets [S28]): $\varnothing \neq \mathcal{B} \neq \{\varnothing\}$ and for all $B, B' \in \mathcal{B}$ it is true that $B \subseteq B'$ iff $B = B'$. In case $\varphi = \bot$, $\mathcal{B} = \{\varnothing\}$ and $A = \varnothing$.
The proof of the observations above is left to the reader.
There are $\binom{n}{k}$ different sets A with cardinality $n - k$.
The number of different Sperner systems $\mathcal{B}$ for a subset of k elements, can easily be shown equal to the number of equivalence classes in $[\wedge, \vee]_k$, hence $\mathsf{D}(k)$. Including the case where $\mathcal{B} = \{\varnothing\}$ we have for each set A with cardinality $n - k$ exactly $\mathsf{D}(k)+1$ different $\mathcal{B}$'s, which proves the number of equivalence classes in $[\vee, \neg\neg, \bot]_n$ to be $\Sigma_{k=0}^{n}\binom{n}{k}(\mathsf{D}(k)+1)$ □

Example 9.3.7
The diagram of $[\vee, \neg\neg, \bot]_2$ and its exact model.





Example 9.3.8
The exact model of $[\vee, \neg\neg, \bot]_3$



10.1 The $[\wedge, \rightarrow]$-fragments

The diagram of $[\wedge, \rightarrow]_n$ will be proved to be a finite Heyting algebra. As Heyting algebras are a special brand of distributive lattices, $[\wedge, \rightarrow]_n$ will have an exact model, which turns out to be an exact Kripke model.
In [D66] Diego proved that the diagram of $[\rightarrow]_n$ is finite for each finite *n*. His algebraic approach can also be used to prove the diagram of $[\wedge, \rightarrow]_n$ to be finite and explain the structure of its exact Kripke models.
Alternative proofs can be found in [B75a] and [JHR91].

Definition 10.1.1 An *implicational semilattice* is a semilattice L (an ordering with meet, see [DP90] or [C77]) where an operation $\rightarrow$ is defined such that:

$$x \wedge y \leq z \quad \Leftrightarrow \quad x \leq y \rightarrow z$$

In case L is a lattice and $\rightarrow$ is defined as above, L is called an *implicational lattice.*
A *Heyting algebra* is an implicational lattice with a bottom element (also known as *pseudo-boolean algebra* as in [RS68]).

Filters in an implicational semilattice (lattice, Heyting algebra) are defined as usual (as in [DP90] for example).

Fact 10.1.2
a. The diagram of the fragment $[\wedge, \rightarrow]_n$ is the free implicational semilattice generated by n elements.
b. If L is an implicational lattice generated by m elements, $m \leq n$, then L is a homomorphic image of $[\wedge, \rightarrow]_n$.

Fact 10.1.3 In an implicational semilattice L:
a. $\top = x \rightarrow x$ is the top element
b. $\uparrow x$ is a filter for every $x \in L$
c. For F a filter in L, x, $x \rightarrow y \in F$ implies $y \in F$
d. For F a filter in L, $\equiv_F$ defined as

$$x \equiv_F y \quad \Leftrightarrow \quad x \rightarrow y \in F \text{ and } y \rightarrow x \in F$$

is an equivalence relation on L.

Definition 10.1.4 If F is a filter in implicational semilattice L then L/F is the quotient of L over the equivalence relation $\equiv_F$ defined in fact 10.2.d.
L/F consists of equivalence classes $|x|_F$ for the equivalence relation $\equiv_F$.
The function h_F: $L \rightarrow L/F$ such that $h(x) = |x|_F$ is called the *canonical homomorphism* for F.

Definition 10.1.5 A filter F in an implicational semilattice L is called *maximal for an element* $f \in L$ if $f \notin F$ and for all filters G such that F is properly contained in F, $f \in G$.
If F is maximal for some f, then F is called *e-maximal*.

Lemma 10.1.6 If L an implicational semilattice, $x, y \in L$ and $x \neq y$ then there is an e-maximal filter F such that either $x \in F$ and $y \notin F$ or $x \notin F$ and $y \in F$.

Pf. Assume not $x \leq y$. Then (using Zorn's Lemma) $\uparrow x$ can be extended to a filter maximal for y. □

Corollary 10.1.7 Every element in an implicational semilattice is uniquely characterized by the set of e-maximal filters it is contained in.

Definition 10.1.8 An implicational semilattice L has a *last-but-one* element m if m is the unique element such that $m <_1 \top$. If L is an implicational semilattice with last-but-one element m then L^- will denote $L \setminus m$ (with inherited order).

Fact 10.1.9 If L is an implicational semilattice with last-but-one element m:

a. if $x < m$ then $m \to x = x$;
b. $h_m(x) = m \to x$ is a surjective homomorphism from L to L^-;
c. L^- is an implicational semilattice.

Lemma 10.1.10 If F a filter in implicational semilattice L, F maximal for f, then L/F is an implicational semilattice with last-but-one element.

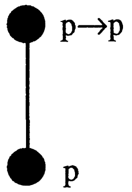
Pf. Let h_F be the canonical homomorphism for F then we prove $h_F(f)$ to be the last-but-one in L/F.
The top of L/F is $h_F(\top)$. As $f \notin F$ (and hence $\top \to f \notin F$) we know $h_F(f) < h_F(\top)$.
Assume $h_F(x) < h_F(\top)$. Then $\{y \in L \mid h_F(y) \in \uparrow h_F(x)\}$ is a filter in L with F as a proper subset. As F is maximal for f we have $h_F(f) \in \uparrow h_F(x)$.
So $h_F(x) \leq h_F(f)$ and $h_F(f)$ is last-but-one in L/F. □

Lemma 10.1.11 If L an implicational semilattice with last-but-one element m and generated by n elements (n finite) then L^- is generated by strictly less then n elements.

Pf. If L is generated by the set of elements P, we will prove $m \in P$.
On the other hand L^- is generated by $P \setminus m$, as is not difficult to see using fact 10.9.a.
Assume $m \notin P$. By induction over the generation of terms x from P we prove:
(IH) $\quad x < m$ or $x = \top$.
<u>$x \in P$:</u> trivial.
<u>$x = y \wedge z$:</u> If $y = \top$ then $x = z$ and so by IH $x < m$ or $x = \top$. (The same for $z = \top$).
Assume $y < m$ and $z < m$. Then of course $y \wedge z < m$.
<u>$x = y \to z$:</u> If $y = \top$ then $x = z$ and IH is also true for x.
Assume $y < m$ and $m \leq y \to z$. Then $y = m \wedge y \leq z$ so $x = y \to z = \top$. □

Theorem 10.1.12 The diagram of $[\wedge, \rightarrow]_n$ is finite.

Pf. The proof is by induction over n. We use L_n as a shorthand for $[\wedge, \rightarrow, P_n]$.
<u>$n = 1$:</u> $L_1 = [\wedge, \rightarrow, p]$ is clearly finite:



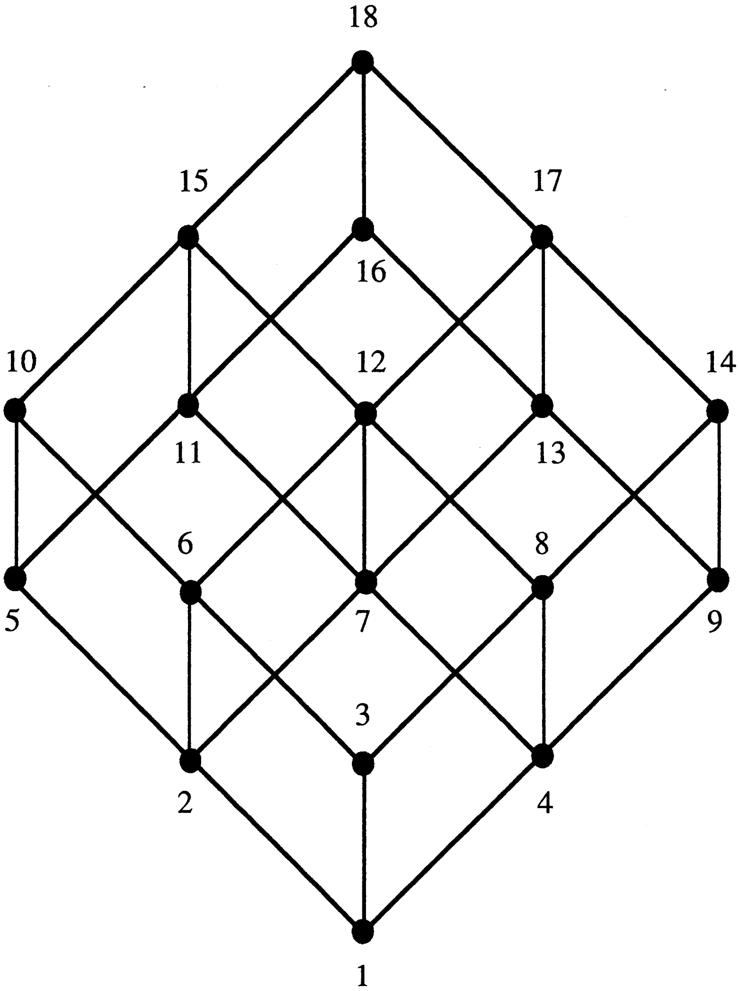
The diagram of $[\wedge, \rightarrow]_1$.

<u>$n = k + 1$:</u> We prove that L_{k+1} has a finite number of filters maximal for some element. From corollary 10.7 we then infer that L_{k+1} is finite.
If F an e-maximal filter in L_{k+1}, L_{k+1}/F has a last-but-one element (lemma 10.1.10).
So $(L_{k+1}/F)^-$ is a homomorphic image of L_k (lemma 10.1.11 and fact 10.1.2.a).
As L_k is assumed to be finite, there are -up to isomorphism- finitely many homomorphic images of L_k.
There is only one (maximal) element difference between L_{k+1}/F and $(L_{k+1}/F)^-$, so there are finitely many different L_{k+1}/F (again up to isomorphism).
If $L_{k+1}/F = L_{k+1}/G$, the canonical mapping h_G is just one of the finitely many ways to map the atoms of P_{k+1} (the set of generators of L_{k+1}/F) in L_{k+1}/F such that the isomorphism holds.
Hence there are only finitely many different e-maximal filters in L_{k+1}/F. □

Corollary 10.1.13 The diagram of $[\wedge, \rightarrow]_n$ is a finite Heyting algebra and hence $[\wedge, \rightarrow]_n$ has an exact model.

Pf. As L_n, the diagram of $[\wedge, \rightarrow]_n$ is finite, each filter F in L_n is principal (i.e. there is a x such that $F = \uparrow x$). The join of φ and ψ in L can now be defined as the equivalence class χ such that $\uparrow\chi = \uparrow\varphi \cap \uparrow\psi$. (Note that $\varphi \cup \psi \vdash \chi \Leftrightarrow \varphi \vdash \chi$ and $\psi \vdash \chi$ for φ, ψ and χ in L_n as required.) So L_n is an implicational lattice (hence distributive).
If P is the set of atoms in $[\wedge, \rightarrow]_n$ then clearly $\wedge P$ is a bottom element, hence L_n is a Heyting algebra. According to definition 4.3 $[\wedge, \rightarrow]_n$ has an exact model. □

Example 10.1.14
The diagram of $[\wedge, \rightarrow]_2$:



1	$p \wedge q$	7	$((p \rightarrow q) \rightarrow q) \wedge ((q \rightarrow p) \rightarrow p)$	13	$(q \rightarrow p) \rightarrow p$
2	p	8	$((q \rightarrow p) \rightarrow p) \rightarrow q$	14	$p \rightarrow q$
3	$(p \rightarrow q) \wedge (q \rightarrow p)$	9	$(q \rightarrow p) \rightarrow q$	15	$((q \rightarrow p) \rightarrow q) \rightarrow q$
4	q	10	$q \rightarrow p$	16	$((p \rightarrow q) \wedge (q \rightarrow p)) \rightarrow p$
5	$(p \rightarrow q) \rightarrow p$	11	$(p \rightarrow q) \rightarrow q$	17	$((p \rightarrow q) \rightarrow p) \rightarrow p$
6	$((p \rightarrow q) \rightarrow q) \rightarrow p$	12	$(((q \rightarrow p) \rightarrow q) \rightarrow q) \wedge (((p \rightarrow q) \rightarrow p) \rightarrow p)$	18	$p \rightarrow p$

In this example the equivalence classes 2, 3, 4, 5 and 9 are the irreducible elements and the suborder of these elements is the exact model of $[\wedge, \rightarrow]_2$.

Notation 10.1.15 The join of two elements φ and ψ in the diagram of $[\wedge, \rightarrow]_n$ will be written as $\varphi \oplus \psi$ (a pseudo-disjunction).

If in the sequel we call an element of $[\wedge, \rightarrow]_n$ irreducible, we mean the corresponding equivalence class to be an $\oplus$-irreducible element of the diagram (the Heyting algebra). According to fact 3.21.d, all formulas of $[\wedge, \rightarrow]_n$ are $\vee$-irreducible.

