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CORRESPONDENCE THEORY FOR EXTENDED MODAL LOGICS

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1 INTRODUCTION

Modal operators record simple, very restricted patterns of relational models through their truth definitions. Such patterns live in classical languages (first-order, second-order, infinitary, ...). Modal correspondence theory studies the relations between modal languages and classical ones. It does so at various levels, depending on the way modal formulas are interpreted. When interpreted on *models* the standard modal language $\mathcal{ML}(\diamond)$, for instance, ends up as a very restricted fragment of a first-order language. When $\mathcal{ML}(\diamond)$ is interpreted on *frames* its propositional variables are universally quantified over, and it ends up as a set of Π^1_1-conditions. In this approach a key issue is: when does a modal Π^1_1-condition reduce to a first-order formula? An important tool here is the Sahlqvist-van Benthem algorithm, which when input a modal formula in $\mathcal{ML}(\diamond)$ of a certain form, reduces it to an equivalent *first-order* property of binary relations via suitable instantiations. Recently this algorithm has been extended by Gabbay & Ohlbach (1992) and Simmons (1992) through the use of Skolem functions.

This paper is concerned with reducibility issues of the kind described above. The paper studies and extends the Sahlqvist-van Benthem and Gabbay-Ohlbach-Simmons algorithms in a very general setting; this is done for various reasons. First, a better understanding of the ins and outs of the algorithms is gained if the analysis is independent of any particular modal calculus. Second, recent years have witnessed a boom in extensions and alterations of the standard modal format; as was noted in (De Rijke 1993*d*), only little is known in the way of general results on transfer or applicability of facts and constructions from standard modal logic to extended ones. A fully general analysis of the above correspondence algorithms reveals their applicability to arbitrary modal logics, and beyond, as will be illustrated in §6 below with examples from a variety of modal and temporal logics, dynamic logic, circumscription and other areas. Third, it's an important tradition in logic to compare different theories and languages; the work reported on below is part of that line of research.

The next section supplies the main preliminaries; it may be skipped on a first reading. §3 defines the central notion of the paper: *correspondence* or *reducibility*; roughly speaking, a formula is reducible for certain variables if it is equivalent to a formula in which those variables don't occur. For most practical purposes actual reductions are obtained by making appropriate substitutions for the forbidden variables. This approach underlies §§4, 5, where we analyze what makes the

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Sahlqvist-van Benthem and Gabbay-Ohlbach-Simmons algorithms work; the analysis involves both a semantic description of the substitution mechanisms, and a syntactic characterization of the formulas allowing such substitutions. A less algorithmic perspective is adopted in §7; there we obtain reducibility results by imposing restrictions on languages and their interpretations. §8 concludes the paper with comments and questions.

Before taking off: a frequent complaint about the actual use of the Sahlqvist-van Benthem algorithm has been its alleged obscurity (Kracht 1993, page 194), (Gabbay & Ohlbach 1992, Section 4.3). To address these complaints we pay special attention to using the algorithms in §§4–6 below.

2 Preliminaries

First we need to be specific about classical logic. For $\boldsymbol{\tau}$ a classical vocabulary, a *(classical) logic* is given by two classes $\text{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ and $\text{Sent}_{\mathcal{L}}[\boldsymbol{\tau}]$ of $\mathcal{L}$-formulas and $\mathcal{L}$-sentences respectively, together with a relation $\models_{\mathcal{L}}$ between structures and $\mathcal{L}$-sentences. $\text{Str}[\boldsymbol{\tau}]$ denotes the class of $\boldsymbol{\tau}$-structures. We assume that for any classical logic $\mathcal{L}$, $\text{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ contains n-placed predicates $\bot_s$ and $\top_s$ ($n \in \mathbb{N}$, s a sort in $\boldsymbol{\tau}$) such that in any model $\mathfrak{A}$, $\bot_s$ is interpreted as the empty set, $\top_s$ as the domain of sort s. Basic model-theoretic notions are introduced as usual.

We assume that we have *membership* or *acceptance* predicates ϵ available, which take as their arguments an n-placed symbol of a 'relational' sort and n terms of the appropriate sorts to form formulas. E.g. if r is a symbol of a binary relational sort, then ϵrxy is a wff; its intended interpretation is that the pair denoted by (x, y) is to belong to the relation denoted by r. Instead of $\epsilon r x_1 \ldots x_n$ we will write $r(x_1, \ldots, x_n)$. Furthermore, equality ($=$) is used only between terms of the individual sort. For a classical logic $\mathcal{L}$, $\Pi^1_1(\mathcal{L})$ denotes the set of formulas with universal quantifier prefix $\forall \ldots$ binding relational symbols of $\mathcal{L}$.

As to modal logic, following (De Rijke 1993*d*) a modal language has a set of (modal) sort symbols and for each sort a set of (propositional) variables, a set of constants, and a set of connectives; in addition it has a set of modal operators. The modal formulas of sort s are built up from atomic symbols p_s, connectives $\bullet$ and modal operators $\#$ according to the rule $\phi ::= p_s \mid \bullet(\phi_{1,s}, \ldots, \phi_{n,s}) \mid \#(\phi_{s_1}, \ldots, \phi_{s_n})$, where it is assumed that $\bullet$, $\#$ return values of sort s. The semantics of a modal operator $\#$ is given by an *$\mathcal{L}$-pattern* $\delta_{\#}$, that is, by an $\mathcal{L}$-formula $\lambda x_{s_1} \ldots x_{s_n}.\, \phi(x_{s_1}, \ldots, x_{s_n}; x_{s_{n+1}}, \ldots x_{s_m})$, where x_{s_i} is a variable of a classical sort s_i, $\phi \in \text{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ for some $\boldsymbol{\tau}$, and $\mathcal{L}$ is a classical logic. Models for modal languages have the form $\mathfrak{M} = (W_s, \ldots, V)$ where $\mathfrak{M}$ is 'rich enough' to interpret the classical vocabulary in which the patterns for our modal operators live, and V is a valuation assigning subsets of W_s to symbols of sort s. Truth of modal formulas is given by $\mathfrak{M}, x \models p_s$ iff $x \in V(p_s)$ for atomic symbols p_s, the obvious clauses for connectives $\bullet$, and $\mathfrak{M}, x \models \#(\phi_1, \ldots, \phi_n)$ iff $\mathfrak{M}, x \models \delta_{\#}(I(\phi_1), \ldots, I(\phi_n))$.

