Institute for Logic, Language and Computation

REMARKS ON UNIFORMLY FINITELY PRECOMPLETE POSITIVE EQUIVALENCES

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Remarks on Uniformly Finitely Precomplete Positive Equivalences

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0. Introduction

A *positive equivalence* $\mathcal{S}$ is an r.e. equivalence relation on non-negative integers ω. Occasionally, especially when more than one equivalence is in sight, imagination will be spared by stipulating that each positive equivalence is an equivalence relation on its own private copy of ω, so that we think of $\mathcal{S}$ as a pair $(\mathrm{Dom}\,\mathcal{S}, \sim_{\mathcal{S}})$, $\mathrm{Dom}\,\mathcal{S}$ being, essentially, ω. Thus, a *morphism* $\mu : \mathcal{S} \to \mathcal{T}$ from a positive equivalence $\mathcal{S}$ to a positive equivalence $\mathcal{T}$ is a mapping $\mu : \mathrm{Dom}\,\mathcal{S}/{\sim_{\mathcal{S}}} \to \mathrm{Dom}\,\mathcal{T}/{\sim_{\mathcal{T}}}$ for which there exists a total recursive function $h : \mathrm{Dom}\,\mathcal{S} \to \mathrm{Dom}\,\mathcal{T}$ s.t. $\mu([x]_{\mathcal{S}}) = [h(x)]_{\mathcal{T}}$ for all $x \in \mathrm{Dom}\,\mathcal{S}$, where $[D]_{\mathcal{R}}$ stands for the closure of an element (a set of elements, a list of elements etc.) D of $\mathrm{Dom}\,\mathcal{R}$ under $\sim_{\mathcal{R}}$. The recursive function h is then said to *represent* μ. Clearly, μ can as well be represented by any r.e. subset H of $\mathrm{Dom}\,\mathcal{S} \times \mathrm{Dom}\,\mathcal{T}$ s.t. for every $x \in \mathrm{Dom}\,\mathcal{S}$ there is a pair $(z, w) \in H$ with $\mu([z]_{\mathcal{S}}) = [w]_{\mathcal{T}}$, for from (an index of) such an H one can effectively construct a representation of μ in the form of a total recursive function.

Positive equivalences together with the morphisms just described constitute a category equivalent to the category of positively numerated sets introduced in Eršov [6, Kapitel II, § 3]. Our paper focuses on one of its remoter reaches.

A positive equivalence is called *precomplete* if for any recursive programme we can effectively compute a number s.t. if this programme happens to converge then its output finds itself in the same equivalence class as that number. Precomplete (positive) equivalences have been extensively studied since the time of Eršov [5].

An application-motivated yet natural generalization of this notion is that of a *uniformly finitely precomplete* positive equivalence introduced by Montagna [8]. Here one requires the same but under the condition that the programme can only output a number in one of the finite number of equivalence classes specified beforehand. (Precise definitions are given in Section 1.)

This being a proper generalization, there are uniformly finitely precomplete positive equivalences that are not precomplete. Such are e.g. the *e-complete* positive equivalences (Bernardi & Montagna [3], Lachlan [7]), an example of which is the provable equivalence relation among sentences of formalized arithmetic (see Bernardi [2]).

The aim of this paper is to document several miscellaneous results and observations on these three classes of equivalences. Section 1 introduces the necessary definitions and notations and contains a rather lengthy exposition of earlier results that we are going to build upon. In Section 2 we address partial isomorphisms (i.e. isomorphisms of domains of proper subobjects) between e- and between precomplete equivalences. These partial isomorphisms are shown to enjoy strong and, in a way, complementary extendability properties. In Section 3 we formulate a property that singles out the e- and the precomplete positive equivalences among all the uniformly finitely precomplete ones. Section 4 shows that the effort spent in Section 3 was not objectless in that there exist uniformly finitely precomplete positive equivalences that are neither e- nor precomplete. The final Section 5 describes sets of fixed points of endomorphisms of uniformly finitely precomplete positive equivalences.

The present paper is reasonably self-contained.

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1. Assessing the heritage

We fix an acceptable numbering $(\varphi_i)_{i\in\omega}$ of unary partial recursive functions (cf. Rogers [10, Exercise 2-10]) and an acceptable numbering $(\nu_i)_{i\in\omega}$ of partial recursive functions of arity zero, which we shall call *recursive numbers.* In what follows, by an *index* of a recursive number or function we shall mean an index in the appropriate one of these two numberings. Acceptability of $(\nu_i)_{i\in\omega}$ implies that the s-1-0 case of the s-m-n Theorem holds for the two numberings: there is a total recursive function s s.t. $\nu_{s(i,x)} \simeq \varphi_i(x)$ for all i, x. Here, as well as everywhere below, $\simeq$ means that the l.h.s. converges iff the r.h.s. does and that the outputs, if any, are equal.

Expressions like $\varphi(x)\downarrow$, $\nu \in W$ for W an r.e. set and similar ones will, apart from their usual role, also stand for the number of steps needed for the corresponding Turing machine to verify that $\varphi(x)$ converges, or that ν converges and its output is in W, respectively. This number of steps is assumed to be ∞ if $\varphi(x)\uparrow$, or if ($\nu\uparrow$ or $\nu \notin W$), so that e.g. the expression $x \sim_S \nu \leq \varphi(x) \in \operatorname{rng} h$ says that ν converges, $x \sim_S \nu$ and the finite number of Turing steps needed to verify this is less than or equal to the number of Turing steps necessary to check $\varphi(x) \in \operatorname{rng} h$, which, in particular, can be ∞, i.e. $\varphi(x)$ does not have to converge, nor does its output, if any, have to lie in $\operatorname{rng} h$. The particular identity of the Turing machines involved will in each case either be clear from the context or irrelevant.

Further, we fix numberings of finite sets of numbers and of finite sets of pairs of numbers s.t. given an x (or (y,z)) and an index of D one can effectively answer the question $x \in ?\ D$ (or $(y,z) \in ?\ D$), and, from an index of D, effectively tell the cardinality of D. Indices in these numberings will be referred to as *strong indices.* The crucial properties are that from x, y we can compute strong indices of $\{x\}$ and $\{(x,y)\}$, and that unions can be performed effectively on strong indices.

For X a finite or infinite set of pairs, we denote by X^{-1} the set $\{(z,y) \mid (y,z) \in X\}$; $\operatorname{dom} X$ and $\operatorname{rng} X$ have their usual meanings, so that $\operatorname{rng} X = \operatorname{dom} X^{-1}$. Let us also agree that $X^1 = X$.

For φ a partial recursive function and an r.e. set W, $\varphi\lceil W$ denotes the partial recursive function which agrees with φ on inputs in W and diverges elsewhere. If, given a strong index of D and an index of φ, we set out to calculate the strong index of $\varphi\lceil D$, we shall only generally succeed if φ converges on all elements of D. $\varphi\lceil n$ will stand for $\varphi\lceil\{0,\ldots,n-1\}$.

Now we turn to positive equivalences. If addressing the $\mathcal{S}$-equivalence classes in plural makes sense then the positive equivalence $\mathcal{S}$ is said to be *non-trivial.*

1.1. DEFINITION. A non-trivial positive equivalence $\mathcal{P}$ is *precomplete* if there exists a total recursive function T, called a *($\mathcal{P}$-)totalizer*, which, when applied to an index of a recursive number ν, produces a number $\nu^T \in \operatorname{Dom}\mathcal{P}$ s.t.

if $\nu\!\downarrow$ then $\nu^T \sim_{\mathcal{P}} \nu$.

The notation ν^T is chosen to be tolerant to the intended abuses of format: thus, e.g. $\varphi^T(x)$ will stand for T applied to an index of the recursive number $\mu \simeq \varphi(x)$ which is obtained effectively from x and an index of φ by the s-1-0 Theorem.

Note that the property of being precomplete is invariant under isomorphisms of positive equivalences: Let $\mathcal{P}$ be precomplete and $\mu : \mathcal{P} \to \mathcal{T}$ an isomorphism and hence a bijection $\operatorname{Dom}\mathcal{P}/{\sim_{\mathcal{P}}} \to \operatorname{Dom}\mathcal{T}/{\sim_{\mathcal{T}}}$, represented by a total recursive h. Clearly, $\mathcal{T}$ is non-trivial. Let k represent μ^{-1} and T be a $\mathcal{P}$-totalizer. It can then be easily seen that the function K defined by $\nu^K \simeq h((k(\nu))^T)$ is a $\mathcal{T}$-totalizer. Similarly, precompleteness is preserved under non-trivial epimorphic images.

A partial recursive function $\delta : \operatorname{Dom}\mathcal{S} \to \operatorname{Dom}\mathcal{S}$ is *($\mathcal{S}$-)diagonal* if $\delta(x) \not\sim_{\mathcal{S}} x$ whenever $\delta(x)\!\downarrow$.

Our starting point is the following theorem of Visser:

1.2. PROPOSITION (Anti Diagonal Normalization (ADN) Theorem, Visser [12]). *Suppose $\mathcal{P}$ is precomplete. There exists a binary total recursive function T, which, when applied to indices of a $\mathcal{P}$-diagonal partial recursive function δ and of a recursive number ν, produces a number $\nu^{T_\delta} \in \operatorname{Dom}\mathcal{P}$ s.t.*

(i) *if $\nu\!\downarrow$ then $\nu^{T_\delta} \sim_{\mathcal{P}} \nu$, and*

(ii) *if $\nu\!\uparrow$ then $\nu^{T_\delta} \notin \operatorname{dom}\delta$.*

PROOF. Let T be a $\mathcal{P}$-totalizer. Given a recursive number ν, construct another recursive number μ by the Recursion Theorem:

$$\mu \simeq \begin{cases} \nu & \text{if } \nu\!\downarrow\, \leq \delta(\mu^T)\!\downarrow, \\ \delta(\mu^T) & \text{if } \delta(\mu^T)\!\downarrow\, < \nu\!\downarrow, \\ \text{divergent} & \text{if neither } \nu \text{ nor } \delta(\mu^T) \text{ converges.} \end{cases}$$

Define $\nu^{T_\delta} = \mu^T$. Clearly, ν^{T_δ} is recursive in the indices of ν and δ.

If μ diverges then we have $\nu\uparrow$ and $\nu^{T_\delta} = \mu^T \notin \operatorname{dom}\delta$.