Lemma 10.1.16 In the diagram of $[\wedge, \rightarrow]_n$ the following are equivalent:

(i) $\uparrow\varphi$ is e-maximal;
(ii) φ is irreducible;
(iii) φ has a unique direct predecessor ψ ($\psi <_1 \varphi$ in the order of $\vdash$).

Pf. (i)$\Rightarrow$(ii): Let $\uparrow\varphi$ be maximal for ψ and $\varphi \vdash \chi \oplus \theta$. If we assume that $\varphi \nvdash \chi$ and $\varphi \nvdash \theta$ then $\psi \in \uparrow\chi$ and $\psi \in \uparrow\theta$. Then we would have $\chi \oplus \theta \vdash \psi$ contradicting $\varphi \nvdash \psi$ ($\psi \notin \uparrow\varphi$). Hence either $\varphi \vdash \chi$ or $\varphi \vdash \theta$.
(ii)$\Rightarrow$(iii): Let $\psi := \oplus\{\chi \in [\wedge, \rightarrow]_n \mid \chi \vdash \varphi$ and $\varphi \nvdash \chi\}$ (as $[\wedge, \rightarrow]_n$ is finite this is a sound definition).
As φ is irreducible clearly $\psi <_1 \varphi$. If $\chi <_1 \varphi$ then $\chi \vdash \psi$ hence $\chi \equiv \psi$.
(iii)$\Rightarrow$(i): If ψ the unique predecessor of φ then $\uparrow\varphi$ is maximal for ψ. □

Lemma 10.1.17 For $\varphi, \psi \rightarrow \chi \in [\wedge, \rightarrow]_n$:
$\varphi \vdash \psi \rightarrow \chi$ iff for all irreducible $\xi \in [\wedge, \rightarrow]_n$ such that $\xi \vdash \varphi$ if $\xi \vdash \psi$ then $\xi \vdash \chi$.

Pf. $\Rightarrow$)Trivial.
$\Leftarrow$) Let ω be the isomorphism mapping equivalence classes of $[\wedge, \rightarrow]_n$ on down sets of irreducibles (in the exact model). By assumption $\omega(\varphi \wedge \psi) \subseteq \omega(\chi)$ and hence $\varphi \wedge \psi \vdash \chi$. This last conclusion proves $\varphi \vdash \psi \rightarrow \chi$. □

Theorem 10.1.18 The exact model of $[\wedge, \rightarrow]_n$ is an exact Kripke model.

Pf. By formula induction we prove that, if $\psi \in [\wedge, \rightarrow]_n$ then for all irreducible $\varphi \in [\wedge, \rightarrow]_n$:
(IH) $\quad \varphi \vdash \psi \quad \Leftrightarrow \quad k_\varphi \Vdash \psi$
ψ atomic: By definition.
$\psi = \chi \wedge \theta$: $\varphi \vdash \chi \wedge \theta \Leftrightarrow \varphi \vdash \chi$ and $\varphi \vdash \theta \overset{(IH)}{\Leftrightarrow} k_\varphi \Vdash \chi$ and $k_\varphi \Vdash \theta \Leftrightarrow k_\varphi \Vdash \chi \wedge \theta$
$\psi = \chi \rightarrow \theta$: Assume $\varphi \vdash \chi \rightarrow \theta$ and let $k_\xi \leq k_\varphi$. As $\xi \vdash \varphi$ from $k_\xi \Vdash \chi$ infer (using IH) that $\xi \vdash \theta$ and (again by IH) that $k_\xi \Vdash \theta$. Hence $k_\varphi \Vdash \chi \rightarrow \theta$.
For the proof in the other direction, assume $k_\varphi \Vdash \chi \rightarrow \theta$. If ξ irreducible in $[\wedge, \rightarrow]_n$ and $\xi \vdash \varphi$ then by using IH one can show that $\xi \vdash \chi$ implies $\xi \vdash \theta$. Using lemma 10.1.16 this proves $\varphi \vdash \chi \rightarrow \theta$. □

To reveal the structure of the exact Kripke models of fragments $[\wedge, \rightarrow]_n$ recall the definition of *atom*$(\varphi) := \{p \in P_n \mid \varphi \vdash p\}$ (definition 3.10). We also will need some lemmas.

Lemma 10.1.19 Let φ be irreducible in the diagram of $[\wedge, \rightarrow]_n$, ψ its unique direct predecessor (lemma 10.1.16) and $\chi \in [\wedge, \rightarrow]_n$ such that

i. $\psi \vdash \chi$
ii. for all $\theta \not\equiv \chi$ if $\theta \vdash \chi$ then $\theta \vdash \psi$.

Then we have: $\quad$ *atom*$(\varphi) =$ *atom*$(\chi) \quad \Rightarrow \quad \varphi \equiv \chi$

Pf. With formula induction we prove for all $\theta \in [\wedge, \rightarrow]_n$
(IH) $\quad \varphi \vdash \theta \Leftrightarrow \chi \vdash \theta$
The lemma then is a simple consequence.
If θ is atomic or a conjunction the proof is trivial.
$\varphi \vdash \sigma \rightarrow \tau$: If $\chi \vdash \sigma$ then using IH also $\varphi \vdash \sigma$. From $\varphi \vdash \tau$, again using IH, infer $\chi \vdash \tau$ and hence $\chi \vdash \sigma \rightarrow \tau$.
So assume $\chi \nvdash \sigma$. As $\chi \wedge \sigma \not\equiv \chi$ use the assumption of the lemma to conclude $\chi \wedge \sigma \vdash \psi$. As $\psi \vdash \varphi$ infer that $\chi \wedge \sigma \vdash \tau$ and hence $\chi \vdash \sigma \rightarrow \tau$.
$\chi \vdash \sigma \rightarrow \tau$: If $\varphi \vdash \sigma$ again the proof of $\chi \vdash \sigma \rightarrow \tau$ is simple.
Assume $\varphi \nvdash \sigma$. As ψ is the unique predecessor of φ, $\varphi \wedge \sigma \vdash \psi$. Again $\psi \vdash \sigma \rightarrow \tau$, so $\varphi \wedge \sigma \vdash \tau$, which proves $\varphi \vdash \sigma \rightarrow \tau$. □

Corollary 10.1.20 If φ irreducible in the diagram of $[\wedge, \rightarrow]_n$ and $\psi \not\equiv \varphi$, $\psi \vdash \varphi$ then $atom(\varphi) \neq atom(\psi)$.

Pf. It is sufficient to prove the corollary for the unique direct predecessor of φ. For this ψ apply lemma 10.1.19 (with ψ in the role of χ). □

Corollary 10.1.21 If ψ the unique direct predecessor of both φ and χ then

$$atom(\varphi) = atom(\chi) \quad \Rightarrow \quad \varphi \equiv \chi$$

For the structure of the exact Kripke model of $[\wedge, \rightarrow]_n$ there are some interesting consequences from lemma 10.1.19 and its corollaries.

Definition 10.1.22 Let $\mathcal{K}$ be an exact Kripke model. For $k_\varphi \in \mathcal{K}$ define $Pred(k_\varphi) := \{k_\psi \in \mathcal{K} \mid k_\psi < k_\varphi\}$

Fact 10.1.23 Let $\mathcal{K}_n$ be the exact Kripke model of $[\wedge, \rightarrow, P_n]$, ω the isomorphism between the diagram of $[\wedge, \rightarrow, P_n]$ and down sets in $\mathcal{K}_n$.

a. In $\mathcal{K}_n$: if $k_\psi < k_\varphi$ then for some atomic p $k_\varphi \nVdash p$ and $k_\psi \Vdash p$

b. If φ irreducible in $[\wedge, \rightarrow]_n$ and ψ its unique direct predecessor, then $\omega(\psi) = Pred(k_\varphi)$

c. For $k_\varphi, k_\psi \in \mathcal{K}_n$:

if $atom(k_\varphi) = atom(k_\psi)$ and $Pred(k_\varphi) = Pred(k_\psi)$ then $k_\varphi = k_\psi$.

Fact 10.1.23.c associates every irreducible element φ in the diagram of $[\wedge, \rightarrow]_n$ with a unique pair $\langle atom(k_\varphi), Pred(k_\varphi)\rangle$ such that $atom(k_\varphi) \subset \bigcap\{atom(k_\psi) \mid k_\psi \in Pred(k_\varphi)\}$. (This is a proper inclusion according to fact 10.23.a.) The following lemma proves the converse also to be true.

Lemma 10.1.24 Let $\mathcal{K}_n$ be the exact Kripke model of $[\wedge, \rightarrow]_n$, S a down set in $\mathcal{K}_n$ and Q a set of atoms in $[\wedge, \rightarrow]_n$ such that $Q \subset \bigcap\{atom(k_\psi) \mid k_\psi \in S\}$.
Then there is an irreducible $\varphi \in [\wedge, \rightarrow]_n$ such that $atom(k_\varphi) = Q$ and $Pred(k_\varphi) = S$.

Pf. As $\mathcal{K}_n$ is an exact model, S corresponds to a formula $\psi \in [\wedge, \rightarrow]_n$.
In the diagram of $[\wedge, \rightarrow]_n$ extend the suborder $\downarrow\psi$ to $(\downarrow\psi)^+$ by adding a new top T'.
Define a homomorphism h between the diagram of $[\wedge, \rightarrow]_n$ and $(\downarrow\psi)^+$ by stipulating:

for atomic p $\quad h(p) = \mathrm{T'}$ $\quad$ if $p \in Q$
$\qquad\qquad\qquad\qquad p \wedge \psi$ otherwise.

(Note that in $(\downarrow\psi)^+$ the meet will correspond to $\wedge$, the join to the pseudo disjunction $\oplus$ in $[\wedge, \rightarrow]_n$, but its 'implication' does not necessarily correspond to $\rightarrow$ but is defined as $\chi \Rightarrow \theta := \oplus\{\xi \in (\downarrow\psi)^+ \mid \xi \wedge \chi \vdash \theta\}$).

Let $ker(h)$ be the kernel of h ($ker(h) := \{\xi \in [\wedge, \rightarrow]_n \mid h(\xi) = \mathrm{T'}\}$). We will prove

(1) $\quad \chi \in ker(h) \Rightarrow \psi \vdash \chi$

(2) $\quad \chi \notin ker(h) \Rightarrow h(\chi) = \chi \wedge \psi$

As $ker(h)$ is a filter in the diagram of $[\wedge, \rightarrow]_n$ which is maximal for ψ (which is not difficult to verify as for all $\theta \notin ker(h)$ we have $\theta \rightarrow \psi \in ker(h)$ and so $\varphi \wedge \theta \vdash \psi$) and the diagram of $[\wedge, \rightarrow]_n$ is finite, there has to be an irreducible $\varphi \in [\wedge, \rightarrow]_n$ such that $\uparrow\varphi = ker(h)$ and ψ is its unique direct predecessor (using lemma 10.1.16). From the definition of h infer $atom(\varphi) = Q$.
To prove (1) and (2) we use formula induction.
In case χ is atomic (1) and (2) are true by definition. The proof in case χ is a conjunctionisstraightforward.

$\chi = \theta{\rightarrow}\xi$: If $\xi \in ker(h)$ then $\psi \vdash \xi$ by induction hypothesis and so $\psi \vdash \theta{\rightarrow}\xi$.
So assume $\xi \notin ker(h)$. Then by IH we have $h(\xi) = \xi\wedge\psi$.
If $\theta \in ker(h)$ then $\theta{\rightarrow}\xi \notin ker(h)$, $\psi \vdash \theta$ and $h(\theta{\rightarrow}\xi) = h(\xi)$.
Hence $h(\theta{\rightarrow}\xi) = \xi\wedge\psi = (\theta{\rightarrow}\xi)\wedge\theta\wedge\psi = (\theta{\rightarrow}\xi)\wedge\psi$.
On the other hand, if we assume $\theta \notin ker(h)$ then $h(\theta) = \theta\wedge\psi$, $h(\xi) = \xi\wedge\psi$.
In that case, if $\theta{\rightarrow}\xi \in ker(h)$ then $h(\theta) \vdash h(\xi)$. So $\theta\wedge\psi \vdash \xi\wedge\psi$ and from $\theta\wedge\psi \vdash \xi$ infer that $\psi \vdash \theta{\rightarrow}\xi$ as required.
If $\xi \notin ker(h)$, $\theta \notin ker(h)$ and $\theta{\rightarrow}\xi \notin ker(h)$ the following derivation can be made:

$$\begin{aligned} h(\theta{\rightarrow}\xi) &= \oplus\{\sigma \in (\downarrow\psi)^+ \mid \sigma\wedge h(\theta) \le h(\xi)\} \text{ and as } \top' \ne h(\theta{\rightarrow}\xi)\text{:} \\ &= \oplus\{\sigma \in \downarrow\psi \mid \sigma\wedge h(\theta) \le h(\xi)\} \\ &= \oplus\{\sigma \in \downarrow\psi \mid \sigma\wedge \theta\wedge\psi \le \xi\wedge\psi\} \\ &= \oplus\{\sigma\wedge\psi \mid \sigma\wedge \theta\wedge\psi \le \xi\wedge\psi\} \\ &= \psi \wedge \oplus\{\sigma \mid \sigma\wedge \theta\wedge\psi \le \xi\wedge\psi\} \\ &= \psi \wedge ((\theta\wedge\psi) \rightarrow (\xi\wedge\psi)) = \psi \wedge (\theta\wedge\psi \rightarrow \xi) = \psi\wedge(\theta{\rightarrow}\xi) \end{aligned}$$ □

Corollary 10.1.25 If k be a minimal element in $\mathcal{K}_n$, $Q = atom(k)$, $\varphi \in [\wedge, \rightarrow, P_n]$ and $\varphi := \bigwedge Q \wedge \bigwedge\{p{\rightarrow}\bigwedge P_n \mid p \in P_n \backslash Q\}$ then $k = k_\varphi$.