The *standard translation* transcribes the truth definition of a modal language into a classical language containing predicate symbols p corresponding to the modal atomic symbol p_s: $ST(p_s) = p(x)$, ST commutes with connectives, and $ST(\#(\phi_1, \ldots, \phi_n)) = \delta_{\#}(ST(\phi_1), \ldots, ST(\phi_n))$. The important connection here is that for all modal formulas ϕ,

$$(W_s, \ldots, V), w \models \phi \ \text{ iff } \ (W_s, \ldots, V(p), \ldots) \models ST(\phi)[w],$$

where $V(p)$ is assigned to the predicate symbol p corresponding to p_s. In the context of the basic modal language $\mathcal{ML}(\Diamond)$ the notion of a *frame* arises when one quantifies over all possible valuations, thus arriving at second-order equivalents of modal formulas:

$$(W, R), x \models \phi \ \text{ iff } \ (W, R) \models \forall \vec{p}\, ST(\phi)[x].$$

This is generalized to arbitrary modal languages by selecting a modal sort s (with non-empty set of variables), and universally quantifying over all variables of that sort, while letting valuations take care of variables of the remaining sorts as before. Thus, we look at (higher-order) formulas of the form $\forall\vec{p}\, ST(\phi)$, where the $\forall\vec{p}$ binds all variables of sort s, rather than at formulas of the form $ST(\phi)$. Our prime question at this point is: when, and if so how, can we get rid of this higher-order quantification?

3 Reducibility

Two examples

3.1. Example. In the basic modal language $\mathcal{ML}(\Diamond)$ the formula $p \to \Diamond p$ is equivalent to the second-order condition $\forall p\,(p(x) \to \exists y\,(Rxy \wedge p(y)))$ when interpreted on frames. By substituting $\lambda u.\, u = x$ for p in the second order formula, it reduces to $\exists y\,(Rxy \wedge y = x)$, or Rxx. This reduction yields an equivalence, one direction of which is just an instantiation; the validity of the other follows from the upward monotonicity of the consequent of $\forall p\,(p(x) \to \exists y\,(Rxy \wedge p(y)))$.

3.2. Example. Recall that propositional dynamic logic (**PDL**) has $\cup$ (union), ; (composition), and * (iteration) as operations on its relational component. Through the standard translation **PDL** ends up as a fragment of $\mathcal{L}_{\omega_1\omega}$; because of the Kleene star * we need to go infinitary here: $ST(\langle a\rangle p) = \exists y\,(\bigvee_n (R_a^n xy) \wedge p(y))$. As an example, on frames the **PDL**-formula $p \wedge [a]p \to \langle b; a^*\rangle p$ is equivalent to the $\Pi^1_1(\mathcal{L}_{\omega_1\omega})$-condition

$$\forall p\,\Big(p(x) \wedge \forall y\,(R_a xy \to p(y)) \to \exists y'z'\,(Rb'_x z' \wedge R_a^* zy \wedge p(y))\Big). \tag{1}$$

Substituting $\lambda u.\,(u = x \vee R_a xu)$ for p in (1) reduces it to the $\mathcal{L}_{\omega_1\omega}$-formula $\exists yz\,(R_b xz \wedge \bigvee_n (R_a^n zy) \wedge (y = x \vee R_a xy))$, or $\exists z\,(R_b xz \wedge R_a^* zx)$. To see this, observe that one direction is again an instantiation; the other follows from the upward monotonicity of the consequent of (1).

Examples 3.1, 3.2 show that in modal higher-order conditions the higher-order quantification can sometimes be removed through suitable substitutions — the question when and if so with which instances such reductions may be done, is analyzed in §§4 and 5.

Basics

Given a (classical) formula β involving variables $p_1, \ldots, p_n$ of some sort s, we want to know whether the Π^1_1-like formula $\forall p_1 \ldots \forall p_n\, \beta$ is equivalent to a formula γ not involving any variables of the sort s.

3.3. Definition. Let $\beta \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, $\gamma \in \mathrm{Form}_{\mathcal{L}'}[\boldsymbol{\tau}']$, for some $\boldsymbol{\tau}, \boldsymbol{\tau}', \mathcal{L}, \mathcal{L}'$. We say that β *corresponds to* γ, or β *is reducible to* γ if for every all $\mathfrak{A} \in \mathrm{Str}[\boldsymbol{\tau} \cup \boldsymbol{\tau}']$, and all $\vec{u} \in A$, we have $\mathfrak{A} \models_{\mathcal{L}'} \beta[\vec{u}]$ iff $\mathfrak{A} \models_{\mathcal{L}} \gamma[\vec{u}]$.

Note that I concentrate on *pointwise* reducibility, that is, on formulas that (may) depend on parameters. In most of the literature on correspondence theory for standard modal logic the emphasize has largely been put on a 'uniform' approach to reducibility, by considering only universally closed formulas; given the local perspective of this paper I have opted for the pointwise version.

In most practical cases Definition 3.3 will apply with $\boldsymbol{\tau} \supseteq \boldsymbol{\tau}'$ and $\mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ usually contains all Π^1_1-like conditions over $\mathcal{L}'$.

3.4. CONVENTION. In the sequel $\boldsymbol{\tau}$ is a fixed classical vocabulary, and s is a sort of $\boldsymbol{\tau}$ such that the only symbols of sort s are the variables $VAR_s = \{p_1, \dots\}$; the elements of VAR_s are called s-variables. A formula is *s-universal* if it is of the form $\forall p_1 \dots \forall p_n \, \beta$, where all s-variables occurring in it are bound by the prefix $\forall p_1 \dots \forall p_n$. If $\forall \vec{p} \, \beta$ is an s-universal formula, we will tacitly assume that the prefix '$\forall \vec{p}$' contains all and only quantifiers binding s-variables. A formula β is *s-free* if it contains no (free or bound) occurrences of s-variables.

Here are some simple reducibility properties of s-universal formulas.