If $\nu\downarrow \leq \delta(\mu^T)\downarrow$ then $\nu^{T_\delta} = \mu^T \sim_{\mathcal{P}} \mu = \nu$ as required.

Were $\delta(\mu^T)\downarrow < \nu\downarrow$ the case, which would, in particular, happen if $\nu\uparrow$ and $\nu^{T_\delta} \in \operatorname{dom}\delta$, we would have $\mu^T \sim_{\mathcal{P}} \mu = \delta(\mu^T)$. But then δ could not be $\mathcal{P}$-diagonal. ∎

An important generalization of the ADN Theorem was found by Sommaruga-Rosolemos. It shares the proof with its predecessor.

1.3. Proposition (General Fixed Point (GFP) Theorem, Sommaruga-Rosolemos [11]). *Suppose $\mathcal{P}$ is precomplete. There exists a binary total recursive function T, which, when applied to indices of a partial recursive function δ and of a recursive number ν, produces a number $\nu^{T_\delta} \in \operatorname{Dom}\mathcal{P}$ s.t.*

(i) *if $\nu\downarrow$, then $\nu^{T_\delta} \sim_{\mathcal{P}} \nu$ or $\nu^{T_\delta} \sim_{\mathcal{P}} \delta(\nu^{T_\delta})$, and*

(ii) *if $\nu\uparrow$, then $\nu^{T_\delta} \notin \operatorname{dom}\delta$ or $\nu^{T_\delta} \sim_{\mathcal{P}} \delta(\nu^{T_\delta})$.* ∎

Below, T_δ will denote the function corresponding to a totalizer T by the ADN and GFP Theorems.

1.4. Proposition (Eršov [5]). *For $\mathcal{P}$ a positive equivalence the following are equivalent:*

(i) *$\mathcal{S}$ is precomplete.*

(ii) *$\mathcal{S}$ is non-trivial and there is a total recursive function f s.t. for all i one has $\varphi_i(f(i))\uparrow$ or $f(i) \sim_{\mathcal{S}} \varphi_i(f(i))$.*

Proof. (ii)$\Rightarrow$(i). Suppose the function f satisfies (ii). We construct a $\mathcal{P}$-totalizer T. Given a recursive number ν, consider the function $\varphi_i(x) \simeq \nu$ and let $\nu^T = f(i)$. Clearly, T is total recursive and is a $\mathcal{P}$-totalizer, for assuming $\nu\downarrow$ we have that φ_i is total and hence $\nu^T = f(i) \sim_{\mathcal{P}} \varphi_i(f(i)) = \nu$ as required.

(i)$\Rightarrow$(ii) (Sommaruga-Rosolemos [11]). We are given a $\mathcal{P}$-totalizer T and have to produce an f as in (ii). Fix a diverging recursive number π and put $f(i) = \pi^{T_{\varphi_i}}$. Since $\pi\uparrow$, we have by the GFP Theorem that either $\varphi_i(\pi^{T_{\varphi_i}})\uparrow$ or $\pi^{T_{\varphi_i}} \sim_{\mathcal{P}} \varphi_i(\pi^{T_{\varphi_i}})$. The same theorem asserts that the function $f(i) = \pi^{T_{\varphi_i}}$ is total recursive q.e.d. ∎

1.5. Definition. A non-trivial positive equivalence $\mathcal{Q}$ is *uniformly finitely precomplete* if there exists a binary total recursive function τ, called a *partial ($\mathcal{Q}$-)totalizer*, which, when applied to an index of a recursive number ν and a finite set $D \subseteq \operatorname{Dom}\mathcal{Q}$ given by a strong index, produces a number $\nu^{\tau(D)} \in \operatorname{Dom}\mathcal{Q}$ s.t.

if $\nu\downarrow$ and $\nu \in [D]_{\mathcal{Q}}$ then $\nu^{\tau(D)} \sim_{\mathcal{Q}} \nu$.

In the sequel we shall be feeding the finite set-arguments to partial totalizers and similar functions in the liberal form of lists of elements and finite sets of elements of Dom $\mathcal{Q}$.

As with precomplete positive equivalences, it is easily verified that uniform finite precompleteness preserves under non-trivial epimorphic images.

An analogue of the GFP Theorem holds for uniformly finitely precomplete positive equivalences:

1.6. PROPOSITION (Relativized GFP (RGFP) Theorem). *Suppose $\mathcal{Q}$ is uniformly finitely precomplete. There exists a ternary total recursive function τ, which, when applied to indices of a partial recursive function δ and of a recursive number ν and a finite set $D \subseteq \operatorname{Dom} \mathcal{Q}$ given by a strong index, produces a number $\nu^{\tau_\delta(D)} \in \operatorname{Dom} \mathcal{Q}$ s.t. under the condition $\operatorname{rng} \delta \subseteq [D]_{\mathcal{Q}}$ we have:*

(i) *if $\nu\downarrow$ and $\nu \in [D]_{\mathcal{Q}}$, then $\nu^{\tau_\delta(D)} \sim_{\mathcal{Q}} \nu$ or $\nu^{\tau_\delta(D)} \sim_{\mathcal{Q}} \delta(\nu^{\tau_\delta(D)})$, and*

(ii) *if $\nu\uparrow$ or $\nu \notin [D]_{\mathcal{Q}}$, then $\nu^{\tau_\delta(D)} \notin \operatorname{dom} \delta$ or $\nu^{\tau_\delta(D)} \sim_{\mathcal{Q}} \delta(\nu^{\tau_\delta(D)})$.*

PROOF. Fix a partial $\mathcal{Q}$-totalizer τ. For ν a recursive number put

$$\mu \simeq \begin{cases} \nu & \text{if } \nu \in [D]_{\mathcal{Q}} \leq \delta(\mu^{\tau(D)})\downarrow, \\ \delta(\mu^{\tau(D)}) & \text{if } \delta(\mu^{\tau(D)})\downarrow < \nu \in [D]_{\mathcal{Q}}, \\ \text{divergent} & \text{if neither } \nu \in [D]_{\mathcal{Q}} \text{ nor } \delta(\mu^{\tau(D)})\downarrow. \end{cases}$$

Define $\nu^{\tau_\delta(D)} = \mu^{\tau(D)}$.

Suppose first $\nu \in [D]_{\mathcal{Q}} \leq \delta(\mu^{\tau(D)})\downarrow$. In this case $\nu \in [D]_{\mathcal{Q}}$ and hence $\nu^{\tau_\delta(D)} = \mu^{\tau(D)} \sim_{\mathcal{Q}} \mu = \nu$ as required.

Next treat the case $\delta(\mu^{\tau(D)})\downarrow < \nu \in [D]_{\mathcal{Q}}$: One has $\mu = \delta(\mu^{\tau(D)}) \in \operatorname{rng} \delta$ and, therefore, $\nu^{\tau_\delta(D)} = \mu^{\tau(D)} \sim_{\mathcal{Q}} \mu = \delta(\mu^{\tau(D)}) = \delta(\nu^{\tau_\delta(D)})$, which satisfies the requirements regardless of the behaviour of ν.

Finally, if μ diverges then we have $\nu\uparrow$ or $\nu \notin [D]_{\mathcal{Q}}$, and $\nu^{\tau_\delta(D)} = \mu^{\tau(D)} \notin \operatorname{dom} \delta$. ∎

The only notational distinction between a partial totalizer τ and the function τ of the RGFP Theorem that we are going to maintain is that of arity.

1.7. COROLLARY (Relativized ADN (RADN) Theorem, Montagna [8]). *Suppose $\mathcal{Q}$ is uniformly finitely precomplete. There exists a ternary total recursive function τ, which, when applied to indices of a $\mathcal{Q}$-diagonal partial recursive function δ and of a recursive number ν and a finite set $D \subseteq \operatorname{Dom} \mathcal{Q}$ given by a strong index, produces a number $\nu^{\tau_\delta(D)} \in \operatorname{Dom} \mathcal{Q}$ s.t. under the condition $\operatorname{rng} \delta \subseteq [D]_{\mathcal{Q}}$ we have:*

(i) *if $\nu\downarrow$ and $\nu \in [D]_{\mathcal{Q}}$ then $\nu^{\tau_\delta(D)} \sim_{\mathcal{Q}} \nu$, and*

(ii) *if $\nu\uparrow$ or $\nu \notin [D]_{\mathcal{Q}}$ then $\nu^{\tau_\delta(D)} \notin \operatorname{dom} \delta$.* ∎

A positive equivalence $\mathcal{M}$ is called *m-complete* if for any positive equivalence $\mathcal{S}$ there exists a monomorphism $\mathcal{S} \rightarrow \mathcal{M}$.

To proceed with the next piece we introduce the following notation:

$$[\![\varphi]\!]_{\mathcal{S}}(x) \simeq \begin{cases} \varphi(y) & \text{for some } y \text{ s.t. } y \sim_{\mathcal{S}} x, \\ \text{divergent} & \text{if no such } y \text{ exists,} \end{cases}$$

where $\mathcal{S}$ is a positive equivalence and φ an arbitrary partial recursive function. Any choice of y in the first clause would do as long as it results in a partial recursive $[\![\varphi]\!]_{\mathcal{S}}$.

More generally, this definition will work for any r.e. set $H \subseteq \mathrm{Dom}\,\mathcal{S} \times \omega$, producing a partial recursive function

$$[\![H]\!]_{\mathcal{S}}(x) \simeq \begin{cases} z & \text{for some } (y,z) \in H \text{ s.t. } y \sim_{\mathcal{S}} x, \\ \text{divergent} & \text{if no such } (y,z) \text{ exists,} \end{cases}$$

so that later we shall be freely using notation like, for example, $[\![\varphi^{-1} \cup \psi]\!]_{\mathcal{S}}$ for partial recursive φ and ψ.