Pf. If k minimal and $atom(k) = Q$ then clearly $k \Vdash \varphi$. On the other hand, if $k \Vdash \varphi$ then $atom(k) = Q$ for $atom(k) \subset P_n$ (lemma 10.1.23).
Assume $l < k$. By corollary 10.1.20, there is a $q \in P_n \backslash atom(k)$ such that $l \Vdash q$. From $l \Vdash \varphi$ infer that $l \Vdash \bigwedge P$, contradicting lemma 10.1.23, hence k is minimal. Such a minimal element k with $atom(k) = Q$ is unique according to lemma 10.1.23. □

The next theorem provides us with an algorithm to calculate the exact Kripke model of $[\wedge,\rightarrow]_n$.

Theorem 10.1.26 The frame of the exact Kripke model $\mathcal{K}_n$ of $[\wedge, \rightarrow, P_n]$ can be constructed stepwise as the union of sets $\mathfrak{L}_k$ of pairs $<Q,S>$ such that $Q \subset P_n$ and S either the empty set or a down set in $\mathfrak{L}_{k-1}$, where $\mathfrak{L}_k$ is defined as:

$$\mathfrak{L}_0 = \{ <Q, \emptyset> \mid Q \subset P \}$$
$$\mathfrak{L}_{k+1} = \mathfrak{L}_k \cup \{ <Q, S> \mid S \subseteq \mathfrak{L}_k \wedge \downarrow S = S \wedge Q \subset \bigcap\{R \mid \exists T . <R, T> \in S\}\}$$

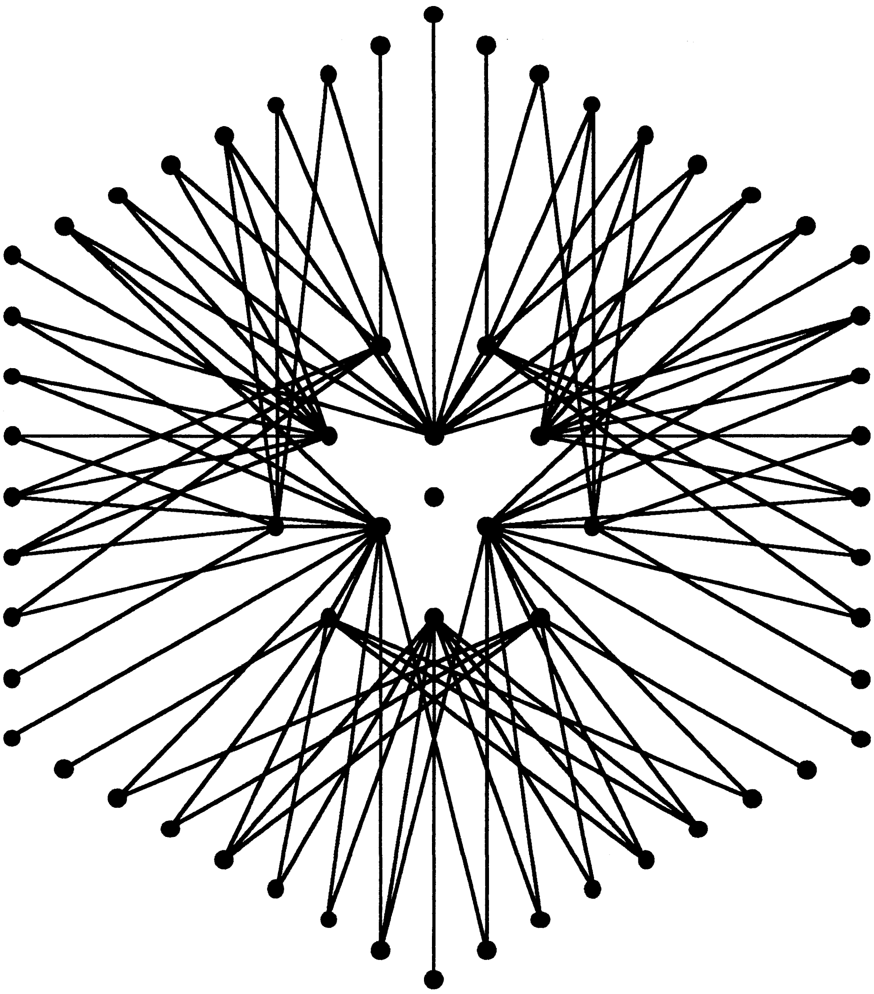
and

$$<Q, S> \le <Q', S'> \text{ iff } <Q, S> \in S' \text{ or } <Q, S> = <Q', S'>$$

On this frame the forcing relation of $\mathcal{K}_n$ is defined as $<Q, S> \Vdash p \Leftrightarrow p \in Q$ for $p \in P_n$.

Pf. Combining fact 10.1.23 and lemma 10.1.24. □

Example 10.1.27
The structure of the exact model of $[\wedge, \to]_3$.



In the centre of this model no atoms are forced. The elements on the first inner triangle force two atoms each, those on the second triangle force one atom each and those on the border do not force any atom at all.

Corollary 10.1.25 gives us formulas for the minimal irreducible elements in $[\wedge, \to]_n$. The characterisation of k_φ for non-minimal elements of $\mathcal{K}_n$ needs some more work.
In the sequel $\mathcal{K}_n = <\mathbf{K}_n, \Vdash>$ will denote the exact Kripke model of $[\wedge, \to, P_n]$ and ω the isomorphism between the diagram of $[\wedge, \to]_n$ and the down sets of $\mathcal{K}_n$.

Definition 10.1.28 For $k_\varphi \in \mathbf{K}_n$ define:

a. $Pred(k_\varphi) := \{k_\psi \mid k_\psi <_1 k_\varphi\}$
b. $N(\varphi)$ is the element in the diagram of $[\wedge, \to]_n$ such that $\omega(N(\varphi)) = \mathbf{K}_n \setminus \uparrow k_\varphi$
c. $oldatom(k_\varphi) := \{p \in P \mid \forall k_\psi \in Pred(k_\varphi).p \in atom(k_\psi)$ and $p \notin atom(k_\varphi)\}$
d. $\Delta oldatom(k_\varphi) := \bigwedge\{p \leftrightarrow q \mid p, q \in oldatom(k_\varphi)\}$

As $\mathcal{K}_n$ is an exact model and $\mathbf{K}_n \setminus \uparrow k_\varphi$ a down set, the formula $N(\varphi)$ exists.
Formulas like $p \leftrightarrow q$ do not belong to $[\wedge, \to]_n$ but are used here as a shorthand for $(p \to q) \wedge (q \to p)$.

Fact 10.1.29 Let k_φ, $k_\chi \in \mathbf{K_n}$, $Pred(k_\varphi) \neq \varnothing$ and $q \in oldatom(k_\varphi)$ then:

a. $\varphi \vdash N(\chi)$ iff $\chi \nvdash \varphi$
b. if k_χ minimal and $p \notin atom(k_\chi)$ then $N(\chi) \equiv \varphi \rightarrow p$
c. $N(\varphi) \equiv \varphi \rightarrow q$
d. for $k_\psi \in Pred(k_\varphi)$: $k_\varphi \Vdash N(\psi) \rightarrow q$

All these facts are easy to check.

Theorem 10.1.30 For $k_\varphi \in \mathbf{K_n}$, $Pred(k_\varphi) \neq \varnothing$ and $q \in oldatom(k_\varphi)$ let Φ be the conjunction of all of the following formulas:

a. $\Lambda atom(k_\varphi)$
b. $\Delta oldatom(k_\varphi)$
c. $N(\psi) \rightarrow q$ for $k_\psi \in Pred(k_\varphi)$
d. $N(\chi)$ for k_χ such that $k_\chi \Vdash \Lambda atom(k_\varphi) \wedge \Lambda oldatom(k_\varphi)$ and not $k_\chi < k_\varphi$.

Then $\varphi \equiv \Phi$.

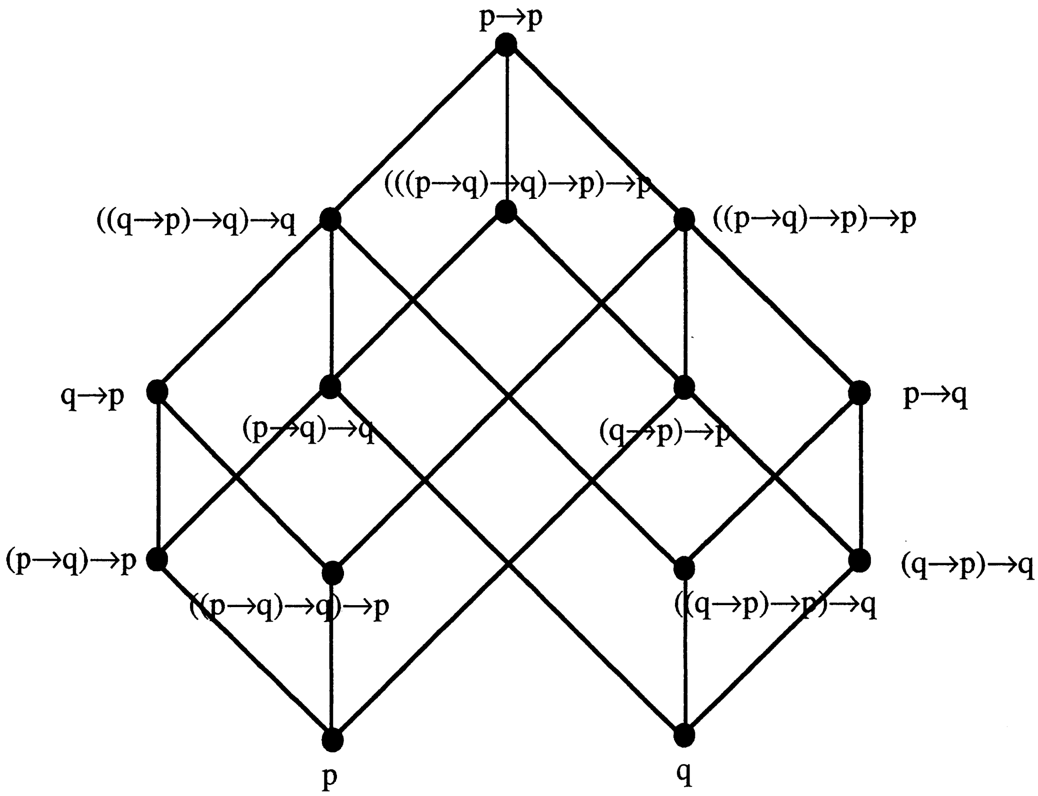
Pf. That $\omega(\Phi) = \downarrow k_\varphi$ is a consequence of $k_\varphi \Vdash \Phi$ and the fact that for $k_\chi \in \mathbf{K_n}$ such that $k_\chi \Vdash \Phi$ $k_\chi \leq k_\varphi$. That $k_\varphi \Vdash \Phi$ is a simple consequence of fact 10.1..29. For example, let k_χ be as described in (d) above. Then clearly $k_\varphi \Vdash N(\chi)$, by 10.1.29.a.
For the proof in the other direction ($\omega(\Phi) \subseteq \downarrow k_\varphi$) assume $k_\chi \Vdash \Phi$.
If $k_\chi \Vdash q$ then $k_\chi \Vdash \Lambda atom(k_\varphi) \wedge \Lambda oldatom(k_\varphi)$ and from (d) and $k_\chi \nVdash N(\chi)$ infer $k_\chi \leq k_\varphi$.
On the other hand assume $k_\chi \nVdash q$. Then $k_\chi \nVdash N(\psi)$ for all $k_\psi \in Pred(k_\varphi)$.
Hence (using 10.1.29.a) for all $k_\psi \in Pred(k_\varphi)$: $k_\psi \leq k_\chi$ and as $k_\psi = k_\chi$ would imply $k_\chi \Vdash q$, in fact $k_\psi < k_\chi$.
So $atom(k_\chi) \subseteq atom(k_\varphi) \cup oldatom(k_\varphi)$. As $k_\chi \Vdash \Delta oldatom(k_\varphi)$ and $k_\chi \Vdash q$ one may conclude that $atom(k_\chi) = atom(k_\varphi)$.
Let $k_\theta \in Pred(k_\chi)$. As $k_\theta \Vdash \Phi$ again, if $k_\theta \Vdash q$ then $k_\theta \leq k_\varphi$. If $k_\theta \nVdash q$, again, for all $k_\psi \in Pred(k_\varphi)$: $k_\psi \leq k_\theta$. As consequence $atom(k_\theta) = atom(k_\chi) = atom(k_\varphi)$.
But no such $k_\theta \in Pred(k_\chi)$ can exist according to corollary 10.1.21.
Hence $k_\theta \in Pred(k_\chi)$ implies $k_\theta \leq k_\varphi$. So $Pred(k_\chi) = Pred(k_\varphi)$ and as $atom(k_\chi) = atom(k_\varphi)$ we have to conclude that in case $k_\chi \Vdash \Phi$ and $k_\chi \nVdash q$ then $k_\chi = k_\varphi$. □

Together with theorem 10.1.27 and fact 10.1.30.b theorem 10.1.31 yields an algorithm to construct the exact model of $[\wedge, \rightarrow]_n$ with a representative of each class of irreducible formulas.