3.5. PROPOSITION. *Let $\forall \vec{p} \, \beta$ be an s-universal formula in* Form[$\boldsymbol{\tau}$]. *Then $\forall \vec{p} \, \beta$ reduces to an s-free formula in* Form[$\boldsymbol{\tau}$] *iff $\forall \vec{p} \, [\neg p_i/p_i]\beta$ does so.*

3.6. PROPOSITION. *Assume $\forall \vec{p} \, \beta, \forall \vec{p}' \, \beta'$ are s-universal formulas in* Form[$\boldsymbol{\tau}$]. *If $\forall \vec{p} \, \beta$ and $\forall \vec{p}' \, \beta'$ reduce to γ and γ', respectively, in* Form[$\boldsymbol{\tau}$], *then $\forall \vec{p}\vec{p}' \, (\beta \wedge \beta')$ reduces to $(\gamma \wedge \gamma')$. If $\forall \vec{p} \, \beta$ and $\forall \vec{p}' \beta'$ have distinct s-variables $\vec{p}$ and $\vec{p}'$, then $\forall \vec{p}\vec{p}' \, (\beta \vee \beta')$ reduces to $(\gamma \vee \gamma')$.*

3.7. PROPOSITION. *Let $\forall \vec{p} \, \beta(\vec{y}; \vec{z})$ be an s-universal formula in* Form[$\boldsymbol{\tau}$] *that is reducible to the s-free $\gamma(\vec{y}; \vec{z}) \in$* Form[$\boldsymbol{\tau}$]. *Assume that $\gamma'(\vec{x}; \vec{y})$ is s-free. Then $\forall \vec{p} \forall \vec{y} \, (\gamma' \to \beta)$ is reducible to $\forall \vec{y} \, (\gamma' \to \gamma)$.*

Proof. Assume $\mathfrak{A} \models \forall \vec{p} \forall \vec{y} \, (\gamma'(\vec{x}; \vec{y}) \to \beta(\vec{y}; \vec{z}))[\vec{u}\vec{w}]$. If $\mathfrak{A} \models \gamma'(\vec{x}; \vec{y})[\vec{u}\vec{v}\vec{w}]$ then obviously $\mathfrak{A} \models \forall \vec{p} \, \beta(\vec{y}; \vec{z})[\vec{u}\vec{v}\vec{w}]$. So, by assumption, $\mathfrak{A} \models \gamma(\vec{y}; \vec{z})[\vec{u}\vec{v}\vec{w}]$. $\dashv$

3.8. REMARK. The class of formulas χ such that $\forall \vec{p} \, \chi$ is reducible to an s-free formula, is not closed under $\neg$. To see this, let $\boldsymbol{\tau}$ contain a binary relation symbol R and a single predicate variable p. Consider the first-order formulas $\beta \equiv \exists y \, (Ryx \wedge p(y))$ and $\beta' \equiv \forall y \, (Ryx \wedge p(y) \to \exists z \, (Ryz \wedge p(z)))$. Then $\forall p \, \beta$ is reducible to a p-free formula, by 4.2, and $\forall p \, \beta'$ is reducible to a p-free formula by 5.3. Hence their conjunction $\forall p \, (\beta \wedge \beta')$ is reducible as well. However, $\forall p \, \neg(\beta \wedge \beta')$, i.e.,

$$\text{(WF)} \qquad \forall p \left(\exists y \, (Ryx \wedge p(y)) \to \exists y \, (Ryx \wedge p(y) \wedge \forall z \, (Ryz \to \neg p(z)))\right)$$

is not reducible to a p-free formula. It may be shown that (WF) expresses that R is well-founded, and hence is not elementary, or, reducible to a p-free formula over $\boldsymbol{\tau}$.

Observe that the converse of the first half of Proposition 3.6 does not hold. As $\forall p \forall x \, (p(x) \to \exists y \, (Rxy \wedge p(y)))$ reduces to $\forall x \, Rxx$, the conjunction of the former formula with (WF) is inconsistent, hence reducible to a p-free formula, although (WF) is not.

4 FINDING THE RIGHT INSTANCES

As was observed before, in many practical cases reducibility results are obtained via suitable substitutions, if at all. In effect, this is the idea underlying the reduction algorithms mentioned in §1. For $\beta \equiv \forall \vec{p} \, \beta'$ an s-universal formula (over a classical vocabulary $\boldsymbol{\tau}$), they find an s-free equivalent of α (again, over $\boldsymbol{\tau}$) by taking suitable s-free *instances* $\gamma_1, \dots, \gamma_n$ of the s-variables $p_1, \dots, p_n$ in β such that

$$\models [\gamma_1/p_1, \dots, \gamma_n/p_n]\beta' \to \forall \vec{p} \, \beta' \tag{2}$$

(the converse implication follows by instantiation). We are interested in combinations of Π^1_1-like s-universal formulas β of the form

$$\forall \vec{p} \, (\alpha \to \pi), \tag{3}$$

where π is monotone, and the antecedent α is a formula 'supplying' the substitution instances γ for $\vec{p}$ that yield the desired reduction of (3) to an s-free formula as in (2). The key-topic below is to make precise in what way the antecedent α supplies the substitution instances. We set down semantic (and syntactic) conditions on formulas that guarantee the existence of such instances, and we describe the instances needed. The results lead to a fully general formulation of the Sahlqvist-van Benthem and Gabbay-Ohlbach-Simmons algorithms in §5.

MONOTONICITY

We first examine the simplest instance of the general schema (3), where α is either $\top$ or $\bot$.

4.1. DEFINITION. Let $\pi(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, let p be an s-variable, for s a sort in $\boldsymbol{\tau}$. We call $\pi(\vec{x})$ *upward (downward) monotone in* p if for all $\mathfrak{A} = (A, p, \dots) \in \mathrm{Str}[\boldsymbol{\tau}]$, and for all $\vec{u} \in A$, and all $p' \supseteq p$ ($p' \subseteq p$), we have that $\mathfrak{A} \models \pi[\vec{u}]$ implies $(A, p', \dots) \models \pi[\vec{u}]$.

The temporal logic formula Pp has $ST(Pp) = \exists y\,(Ryx \wedge p(y))$, which is monotone in p.