1.8. PROPOSITION (Bernardi & Sorbi [4], Montagna [8]). *Any uniformly finitely precomplete positive equivalence $\mathcal{Q}$ is m-complete. In particular, the number of $\mathcal{Q}$-equivalence classes is infinite.*

PROOF. Since $\mathcal{Q}$ is non-trivial, there exist $a, b \in \mathrm{Dom}\,\mathcal{Q}$ with $a \not\sim_{\mathcal{Q}} b$. Fix one such pair. Given a positive equivalence $\mathcal{S}$, we define a function $h : \mathrm{Dom}\,\mathcal{S} \to \mathrm{Dom}\,\mathcal{Q}$ and a sequence of (indices of) partial recursive functions $\delta_x : \mathrm{Dom}\,\mathcal{Q} \to \mathrm{Dom}\,\mathcal{Q}$ by simultaneous recursion:

$$h(x) \simeq [\![h\lceil x]\!]_{\mathcal{S}}^{\tau_{\delta_x}(\mathrm{rng}\,h\lceil x;a,b)}(x),$$

$$\text{where} \qquad \delta_x(y) \simeq \begin{cases} a & \text{if } y \in [\mathrm{rng}\,h\lceil x]_{\mathcal{Q}} \leq y \sim_{\mathcal{Q}} a, \\ b & \text{if } y \sim_{\mathcal{Q}} a < y \in [\mathrm{rng}\,h\lceil x]_{\mathcal{Q}}, \\ \text{divergent} & \text{otherwise.} \end{cases}$$

Clearly, h is total.

By induction on x we prove the following:

(i) If $y, z < x$, then $y \sim_{\mathcal{S}} z$ iff $h(y) \sim_{\mathcal{Q}} h(z)$, and

(ii) $a \notin [\mathrm{rng}\,h\lceil x]_{\mathcal{Q}}$.

By (ii) of the IH we have

$$\delta_x(y) \simeq \begin{cases} a & \text{if } y \in [\mathrm{rng}\,h\lceil x]_{\mathcal{Q}}, \\ b & \text{if } y \sim_{\mathcal{Q}} a, \\ \text{divergent} & \text{otherwise.} \end{cases}$$

From the same premiss it follows that δ_x is $\mathcal{Q}$-diagonal and hence, by the RADN Theorem, $h(x) \not\sim_{\mathcal{Q}} a \in \mathrm{dom}\,\delta_x$ implying $a \notin [\mathrm{rng}\,h\lceil x{+}1]_{\mathcal{Q}}$.

If for some $y < x$ we have $y \sim_{\mathcal{S}} x$ then $[\![h\lceil x]\!]_{\mathcal{S}}(x) = h(z)$ for some $z \sim_{\mathcal{S}} x$ s.t. $z < x$ and, since δ_x is diagonal, $h(x) \sim_{\mathcal{Q}} [\![h\lceil x]\!]_{\mathcal{S}}(x)$ by the RADN Theorem. By (i) of the IH one then has $h(x) \sim_{\mathcal{Q}} [\![h\lceil x]\!]_{\mathcal{S}}(x) = h(z) \sim_{\mathcal{Q}} h(y)$.

In the case that no such y exists, and so $[\![h\lceil x]\!]_{\mathcal{S}}(x)\uparrow$, the RADN Theorem guarantees that $h(x) \notin \mathrm{dom}\,\delta_x \supseteq [\mathrm{rng}\,h\lceil x]_{\mathcal{Q}}$ whence $h(x) \not\sim_{\mathcal{Q}} h(y)$ for all $y < x$. Thus (i) of the induction step is also established.

From (i) it now readily follows that h represents a monomorphism $\mathcal{S} \to \mathcal{Q}$. ∎

1.9. DEFINITION. A non-trivial positive equivalence $\mathcal{E}$ is *e-complete* if there exists a binary total recursive function τ_+, called a *precision partial ($\mathcal{E}$-)totalizer*, which, when applied to an index of a recursive number ν and (a strong index of) a finite set $D \subseteq \mathrm{Dom}\,\mathcal{E}$, outputs a number $\nu^{\tau_+(D)} \in \mathrm{Dom}\,\mathcal{E}$ s.t.

(i) if $\nu\downarrow$ and $\nu \in [D]_{\mathcal{E}}$, then $\nu^{\tau_+(D)} \sim_{\mathcal{E}} \nu$, and

(ii) if $\nu\uparrow$ or $\nu \notin [D]_{\mathcal{E}}$, then $\nu^{\tau_+(D)} \notin [D]_{\mathcal{E}}$.

Once again, one can check that e-completeness is preserved by isomorphisms, although it does not generally preserve under (non-trivial) epimorphic images.

1.10. PROPOSITION (Montagna [8] and Bernardi & Montagna [3]). *Let $\mathcal{E}$ be a positive equivalence. The following are equivalent:*

(i) *$\mathcal{E}$ is e-complete.*

(ii) *$\mathcal{E}$ is uniformly finitely precomplete and there exists a total recursive $\mathcal{E}$-diagonal function δ, called an ($\mathcal{E}$)-shift.*

(iii) *$\mathcal{E}$ is uniformly finitely precomplete and there exists a total recursive function Δ, called a collective ($\mathcal{E}$)-shift, s.t. for any finite subset D of $\operatorname{Dom}\mathcal{E}$ given by a strong index, we have $\Delta(D) \notin [D]_{\mathcal{E}}$.*

PROOF. (i)$\Rightarrow$(ii). Clearly, $\mathcal{E}$ is uniformly finitely precomplete for any precision partial totalizer is a partial totalizer.

Let τ_+ be a precision partial $\mathcal{E}$-totalizer and π a divergent recursive number. Put $\delta(x) = \pi^{\tau_+(x)}$. Since $\pi\uparrow$, we have $\pi^{\tau_+(x)} \notin [x]_{\mathcal{E}}$, that is, $\delta(x) \not\sim_{\mathcal{E}} x$. Thus δ is an $\mathcal{E}$-shift.

(ii)$\Rightarrow$(iii). We have to obtain a collective $\mathcal{E}$-shift Δ from a given $\mathcal{E}$-shift δ.

For D a finite set given by a strong index and a diverging recursive number π, put

$$\Delta(D) = \pi^{\tau_\gamma(\operatorname{rng}\delta\lceil D)},$$

$$\text{where} \qquad \gamma(y) \simeq \begin{cases} \delta(x) & \text{for some } x \text{ s.t. } y \sim_{\mathcal{E}} x \in D, \\ \text{divergent} & \text{if no such } x \text{ exists,} \end{cases}$$

τ being a partial $\mathcal{E}$-totalizer. Observe that γ is $\mathcal{E}$-diagonal, and hence the RADN Theorem guarantees that $\Delta(D) \notin \operatorname{dom}\gamma = [D]_{\mathcal{E}}$ as required.

(iii)$\Rightarrow$(i). We have got a partial $\mathcal{E}$-totalizer τ and a collective $\mathcal{E}$-shift Δ. Here is how to produce a precision partial $\mathcal{E}$-totalizer: Let

$$\nu^{\tau_+(D)} = \nu^{\tau_\beta(D;\Delta(D))}, \qquad \text{where} \qquad \beta(y) \simeq \begin{cases} \Delta(D) & \text{if } y \in [D]_{\mathcal{E}}, \\ \text{divergent} & \text{otherwise} \end{cases}$$

and observe with the help of the RADN Theorem that since β is diagonal, τ_+ is a precision partial $\mathcal{E}$-totalizer. ∎

1.11. COROLLARY (Bernardi–Montagna Fixed Point Property, Bernardi & Montagna [3]). *If a uniformly finitely precomplete positive equivalence $\mathcal{Q}$ is not e-complete then for every total recursive function $\varphi : \operatorname{Dom}\mathcal{Q} \to \operatorname{Dom}\mathcal{Q}$ there exists a $\mathcal{Q}$-fixed point x: $x \sim_{\mathcal{Q}} \varphi(x)$.* ∎

Note that since the equivalence relation $\sim_{\mathcal{Q}}$ of the above Corollary is r.e., the fixed point in question can be effectively calculated by a recursive function: $f(i) \sim_{\mathcal{Q}} \varphi_i(f(i))$, but $f(i)$ does not have to converge unless φ_i is total (compare this with Proposition 1.4).

Most of the considerations of the present paper would have hardly been properly grounded were it not for the

1.12. FACT. *Pre- and e-complete positive equivalences exist.*

COMMENT. The reader may consult Eršov [5], Visser [12], Bernardi & Montagna [3] and Visser [12], Bernardi & Montagna [3], Lachlan [7] for (natural) examples of pre- and e-complete positive equivalences respectively. ∎

Finally, we would like to mention (without proof) a result which shows that the e- and the precomplete equivalences occupy somewhat polar positions within the class of uniformly finitely precomplete ones.

1.13. PROPOSITION (Bernardi & Montagna [3]). *For any e-complete $\mathcal{E}$, uniformly finitely precomplete $\mathcal{Q}$ and precomplete $\mathcal{P}$, there are epimorphisms $\mathcal{E} \to \mathcal{Q}$ and $\mathcal{Q} \to \mathcal{P}$. In particular, the uniformly finitely precomplete positive equivalences are precisely the non-trivial factorobjects of the e-complete ones.* ∎

2. Extending partial isomorphisms

Partial morphisms $\xi : \mathcal{S} \to \mathcal{T}$ between positive equivalences $\mathcal{S}$ and $\mathcal{T}$ are partial mappings $\mathrm{Dom}\,\mathcal{S}/\!\sim_{\mathcal{S}} \to \mathrm{Dom}\,\mathcal{T}/\!\sim_{\mathcal{T}}$ that are *represented* by some partial recursive function $h : \mathrm{Dom}\,\mathcal{S} \to \mathrm{Dom}\,\mathcal{T}$ s.t for all $x \in \mathrm{Dom}\,\mathcal{S}$, $h(x)$ converges iff $\xi([x]_{\mathcal{S}})$ is defined, and, if so, $\xi([x]_{\mathcal{S}}) = [h(x)]_{\mathcal{T}}$. More generally, partial morphisms can be represented by appropriate r.e. subsets of $\mathrm{Dom}\,\mathcal{S} \times \mathrm{Dom}\,\mathcal{T}$.

A partial morphism ξ is *finite* if so is the set of equivalence classes on which ξ is defined. Note that any finite partial morphism can be represented by a finite set of pairs of representatives of the equivalence classes. If such a set is given by a strong index, one speaks of a *strong representation.*

A morphism $\mu : \mathcal{S} \to \mathcal{T}$ is an *extension* of a partial morphism $\xi : \mathcal{S} \to \mathcal{T}$ if $\xi \subseteq \mu$ as functions on equivalence classes.