10.2 The [→]-fragments

The [→]-fragments are treated here as subfragments of the [→, ∧]-fragments. In the sequel we will prove that for $n \geq 2$ the fragment $[\rightarrow]_n$ does not have an exact model of its own. But of course the diagram of $[\rightarrow]_n$ can be calculated using $\mathcal{K}_n = \langle \mathbf{K_n}, \vdash \rangle$, the exact model of $[\wedge, \rightarrow]_n$. In exhibiting some of the structure of the [→]-fragments it turns out that knowledge of the structure of the [∧, →]-fragments is very useful.

Example 10.2.1 The diagram of $[\rightarrow]_2$.



Fact 10.2.2

a. For all $\varphi \in [\rightarrow]_n$ there is an atomic $p \in [\rightarrow]_n$ and a $\psi \in [\wedge, \rightarrow]_n$ such that $\varphi \equiv \psi \rightarrow p$.

b. If $p, \psi \in [\wedge, \rightarrow]_n$ and p atomic, then there is a $\varphi \in [\rightarrow]_n$ such that $\varphi \equiv \psi \rightarrow p$.

Theorem 10.2.3 If $n \geq 2$ then there is no meet or join definable in the diagram of $[\rightarrow]_n$.

Pf. Assume p and q to be among the atomic formulas of $[\rightarrow]_n$ and let $\otimes$ be the (supposed) meet in the diagram of $[\rightarrow]_n$, $\oplus$ the (supposed) join.
If $\varphi \in [\rightarrow]_n$ such that $\varphi \equiv p \otimes q$ and $r, \psi \in [\wedge, \rightarrow]_n$ such that r atomic and $\varphi \equiv \psi \rightarrow r$, then $r \vdash \varphi$ and $\varphi \vdash p$. As p and r are atomic infer $r = p$.
The same holds for q and hence $p = q$. So if $p \neq q$ there is no meet of p and q possible in the diagram of $[\rightarrow]_n$.
If $\varphi \in [\rightarrow]_n$ such that $\varphi \equiv p \oplus q$ and $r, \psi \in [\wedge, \rightarrow]_n$ such that r atomic and $\varphi \equiv \psi \rightarrow r$, then $\psi \rightarrow r \vdash (p \rightarrow q) \rightarrow q$ and $\psi \rightarrow r \vdash (q \rightarrow p) \rightarrow p$.

Assuming $r \neq p$, one infers from $r \vdash (p \to q) \to q$ that $r \vdash q$ (by substitution of q for p). As r and q are atomic, this yields $r = q$ and $p \oplus q \equiv \psi \to q$. From $p \vdash p \oplus q$ it follows that $\psi \vdash p \to q$ and hence $(p \to q) \to q \vdash \psi \to q$. But this would imply that $(p \to q) \to q \vdash (q \to p) \to p$, which is not the case if $p \neq q$.
On the other hand the assumption $p = r$ will lead to a contradiction in the same vein as did $r = q$ above.
So, for $n > 1$, there is no join possible in the diagram of $[\to]_n$. □

Corollary 10.2.4 If $n \geq 2$ then the diagram of $[\to]_n$ is not a lattice and $[\to]_n$ does not have an exact model.

Of course in case of $n = 1$ the diagram of $[\to]_1$ is a very simple (finite distributive) lattice.

For calculations with implications in $[\wedge, \to]_n$ it is often easier to use $\bar{\omega}$, the dual of the isomorphism ω between the diagram of $[\wedge, \to]_n$ and the down sets of $\mathcal{K}_n$. Note that $\bar{\omega}$ is an isomorphism between the diagram of $[\wedge, \to]_n$ and the up sets of $\mathcal{K}_n$.

Notation 10.2.5 If $\varphi \in [\wedge, \to]_n$ then $\bar{\omega}(\varphi) = \omega(\varphi)^c = \mathbf{K_n} \backslash \omega(\varphi)$.

Lemma 10.2.6 For $\varphi, \psi \in [\wedge, \to]_n$:

a. $\bar{\omega}(\varphi) \subseteq \bar{\omega}(\psi) \Leftrightarrow \psi \vdash \varphi$
b. $\bar{\omega}(\varphi \wedge \psi) = \bar{\omega}(\varphi) \cup \bar{\omega}(\psi)$
c. $\bar{\omega}(\varphi \to \psi) = \uparrow(\bar{\omega}(\psi) \backslash \bar{\omega}(\varphi))$

Pf. As (a) and (b) are almost trivial only (c) will be proved.
Recall from fact 4.7 that $\omega(\varphi \to \psi) = (\omega(\varphi)^c \cup \omega(\psi))^\circ$. As $X^\circ = (\uparrow(X^c))^c$ (fact 3.7) infer that $\bar{\omega}(\varphi \to \psi) = ((\omega(\varphi)^c \cup \omega(\psi))^\circ)^c = \uparrow(\bar{\omega}(\psi) \backslash \bar{\omega}(\varphi))$ □

Notation 10.2.7 If $\varphi \in [\wedge, \to]_n$ then $lcv(\varphi) = \{k \in \mathbf{K_n} \mid k$ is minimal in $\bar{\omega}(\varphi)\}$.

The notation $lcv(\varphi)$, the *lower carrier valuation* of φ, was introduced in [B75a].

Lemma 10.2.8 For $\psi \in [\wedge, \to]_n$ the number of elements in the diagram of $[\wedge, \to]_n$ that have a representative of the form $\varphi \to \psi$ (where $\varphi \in [\wedge, \to]_n$) is equal to the cardinality of the power set of $lcv(\psi)$.

Pf. As $\bar{\omega}(\varphi \to \psi) = \uparrow(\bar{\omega}(\psi) \backslash \bar{\omega}(\varphi)) = \uparrow(lcv(\psi) \backslash \bar{\omega}(\varphi))$, every $\varphi \to \psi$ corresponds exactly with a subset in $lcv(\psi)$.
If A is a subset of $lcv(\psi)$ then there is a formula $\chi \in [\wedge, \to]_n$ such that $\bar{\omega}(\chi) = \uparrow(lcv(\psi) \backslash A)$ but then $\bar{\omega}(\chi \to \psi) = \uparrow A$. Which proves every subset of $lcv(\psi)$ to correspond exactly with an equivalence class representable by a formula of the form $\varphi \to \psi$. □

The number of elements in the diagram of $[\rightarrow]_n$ can be calculated using the structure of $\mathcal{K}_n$ and the following theorem.

Theorem 10.2.9 Let $\bigcup_{i\leq n}\{p_i\}$ be the set of atoms in $[\rightarrow]_n$ and $N_{n,k} = |\bigcap_{i\leq n} lcv(p_i)|$, then the number of equivalence classes in $[\rightarrow]_n$ is: $\Sigma_{k=1}^{n}(-1)^{k-1}\binom{n}{k}2^{N_{n,k}}$

Pf. According to fact 10.2.2 each class in $[\rightarrow]_n$ may be represented by a formula of the form $\varphi\rightarrow p$ (where p is atomic). Using lemma 10.2.8, for each p the number of classes representable by a formula $\varphi\rightarrow p$ corresponds to the cardinality of the power set of $lcv(p)$.
The problem of calculating the number of subsets in a union of non-disjunct subsets (as $\varphi\rightarrow p \equiv \psi\rightarrow q$ is possible) is solved by the summation given, using the n-fold symmetry in $\mathcal{K}_n$. □

Corollary 10.2.10 The number of elements in the diagram of $[\rightarrow]_3$ is:
$3.2^{23} - 3.2^{3} + 1.2^{1} = 25\,165\,802$

Pf. Use $\mathcal{K}_3$ (example 10.1.27) and determine $lcv(p)$, $lcv(q)$, $lcv(r)$ and their intersections. After that, the calculation is a simple application of the theorem. □

Gerard Renardel de Lavalette calculated the cardinality of the diagram of $[\rightarrow]_4$:
$2^{623\,662\,965\,552\,393} - 50\,331\,618$.
Theorem 10.2.9 is a simple generalisation of the method used by him (and based on the technique that was developed by De Bruijn in [B75a]).

11 The $[\wedge, \to, \neg]$-fragments

The diagram of $[\wedge, \to, \neg]_n$ is a homomorphic image of the diagram of $[\wedge, \to]_{n+1}$, as will be proved in this section.
Using this homomorphism, the structure of the exact model of $[\wedge, \to, \neg]_n$ is easily derived.
The above mentioned homomorphism will be introduced first will simply be denoted as **F** in this section.

Notation 11.1 Let P to be the set of atoms in $[\wedge, \to, \neg]_n$ and $P \cup \{z\}$ the set of atoms in $[\wedge, \to]_{n+1}$. Then, for $\varphi \in [\wedge, \to]_{n+1}$, $\mathbf{F}(\varphi) := \varphi[z := \bot]$.

Recall the notion of a P-model from definition 5.1.9.
The following facts can derived by careful inspection of the proof of the completeness for IpL of (finite) Kripke models (as in theorem 6.1.1 in [TD88a]) and a simple formula induction.

Fact 11.2 Let φ be a IpL formula such that for every atomic subformula p of φ: $p \in P$ and $z \notin P$. Then:

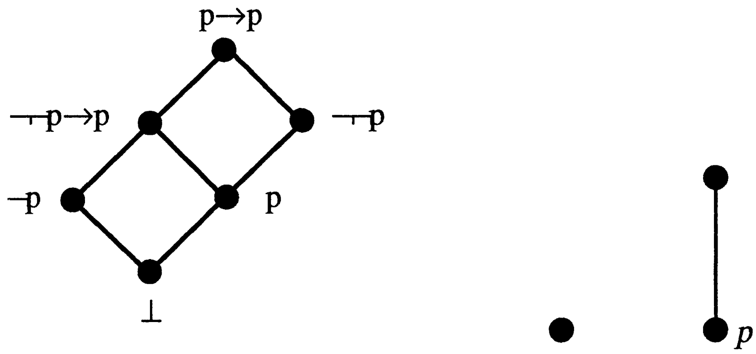
a. $\vdash \varphi \quad \Leftrightarrow \quad$ for all P-models $\mathcal{K}$: $\mathcal{K} \Vdash \varphi$
b. If $\mathcal{K}$ a P-model: $\mathcal{K} \Vdash \varphi \quad \Leftrightarrow \quad \mathcal{K} \Vdash \varphi[\bot := z]$

Lemma 11.3 **F** is a surjective homomorphism from the diagram of $[\wedge, \to]_{n+1}$ to the diagram of $[\wedge, \to, \neg]_n$.

Pf. That $\mathbf{F}(\varphi \wedge \psi) = \mathbf{F}(\varphi) \wedge \mathbf{F}(\psi)$ and $\mathbf{F}(\varphi \to \psi) = \mathbf{F}(\varphi) \to \mathbf{F}(\psi)$ is a simple consequence of the definition of **F** (notation 11.1).
To prove **F** monotone in the order of the diagrams, that is: $\varphi \vdash \psi \Rightarrow \mathbf{F}(\varphi) \vdash \mathbf{F}(\psi)$, it is sufficient to prove: $\vdash \varphi \Rightarrow \vdash \mathbf{F}(\varphi)$ (for all $\varphi \in [\wedge, \to]_{n+1}$).
So assume $\vdash \varphi$. By fact 11.2.a φ is forced by all $P \cup \{z\}$-models and hence by all P-models. According to fact 11.2.b all P-models will force $\mathbf{F}(\varphi)$ and, again by 11.2.a, $\vdash \mathbf{F}(\varphi)$.
Finally, to prove **F** surjective is simple, as $\mathbf{F}(\varphi[\bot := z]) = \varphi$ for all $\varphi \in [\wedge, \to, \neg]_n$. □

Corollary 11.4 The diagram of $[\wedge, \to, \neg]_n$ is a finite Heyting algebra and hence $[\wedge, \to, \neg]_n$ has an exact model.