4.2. PROPOSITION. *Let* $\pi(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ *be upward monotone in* p_i. *Assume* $\forall p_1 \dots \forall p_n\, \pi(\vec{x})$ *is* s*-universal. Then* $\forall p_1 \dots \forall p_n\, \pi(\vec{x})$ *is reducible to an* s*-free formula iff the formula* $\forall p_1 \dots \forall p_{i-1}$ $\forall p_{i+1} \dots \forall p_n\, [\bot/p_i]\pi(\vec{x})$ *is.*

Proof. If $\mathfrak{A} \models \forall p_1 \dots \forall p_n\, \pi[\vec{u}]$ then $\mathfrak{A} \models \forall p_1 \dots \forall p_{i-1} \forall p_{i+1} \dots \forall p_n\, [\bot/p_i]\pi[\vec{u}]$, for any $\mathfrak{A} \in \mathrm{Str}[\boldsymbol{\tau}]$. Using the fact that π is upward monotone in p_i, one sees that the converse implication holds as well. $\dashv$

4.3. PROPOSITION. *Let* $\pi(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ *be downward monotone in* p_i. *Assume* $\forall p_1 \dots \forall p_n\, \pi(\vec{x})$ *is* s*-universal. Then* $\forall p_1 \dots \forall p_n\, \pi(\vec{x})$ *is reducible to an* s*-free formula iff the formula* $\forall p_1 \dots \forall p_{i-1}$ $\forall p_{i+1} \dots \forall p_n\, [\top/p_i]\pi(\vec{x})$ *is.*

4.4. COROLLARY. *Assume that for every* s*-variable* p, $\pi \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ *is either upward or downward monotone in* p. *Then, if* $\forall \vec{p}\, \pi$ *is* s*-universal, it reduces to a* p*-free formula in* $\mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ *via a suitable instantiation.*

Observe that the only instantiations needed in Corollary 4.4 are $\bot$ and $\top$.

As an example, the temporal formula *Until*$(\top, p)$ translates into $\exists y\,(Rxy \wedge \forall z\,(Rxz \wedge Rzy \to p(z)))$, which is upward monotone in p. Substituting $\bot$ for p, we find that on frames *Until*$(\top, p)$ is equivalent to $\exists y\,(Rxy \wedge \neg\exists z\,(Rxz \wedge Rzy))$.

To actually *use* semantic properties of formulas, a syntactic characterization of all and only the formulas having the properties comes in handy. For monotonicity this involves positive and negative occurrences. An occurrence of a symbol is said to be *positive* iff it is within the scope of an even number of negation signs; otherwise an occurrence is called *negative*.

4.5. THEOREM. *Let* $\beta \in \mathrm{Form}_{\mathcal{L}_{\omega\omega}}[\boldsymbol{\tau}]$, $p \in VAR_s$. *Then* β *is upward (downward) monotone in* p *iff* β *is equivalent to a formula in* $\mathrm{Form}_{\mathcal{L}_{\omega\omega}}[\boldsymbol{\tau}]$ *in which all occurrences of* p *are positive (negative).*

Proof. We prove the characterization of upward monotonicity only. A quick proof using Lyndon Interpolation runs as follows. For a new relation symbol p' let $\boldsymbol{\tau}'$ be $\boldsymbol{\tau}$ extended with p'; $\boldsymbol{\tau}'$-structures then take the form $(\mathfrak{A}, X)$, where $\mathfrak{A}$ is a $\boldsymbol{\tau}$-structure, and X is a relation over an appropriate domain in $\mathfrak{A}$ which interprets p'. The assumption that β is upward monotone in p amounts to $\beta(p')$, $\forall \vec{x}\,(p'(\vec{x}) \to p(\vec{x})) \models \beta(p)$, for a new relation symbol p'. Let γ be an appropriate Lyndon-interpolant. As p' occurs only on the left-hand side of the $\models$-sign, γ does not contain p'; and as p occurs only positively on the left-hand side, p is positive in γ. Hence,

γ is the required equivalent. $\dashv$

The result extends to many other logics, including all logics that have Lyndon Interpolation such as $\mathcal{L}_{\omega_1\omega}$.

CONTINUITY: THE BASIC CASE

We now allow the scheme (2) to contain continuous antecedent formulas.

4.6. DEFINITION. Let $\alpha(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$. Then $\alpha(\vec{x})$ is called *continuous in* $p \in VAR_s$ if for all $\mathfrak{A} = (A, \bigcup_i T_i, \ldots) \in \mathrm{Str}[\boldsymbol{\tau}]$, where $\bigcup_i T_i$ interprets p, and for all $\vec{u} \in A$, we have $\mathfrak{A} \models \alpha[\vec{u}]$ iff $(A, T_i, \ldots) \models \alpha[\vec{u}]$, for some i.

As an example, both $\exists y\,(p(y) \wedge q(y))$ and $\exists y\,(p(y) \wedge \neg q(y))$ are continuous in p; their conjunction is not, however. Hence the class of continuous formulas is not closed under $\wedge$.

As a further example, let $\mathfrak{B}$ be a complete Boolean algebra with operators (BAO), and let f be a completely additive n-ary operator on $\mathfrak{B}$. According to the well-known duality between BAO's and modal frames, f can be represented as a relation R_f on such frames (cf. (Jónsson & Tarski 1952, De Rijke & Venema 1991)). Then, the modal operator $\Diamond_f$, defined by

$$\Diamond_f(p_1, \ldots, p_n) = \{\, x : \exists y_1 \in p_1 \ldots \exists y_n \in p_n\; Rxy_1 \ldots y_n \,\},$$

is a continuous operator. This connection can be made into full-fledged representation: a formula $\beta(p_1, \ldots, p_n; x_1, \ldots x_m)$ is continuous in $\vec{p}$ iff in each model $\mathfrak{A} = (A, \ldots)$ the set $\{\, \vec{a} : \mathfrak{A} \models \beta[\vec{a}] \,\}$ can be represented as the R-image of $p_1, \ldots, p_n$, for some $R \subseteq A^{n+m}$.