A partial morphism $\xi : \mathcal{S} \to \mathcal{T}$ is a *partial isomorphism* if the partial mapping ξ is one-one, and there are $\mathcal{S}$-equivalence classes on which ξ is undefined if and only if there are $\mathcal{T}$-equivalence classes with no ξ-preimage.

It has been known for some time that all e-complete positive equivalences are isomorphic (Montagna [8], Lachlan [7]) as well as all precomplete ones (Lachlan [7]). We shall demonstrate that the isomorphisms involved can be constructed as extensions of a large class of partial isomorphisms. In the precomplete case, our proof is a straightforward modification of the original isomorphism proof. The proof of the e-complete case does not differ from the original one.

2.1. THEOREM (after Montagna [8]). *Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be e-complete. Suppose a finite* *set $F \subseteq \mathrm{Dom}\,\mathcal{E}_0 \times \mathrm{Dom}\,\mathcal{E}_1$ strongly represents a finite partial isomorphism $\xi : \mathcal{E}_0 \longrightarrow \mathcal{E}_1$. Then we can construct (a representation of) an isomorphism $\mu : \mathcal{E}_0 \longrightarrow \mathcal{E}_1$ extending ξ effectively in a strong index of F.*

PROOF (Montagna [8]). Starting from a set F as in the statement, we compile an r.e. set of pairs $H \supseteq F$ representing an isomorphism μ which then obviously extends ξ. This is done by a standard back-and-forth argument of which, since the situation is perfectly symmetric, we present just one step, namely, the first one.

Since F strongly represents a partial isomorphism, we have for all $(x, w), (y, z) \in F$ that $x \sim_{\mathcal{E}_0} y$ iff $w \sim_{\mathcal{E}_1} z$. Let u be an arbitrary element of $\mathrm{Dom}\,\mathcal{E}_0 - \mathrm{dom}\,F$. Let τ^1_+ be a precision partial $\mathcal{E}_1$-totalizer and define v by

$$v = [\![F]\!]_{\mathcal{E}_0}^{\tau^1_+(\mathrm{rng}\,F)}(u).$$

Note that if $u \sim_{\mathcal{E}_0} x$ for $(x, w) \in F$ then $[\![F]\!]_{\mathcal{E}_0}(u) \sim_{\mathcal{E}_0} w \in \mathrm{rng}\,F$ and hence $v \sim_{\mathcal{E}_1} w$. If u is in an $\mathcal{E}_0$-equivalence class alien to F then $v \notin [\mathrm{rng}\,F]_{\mathcal{E}_1}$. Thus $F \cup \{(u, v)\}$ also represents a finite partial isomorphism and hence can be enumerated in H. ∎

2.2. COROLLARY (Montagna [8], Lachlan [7]). *Any two e-complete positive equivalences are isomorphic.* ∎

2.3. COROLLARY. *Let $\mathcal{E}$ be an e-complete and $\mathcal{S}$ an arbitrary positive equivalence. One can find monomorphic extensions of finite partial isomorphisms $\xi : \mathcal{S} \longrightarrow \mathcal{E}$ effectively in strong representations of ξ.*

PROOF. By Proposition 1.8 there exists a monomorphism $\eta : \mathcal{S} \longrightarrow \mathcal{E}$. From our premisses it follows that $\xi \circ \eta^{-1}$ is a finite partial automorphism $\mathcal{E} \longrightarrow \mathcal{E}$ whose strong representation one can find effectively from that of ξ. By Theorem 2.1 we can effectively obtain a representation of a full automorphism $\mu : \mathcal{E} \longrightarrow \mathcal{E}$ extending $\xi \circ \eta^{-1}$. It is then easily verified that $\mu \circ \eta : \mathcal{S} \longrightarrow \mathcal{E}$ is the required monomorphic extension of ξ. ∎

The property asserted by Corollary 2.3 of e-complete positive equivalences is used in Lachlan [7] to define the notion of e-completeness. The following Proposition shows that a slightly weaker property already distinguishes the e-complete positive equivalences among the uniformly finitely precomplete ones. In particular, Theorem 2.1 does not hold for precomplete positive equivalences.

2.4. PROPOSITION. *Suppose a uniformly finitely precomplete positive equivalence $\mathcal{E}$ is such that monomorphic extensions of finite partial automorphisms $\xi : \mathcal{E} \longrightarrow \mathcal{E}$ are found effectively in strong representations of the latter. Then $\mathcal{E}$ is e-complete.*

PROOF. Let $\mathcal{E}$ be uniformly finitely precomplete and suppose a recursive function ι provides indices of monomorphic extensions of finite partial automorphisms strongly represented by its argument. Fix $a, b \in \mathrm{Dom}\,\mathcal{E}$ s.t. $a \not\sim_{\mathcal{E}} b$. The function $\delta(x) \simeq \varphi_{\iota(\{(a,x)\})}(b)$ is then an $\mathcal{E}$-shift and hence, by Proposition 1.10, $\mathcal{E}$ is e-complete. ∎

Despite the failure of effective extendability for precomplete positive equivalences, we are going to see that one can prolong not just finite, but arbitrary partial isomorphisms between these. First, however, we point out that this is not the case for e-complete equivalences.

2.5. PROPOSITION. *Suppose $\mathcal{E}$ is e-complete. There is a partial automorphism of $\mathcal{E}$ which can not be extended to a full endomorphism.*

PROOF. Let $\mathcal{I}$ be the minimal positive equivalence: $x \sim_{\mathcal{I}} y$ iff $x = y$. We shall produce two monomorphisms $\eta, \theta : \mathcal{I} \to \mathcal{E}$ s.t. the partial automorphism $\theta \circ \eta^{-1} : \mathcal{E} \to \mathcal{E}$ is not extendable.

Let $\mathcal{P}$ be a precomplete positive equivalence. Since by Proposition 1.8 both $\mathcal{P}$ and $\mathcal{E}$ are m-complete, there are monomorphisms $\kappa : \mathcal{I} \to \mathcal{P}$ and $\lambda : \mathcal{P} \to \mathcal{E}$. Let $\eta = \lambda \circ \kappa$.

To construct θ, let Δ be a collective $\mathcal{E}$-shift. Define

$$t(x) \simeq \Delta(\operatorname{rng} t\lceil x; 0, \ldots, x).$$

Clearly, t is total. Furthermore, it represents a monomorphism $\mathcal{I} \to \mathcal{E}$ for if $x \not\sim_{\mathcal{I}} y$, that is, $x \neq y$, then either $t(x) \in \operatorname{rng} t\lceil y$ or $t(y) \in \operatorname{rng} t\lceil x$ and, therefore, $t(x) \not\sim_{\mathcal{E}} t(y)$. Let θ be the monomorphism represented by t.

Suppose $\mu : \mathcal{E} \to \mathcal{E}$ were an endomorphism extending $\theta \circ \eta^{-1}$. Let then a total recursive function $h : \operatorname{Dom} \mathcal{P} \to \operatorname{Dom} \mathcal{E}$ represent $\mu \circ \lambda$.

Consider the function $\delta : \operatorname{Dom} \mathcal{P} \to \operatorname{Dom} \mathcal{P}$:

$$\delta(x) \simeq [\![h^{-1}]\!]_{\mathcal{E}} \circ t(h(x)).$$

Note that since μ extends $\theta \circ \eta^{-1}$, we have $\theta = \mu \circ \eta = \mu \circ \lambda \circ \kappa$ and hence $\operatorname{rng} t \subseteq [\operatorname{rng} h]_{\mathcal{E}} = \operatorname{dom} [\![h^{-1}]\!]_{\mathcal{E}}$ because t represents θ and h represents $\mu \circ \lambda$. Therefore, $[\![h^{-1}]\!]_{\mathcal{E}} \circ t$ and hence δ are total.

Further, observe that we have $h \circ \delta(x) \sim_{\mathcal{E}} h \circ [\![h^{-1}]\!]_{\mathcal{E}} \circ t(h(x)) \sim_{\mathcal{E}} t(h(x))$, whereas $t(h(x)) \not\sim_{\mathcal{E}} h(x)$ by the definition of t. This gives $h \circ \delta(x) \not\sim_{\mathcal{E}} h(x)$ and so $\delta(x) \not\sim_{\mathcal{P}} x$ for h represents a morphism $\mathcal{P} \to \mathcal{E}$. Thus δ is a $\mathcal{P}$-shift which is impossible since $\mathcal{P}$ is precomplete.

The contradiction proves that no endomorphism μ extending $\theta \circ \eta^{-1}$ exists. ∎

2.6. THEOREM (after Lachlan [7]). *Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be precomplete. Then any partial isomorphism $\mathcal{P}_0 \to \mathcal{P}_1$ has an extension to an isomorphism $\mathcal{P}_0 \to \mathcal{P}_1$.*

PROOF. Let a partial isomorphism $\xi : \mathcal{P}_0 \to \mathcal{P}_1$ be represented by an r.e. set $h \subseteq \operatorname{Dom} \mathcal{P}_0 \times \operatorname{Dom} \mathcal{P}_1$. We assume that there are $\mathcal{P}_0$-equivalence classes not mapped by ξ as well as $\mathcal{P}_1$-equivalence classes left unmapped to by ξ, for otherwise ξ is a full-blown isomorphism in which case nothing needs to be done.

We are going to describe a construction commencing in recursive Stages meant to bring forth a representation $H \subseteq \operatorname{Dom} \mathcal{P}_0 \times \operatorname{Dom} \mathcal{P}_1$ of the desired extension of ξ.

In various lower indices below, $\mathcal{P}_i$ will be persistently replaced by i, so that e.g. $\sim_i$ stands for $\sim_{\mathcal{P}_i}$.

Before and after each Stage, for both i we have a finite collection of *i-clusters*. These are pairwise disjoint finite sets of elements of $\operatorname{Dom} \mathcal{P}_i$ and they are meant to simulate the $\mathcal{P}_i$-equivalence classes. Elements of $\operatorname{Dom} \mathcal{P}_i$ that are not found in any i-cluster are called *unattended*. Each i-cluster is ascribed one of the three statuses: *homegrown*, *imported*, or *pre-engaged*. Each homegrown i-cluster is assigned an index (of a recursive number). Furthermore, there is a bijection * between the i- and the $(1-i)$-clusters. An i-cluster A is homegrown if and only if the $(1-i)$-cluster A^* is imported and A is pre-engaged if and only if A^* is. (We stipulate that * works both ways so that $^{**} = \mathrm{id}$.)