Example 11.5 The diagram of $[\wedge, \to, \neg]_1$ and its exact model.



Lemma 11.6 $\uparrow(z\rightarrow\wedge P)$ is the kernel of **F** (in the diagram of $[\wedge,\rightarrow]_{n+1}$).

Pf. Let *ker* (**F**) be the kernel of **F**. That $\uparrow(z\rightarrow\wedge P)\subseteq ker(\mathbf{F})$ is a simple consequence of $\mathbf{F}(z\rightarrow\wedge P)=\bot\rightarrow\wedge P\equiv\top$.
To prove the inclusion in the other direction, let $\vdash\mathbf{F}(\varphi)$.
By fact 11.2.a $\mathbf{F}(\varphi)$ is forced by all P-models. Hence $\mathbf{F}(\varphi)$ is forced by those submodels $\downarrow k_\xi$ of $\mathcal{K}_{n+1}=<\mathbf{K}_{n+1},\Vdash>$, (the exact model of $[\wedge,\rightarrow]_{n+1}$) that are P-models.
For $k_\xi\in\mathbf{K}_{n+1}$: $k_\xi\nVdash z\wedge\wedge P$, $\downarrow k_\xi$ is a P-model iff $k_\xi\Vdash z\rightarrow\wedge P$. So infer by fact 11.2.b that if $k_\xi\Vdash z\rightarrow\wedge P$ then also $k_\xi\Vdash\varphi$.
As $\mathcal{K}_{n+1}$ is the exact model of $[\wedge,\rightarrow]_{n+1}$ this proves $z\rightarrow\wedge P\vdash\varphi$.
Hence $ker(\mathbf{F})\subseteq\uparrow(z\rightarrow\wedge P)$. □

Corollary 11.7 The submodel of $\mathcal{K}_{n+1}$ corresponding to $\omega(z\rightarrow\wedge P)$ is the exact Kripke model of $[\wedge,\rightarrow,\neg]_n$.

Pf. By lemma 11.5 and fact 11.2.b; for $\varphi,\psi\in[\wedge,\rightarrow,\neg]_n$:

$$\varphi\vdash\psi \iff \mathbf{F}(\varphi[\bot:=z])\vdash\mathbf{F}(\varphi[\bot:=z])$$
$$\iff z\rightarrow\wedge P\vdash(\varphi\rightarrow\psi)[\bot:=z]$$
$$\iff z\rightarrow\wedge P\vdash(\varphi\rightarrow\psi) \qquad \text{(as in the proof of 11.5).}$$

So if ν is the mapping of equivalence classes of $[\wedge,\rightarrow,\neg]_n$ to down sets of $\omega(z\rightarrow\wedge P)$, defined as the restriction of ω to $\omega(z\rightarrow\wedge P)$ then ν is clearly a homomorphism from the diagram of $[\wedge,\rightarrow,\neg]_n$ to down sets of $\omega(z\rightarrow\wedge P)$.
To prove ν surjective, let U be a downset in $\omega(z\rightarrow\wedge P)$.
Then $U=\omega(\varphi)$ for some $\varphi\in[\wedge,\rightarrow]_{n+1}$. Recall that $\omega(z\rightarrow\wedge P)$ is a P-model and hence $\omega(\varphi)=\nu(\mathbf{F}(\varphi))$. As $\nu(\varphi)=\{\,k_\xi\in\omega(z\rightarrow\wedge P)\mid k_\xi\Vdash\varphi\}$ clearly $\omega(z\rightarrow\wedge P)$ is an exact Kripke model. □

Theorem 11.8 The frame of the exact Kripke model $\mathcal{N}_n$ of $[\wedge,\rightarrow]_n$ can be constructed stepwise as the union of sets $\mathcal{E}_k$ of tuples $<Q,S>$ such that $Q\subset P$, P the set of atoms in $[\wedge,\rightarrow]_n$, and S either the empty set or a down set in $\mathcal{E}_{k-1}$. Where $\mathcal{E}_k$ is defined as:

$$\mathcal{E}_0=\{\,<Q,\varnothing>\mid Q\subseteq P\,\}$$
$$\mathcal{E}_{k+1}=\mathcal{E}_k\cup\{\,<Q,S>\mid S\subseteq\mathcal{E}_k\wedge\downarrow S=S\wedge Q\subset\bigcap\{R\mid\exists T\,.<R,T>\in S\,\}\}$$

and

$$<Q,S>\ \le\ <Q',S'> \text{ iff } <Q,S>\in S' \text{ or } <Q,S>=<Q',S'>$$

On this frame the forcing relation of $\mathcal{N}_n$ is defined as $<Q,S>\Vdash p\iff p\in Q$ for atomic formulas $p\in P$.

Pf. From corollary 11.7 it is clear that the construction described by theorem 10.1.26, if restricted by excluding all elements connected to those forcing a chosen atomic formula, will yield the exact model $\mathcal{N}_n$. This is exactly what the construction described in this theorem accomplishes. □

Again, as in corollary 10.1.25, it is not difficult to find the formulas corresponding with the minimal elements of $\mathcal{N}_n$.

Fact 11.9 If k be a minimal element in $\mathcal{N}_n$, $Q=atom(k)$, P_n the set of atoms in $[\wedge,\rightarrow,\neg]_n$ and $\varphi:=\wedge Q\wedge\wedge\{\neg p\mid p\in P_n\setminus Q\}$ then $k=k_\varphi$.

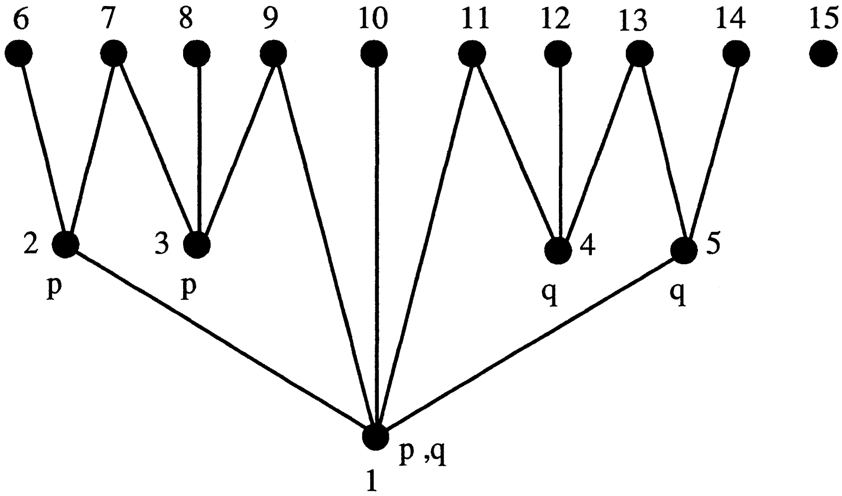
Theorem 11.10 For $k_\varphi \in \mathcal{N}_n$, $Pred(k_\varphi) \neq \varnothing$ and $q \in oldatom(k_\varphi)$ let Φ be the conjunction of all of the following formulas:

a. $\Lambda atom(k_\varphi)$
b. $\Delta oldatom(k_\varphi)$
c. $N(\psi) \to q$ for $k_\psi \in Pred(k_\varphi)$
d. $N(\chi)$ for k_χ such that $k_\chi \Vdash \Lambda atom(k_\varphi) \wedge \Lambda oldatom(k_\varphi)$ and not $k_\chi < k_\varphi$.

Then $\varphi \equiv \Phi$.

Pf. In fact theorem 11.10 is exactly the same as 10.1.30 and so is its proof. □

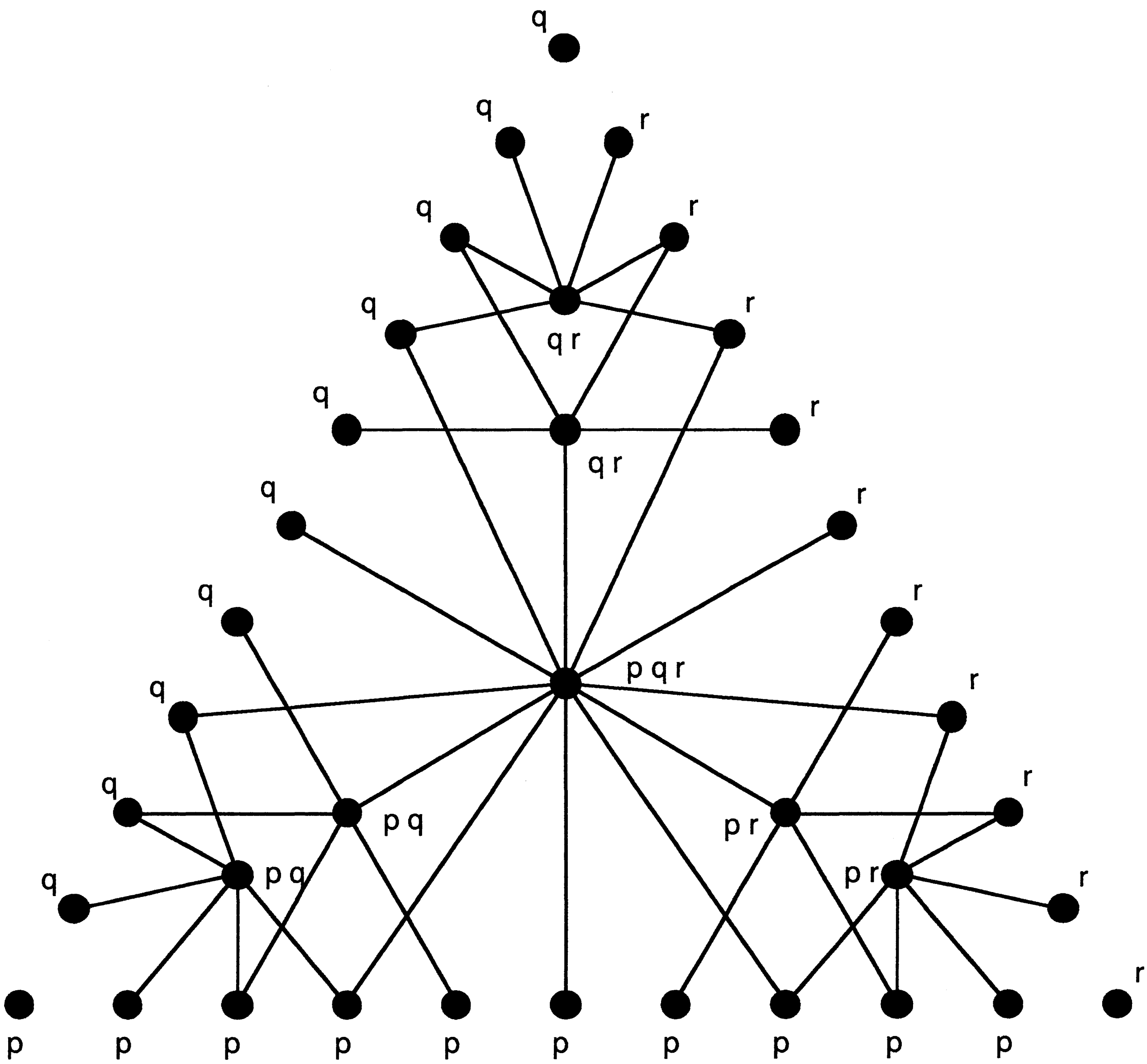
Example 11.11 The exact model of $[\wedge, \to, \neg]_2$ and its formulas.



1. $p \wedge q$
2. $p \wedge \neg\neg q$
3. $p \wedge \neg q$
4. $q \wedge \neg p$
5. $q \wedge \neg\neg p$
6. $\neg\neg q \wedge ((p \to q) \to p)$
7. $(\neg\neg q \to p) \wedge ((\neg\neg q \to q) \to p)$
8. $\neg(p \to q)$
9. $(q \to p) \wedge (\neg q \to p) \wedge (\neg\neg q \to q)$
10. $(p \leftrightarrow q) \wedge \neg\neg p$
11. $(p \to q) \wedge (\neg p \to q) \wedge (\neg\neg p \to p)$
12. $\neg(q \to p)$
13. $(\neg\neg p \to q) \wedge ((\neg\neg p \to p) \to q)$
14. $\neg\neg p \wedge ((q \to p) \to q)$
15. $\neg p \wedge \neg q$

Example 11.12 The structure of the exact model of $[\wedge, \to, \neg]_2$.
The exact model of $[\wedge, \to, \neg]_2$ has 6423 points. The $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ levels of its construction (see theorem 11.8) are depicted on the opposite page. The $\mathcal{L}_3$-level with 6386 would complete this model.
From the structure of this exact model the number of classes in $[\wedge, \to, \neg]_2$ can be computed. The 1923 digits of the outcome of this computation can be found in the tabel in the apendix of this report.