4.7. PROPOSITION. *Let $\beta(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then β is continuous in p iff for all $\boldsymbol{\tau}$-structures $\mathfrak{A} = (A, T, \ldots)$, where T interprets p, we have: $\mathfrak{A} \models \beta[\vec{u}]$ iff either $(A, \emptyset, \ldots) \models \beta[\vec{u}]$ or for some $\vec{t} \in T$, $(A, \{\vec{t}\,\}, \ldots) \models \beta[\vec{u}]$.*

Proof. For the *if* direction consider the set $\bigcup_i T_i$. We have $(A, \bigcup_i T_i, \ldots) \models \beta[\vec{u}]$ iff either $(A, \emptyset, \ldots) \models \beta[\vec{u}]$ or for some $\vec{t} \in \bigcup_i T_i$, $(A, \vec{t}, \ldots) \models \beta[\vec{u}]$, iff for some i such that $\vec{t} \in T_i$, $(A, \{\vec{t}\,\}, \ldots) \models \beta[\vec{u}]$. For the *only-if* direction observe that $(A, T, \ldots) \models \beta[\vec{u}]$ iff $(A, \bigcup_{\vec{t} \in T}\{\vec{t}\,\} \cup \emptyset, \ldots) \models \beta[\vec{u}]$ iff either $(A, \emptyset, \ldots) \models \beta[\vec{u}]$, or for some $\vec{t}$, $(A, \{\vec{t}\,\}, \ldots) \models \beta[\vec{t}]$, as required. $\dashv$

In general, continuity of a formula in $p_1, \ldots, p_k$ can be equivalently stated as 2^k possibilities; because of this 'explosion' we don't state results on continuity in full generality.

4.8. LEMMA. *Let $\pi(\vec{x}; \vec{y}; \vec{z}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ be upward monotone in p, and assume $\alpha(\vec{x}; \vec{y}; \vec{z}') \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ is continuous in p. Then $\forall p \forall \vec{y}\,(\alpha \to \pi)$ is reducible to a p-free formula via suitable instantiations.*

Proof. The instances we need here are of the form $\lambda\vec{z}.\,\vec{z} = \vec{y}$, $\lambda\vec{z}.\,\vec{z} = \vec{z}$, or $\lambda\vec{z}.\,\vec{z} \neq \vec{z}$ depending on whether p occurs in α and π, only in α, or only in π. Assume first that p occurs both in α and π. Then

$$\models \Big(\forall \vec{y}\,[(\lambda\vec{y}'.\,\vec{y}' = \vec{y})/p]\forall \vec{z}\,(\alpha \to \pi)\Big) \to \Big(\forall p \forall \vec{z}\,(\alpha \to \pi)\Big).$$

To see this, assume $(\mathfrak{A}, T) \models \forall \vec{y}\,[(\lambda\vec{y}'.\vec{y}' = \vec{y})/p](\alpha \to \pi), \alpha[\vec{u}; \vec{z} \mapsto \vec{w}]$, where T interprets p, and $\vec{z} \mapsto \vec{w}$ means that $\vec{w}$ is assigned to $\vec{z}$. Now, $(\mathfrak{A}, T) \models \alpha[\vec{u}; \vec{z} \mapsto \vec{w}]$ implies that for some $\vec{t} \in T$, $(\mathfrak{A}, \{\vec{t}\,\}) \models \alpha[\vec{u}; \vec{z} \mapsto \vec{w}]$, by 4.7 and monotonicity. Hence $\mathfrak{A} \models [(\lambda\vec{y}'.\vec{y}' = \vec{y})/p]\alpha[\vec{u}; \vec{y} \mapsto \vec{t}; \vec{z} \mapsto \vec{w}]$, so $\mathfrak{A} \models [(\lambda\vec{y}'.\vec{y}' = \vec{y})/p]\pi[\vec{u}; \vec{y} \mapsto \vec{t}; \vec{z} \mapsto \vec{w}]$. But then, by monotonicity, $(\mathfrak{A}, T) \models \pi[\vec{u}; \vec{z} \mapsto \vec{w}]$, as required.

Next, if p occurs only in α, then $\alpha \to \pi$ is downward monotone in p. Hence $\forall p \forall \vec{y}\, (\alpha \to \pi)$ reduces to a p-free formula by instantiating with $\lambda \vec{z}.\, \vec{z} = \vec{z}$ as in 4.3. The case that p occurs only in π is entirely analogous. $\dashv$

How can we apply Lemma 4.8 to obtain reducibility results in 'real life' modal formalisms? §5 contains a double answer in the form of the Sahlqvist-van Benthem and Gabbay-Ohlbach-Simmons algorithms. For readers unable to wait until then, the following is a bare-bones sketch of how to proceed:

- Translate your modal formula into classical logic, preferably into a formula of the form $\forall p \forall \vec{y}\, (\alpha \to \pi)$.
- Perform some cleaning up in the antecedent of the translation to reveal the substitutions needed. As may be seen from the proof of Lemma 4.8, for continuous α the required substitution instances are singletons.
- Perform the substitution, and do some cleaning up.

Here are two examples; formulas supplying the substitutions are underlined.

EXAMPLE. Consider the formula $\Diamond p \to \Box p$ in $\mathcal{ML}(\Diamond)$.

- Higher-order translation: $\forall p\, (\exists y\, (Rxy \wedge p(y)) \to \forall z\, (Rxz \to p(z)))$,
- after rewriting: $\forall p \forall y\, (Rxy \wedge \underline{p(y)} \to \forall z\, (Rxz \to p(z)))$, which has an antecedent continuous in p, and a consequent upward monotone in p,
- substituting $\lambda u.\, u = y$ for p reduces this to $\forall y\, (Rxy \to \forall z\, (Rxz \to z = y))$.

EXAMPLE. In Venema (1991)'s modal logic of converse and composition, one has a binary modal operator $\circ$ based on a ternary relation C. Consider the formula $(a \circ b) \to (b \circ a)$.

- Higher-order translation: $\forall ab\, (\exists yz\, (Cxyz \wedge a(y) \wedge b(z)) \to \exists y'z'\, (Cxy'z' \wedge b(y') \wedge a(z'))$.
- after rewriting: $\forall ab \forall yz\, (Cxyz \wedge \underline{a(y)} \wedge \underline{b(z)} \to \exists y'z'\, (Cxy'z' \wedge b(y') \wedge a(z'))$, which has an antecedent continuous in a, b, and a consequent upward monotone in a, b,
- substituting $\lambda u.\, u = y$ for a, $\lambda u.\, u = z$ for b reduces this to $\forall yz\, (Cxyz \to Cxzy)$.

To facilitate locating the right substitution instance it is useful to syntactically characterize the continuous formulas.

4.9. DEFINITION. Let $\beta \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then β is called *distributive in* p if it is of the form $\exists \vec{x}\, (p(\vec{x}) \wedge \beta') \vee \gamma$, where β', γ are p-free.

An example from **PDL**: $\langle a \rangle \langle b^* \rangle p$ translates into $\exists yz\, (p(z) \wedge R_a xy \wedge \bigvee_n (R_b^n yz))$ — a formula that is distributive in p.