Before Stage 0 there are no clusters and so all elements of $\mathrm{Dom}\,\mathcal{P}_0$ and $\mathrm{Dom}\,\mathcal{P}_1$ are unattended and * is void.

At each Stage of the construction, new clusters may be created, previously unattended elements may be added to a cluster, two already existing i-clusters may merge to form a new i-cluster and the status of a cluster may change. The bijection * is then updated in such a way that along with any newly born i-cluster A we also create a new $(1-i)$-cluster B and put $A^* = B$; if a new element is added to a cluster A then $(A \cup \{a\})^* = A^*$; if two i-clusters A and C merge then A^* and C^* also do and we have $(A \cup C)^* = A^* \cup C^*$; changes in status do not affect *. This will ensure that once at some Stage for $a \in \mathrm{Dom}\,\mathcal{P}_i$, $b \in \mathrm{Dom}\,\mathcal{P}_{1-i}$ there are clusters A and B respectively with $a \in A$, $A^* = B$ and $b \in B$, then two clusters satisfying these conditions can be found at any later Stage.

Let T^0 and T^1 be $\mathcal{P}_0$- and $\mathcal{P}_1$-totalizers respectively. For $i = 0, 1$ fix elements $x_i, y_i \in \mathrm{Dom}\,\mathcal{P}_i$ s.t. $x_i \in \mathrm{dom}\,h^{1-2i}$ and $[y_i]_{\mathcal{E}} \cap \mathrm{dom}\,h^{1-2i} = \emptyset$. Note that $x_i \not\sim_i y_i$.

As we proceed to describe the Stages of the construction, the reader is invited to inductively check that, apart from the fact that the construction adheres to the rules indicated, the following clauses hold before and after each Stage:

(i) Any i-cluster only contains $\mathcal{P}_i$-equivalent elements of $\mathrm{Dom}\,\mathcal{P}_i$.

(ii) For any pre-engaged 0- and 1-clusters A and B with $A^* = B$ we have $\xi([A]_0) = [B]_1$, and

(iii) For any homegrown i-custer A and its associated index j, $\nu_j^{T^{1-i}} \in A^*$.

While clause (iii) should be unproblematic, we shall be explaining wherever necessary why the instructions of each Stage violate neither (i) nor (ii).

S t a g e $6n + i$ $(i = 0, 1)$.

Find the smallest unattended number a in $\mathrm{Dom}\,\mathcal{P}_i$. Fix an index j. (Since the construction may later define the value of ν_j, this involves an appeal to the Recursion Theorem.) Calculate $b = \nu_j^{T^{1-i}} \in \mathrm{Dom}\,\mathcal{P}_{1-i}$.

If b is unattended, let $\{a\}$ and $\{b\}$ be a pair of new clusters with $\{a\}^* = \{b\}$, $\{a\}$ homegrown and $\{b\}$ imported. Associate the index j to $\{a\}$.

If b is inside a $(1-i)$-cluster B, consider B's current status:

C a s e A. B is homegrown.

Let then k be the index associated with B. Define $\nu_k = a$. Let a join the imported i-cluster B^* to form a new homegrown i-cluster $B^* \cup \{a\}$ with j the associated index. The $(1-i)$-cluster B becomes imported and forms a *-pair with the new i-cluster $B^* \cup \{a\}$. (Note that we then have $a = \nu_k \sim_i \nu_k^{T^i} \in B^*$ so that the new i-cluster remains inside a single $\mathcal{P}_i$-equivalence class.)

C a s e B. B is imported.

Let k be the index associated with the homegrown i-cluster B^*. Define $\nu_k = x_{1-i}$ and $\nu_j = y_{1-i}$. (This implies $x_{1-i} = \nu_k \sim_{1-i} \nu_k^{T^{1-i}} \in B \ni b = \nu_j^{T^{1-i}} \sim_{1-i} \nu_j = y_{1-i}$ making $x_{1-i} \sim_{1-i} y_{1-i}$ which contradicts the choice of x_{1-i} and y_{1-i}. Thus the Recursion Theorem prevents Case B from ever happening. Formally, here and in similar impossible Cases below we should add an instruction to halt the construction at this Stage.)

C a s e C. B is pre-engaged.

Put $\nu_j = y_{1-i}$. (Hence $y_{1-i} = \nu_j \sim_{1-i} \nu_j^{T^{1-i}} = b \in B \subseteq [\operatorname{dom} h^{2i-1}]_{1-i}$ contradicting the choice of y_{1-i}. Thus Case C never happens.)

Go to the next Stage.

S t a g e $6n+2+i$ $(i=0,1)$.

Look for two numbers a, c in distinct i-clusters, call these A and C respectively, s.t. $a \sim_i c$ is verified in $\leq n$ Turing steps with $\max(a,c)$ the smallest among such.

Go to the next Stage if a and c are not found. If they are, declare $A \cup C$ to be a new i-cluster *-paired with the new $(1-i)$-cluster $A^* \cup C^*$. (Note that since $A \ni a \sim_i c \in C$, the new i-cluster lies within a single $\mathcal{P}_i$-equivalence class.) For other assignments consider the following Cases:

C a s e A. Both A and C are homegrown.

Let j and k be the indices associated with A and C respectively. Define $\nu_k = \nu_j^{T^{1-i}}$. Call the i-cluster $A \cup C$ homegrown with j the associated index and declare the $(1-i)$-cluster $A^* \cup C^*$ imported. (Note that the elements of A^* and C^* are $\mathcal{P}_{1-i}$-equivalent for $A^* \ni \nu_j^{T^{1-i}} = \nu_k \sim_{1-i} \nu_k^{T^{1-i}} \in C^*$.)

C a s e B. Both A and C are imported.

Let j and k be associated with A^* and C^* respectively. Put $\nu_j = x_i$ and $\nu_k = y_i$. (This will prevent Case B from happening for $x_i = \nu_j \sim_i \nu_j^{T^i} \in A \ni a \sim_i c \in C \ni \nu_k^{T^i} \sim_i \nu_k = y_i$ implies $x_i \sim_i y_i$, a contradiction.)

C a s e C. Both A and C are pre-engaged.

$A \cup C$ and $A^* \cup C^*$ are defined to be pre-engaged. (We have $[A]_i = [C]_i$ and hence it follows from the inductive clause (ii) that $[A^*]_{1-i} = [C^*]_{1-i}$ and that the new clusters are mapped one to the other by ξ.)

C a s e D. One of the clusters, say A, is homegrown and the other, C, is imported.

Declare $A \cup C$ imported and $A^* \cup C^*$ homegrown. Associate with $A^* \cup C^*$ the index that has been associated with C^*. Suppose j is associated with the homegrown A. Put $\nu_j = c$ for some $c \in C^*$. (This ensures $A^* \ni \nu_j^{T^{1-i}} \sim_{1-i} \nu_j \in C^*$, so A^* and C^* are within the same $\mathcal{P}_{1-i}$-equivalence class.)

C a s e E. A is homegrown and C is pre-engaged.

Both $A \cup C$ and $A^* \cup C^*$ become pre-engaged. For j the index associated with A we put $\nu_j = c$ for some $c \in C^*$. (This provides for $A^* \ni \nu_j^{T^{1-i}} \sim_{1-i} \nu_j \in C^*$, and hence $A^* \sim_{1-i} C^*$.)

C a s e F. A is imported and C is pre-engaged.

Let j be the index associated with A^*. Put $\nu_j = y_i$. (This results in $y_i = \nu_j \sim_i \nu_j^{T^i} \in A \ni a \sim_i c \in [\operatorname{dom} h^{1-2i}]_i$ contradicting the assumption on y_i. So Case F is safe from happening.)

Go to the next Stage.

S t a g e $6n+4+i$ $(i=0,1)$.

Search for the least number $a \in \operatorname{Dom}\mathcal{P}_i$ s.t. a is in a homegrown or imported i-cluster and $a \in \operatorname{dom} h^{1-2i}$ is detected in $\leq n$ Turing steps.

Go to the next Stage if no such a is found. Otherwise, let A be the i-cluster with $a \in A$ and consider the following Cases:

C a s e A. A is homegrown.

Rule that both A and A^* are pre-engaged and for j the index associated with A put $\nu_j = [\![h^{1-2i}]\!]_i(a)$. (Hence $\nu_j \sim_{1-i} \nu_j^{T^{1-i}} \in A^*$ so that $[\![h^{1-2i}]\!]_i(a) \in [A^*]_{1-i}$ and, therefore, one of the equivalence classes $[A]_i$ and $[A^*]_{1-i}$ is taken to the other by ξ as required.)

C a s e B. A is imported.

Let j be associated with A^*. Put $\nu_j = y_i$. (Then $y_i = \nu_j \sim_i \nu_j^{T^i} \in A \ni a \in \operatorname{dom} h^{1-2i}$, contrary to the choice of y_i. Thus Case B never happens.)

Go to the next Stage.

The description of the construction is now complete. We put H to be the set of all such pairs $(a,b) \in \operatorname{Dom}\mathcal{P}_0 \times \operatorname{Dom}\mathcal{P}_1$ that at some Stage of the construction a and b are found in clusters connected with each other by *.

The systematic approach of Stages $6n+i$ takes care that any element of $\operatorname{Dom}\mathcal{P}_i$ gets sooner or later into some i-cluster, so that $\operatorname{dom} H^{1-2i} = \operatorname{Dom}\mathcal{P}_i$. Stages $6n+2+i$ guarantee that any $a, c \in \operatorname{Dom}\mathcal{P}_i$ s.t. $a \sim_i c$ eventually find their way inside one and the same i-cluster. Together with clause (i) preceding the description of our construction this implies that H represents an isomorphism $\mu : \mathcal{P}_0 \to \mathcal{P}_1$. Stages $6n+4+i$ ensure that every element a of $[\operatorname{dom} h^{1-2i}]_i$ becomes in due course a member of a pre-engaged i-cluster while clause (ii) says that $\xi^{1-2i}([a]_i) = \mu^{1-2i}([a]_i)$, entailing $\xi \subseteq \mu$ q.e.d. ∎

2.7. Corollary (Lachlan [7]). *All precomplete positive equivalences are isomorphic.* ∎

2.8. Corollary. *Let $\mathcal{P}$ be a precomplete and $\mathcal{S}$ an arbitrary positive equivalence. Then to any partial isomorphism $\xi : \mathcal{S} \to \mathcal{P}$ there exists a monomorphic extension $\mu : \mathcal{S} \to \mathcal{P}$.*

Proof. Similar to that of Corollary 2.3. ∎

3. Fixed points and diagonals

Consider the following condition on a positive equivalence $\mathcal{S}$:

(+) There is a binary total recursive function $\gamma : \omega \times \operatorname{Dom}\mathcal{S} \to \operatorname{Dom}\mathcal{S}$ s.t.

if $\gamma(i,x) \sim_{\mathcal{S}} x$, then $\varphi_i(\gamma(i,x))\uparrow$ or $\varphi_i(\gamma(i,x)) \sim_{\mathcal{S}} \gamma(i,x) \sim_{\mathcal{S}} x$.