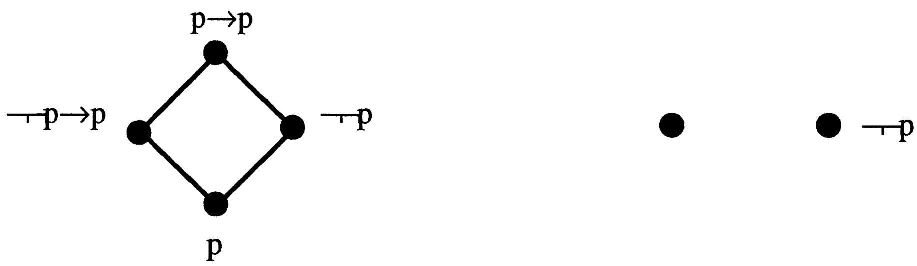
Inventory of IpL fragments



12 The [$\wedge$, $\to$, $\neg\neg$]-fragments

As in case of earlier section on [$\wedge$, $\vee$, $\neg\neg$] fragments (section 7) the [$\wedge$, $\to$, $\neg\neg$] fragments do have exact models that can be extended to Kripke completions.
In this section we prove that for $[\wedge, \to, \neg\neg]_n$ the exact model needs only to be extended by one element, to obtain a Kripke model $\mathcal{K}$ that is complete for $[\wedge, \to, \neg\neg]_n$ (i.e. for all $\varphi, \psi \in [\wedge, \to, \neg\neg]$: $\varphi \vdash \psi \Leftrightarrow \mathcal{K} \Vdash \varphi \to \psi$).

Example 12.1 The diagram of [p, $\wedge$, $\to$, $\neg\neg$] and its exact model (right).



Notation 12.2 In the sequel of this section, let $\mathcal{N}_n$ =<$\mathbf{N_n}$, $\vdash$> the exact model of $[\wedge, \to, \neg]_n$ and $\Phi_n := \bigwedge_{i \le n}(\bigwedge_{j \ne i} \neg p_j \wedge \neg\neg p_i \to p_i)$.

The main result of this section will be the proof that $\omega(\Phi_n) \backslash \omega(\Lambda P_n)$ in $\mathcal{N}_n$ is the exact model of $[\wedge, \to, \neg\neg]_n$ and $\omega(\Phi_n)$ is a complete Kripke extension for it.

Lemma 12.3 For $k_\chi \in \mathbf{N_n}$, if $k_\chi \nVdash \Phi_n$ then k_χ has one predecessor in $\mathcal{N}_n$, a terminal node where exactly one atom is forced.

Pf. Let $k_\chi \nVdash \Phi_n$. Without loss of generality assume $k_\chi \Vdash \neg p_i$ if $i \ne n$, $k_\chi \Vdash \neg\neg p_n$ and $k_\chi \nVdash p_n$ Then k_χ is no terminal node, otherwise $k_\chi \Vdash \neg\neg p_n \to p_n$.
If $k_\xi < k_\chi$ then (as a consequence of theorem 11.8 and as $n > 1$) there is an atom in P_n such that $p \in atom(k_\xi) \backslash atom(k_\chi)$. As $k_\xi \Vdash \neg p_i$ if $i \ne n$ this proves k_ξ forces only p_n. In $\mathcal{N}_n$ such a k_ξ is unique (see theorem 11.8). □

Lemma 12.4 For $k_\chi \in \mathbf{N_n}$ if $k_\chi \nVdash \Phi_n$ there is a $k_\theta \in \mathbf{N_n}$, $k_\theta \Vdash \Phi_n$, such that for all $\varphi \in [\wedge, \to, \neg\neg]_n$ in $\mathcal{N}_n$: $k_\chi \Vdash \varphi \Leftrightarrow k_\theta \Vdash \varphi$.

Pf. Let $k_\chi \in \mathbf{N_n}$ and $k_\chi \nVdash \Phi_n$. Using lemma 12.3 (again without loss of generality) we may assume k_χ is directly above the terminal node k_ξ such that $atom\,(k_\xi) = \{p_n\}$. By theorem 11.8 (and as $n > 1$) there is a k_θ with exactly two predecessors, k_ξ and the terminal node of $\mathcal{N}_n$ forcing all atoms of P_n. By lemma 12.3 $k_\theta \Vdash \Phi_n$ and by a simple formula induction one proofs for $\varphi \in [\wedge, \to, \neg\neg]_n$: $k_\chi \Vdash \varphi \Leftrightarrow k_\theta \Vdash \varphi$. □

Corollary 12.5 If $\varphi \in [\wedge, \to, \neg\neg]_n$ then $\Phi_n \vdash \varphi \Leftrightarrow \vdash \varphi$.

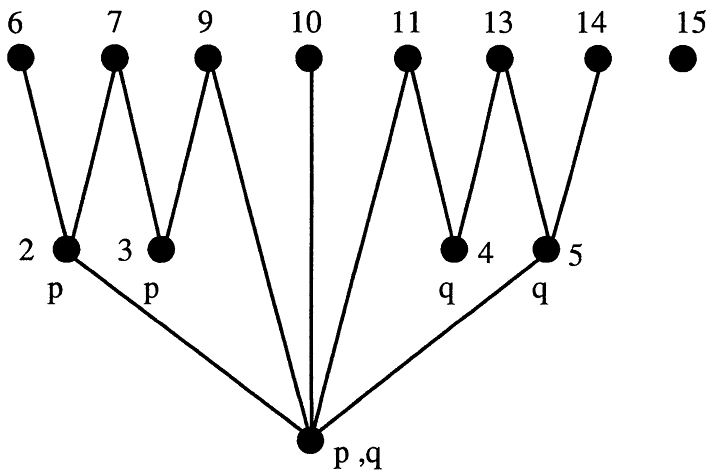
Pf. Assume $\Phi_n \vdash \varphi$ and k $\in \mathbf{N_n}$. If $k_\chi \nVdash \Phi_n$, by lemma 12.4, there is a $k_\theta \in \mathbf{N_n}$ such that $k_\theta \Vdash \Phi_n$ and k_θ is equivalent with k_χ for all $[\wedge, \to, \neg\neg]_n$ formulas. As $k_\theta \Vdash \varphi$ also $k_\chi \Vdash \varphi$. Hence in $\mathcal{N}_n$: $\omega(\varphi) = \mathbf{N_n}$, so $\vdash \varphi$. The other direction is of course trivial. □

Theorem 12.6 In $\mathbf{N}_n$, the exact model of $[\wedge, \to, \neg]_n$, $\omega(\Phi_n)\backslash\omega(\Lambda P_n)$ is the exact model of $[\wedge, \to, \neg\neg]_n$ and $<\omega(\Phi_n), \vdash>$ is a Kripke model such that for all $\varphi, \psi \in [\wedge, \to, \neg]_n$: $\varphi \vdash \psi \Leftrightarrow <\omega(\Phi_n), \vdash> \Vdash \varphi \to \psi$.

Pf. By corollary 12.5 for $\varphi, \psi \in [\wedge, \to, \neg\neg]_n$: $\Phi_n \vdash \varphi \to \psi \Leftrightarrow \varphi \vdash \psi$.
Hence, as ΛP_n is the bottom of the diagram of $[\wedge, \to, \neg\neg]_n$, every equivalence class in this fragment corresponds to a down set of $\omega(\Phi_n)\backslash\omega(\Lambda P_n)$.
To prove that every down set in $\mathbf{N}_n$ corresponds to an equivalence class in $[\wedge, \to, \neg\neg]_n$, a translation $\alpha(\varphi)$ is defined such that $\alpha(\varphi) \in [\wedge, \to, \neg\neg]_n$ and for k_χ in $\mathbf{N}_n$: if $k_\chi \not\Vdash \Lambda P_n$ and $k_\chi \Vdash \Phi_n$ then $k_\chi \vdash \varphi \leftrightarrow \alpha(\varphi)$.
We define α inductively by stipulating for $\varphi \in [\wedge, \to, \neg\neg]_n$ that $\alpha(\neg\varphi) = \neg\neg\varphi \to \Lambda P_n$ (and $\alpha(\varphi) = \varphi$ in case of atoms, conjunctions, implications and double negations).
Of course $\neg\varphi \vdash \neg\neg\varphi \to \Lambda P_n$.
For the other direction, let $k_\chi \neq \Lambda P_n$ and $k_\chi \Vdash \neg\neg\varphi \to \Lambda P_n$.
If $k_\chi \not\Vdash \neg\Lambda P_n$ then in the exact model of $[\wedge, \to, \neg]_n$ there is a k_ξ such that $k_{\Lambda P} <_1 k_\xi \leq k_\chi$. It is easy to check that $k_\xi \Vdash \neg\neg\varphi$ for all $\varphi \in [\wedge, \to, \neg\neg]_n$.
As $k_\xi \not\Vdash \Lambda P_n$ we have to contradict our previous assumption and so $k_\chi \Vdash \neg\Lambda P_n$.
As a consequence $k_\chi \Vdash \neg\varphi$. □

Note that theorem 12.6 in combination with theorem 11.8 and lemma 12.3 defines a transformation of the exact model $\mathbf{N}_n$ of $[\wedge, \to, \neg]_n$ to a Kripke model extending the exact model of $[\wedge, \to, \neg\neg]_n$. This transformation is just the elimination of those elements which have as their only predecessor a terminal node forcing exactly one atom. Compare example 11.11 with 12.7:

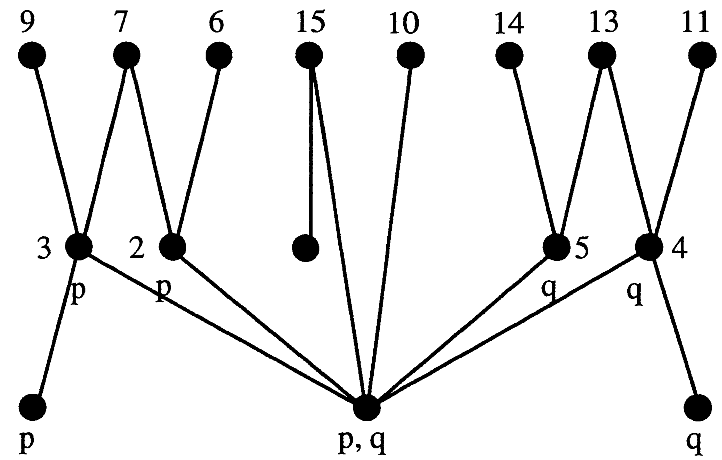
Example 12.7 A complete Kripke model extending the exact model of $[p, q, \wedge, \to, \neg\neg]$.



2. $p \wedge \neg\neg q$
3. $p \wedge (\neg\neg q \to q)$
4. $q \wedge (\neg\neg p \to p)$
5. $\neg\neg p \wedge q$
6. $((p \to q) \to p) \wedge \neg\neg q$
7. $(\neg\neg q \to p) \wedge ((\neg\neg q \to q) \to p)$
9. $(p \to \neg\neg q) \to p \wedge q$
10. $(p \leftrightarrow q) \wedge \neg\neg p$
11. $(q \to \neg\neg p) \to p \wedge q$
13. $(\neg\neg p \to q) \wedge ((\neg\neg p \to p) \to q)$
14. $((q \to p) \to q) \wedge \neg\neg p$
15. $((q \to p) \to \neg\neg p) \to p \wedge q$

According to theorem 5.3.7 the exact model of $[\wedge, \rightarrow, \neg\neg]_n$ also has a Kripke completion.

Example 12.8 The Kripke completion of $[p, q, \wedge, \rightarrow, \neg\neg]$ (the numbers of the elements of the exact model correspond to those in the previous example).