4.10. THEOREM. *Let* $\beta \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$*, and let* p *be an* s*-variable. Then* β *is continuous in* p *iff* β *is equivalent to a formula that is distributive in* p.

Proof. I only prove the *only-if* direction. Let β be continuous in p. Let $\mathfrak{A} = (A, T, \ldots) \models \beta[\vec{u}]$, where T interprets p. Then, by continuity and 4.7,

$$(A, T, \ldots) \models \Big(\exists \vec{x}\, (p(\vec{x}) \wedge [(\lambda \vec{y}.\, \vec{y} = \vec{x})/p]\beta) \vee [(\lambda \vec{y}.\, \vec{y} \neq \vec{y})/p]\beta\Big)[\vec{u}].$$

Let γ denote the latter formula. Then γ has the required syntactic form. Moreover, as γ does not depend on $\mathfrak{A}$ or $\vec{u}$, we have that $\models \beta \to \gamma$; but by the continuity of β and 4.7 this can be strengthened to $\models \beta \leftrightarrow \gamma$, as required. $\dashv$

If in $\forall \vec{p}\, (\alpha \to \pi)$ the antecedent α is distributive in p, then it is continuous in p by Theorem

4.10 — hence the required substitution instance is simply $\lambda\vec{u}.\,\vec{u} = \vec{y}$, where $\vec{y}$ is the unique occurrence $p(\vec{y})$ of p in α.

Generalizing continuity: small subsets

The important features of continuous formulas are that their semantic value may be computed locally (on singletons), and that they are upward monotone. We now generalize from the basic case by maintaining upward monotonicity but liberalizing local computability to 'depends only on *small* sets;' after that we replace the latter with 'depends only on a *definable* set.'

4.11. Definition. Let $\beta(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, and let p be an s-variable. For λ a cardinal, β is called *λ-continuous in p*, if for all $\mathfrak{A} = (A, \bigcup_{i\in I} T_i, \ldots) \in \mathrm{Str}[\boldsymbol{\tau}]$, where $\bigcup_{i\in I} T_i$ interprets p, and for all $\vec{u} \in A$, we have $\mathfrak{A} \models \beta[\vec{u}]$ iff there is an $I_0 \subseteq I$ with $|I_0| < \lambda$ and $(A, \bigcup_{i\in I_0} T_i, \ldots) \models \beta[\vec{u}]$.

Further, $\beta(\vec{x})$ is called *globally λ-continuous in p* if there is a $\kappa < \lambda$ such that for all $\mathfrak{A} = (A, \bigcup_{i\in I} T_i, \ldots) \in \mathrm{Str}[\boldsymbol{\tau}]$, and for all $\vec{u} \in A$, we have $\mathfrak{A} \models \beta[\vec{u}]$ iff for some $I_0 \subseteq I$, $|I_0| \leq \kappa$ and $(A, \bigcup_{i\in I_0} T_i, \ldots) \models \beta[\vec{u}]$.

In Roorda (1993)'s modal approach to Lambek calculus the formula $\triangle(p \wedge q, p \wedge \neg q)$ translates into $\exists yz\,(Cxyz \wedge p(y) \wedge q(y) \wedge p(z) \wedge \neg q(z))$. This formula is not continuous in p; it is 3-continuous in p.

4.12. Proposition. *Let $\beta(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then β is λ-continuous in p iff for every $\mathfrak{A}$ we have $\mathfrak{A} = (A, T, \ldots) \models \beta[\vec{u}]$ iff for some $T_0 \subseteq T$ with $|T_0| < \lambda$, $(A, T_0, \ldots) \models \beta[\vec{u}]$.*

Recall that $\mathcal{L}$ has the *Löwenheim-Skolem property down to κ* if each satisfiable $\mathcal{L}$-formula has a model of power $\leq \kappa$. (The power of a $\boldsymbol{\tau}$-structure $\mathfrak{A}$ is defined as $|A|$ in the one-sorted case, and as $\sum_{s\in\boldsymbol{\tau}} |A_s|$ in the many-sorted case.)

We say that $\beta(p)$ *commutes with unions of non-decreasing chains of sets of length λ* if for every a non-decreasing chain of sets $\{\, T_i \,\}_{i<\lambda}$ we have $\mathfrak{A} = (A, \bigcup_{i<\lambda} T_i, \ldots) \models \beta[\vec{u}]$ iff for some $\kappa < \lambda$, $(A, T_\kappa, \ldots) \models \beta[\vec{u}]$.

4.13. Proposition. *Assume $\mathcal{L}$ has the Löwenheim-Skolem property down to λ. Let β be an $\mathcal{L}$-formula, and let p be an s-variable. Then β is λ-continuous in p iff β commutes with unions of non-decreasing chains of sets of length λ.*

Proof. The *only if* direction: assume $\{\, T_i \,\}_{i<\lambda}$ is a non-decreasing chain of sets such that $(A, \bigcup_{i<\lambda} T_i, \ldots) \models \beta[\vec{u}]$, where $\bigcup_{i<\lambda} T_i$ interprets p. By λ-continuity this is equivalent to: for some $\kappa < \lambda$, $(A, \bigcup_{i<\kappa} T_i, \ldots) \models \beta[\vec{u}]$. As the T_i's form a non-decreasing chain, this is equivalent to $(A, T_\kappa, \ldots) \models \beta[\vec{u}]$, as required.