Let us verify that (+) preserves under isomorphisms. Suppose γ is the function witnessing (+) for $\mathcal{S}$ and let total recursive functions h and k represent isomorphisms

$\mu : \mathcal{S} \longrightarrow \mathcal{T}$ and μ^{-1} respectively. For a partial recursive function $\varphi_i : \operatorname{Dom}\mathcal{T} \longrightarrow \operatorname{Dom}\mathcal{T}$ define $\varphi_{i^*}(z) \simeq k \circ \varphi_i \circ h(z)$. Note that * is an effective operation on indices.

Let $\beta(i,x) \simeq h \circ \gamma(i^*, k(x))$. We check that the total recursive function β witnesses (+) for $\mathcal{T}$:

First observe that $\varphi_{i^*} \circ \gamma(i^*, k(x)) \simeq k \circ \varphi_i \circ h \circ \gamma(i^*, k(x)) \simeq k \circ \varphi_i \circ \beta(i,x)$.

Suppose $\beta(i,x) \sim_{\mathcal{T}} x$. Then $k \circ \beta(i,x) = k \circ h \circ \gamma(i^*, k(x)) \sim_{\mathcal{S}} k(x)$. Since $k \circ h$ represents the identity on $\mathcal{S}$, we have $\gamma(i^*, k(x)) \sim_{\mathcal{S}} k(x)$, and so, by (+) of $\mathcal{S}$, $\varphi_{i^*} \circ \gamma(i^*, k(x))\uparrow$ or $\varphi_{i^*} \circ \gamma(i^*, k(x)) \sim_{\mathcal{S}} k(x)$.

$\varphi_{i^*} \circ \gamma(i^*, k(x))\uparrow$ means $k \circ \varphi_i \circ \beta(i,x)\uparrow$. Therefore, since k is total, $\varphi_i(\beta(i,x))\uparrow$. If $\varphi_{i^*} \circ \gamma(i^*, k(x)) \sim_{\mathcal{S}} k(x)$ then $\varphi_i(\beta(i,x)) \sim_{\mathcal{T}} h \circ k \circ \varphi_i \circ \beta(i,x) = h \circ \varphi_{i^*} \circ \gamma(i^*, k(x)) \sim_{\mathcal{T}} h \circ k(x) \sim_{\mathcal{T}} x$. This establishes (+) for $\mathcal{T}$.

3.1. FACT. *Precomplete and e-complete positive equivalences enjoy (+).*

PROOF. One takes $\gamma(i,x) = f(i)$ for the function f of Proposition 1.4 in the precomplete case and, for e-complete equivalences, $\gamma(i,x) = \delta(x)$ for δ a shift. ∎

A property considerably stronger than (+) holds for e-complete positive equivalences:

3.2. PROPOSITION. *Let $\mathcal{E}$ be e-complete. Given a strong index of a finite set $D \subseteq \operatorname{Dom}\mathcal{E}$ and (an index of) a partial recursive function $\varphi : \operatorname{Dom}\mathcal{E} \longrightarrow \operatorname{Dom}\mathcal{E}$, one can effectively find a number $n \in \operatorname{Dom}\mathcal{E}$ s.t.*

if $n \in [D]_{\mathcal{E}}$ or $\varphi(n) \in [D]_{\mathcal{E}}$, then $\varphi(n) \sim_{\mathcal{E}} n$.

FIRST PROOF. Fix a total recursive function h representing a monomorphism $\mathcal{E} \longrightarrow \mathcal{P}$ with $\mathcal{P}$ precomplete. Note that $h\lceil D$ represents a finite partial monomorphism and hence so does $(h\lceil D)^{-1}$. By Corollary 2.3, a representation g of a monomorphism $\mu : \mathcal{P} \longrightarrow \mathcal{E}$ extending the partial one represented by $(h\lceil D)^{-1}$ can be found effectively in D. Since $\mathcal{P}$ is precomplete, by Proposition 1.4 we can effectively find an $m \in \operatorname{Dom}\mathcal{P}$ s.t. $h \circ \varphi^{\tau_+(D)} \circ g(m) \sim_{\mathcal{P}} m$, where τ_+ is a precision partial $\mathcal{E}$-totalizer. Note that one then has $g \circ h \circ \varphi^{\tau_+(D)} \circ g(m) \sim_{\mathcal{E}} g(m)$. We claim that we can put $n = g(m)$:

Suppose $n = g(m) \in [D]_{\mathcal{E}}$. Since $g \circ h$ is, modulo $\sim_{\mathcal{E}}$, the identity on $[D]_{\mathcal{E}}$, we have $\varphi^{\tau_+(D)}(g(m)) \sim_{\mathcal{E}} g(m) \in [D]_{\mathcal{E}}$ and hence $\varphi(g(m))\downarrow$ and $\varphi(g(m)) \sim_{\mathcal{E}} \varphi^{\tau_+(D)}(g(m)) \sim_{\mathcal{E}} g(m)$ as required. If $\varphi(n) = \varphi(g(m)) \in [D]_{\mathcal{E}}$ then $\varphi(g(m)) \sim_{\mathcal{E}} \varphi^{\tau_+(D)}(g(m)) \sim_{\mathcal{E}} g \circ h \circ \varphi^{\tau_+(D)} \circ g(m) \sim_{\mathcal{E}} g(m)$ as was to be shown. ∎

SECOND PROOF. Define a recursive number

$$\nu \simeq \begin{cases} \varphi(\nu^{\tau_+(D)}) & \text{if } \varphi(\nu^{\tau_+(D)}) \in [D]_{\mathcal{E}} \leq \nu^{\tau_+(D)} \in [D]_{\mathcal{E}}, \\ \delta(\nu^{\tau_+(D)}) & \text{if } \nu^{\tau_+(D)} \in [D]_{\mathcal{E}} < \varphi(\nu^{\tau_+(D)}) \in [D]_{\mathcal{E}}, \\ \text{divergent} & \text{otherwise,} \end{cases}$$

where τ_+ is a precision partial $\mathcal{E}$-totalizer and δ an $\mathcal{E}$-shift. We show that $n = \nu^{\tau_+(D)}$ satisfies the requirements of the Proposition.

Indeed, if $\nu^{\tau_+(D)} \in [D]_{\mathcal{E}}$ or $\varphi(\nu^{\tau_+(D)}) \in [D]_{\mathcal{E}}$, then either $\varphi(\nu^{\tau_+(D)}) \in [D]_{\mathcal{E}} \leq \nu^{\tau_+(D)} \in [D]_{\mathcal{E}}$ or $\nu^{\tau_+(D)} \in [D]_{\mathcal{E}} < \varphi(\nu^{\tau_+(D)}) \in [D]_{\mathcal{E}}$.

In the first case we have $\nu = \varphi(\nu^{\tau_+(D)}) \in [D]_{\mathcal{E}}$ and hence $\nu^{\tau_+(D)} \sim_{\mathcal{E}} \nu = \varphi(\nu^{\tau_+(D)})$ which is all right. The second case can not happen for then $[D]_{\mathcal{E}} \ni \nu^{\tau_+(D)} \sim_{\mathcal{E}} \nu =$

$\delta(\nu^{\tau+(D)})$, δ failing to be diagonal. ∎

Finally, we show that the e- and the precomplete positive equivalences are the only uniformly finitely precomplete ones satisfying (+).

3.3. PROPOSITION. *Suppose Q is uniformly finitely precomplete and satisfies (+). Then Q is either e- or precomplete.*

PROOF. Assume Q is not e-complete and so, by the Bernardi–Montagna Fixed Point Property, every total recursive function has a Q-fixed point. Given an index i for a partial recursive function, consider the function $\lambda x.\gamma(i,x)$, where γ is the total recursive function featured in (+). Let y be its Q-fixed point: $\gamma(i,y) \sim_Q y$. Then, by (+), $\varphi_i(\gamma(i,y))\uparrow$ or $\varphi_i(\gamma(i,y)) \sim_Q \gamma(i,y)$. Note that y is effective in i, so we have constructed a total recursive function $f(i) = \gamma(i,y)$ satisfying clause (ii) of Proposition 1.4, thus establishing that Q is precomplete. ∎

4. Between e- and pre-

While examples of e- and precomplete positive equivalences occur in everyday life, the existence of uniformly finitely precomplete positive equivalences that are demonstrably outside these two classes has, as far as I know, been avoiding human experience. This circumstance appears to be somewhat regrettable in that it would, for example, be nice to know whether the applicability range of Corollary 1.11 is materially wider than that of (i) ⇒ (ii) of Proposition 1.4.

The Bernardi–Montagna Fixed Point Property of Corollary 1.11 is in fact very close to precompleteness, as is seen from (ii) ⇒ (i) of Proposition 1.4. However, Badaev [1] shows that there exist countably many pairwise non-isomorphic (hence most of them not precomplete) positive equivalences enjoying this property, namely that every total recursive function has a fixed point modulo the equivalence relation; Badaev calls such positive equivalences *weakly precomplete.* The next Theorem shows that there is a weakly precomplete positive equivalence among uniformly finitely precomplete ones that is not precomplete.

4.1. THEOREM. *There is a uniformly finitely precomplete positive equivalence $\mathcal{R}$ that is neither e- nor precomplete.*

PROOF. Let us fix an e-complete positive equivalence $\mathcal{E}$ together with its collective shift Δ. We are going to construct the desired equivalence $\mathcal{R}$ as a factor of $\mathcal{E}$ so that the uniform finite precompleteness of $\mathcal{R}$ is automatic. In constructing $\mathcal{R}$ we take care, in a priority-like fashion, of the following two infinite series of requirements:

P_d: If φ_d is total then it should have an $\mathcal{R}$-fixed point.