Table

#atoms:	1	2	3
[∧]	1	3	7
[∨]	1	3	7
[∧,∨]	1	4	18
[¬]	3	6	9
[¬¬]	2	4	6
[∧,¬]	5	23	311
[∨,¬]	7	385	$>2^{70}$
[∧,∨,¬]	7	626	$>2^{70}$
[∧,¬¬]	2	8	26
[∨,¬¬]	2	9	40
[∧,∨,¬¬]	2	19	1 889
[→]	2	14	25 165 802
[∧,→]	2	18	623 662 965 552 330
[∨,→]	2	∞	∞
[∧,∨,→]	2	∞	∞
[→,¬]	6	518	$3*2^{2\,148}-546$ [1]
[∧,→,¬]	6	2134	(A)
[∨,→,¬]	11	∞	∞
[∧,∨,→,¬]	11	∞	∞
[→,¬¬]	4	252	$3*2^{689}-380$ [1]
[∧,→,¬¬]	4	676	$>2^{6\,383}$
[∨,→,¬¬]	5	∞	∞
[∧,∨,→,¬¬]	5	∞	∞

The number (A) was calculated with a Mathematica-program by G. Renardel de Lavalette:

2385351090480492390853646413339133747025615299710901627960612470750032688502816063337432610285
1405827074085958557851857316972228706343515481647745510067300534461520514807499786875488139392
3444865679964852452325439433729138822091609839191386759807380638954594760890360815576879124178
1137739941904366215669247582299927405773012313171434650148857286106293669904259609272537857286
8491272712012675687555199920889937803673124068400811155686757146749638659745341966397343525240
3693430417730445657028232115210122043282697803859354958719561298318118789345879838234751135199
9075002797617209798908503195548980385712881263878902567993333287059490507689711521909112697578
0798055001237127554396205209442741592500688474353052965956619945719436136715051701844142763673
5090517116806133764983542543661798774036708731768419238640888313490745738623900865231463501660
1573717671230516190181144414611500132249202791000647249183864045852362977616094414762999993096
1650177344426785162274523755062900873646045131468625073787337004447927030524156059024181381821
7796310418774113313134437935316573299754930440874865583477327660932604455374223117461731779709
9352819020183117687000447391980301631226240785745841846388391474813267681770574727454215143982
4203308859517469910490858730437804621971785778601804184276982651560872130379381608212457177138
1482585476984496988963204119188869627498008746051248577769359343606951739527723134540782809898
0787933234903875965383757843614426540104617022254343668201611284496543949479906553242581974948
2642048348803493746058787080750389552034698832805380268905837851783083939857171884062118390901
4410826752614933525047420939070983048354586384109943140177530505815812025437869779793845598990
0962307329635934966082733645869281440647431698737494310390629154854362968976529657537647190442
2451714399007297503766371496442187703134084481925540030231140403567182306662271616684332089066
7512953392966722394434024845812798019917566

[1] According to a private communication by G. Renardel de Lavalette

Literature

[B75a] N.G. de Bruijn, *Exact finite models for minimal propositional calculus over a finite alphabet,* Technological University Eindhoven, Report 75-WSK-02, 1975

[B75b] N.G. de Bruijn, *An Algol program for deciding derivability in minimal propositional calculus with implication and conjunction over a three letter alphabet,* Technological University Eindhoven, Memorandum 1975-06, 1975

[B85] J.F.A.K. van Benthem, *Modal Logic and Classical Logic* , Napoli, 1983

[C77] H.B. Curry, *Foundations of Mathematical Logic,* second edition, New York, 1977

[CD65] J. N. Crossley and M. A. E. Dummett, *Formal Systems and recursive functions, North Holland,* Amsterdam, 1965.

[D59] M. Dummett, *A propositional calculus with denumerable matrix,* JSL 24, pp 97 - 106, 1959

[D66] A. Diego, *Sur les Algèbres de Hilbert,* Gautier-Villars, Paris, 1966.

[DP90] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order,* Cambridge University Press, Cambridge, 1990

[G32] K. Gödel, *Zum intuitionistischen Aussagenkalküls* , Anzeiger der Akademie der Wissenschaften in Wien 69, 65-66, 1932

[G81] D.M. Gabbay, *Semantical investigations in Heyting's intuitionistic logic,* Dordrecht, 1981

[H66] T. Hosoi, *The axiomatization of the intermediate propositional systems S_n of Gödel* , Journal of the Faculty of Science of the University of Tokio 13, 183 - 187, 1966

[H80] A. Hendriks, *Logische Automaten, Beslissen en Bewijzen met de Computer,* Master's thesis, University of Amsterdam, 1980

[J36] S. Jaskowski, *Recherches sur le système de la logique intuitionniste,* in: Actes du Congrès International de Philosophie Scientifique (Vol. VI), Hermann, Paris, 58-61, 1936

[J70] D.H.J. de Jongh, *A characterisation of the intuitionistic propositional calculus* in: A. Kino, J. Myhill, R.E. Vesley (eds.), Intuitionism and Proof Theory, North-Holland, Amsterdam, 211-217, 1970

[J80] D.H.J. de Jongh, *A class of intuitionistic connectives,* in: J. Barwise, H.J. Keisler, K. Kunen (eds.), The Kleene Symposium, North-Holland, Amsterdam, 103-111, 1980

[JHR91] D. de Jongh, L. Hendriks, G.R. Renardel de Lavalette, *Computations in Fragments of Intuitionistic Propositional Logic,* Journal of Automated Reasoning 7, 537-561, 1991

[JT66] D.H.J. de Jongh, A.S. Troelstra, *On the connection of partially ordered sets with some pseudo-Boolean algebras,* Indag. Math. 28 no 3, 317-329, 1966

[K62] S.C. Kleene, *Disjuncton and Existence Under Implications in Elementary Intuitionistic Formalisms,* JSL 27/1, 11 - 18, 1962
[K65] S. A. Kripke, *Semantical analysis of intuitionistic logic I,* in [CD65], 92 - 130.
[K69] D. Kleitman, *On Dedekind's problem: the number of monotone Boolean functions,* Proceedings of the AMS 21, 677 - 682, 1969
[K88] A. Kisielewicz, *A solution of Dedekind's problem on the number of isotone Boolean functions,* J.reine angew. Math. 386, 139 -144, 1988

[MT48] J.C.C. McKinsey, A. Tarski, *Some theorems about sentential calculi of Lewis and Heyting,* JSL 13, 1 - 15, 1948.

[N60] I. Nishimura, *On formulas of one variable in intuitionistic propositional calculus,* JSL 25, 327-331, 1960

[R49] N.S. Rieger, *On the Lattice Theory of Brouwerian Propositional Logic,* Acta Fac. Rerum Nat. Univ. Carolinae 189, 3-40, 1949
[R74] H.Rasiowa, *An Algebraic Approach to Non-classical Logics,* North-Holland, Amsterdam, 1974
[R85] H. van Riemsdijk, *Het genereren van Diagrammen,* Master's thesis, University of Amsterdam, 1985
[R86] P.H. Rodenburg, *Intuitionistic Correspondence Theory,* Ph.D. thesis, University of Amsterdam, 1986
[R89] G.R. Renardel de Lavalette, *Interpolation in fragments of intuitionistic propostional logic,* JSL 54, pp 1419-1430, 1989
[RS68] H. Rasiowa, R. Sikorski, *The Mathematics of Methamathematics,* Warszawa, 1968

[S13] Th.Skolem, *Om konstitusjonen av den identiske kalkyls grupper,* Proc. Scan. Math. Con., Kristiania , 149 - 163, 1913
[S19] Th. Skolem, *Untersuchungen über die Axiome des Klassenkalkuls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen,* Skrifter Videnskapsselskapet Kristiania, I. Mat.-naturv. klasse 1919, No. 3 37 pp, 1919
[S28] E. Sperner, *Ein Satz über Untermengen einer endlichen Menge,* Mathematische Zeitschrift, XXVII, pp 544 - 548, 1928.
[S36] Th. Skolem, *Über gewisse 'Verbände' oder 'Lattices',* Avhandlinger Norske Videnskaps-Akademi i Oslo I. Mat.-naturv. klasse 1936, no. 7, pp 1 - 16, 1936
[S37] M. H. Stone, *Topological representation of distributive lattices and Brouwerian Logics,* Cas. Mat. Fys. 67, 1 - 25, 1937
[S73] N.J.A. Sloane, *A Handbook of Integer Sequences,* New York, 1973

[T38] A. Tarski, *Der Aussagenkalkül und die Topologie,* Fundamenta Mathematicae 31, 103-134, 1938
[T62] I. Thomas, *Finite limitations on Dummett's LC* , Notre Dame Journal of Formal Logic, Volume III/3, 170 - 174, 1962
[T65] A.S. Troelstra, *Finite logics* , Indag. Math. 27, 141 - 152, 1965
[TD88a] A.S. Troelstra , D. van Dalen, *Constructivism in mathematics, an introduction* (Vol. I), North-Holland, Amsterdam etc., 1988
[TD88b] A.S. Troelstra , D. van Dalen, *Constructivism in mathematics, an introduction* (Vol. II), North-Holland, Amsterdam etc., 1988

[U73] A. Urquhart, *Free Heyting algebras,* Algebra Universalis 3, 94 - 97, 1973

[U74] A. Urquhart, *Implicational formulas in intuitionistic logic,* JSL 39/4, 661 - 664, 1974 .

Index

The ILLC Prepublication Series

1990 *Logic, Semantics and Philosophy of Language*
LP-90-01 Jaap van der Does — A Generalized Quantifier Logic for Naked Infinitives
LP-90-02 Jeroen Groenendijk, Martin Stokhof — Dynamic Montague Grammar
LP-90-03 Renate Bartsch — Concept Formation and Concept Composition
LP-90-04 Aarne Ranta — Intuitionistic Categorial Grammar
LP-90-05 Patrick Blackburn — Nominal Tense Logic
LP-90-06 Gennaro Chierchia — The Variablity of Impersonal Subjects
LP-90-07 Gennaro Chierchia — Anaphora and Dynamic Logic
LP-90-08 Herman Hendriks — Flexible Montague Grammar
LP-90-09 Paul Dekker — The Scope of Negation in Discourse, towards a Flexible Dynamic Montague grammar
LP-90-10 Theo M.V. Janssen — Models for Discourse Markers
LP-90-11 Johan van Benthem — General Dynamics
LP-90-12 Serge Lapierre — A Functional Partial Semantics for Intensional Logic
LP-90-13 Zhisheng Huang — Logics for Belief Dependence
LP-90-14 Jeroen Groenendijk, Martin Stokhof — Two Theories of Dynamic Semantics
LP-90-15 Maarten de Rijke — The Modal Logic of Inequality
LP-90-16 Zhisheng Huang, Karen Kwast — Awareness, Negation and Logical Omniscience
LP-90-17 Paul Dekker — Existential Disclosure, Implicit Arguments in Dynamic Semantics

Mathematical Logic and Foundations
ML-90-01 Harold Schellinx — Isomorphisms and Non-Isomorphisms of Graph Models
ML-90-02 Jaap van Oosten — A Semantical Proof of De Jongh's Theorem
ML-90-03 Yde Venema — Relational Games
ML-90-04 Maarten de Rijke — Unary Interpretability Logic
ML-90-05 Domenico Zambella — Sequences with Simple Initial Segments
ML-90-06 Jaap van Oosten — Extension of Lifschitz' Realizability to Higher Order Arithmetic, and a Solution to a Problem of F. Richman
ML-90-07 Maarten de Rijke — A Note on the Interpretability Logic of Finitely Axiomatized Theories
ML-90-08 Harold Schellinx — Some Syntactical Observations on Linear Logic
ML-90-09 Dick de Jongh, Duccio Pianigiani — Solution of a Problem of David Guaspari
ML-90-10 Michiel van Lambalgen — Randomness in Set Theory
ML-90-11 Paul C. Gilmore — The Consistency of an Extended NaDSet

Computation and Complexity Theory
CT-90-01 John Tromp, Peter van Emde Boas — Associative Storage Modification Machines
CT-90-02 Sieger van Denneheuvel, Gerard R. Renardel de Lavalette — A Normal Form for PCSJ Expressions
CT-90-03 Ricard Gavaldà, Leen Torenvliet, Osamu Watanabe, José L. Balcázar — Generalized Kolmogorov Complexity in Relativized Separations
CT-90-04 Harry Buhrman, Edith Spaan, Leen Torenvliet — Bounded Reductions
CT-90-05 Sieger van Denneheuvel, Karen Kwast — Efficient Normalization of Database and Constraint Expressions
CT-90-06 Michiel Smid, Peter van Emde Boas — Dynamic Data Structures on Multiple Storage Media, a Tutorial
CT-90-07 Kees Doets — Greatest Fixed Points of Logic Programs
CT-90-08 Fred de Geus, Ernest Rotterdam, Sieger van Denneheuvel, Peter van Emde Boas — Physiological Modelling using RL
CT-90-09 Roel de Vrijer — Unique Normal Forms for Combinatory Logic with Parallel Conditional, a case study in conditional rewriting

Other Prepublications
X-90-01 A.S. Troelstra — Remarks on Intuitionism and the Philosophy of Mathematics, Revised Version
X-90-02 Maarten de Rijke — Some Chapters on Interpretability Logic
X-90-03 L.D. Beklemishev — On the Complexity of Arithmetical Interpretations of Modal Formulae
X-90-04 — Annual Report 1989
X-90-05 Valentin Shehtman — Derived Sets in Euclidean Spaces and Modal Logic
X-90-06 Valentin Goranko, Solomon Passy — Using the Universal Modality: Gains and Questions
X-90-07 V.Yu. Shavrukov — The Lindenbaum Fixed Point Algebra is Undecidable
X-90-08 L.D. Beklemishev — Provability Logics for Natural Turing Progressions of Arithmetical Theories
X-90-09 V.Yu. Shavrukov — On Rosser's Provability Predicate
X-90-10 Sieger van Denneheuvel, Peter van Emde Boas — An Overview of the Rule Language RL/1
X-90-11 Alessandra Carbone — Provable Fixed points in $I\Delta_0+\Omega_1$, revised version
X-90-12 Maarten de Rijke — Bi-Unary Interpretability Logic
X-90-13 K.N. Ignatiev — Dzhaparidze's Polymodal Logic: Arithmetical Completeness, Fixed Point Property, Craig's Property
X-90-14 L.A. Chagrova — Undecidable Problems in Correspondence Theory
X-90-15 A.S. Troelstra — Lectures on Linear Logic