For the converse, assume that β commutes with unions of non-decreasing chains of length λ. Let $(A, T, \ldots) \models \beta[\vec{u}]$. We may assume that $|A| \leq \lambda$. Then $T = \bigcup_{i<\lambda} T_i$, where $T_0 \subseteq T_1 \subseteq \cdots$ all have $|T_i| < \lambda$. Hence, $(A, T_\kappa \ldots) \models \beta[\vec{u}]$, for some $\kappa < \lambda$, which is sufficient by 4.12. Conversely, if $(A, T_0, \ldots) \models \beta[\vec{u}]$, for some $T_0 \subseteq T$ with $|T_0| < \lambda$, define $T_i = T$ $(0 < i < \lambda)$. Then, by the assumption on β, $(A, T, \ldots) \models \beta[\vec{u}]$. ⊣

4.14. Proposition. *Assume $\mathcal{L}$ is λ-compact. Let $\beta \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then β is λ-continuous in p iff it is globally λ-continuous in p.*

Proof. I only prove the direction from left to right. If β is λ-continuous, then

$$\models \beta \leftrightarrow \bigvee_{\kappa<\lambda} \underbrace{\exists\vec{y}_0 \ldots \exists\vec{y}_\nu \ldots}_{\kappa} \Big(\bigwedge_{i\leq\kappa} p(\vec{y}_i) \wedge [(\lambda\vec{z}.\ \bigvee_{i\leq\kappa} \vec{z} = \vec{y}_i)/p]\beta \Big).$$

By compactness there is a $\kappa_0 < \lambda$ such that

$$\models \beta \to \bigvee_{\mu \le \kappa_0} \underbrace{\exists \vec{y}_0 \ldots \exists \vec{y}_\nu \ldots}_{\mu} \Big(\bigwedge_{i \le \mu} p(\vec{y}_i) \wedge [(\lambda \vec{z}. \bigvee_{i \le \mu} \vec{z} = \vec{y}_i)/p]\beta \Big). \tag{4}$$

As β is upward monotone in p, the implication in (4) must be an equivalence. $\dashv$

4.15. EXAMPLE. In $\mathcal{L}_{\omega\omega}$ ω-continuity and global ω-continuity coincide, according to 4.14. Thus, we need to go beyond $\mathcal{L}_{\omega\omega}$ to find an example of a formula that is ω-continuous, but not globally. In $\mathcal{L}_{\omega_1\omega}$ let β be the statement 'at most n elements satisfy p', and put $\beta := \bigvee_n \beta$. Then β is ω-continuous, but not globally so.

Likewise, in *weak second order logic* $\mathcal{L}^{w2}$, where the relation variables range over *finite* sets only, the statement $\exists q \forall x\, (q(x) \to p(x))$ is locally, but obviously not globally ω-continuous.

In the setting of Boolean algebras with operators, the operators f defined by globally ω-continuous formulas are also known as *(completely) ω-additive* ones: $f(\Sigma U) = \Sigma\{ f(\Sigma(T)) : T \subseteq U, |T| \le m\}$, for some $m \in \omega$ (cf. (Henkin 1970)).

4.16. LEMMA. *Assume that $\pi(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ is upward monotone in p. Let $\alpha(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ be globally ω-continuous in p. Then $\forall \vec{p}\, (\alpha(\vec{x}) \to \pi(\vec{x}))$ reduces to a p-free formula via suitable instantiations.*

Proof. The instances we need here are of the form $\lambda \vec{z}. \bigvee_{i \le n} \vec{z} = \vec{y}_i$ ($n < \omega$), $\lambda \vec{z}. \vec{z} = \vec{z}$, and $\lambda \vec{z}. \vec{z} \ne \vec{z}$, depending on whether p occurs both in α and π, only in α or only in π. The latter two cases are analogous to the corresponding cases in 4.8. So assume p occurs both in α and π. Let $n < \omega$ be the upper bound given by global ω-continuity. Then the following is universally valid:

$$\Big(\bigwedge_{0 \le i \le n} \forall \vec{y}_0 \ldots \forall \vec{y}_i\, [(\lambda \vec{y}'. \bigvee_i \vec{y}_i = \vec{y}')/p] \forall \vec{z}\, (\alpha(\vec{x}) \to \pi(\vec{x})) \Big) \to \Big(\forall p \forall \vec{z}\, (\alpha \to \pi) \Big).$$

This may be seen by using 4.12 and arguing as in 4.8. This suffices. $\dashv$

To be able to restate 4.16 for arbitrary $\lambda > \omega$ we need to assume that $\mathcal{L}$ is closed under quantifier strings and disjunctions of arbitrary length $< \lambda$.

By 4.14 the requirement in 4.16 that α be a globally ω-continuous formula may be weakened to ω-continuity whenever $\mathcal{L}$ is $\aleph_0$-compact.

EXAMPLE. Consider the formula $\Diamond p \wedge \Diamond\Diamond p \to \Box p$ in $\mathcal{ML}(\Diamond)$.

- Higher-order equivalent:

$$\forall p\, (\exists y\, (Rxy \wedge p(y)) \wedge \exists y'\, (Rxy' \wedge \exists y''\, (Ry'y'' \wedge p(y'')) \to \forall z\, (Rxz \to p(z))),$$

- after rewriting:

$$\forall p \forall y y' y''\, (Rxy \wedge \underline{p(y)} \wedge Rxy' \wedge Ry'y'' \wedge \underline{p(y'')} \to \forall z\, (Rxz \to p(z))),$$

- substituting $\lambda u. (u = y \vee u = y')$ for p reduces this to

$$\forall y y' y''\, (Rxy \wedge Rxy' \wedge Ry'y'' \to \forall z\, (Rxz \to (z = y \vee z = y''))).$$

EXAMPLE. Van der Hoek & De Rijke (1992, 1993) study systems of graded modal logic containing modal operators $\langle R\rangle_k p$ whose translation reads

$$\exists x_0 \dots x_k \,(\bigwedge_i Rxx_i \wedge \bigwedge_{0\leq i\neq j\leq k} (x_i \neq x_j) \wedge \bigwedge_i p(x_i));$$

the latter is clearly not continuous, but it is $k+1$-continuous. Consider the graded modal formula $p \wedge \langle R\rangle_k q \to \langle R\rangle_0 (q \wedge \langle R\rangle_0 p)$.

– Higher-order equivalent:

$$\forall pq \forall x_0 \dots x_k \Big(p(x) \wedge \bigwedge_i Rxx_i \wedge \bigwedge_{0\leq i\neq j\leq k} (x_i \neq x_j) \wedge \bigwedge_i q(x_i) \to \exists y\,(Rxy \wedge q(y) \wedge \exists z\,(Ryz \wedge p(z)))\Big), \quad (5)$$

which is of the form prescribed by Lemma 4.16,

– substituting $\lambda u.\, u = x$ for p, and $\lambda u.\, \bigvee_{i\leq k}(u = x_i)$ for q reduces (5) to

$$\forall x_0 \dots x_k \,(\bigwedge_i Rxx_i \wedge \bigwedge_{i\neq j} (x_i \neq x_j) \to \bigvee_{i\leq k} Rx_i x).$$

4.17. DEFINITION. Let $\beta \in \mathrm{Form}_{\mathcal{L}_{\omega\omega}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then β is called *ω-distributive in p* if is built up from p-free formulas and atomic formulas $p(\vec{x})$ using only $\wedge$, $\vee$ and $\exists$.