N_e: If φ_e is total then there should be an index i s.t. $\varphi_e(i) \not\sim_{\mathcal{R}} \nu_i$.

Here d and e range over ω. The priority ranking, which is useful to keep in mind, is

$P_0, N_0, P_1, N_1, \ldots$ The positive requirements P_d, if they are met, see to it that $\mathcal{R}$ has the Bernardi–Montagna Fixed Point Property so that $\mathcal{R}$ can not be e-complete, while the negative requirements N_e insist that no total recursive function be an $\mathcal{R}$-totalizer so that $\mathcal{R}$ can be neither precomplete nor trivial.

The construction of $\mathcal{R}$ proceeds in Stages, before and after each of which each positive and negative requirement is in one of the three states: a positive requirement can either be *vexed*, *allocated*, or *settled*; a negative one is *vexed*, *targeted*, or *accomodated*. Each requirement preserves its state through any Stage unless this state is explicitly changed by the instructions of that Stage. At each moment almost all requirements are vexed.

Further, there are three kinds of *labels* Z_d, X_e and Y_e, with $d, e \in \omega$ that can be attached to an element of $\operatorname{Dom} \mathcal{E}$. A label Z_d is attached the moment the positive requirement P_d becomes allocated. Later, the same requirement P_d may become vexed, in which case the label Z_d is temporarily removed from the playfield to reappear at a later Stage and, possibly, at a different location. If the requirement P_d changes its state from allocated to settled then the label Z_d is not removed, but becomes a *sticker* z_d, which means that it will stay where it is forever. At the same moment another sticker s_d appears at a certain element of $\operatorname{Dom} \mathcal{E}$. The labels X_e and Y_e are only present if the current state of N_e is accomodated. If, later, N_e chances to change its state, these labels are removed until N_e becomes accomodated again and so on.

On top of that, the moment a negative requirement gets targeted, an index (of a recursive number) is associated to it and stays with it until the requirement becomes vexed.

We shall be using the names X_e, Y_e and z_d not just for the labels and stickers themselves, but also for the numbers that these labels or stickers are (currently) attached to, confusion being unlikely. The notation D_m is reserved for the finite set of (current) positions of all labels and stickers that, immediately before Stage m, label or stick to some element of $\operatorname{Dom} \mathcal{E}$.

Before Stage 0 all requirements are vexed, neither any labels nor stickers are present and so $D_0 = \emptyset$.

Now that all the characters are introduced, we stage the construction:

S t a g e $4n$.

Pick the vexed positive requirement with the smallest index among such. Call it P_d.

By Proposition 3.2 find a number $m \in \operatorname{Dom} \mathcal{E}$ s.t. either of the two conditions $m \in [D_{4n}]_{\mathcal{E}}$ and $\varphi_d(m) \in [D_{4n}]_{\mathcal{E}}$ implies $\varphi_d(m) \sim_{\mathcal{E}} m$. Attach the label Z_d to m. Declare all allocated positive requirements $P_{d'}$ with $d' > d$ vexed. Declare P_d allocated.

S t a g e $4n + 1$.

Choose the smallest negative vexed requirement N_e.

Associate an index i (of arecursive number) to it. (Note that we may later define the value of ν_i.) Call N_e targeted.

S t a g e $4n + 2$.

Let P_d be the smallest positive allocated requirement s.t. $\varphi_d(Z_d)$ converges in $\leq n$ Turing steps. (Just go to the next Stage if there is no such P_d).

From now on the label Z_d becomes a sticker z_d and we put the sticker s_d on the number $\varphi_d(z_d)$. Declare all allocated positive requirements $P_{d'}$ with $d' > d$ and all accomodated requirements N_e with $e \geq d$ vexed. Put P_d itself in settled state.

S t a g e $4n+3$.

Find the smallest targeted requirement N_e s.t. $\varphi_e(i)$ converges in $\leq n$ Turing steps, where i is the index associated with N_e. Go to the next Stage if there are none.

Put $\nu_i = \Delta(D_{4n+3}; \varphi_e(i))$. Attach labels X_e and Y_e to $\varphi_e(i)$ and ν_i respectively. (Note that one then has $X_e \not\sim_{\mathcal{E}} Y_e$.) Call N_e accomodated. Declare all allocated requirements P_d with $d > e$ vexed.

The description of the procedure is now complete. We let $\sim_{\mathcal{R}}$ be the minimal equivalence relation containing $\sim_{\mathcal{E}}$ and s.t. $z_d \sim_{\mathcal{R}} s_d$ whenever these stickers are deployed by the construction just described.

C l a i m 0. *Every requirement, positive or negative, can change its state only finitely often, and the final state of any requirement can not be vexed.*

This is established by induction on the priority ranking of requirements. For this it is sufficient to note that a positive (negative) requirement goes from vexed to allocated to settled (from vexed to targeted to accomodated) unless some requirement of higher priority changes its state, in which case the requirement in question may become vexed.

Thus every requirement reaches a final state. Moreover, inspection of Stages $4n$ and $4n+1$ reveals that this state can not be vexed, for vexed requirements are systematically allocated or targeted.

Claim 0 also implies that each label is attached and removed at most finitely often, and that the same holds for indices of recursive numbers associated to negative requirements. By i_e we shall denote the last index ever associated to the requirement N_e, and we use the notation $i_e\Downarrow$ for the number of the Stage in our construction at which i_e was associated with N_e.

Similarly, z_d and $z_d\Downarrow$ will from now on stand for the final position of the label Z_d and the number of the Stage at which this happened, regardless of whether or not Z_d remains a label forever or becomes in future life a sticker. If the final state of P_d is settled then s_d and $s_d\Downarrow$ are the position and the date of birth respectively of the sticker s_d. Finally, $x_e = \varphi_e(i_e)$ and $y_e = \nu_{i_e}$ are the final positions of labels X_e and Y_e in case the requirement N_e is eventually accomodated, with $y_e\Downarrow$ the number of the Stage when this final accomodation took place.

C l a i m 1. *For $d \neq d'$ one can not have $z_d \sim_{\mathcal{E}} z_{d'}$ unless $z_d \sim_{\mathcal{E}} s_d$ or $z_{d'} \sim_{\mathcal{E}} s_{d'}$.*

Suppose w.l.o.g. $z_d\Downarrow < z_{d'}\Downarrow$. Then $z_d \in D_{z_{d'}\Downarrow}$ and therefore by inspection of Stages $4n$ it is seen that $z_{d'} \sim_{\mathcal{E}} z_d \in [D_{z_{d'}\Downarrow}]_{\mathcal{E}}$ can only happen if $z_{d'} \sim_{\mathcal{E}} \varphi_{d'}(z_{d'}) = s_{d'}$.

C l a i m 2. *For N_e eventually accomodated, one can not have $y_e \sim_{\mathcal{E}} z_d$ unless $z_d \sim_{\mathcal{E}} s_d$.*

If $y_e\Downarrow < z_d\Downarrow$ then $y_e \in D_{z_d\Downarrow}$ and, as in Claim 1, $y_e \sim_{\mathcal{E}} z_d$ implies $z_d \sim_{\mathcal{E}} s_d$. If $z_d\Downarrow < y_e\Downarrow$ then $z_d \in D_{y_e\Downarrow}$. So $y_e = \nu_{i_e} = \Delta(D_{y_e\Downarrow}; x_e) \not\sim_{\mathcal{E}} z_d$.

C l a i m 3. *Suppose P_d and $\mathrm{P}_{d'}$ are both eventually settled and $z_d \not\sim_{\mathcal{E}} s_d \sim_{\mathcal{E}} z_{d'} \not\sim_{\mathcal{E}} s_{d'}$. Then $z_d\Downarrow < z_{d'}\Downarrow < s_{d'}\Downarrow < s_d\Downarrow$.*

We clearly have $z_{d'}\Downarrow < s_{d'}\Downarrow$ and $z_d\Downarrow < s_d\Downarrow$, for before a positive requirement is settled, it has to be allocated.

Let us show $z_d\Downarrow < z_{d'}\Downarrow$. If the opposite, $z_{d'}\Downarrow < z_d\Downarrow$, held then $z_{d'} \in D_{z_d\Downarrow}$ and the instructions of Stages $4n$ would allocate Z_d in such a way that $s_d = \varphi_d(z_d) \sim_{\mathcal{E}} z_{d'} \in [D_{z_d\Downarrow}]_{\mathcal{E}}$ only if $z_d \sim_{\mathcal{E}} \varphi_d(z_d) = s_d$, quod non.

Next we shall see that $d < d'$. For suppose otherwise, $d' > d$. As is already established, we either have $z_d\Downarrow < s_d\Downarrow < z_{d'}\Downarrow$ or $z_d\Downarrow < z_{d'}\Downarrow < s_d\Downarrow$. In the former case $s_d \in D_{z_{d'}\Downarrow}$, so $z_{d'}$ can not be $\mathcal{E}$-equivalent to s_d because $z_{d'} \not\sim_{\mathcal{E}} s_{d'}$ (and this contradicts the assumptions of the Claim). In the latter case we have that the state of P_d immediately before Stage $z_{d'}\Downarrow$ is allocated and, since P_d has lower priority, it becomes vexed at this Stage contradicting $z_d\Downarrow < z_{d'}\Downarrow$. Thus $d < d'$.

Finally, if $s_d\Downarrow < s_{d'}\Downarrow$ were the case then the instructions of Stage $s_d\Downarrow$ would make $\mathrm{P}_{d'}$ vexed at this Stage for $d < d'$, i.e. $\mathrm{P}_{d'}$ has lower priority, and hence $s_d\Downarrow < z_{d'}\Downarrow$, whence it follows by inspection of Stages $4n$ that since $s_d \in D_{z_{d'}\Downarrow}$, $z_{d'} \sim_{\mathcal{E}} s_d$ could only happen if $z_{d'} \sim_{\mathcal{E}} s_{d'}$, quod non. Therefore, $s_{d'}\Downarrow < s_d\Downarrow$ and the Claim is established.

Our aim is to prove that all the requirements P_d, N_e are met. For P_d this is easy: Consider z_d. If φ_d is total then $\varphi_d(z_d)\downarrow$ and for a certain n, Stage $4n+2$ will make a sticker out of the label Z_d and deploy a sticker s_d so that $z_d \sim_{\mathcal{R}} s_d = \varphi_d(z_d)$ as required.