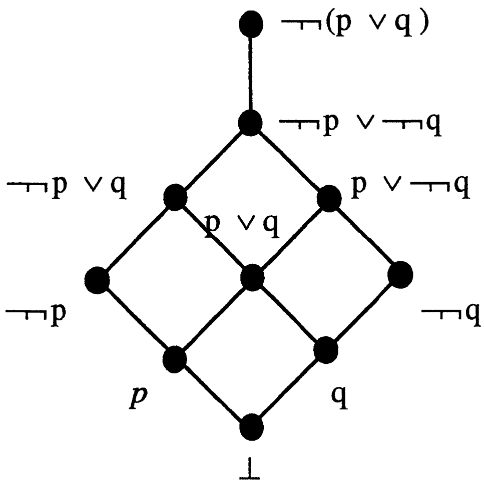
1991 *Logic, Semantics and Philosophy of Langauge*
LP-91-01 Wiebe van der Hoek, Maarten de Rijke — Generalized Quantifiers and Modal Logic
LP-91-02 Frank Veltman — Defaults in Update Semantics
LP-91-03 Willem Groeneveld — Dynamic Semantics and Circular Propositions
LP-91-04 Makoto Kanazawa — The Lambek Calculus enriched with Additional Connectives
LP-91-05 Zhisheng Huang, Peter van Emde Boas — The Schoenmakers Paradox: Its Solution in a Belief Dependence Framework
LP-91-06 Zhisheng Huang, Peter van Emde Boas — Belief Dependence, Revision and Persistence
LP-91-07 Henk Verkuyl, Jaap van der Does — The Semantics of Plural Noun Phrases
LP-91-08 Víctor Sánchez Valencia — Categorial Grammar and Natural Reasoning
LP-91-09 Arthur Nieuwendijk — Semantics and Comparative Logic
LP-91-10 Johan van Benthem — Logic and the Flow of Information

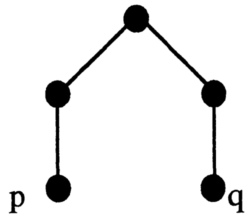
Mathematical Logic and Foundations
ML-91-01 Yde Venema — Cylindric Modal Logic
ML-91-02 Alessandro Berarducci, Rineke Verbrugge — On the Metamathematics of Weak Theories
ML-91-03 Domenico Zambella — On the Proofs of Arithmetical Completeness for Interpretability Logic
ML-91-04 Raymond Hoofman, Harold Schellinx — Collapsing Graph Models by Preorders
ML-91-05 A.S. Troelstra — History of Constructivism in the Twentieth Century
ML-91-06 Inge Bethke — Finite Type Structures within Combinatory Algebras
ML-91-07 Yde Venema — Modal Derivation Rules
ML-91-08 Inge Bethke — Going Stable in Graph Models
ML-91-09 V.Yu. Shavrukov — A Note on the Diagonalizable Algebras of PA and ZF
ML-91-10 Maarten de Rijke, Yde Venema — Sahlqvist's Theorem for Boolean Algebras with Operators
ML-91-11 Rineke Verbrugge — Feasible Interpretability
ML-91-12 Johan van Benthem — Modal Frame Classes, revisited

Computation and Complexity Theory
CT-91-01 Ming Li, Paul M.B. Vitányi — Kolmogorov Complexity Arguments in Combinatorics
CT-91-02 Ming Li, John Tromp, Paul M.B. Vitányi — How to Share Concurrent Wait-Free Variables
CT-91-03 Ming Li, Paul M.B. Vitányi — Average Case Complexity under the Universal Distribution Equals Worst Case Complexity
CT-91-04 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence
CT-91-05 Sieger van Denneheuvel, Karen Kwast — Weak Equivalence for Constraint Sets
CT-91-06 Edith Spaan — Census Techniques on Relativized Space Classes
CT-91-07 Karen L. Kwast — The Incomplete Database
CT-91-08 Kees Doets — Levationis Laus
CT-91-09 Ming Li, Paul M.B. Vitányi — Combinatorial Properties of Finite Sequences with high Kolmogorov Complexity
CT-91-10 John Tromp, Paul Vitányi — A Randomized Algorithm for Two-Process Wait-Free Test-and-Set
CT-91-11 Lane A. Hemachandra, Edith Spaan — Quasi-Injective Reductions
CT-91-12 Krzysztof R. Apt, Dino Pedreschi — Reasoning about Termination of Prolog Programs

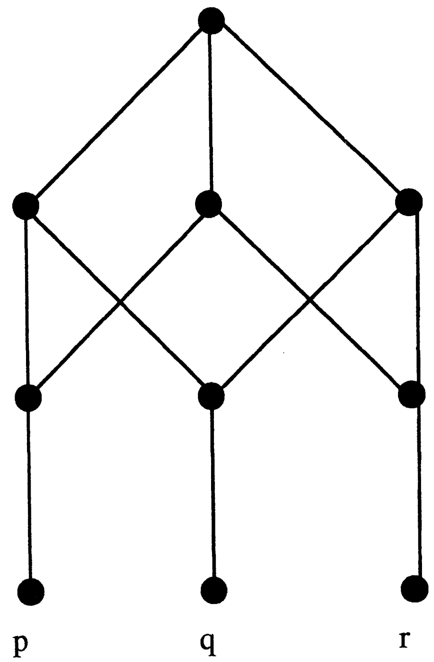
Computational Linguistics
CL-91-01 J.C. Scholtes — Kohonen Feature Maps in Natural Language Processing
CL-91-02 J.C. Scholtes — Neural Nets and their Relevance for Information Retrieval

Example 9.3.7
The diagram of $[\vee, \neg\neg, \bot]_2$ and its exact model.

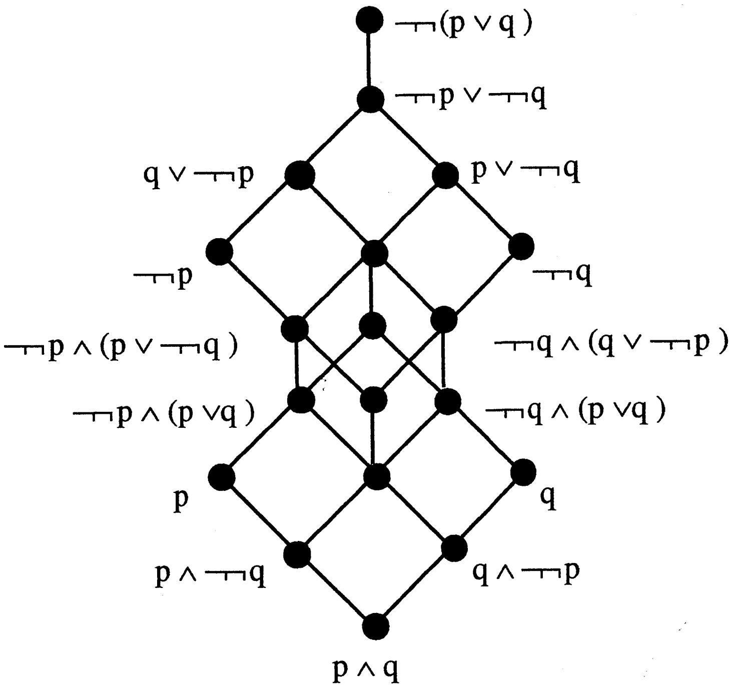


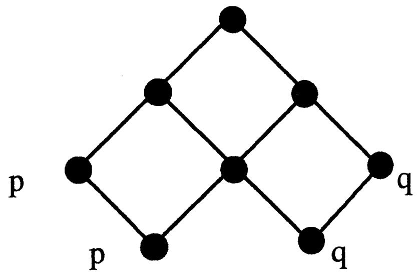


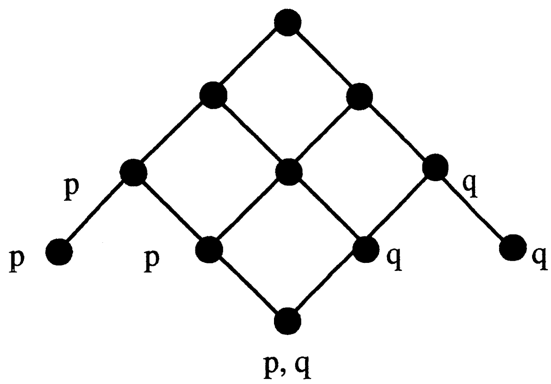
Example 9.3.8
The exact model of $[\vee, \neg\neg, \bot]_3$



Example 9.1.5
The diagram of [$\wedge, \vee, \neg\neg$, p, q], its exact model and the Kripke completion of its exact model (without $k_{\varnothing}$):



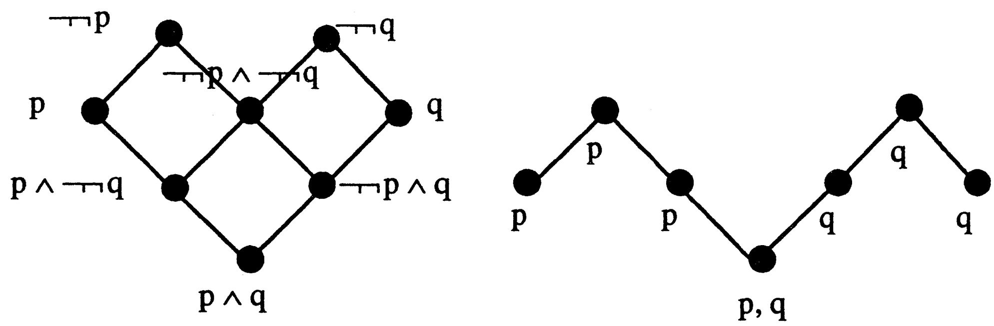




Theorem 9.2.4 If $\mathcal{E}_n = \langle E_n, \vdash \rangle$ is the exact model of $[\wedge, \neg\neg, \mathrm{T}, P_n]$ and $\mathcal{K}_n = \langle W_n, \vdash \rangle$ is an extension of $\mathcal{E}_n$ where $W_n = E_n \cup \{\bigwedge P_n \backslash Q \mid |Q| \leq 1\}$, then $\mathcal{K}_n$ is equivalent to the Kripke completion of $\mathcal{E}$ forl$[\wedge, \neg\neg, \mathrm{T}, P_n]$-formulas

Pf. Clearly $\mathcal{K}_n$ is a part of the end extension of $\mathcal{E}_n$ and if $Q = P_n$ or $Q = P_n \backslash \{p\}$ then no element of $\mathcal{E}_n$ will reduce to k_Q in the Kripke completion.
If $Q = P_n \backslash \{p, q\}$ for some $p \neq q$ then φ_Q does not imply any of the formulas in E_n. For assume that $\varphi_Q \vdash \bigwedge P_n \backslash \{p\}$ then $\varphi_Q \vdash q$ and also $\varphi_Q \vdash \neg q$, but of course φ_Q is consistent.
By lemma 5.3.8 the Kripke completion without such k_Q's is equivalent to the Kripke completion itself, as far as the $[\wedge, \neg\neg, \mathrm{T}, P_n]$-formulasareconcerned. □

Example 9.2.5 The diagram of $[\wedge, \neg\neg, p, q]$ and a Kripke completion for $[\wedge, \neg\neg, \mathrm{T}, p, q]$



Theorem 9.2.4 If $\mathfrak{E}_n = < E_n, \vdash >$ is the exact model of $[\wedge, \neg\neg, \mathrm{T}, P_n]$ and $\mathfrak{K}_n = < W_n, \vdash >$ is an extension of $\mathfrak{E}_n$ where $W_n = E_n \cup \{\Lambda P_n \backslash Q \mid |Q| \leq 1\}$, then $\mathfrak{K}_n$ is equivalent to the Kripke completion of $\mathfrak{E}$ forl$[\wedge, \neg\neg, \mathrm{T}, P_n]$-formulas

Pf. Clearly $\mathfrak{K}_n$ is a part of the end extension of $\mathfrak{E}_n$ and if $Q = P_n$ or $Q = P_n \backslash \{p\}$ then no element of $\mathfrak{E}_n$ will reduce to k_Q in the Kripke completion.
If $Q = P_n \backslash \{p, q\}$ for some $p \neq q$ then φ_Q does not imply any of the formulas in E_n. For assume that $\varphi_Q \vdash \Lambda P_n \backslash \{p\}$ then $\varphi_Q \vdash q$ and also $\varphi_Q \vdash \neg q$, but of course φ_Q is consistent.
By lemma 5.3.8 the Kripke completion without such k_Q's is equivalent to the Kripke completion itself, as far as the $[\wedge, \neg\neg, \mathrm{T}, P_n]$-formulasareconcerned. □

Example 9.2.5 The diagram of $[\wedge, \neg\neg, p, q]$ and a Kripke completion for $[\wedge, \neg\neg, \mathrm{T}, p, q]$