We turn to N_e. Assuming φ_e total we have $\varphi_e(i_e)\downarrow$ and hence x_e and y_e are eventually defined via Stage $4n+3$ for an appropriate n. We are going to show that $x_e \not\sim_{\mathcal{R}} y_e$ for then $\varphi_e(i_e) = x_e \not\sim_{\mathcal{R}} y_e = \nu_{i_e}$, which satisfies N_e.

First note that $x_e \not\sim_{\mathcal{E}} \Delta(D_{y_e\Downarrow}; x_e) = y_e$ and so the only possibility for $x_e \sim_{\mathcal{R}} y_e$ is that there exists a finite sequence of pairs $((a_0, b_0), (a_1, b_1), \dots, (a_n, b_n))$ s.t. $b_i \sim_{\mathcal{E}} a_{i+1}$ for all $i < n$, $a_0 \sim_{\mathcal{E}} x_e$ and $b_n \sim_{\mathcal{E}} y_e$, and for all $j \leq n$ there is an eventually settled positive requirement P_{d_j} s.t. $\{a_j, b_j\} = \{z_{d_j}, s_{d_j}\}$. We clearly can assume $z_{d_j} \not\sim_{\mathcal{E}} s_{d_j}$ for otherwise we can just delete the pair (a_j, b_j) from the sequence.

By induction on j from n downto 0, using the above assumption and Claims 1 and 2, one obtains that $(a_j, b_j) = (z_{d_j}, s_{d_j})$ for all $j \leq n$, so that $s_{d_j} \sim_{\mathcal{E}} z_{d_{j+1}}$, $z_{d_0} \sim_{\mathcal{E}} x_e$ and $s_{d_n} \sim_{\mathcal{E}} y_e$. n applications of Claim 3 give then that

$$z_{d_0}\Downarrow < \cdots < z_{d_n}\Downarrow < s_{d_n}\Downarrow < \cdots < s_{d_0}\Downarrow.$$

Next we show $z_{d_n}\Downarrow < y_e\Downarrow < s_{d_n}\Downarrow$. Indeed, we can not have $y_e\Downarrow < z_{d_n}\Downarrow$ because then $y_e \in D_{z_{d_n}\Downarrow}$ and, by the instructions of Stages $4n$, we only can have $\varphi_{d_n}(z_{d_n}) = s_{d_n} \sim_{\mathcal{E}} y_e \in [D_{z_{d_n}\Downarrow}]_{\mathcal{E}}$ if $z_{d_n} \sim_{\mathcal{E}} \varphi_{d_n}(z_{d_n}) = s_{d_n}$ which, as we have agreed, is not the case. Neither can we have $s_{d_n}\Downarrow < y_e\Downarrow$ for if so then $s_{d_n} \in D_{y_e\Downarrow}$ and hence $y_e = \nu_{i_e} = \Delta(D_{y_e\Downarrow}; x_e) \not\sim_{\mathcal{E}} s_{d_n}$ contrary to what we have seen above.

Let us now consider the relative priority ranking of P_{d_0} and N_e.

Suppose $d_0 \leq e$. In this case N_e becomes vexed the moment s_{d_0} is deployed and we can not have $y_e\Downarrow < s_{d_0}\Downarrow$, contrary to what we have established. Try $e < d_0$. Since $z_{d_0}\Downarrow < y_e\Downarrow$, we have that before Stage $y_e\Downarrow$ the requirement P_{d_0} is allocated, so Stage $y_e\Downarrow$ puts P_{d_0} in vexed state and one can not have $z_{d_0}\Downarrow < y_e\Downarrow$, again, contradicting earlier considerations.

The contradiction proves that no finite sequence of pairs of stickers stretches from x_e to y_e and, therefore, $x_e \not\sim_{\mathcal{R}} y_e$ q.e.d. ∎

The behaviour of labels X_e and Y_e in the proof of Theorem 4.1 is in effect a limiting computation of the point at which the function φ_e, if total, fails to be an $\mathcal{R}$-totalizer: $x_e = \varphi_e(i_e) \not\sim_{\mathcal{R}} \nu_{i_e} = y_e$. That one can not pinpoint this failure in a more straightforward way is seen from the following

4.2. PROPOSITION. *A uniformly finitely precomplete positive equivalence Q is e-complete iff the failures of Q-totalization are effective, i.e. there exists a total recursive function q s.t. for all i, if φ_i is total, then $\nu_{q(i)}\downarrow$ and $\varphi_i(q(i)) \not\sim_Q \nu_{q(i)}$.*

PROOF. (only if). Let Q be e-complete and δ a Q-shift.

Given an i, define $\nu_k \simeq \delta(\varphi_i(k))$ and let $q(i) = k$. Suppose φ_i is total. Then $\nu_k = \delta(\varphi_i(k))\downarrow$ and $\varphi_i(q(i)) = \varphi_i(k) \not\sim_Q \nu_k = \nu_{q(i)}$.

(if). Suppose we are given a total recursive function q as in the statement. We construct a Q-shift δ.

Given an x, let c be s.t. $\varphi_c(y) = x$, all y. Let $\delta(x) = \nu_{q(c)}$. Observe that since c indexes a total recursive function, we have $\nu_{q(c)}\downarrow$ and $\delta(x) = \nu_{q(c)} \not\sim_Q \varphi_c(q(c)) = x$. ∎

Incidentally, note that we can not have $\nu_{q(i)}\downarrow$ for all i in the statement of Proposition 4.2 for, were this to be the case, we would get $\nu_{q(s)} \not\sim_Q \varphi_s(q(s)) = \nu_{q(s)}$ for s the index of the universal partial recursive function $\varphi_s(x) \simeq \nu_x$, which is absurd.

5. Fixed points of endomorphisms

Here we characterize those subsets of $\operatorname{Dom} Q$ for Q uniformly finitely precomplete, that can be obtained as the sets of fixed points of an endomorphism of Q. For precomplete Q, this question is implicitly raised by the Main Lemma in Montagna [9].

5.1. PROPOSITION. *Let Q be a uniformly finitely precomplete positive equivalence and let W be an r.e. subset of $\operatorname{Dom} Q$ closed under $\sim_Q$. The following are equivalent:*

(i) *There is an endomorphism μ of Q s.t. $\{ x \in \operatorname{Dom} Q \mid \mu([x]_Q) = [x]_Q \} = W$.*

(ii) *W is nonempty unless Q is e-complete.*

PROOF. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Consider first the case of non-empty W. Fix $a \in W$. We define a total recursive function h to represent the desired μ:

$$h(x) \simeq [\![\mathrm{id}\lceil W \cup h\lceil x]\!]_Q^{\tau_{a_x}(\operatorname{rng} h\lceil x;a,x)}(x),$$

$$\text{where} \qquad a_x(y) \simeq \begin{cases} a & \text{if } y \sim_Q x, \\ \text{divergent} & \text{otherwise.} \end{cases}$$

Here τ is a partial Q-totalizer and its lower index is appended via the RGFP Theorem.

We shall demonstrate by induction on x:

(i) If $y, z < x$ and $y \sim_Q z$, then $h(y) \sim_Q h(z)$, and

(ii) If $y < x$, then $h(y) \sim_Q y$ iff $y \in W$.

This clearly implies that h represents an endomorphism μ whose fixed points are precisely the members of W.

Let us now proceed with the induction step. There are two Cases.

C a s e 1. $x \notin W$.

If there is a $y < x$ with $y \sim_Q x$ then, by (i) of the IH, $[\![h\lceil x]\!]_Q(x) \sim_Q h(y)$, $(\mathrm{id}\lceil W)(x)\uparrow$ and, since a_x is diagonal, by the RADN Theorem there holds $h(y) \sim_Q [\![h\lceil x]\!]_Q(x) = [\![\mathrm{id}\lceil W \cup h\lceil x]\!]_Q(x) \sim_Q h(x)$. (ii) follows then by (ii) of the IH.

If no such y exists then $[\![\mathrm{id}\lceil W \cup h\lceil x]\!]_Q(x)\uparrow$, which by the RADN Theorem implies $h(x) \notin \mathrm{dom}\, a_x = [x]_Q$, that is, $h(x) \not\sim_Q x$ as was to be shown.

C a s e 2. $a \not\sim_Q x \in W$.

Here we have $(\mathrm{id}\lceil W)(x) = x$, and $[\![h\lceil x]\!]_Q(x) \sim_Q x$ or $[\![h\lceil x]\!]_Q(x)\uparrow$ by the IH. Therefore, since a_x is diagonal, the RADN Theorem gives $h(x) \sim_Q [\![\mathrm{id}\lceil W \cup h\lceil x]\!]_Q(x) \sim_Q x$ so (ii) is established. (i) follows from (ii) of the IH.

C a s e 3. $a \sim_Q x \in W$.

Now $[\![\mathrm{id}\lceil W \cup h\lceil x]\!]_Q(x) \sim_Q x$ by the IH, and, for all y, $a_x(y) \sim_Q y$ implies $y \sim_Q a \sim_Q x$. By the RGFP Theorem this results in $h(x) \sim_Q [\![\mathrm{id}\lceil W \cup h\lceil x]\!]_Q(x) \sim_Q x$, or $a_x(h(x)) \sim_Q h(x)$, which entails $h(x) \sim_Q x$ all the same. (ii) is proven. (i) follows by IH.

Thus we have constructed the required $\mu : Q \to Q$ under the assumption $W \neq \emptyset$.

Finally, we consider the situation when $W = \emptyset$ and Q is e-complete. We have to produce a representation h of an endomorphism $Q \to Q$ without any fixed points. Although such endomorphisms are known from a particular example of e-complete positive equivalence (cf. Bernardi [2] or Bernardi & Montagna [3]), we give a coordinate-free construction. Here it is:

$$h(x) \simeq [\![h\lceil x]\!]_Q^{\tau_+(\mathrm{rng}\, h\lceil x;x)}(x),$$

where τ_+ is a precision partial Q-totalizer. One easily verifies the inductive clauses

(i) If $y, z < x$ and $y \sim_Q z$, then $h(y) \sim_Q h(z)$, and

(ii) If $y < x$ then $h(y) \not\sim_Q y$

that clearly imply that h represents a fixed point-free μ. ∎

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