4.18. THEOREM. *Let $\beta \in \mathrm{Form}_{\mathcal{L}_{\omega\omega}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then β is ω-continuous in p iff it is equivalent to a formula that is ω-distributive in p.*

Proof. This is immediate from 4.14. ⊣

What about λ-continuity for $\lambda > \omega$? As with 4.16 more general versions of 4.18 may be obtained by requiring suitable syntactic closure conditions and using appropriate versions of compactness.

A potentially more interesting issue is this: what are the ω-continuous formulas in extensions of $\mathcal{L}_{\omega\omega}$? In the case of $\mathcal{L}_{\omega_1\omega}$ the answer is almost immediate from 4.14: an $\mathcal{L}_{\omega_1\omega}$-formula is ω-continuous in p iff it is equivalent to a formula constructed from p-free formulas and atomic formulas $p(\vec{x})$ using only $\wedge$, $\bigvee$, $\exists$.

As with continuous formulas the characterization result for ω-continuous formulas is useful in locating the required substitution instances; they are finite disjunctions of the form $\lambda\vec{u}.\, (\bigvee_{i\leq n}(\vec{u} = \vec{y}_i))$.

GENERALIZING CONTINUITY: DEFINABLE SUBSETS

The next obvious way to generalize the notion of continuity is to demand that β holds of p not iff it holds of a singleton in p, but iff it holds of a definable subset of p. In this approach we fix some set X from which the possible definitions of subsets of p may be taken. As in earlier cases, both *local* and *global* versions are possible.

4.19. DEFINITION. A subset $X \subseteq A$ is *$\mathcal{L}$-definable* in $\mathfrak{A}$ if there is an $\mathcal{L}$-formula $\gamma(\vec{x};\vec{y})$ and elements $\vec{t} \in \mathfrak{A}$ such that $X = \{\,\vec{t} : \mathfrak{A} \models \gamma[\vec{u};\vec{t}]\,\}$. If s is a sort in $\boldsymbol{\tau}$, a subset which is definable is *s-free definable* if it has an s-free definition.

The following may be somewhat hard to digest at first. The reward will be considerable, however, as the following will allow us to obtain reducibility results encompassing and vastly extending our earlier results.

4.20. DEFINITION. Let $\beta(\vec{x}) \in \mathrm{Form}_{\mathcal{L}'}[\boldsymbol{\tau}]$, and let p be an s-variable. Then $\beta(\vec{x})$ is *$\mathcal{L}$-definably continuous* in p if for all $\mathfrak{A} = (A, T, \ldots) \in \mathrm{Str}[\boldsymbol{\tau}]$, where T interprets p, and for all $\vec{u} \in A$ we have $\mathfrak{A} \models \beta(\vec{x})[\vec{u}]$ iff for some s-free $\mathcal{L}$-definable subset $X_\gamma = \{\, \vec{t} : \mathfrak{A} \models \gamma(\vec{x}; \vec{y})[\vec{u}\vec{t}] \,\}$ of T we have $(A, X_\gamma, \ldots) \models \beta(\vec{x})[\vec{u}]$.

We call $\beta(\vec{x})$ *$\mathcal{L}$-definably continuous* in p *with additional parameters* if for all structures $\mathfrak{A} = (A, T, \ldots) \in \mathrm{Str}[\boldsymbol{\tau}]$, where T interprets p, and for all $\vec{u} \in A$ we have $\mathfrak{A} \models \beta[\vec{u}]$ iff for some subset $X_\gamma = \{\, \vec{t} : \mathfrak{A} \models \gamma(\vec{x}; \vec{y}; \vec{z})[\vec{u}\vec{t}\vec{w}] \,\}$ of T that is s-free and $\mathcal{L}$-definable, we have $(A, X_\gamma, \ldots) \models \beta[\vec{u}]$.

Further, $\beta(\vec{x})$ is *globally $\mathcal{L}$-definably continuous* if there is a *fixed* finite stock of $\mathcal{L}$-formulas $\gamma_0(\vec{x}; \vec{y}), \ldots, \gamma_n(\vec{x}; \vec{y})$ such that for all $\mathfrak{A} = (A, T, \ldots)$ and $\vec{u}$ in A, we have $\mathfrak{A} \models \beta(\vec{x})[\vec{u}]$ iff for some i $(0 \leq i \leq n)$ $(A, \{\, \vec{t} : \mathfrak{A} \models \gamma_i(\vec{x}; \vec{y})[\vec{u}\vec{t}] \,\}, \ldots) \models \beta(\vec{x})[\vec{u}\vec{t}]$. A global version of $\mathcal{L}$-definable continuity with parameters is defined analogously.

4.21. EXAMPLE. Let $\beta \equiv \forall y\, (\exists z\, (Rxz \wedge Rzy) \to p(y))$; then β is $\mathcal{L}_{\omega\omega}$-definably continuous: $\mathfrak{A} = (A, T, \ldots) \models \beta[u]$, where T interprets p, implies $(A, \{\, v : \mathfrak{A} \models \exists z\, (Rxz \wedge Rzy)[uv] \,\}, \ldots) \models \beta[u]$; the converse implication follows from the fact that β is monotone in p.

For a first-order formula that is *not* $\mathcal{L}_{\omega\omega}$-definably continuous, consider

$$\beta \equiv \forall y\, \Big(Rxy \to [\exists z\, (Ryz \wedge p(z)) \wedge \exists z\, (Ryz \wedge \neg p(z))]\Big),$$

and let $\mathfrak{A} = (\mathbb{N}, \leq, \{\, 2n : n \in \mathbb{N} \,\})$, where $\leq$ interprets R and $\{\, 2n : n \in \mathbb{N} \,\}$ interprets p. Then $\mathfrak{A} \models \beta[0]$. The only $\mathcal{L}_{\omega\omega}$-definable subsets of $\mathbb{N}$ (in terms of R, $=$) are the finite and co-finite sets. But clearly, for no finite or co-finite subset X of $\{\, 2n : n \in \mathbb{N} \,\}$, $(\mathbb{N}, \leq, X) \models \beta[0]$.

4.22. PROPOSITION. *Let $\beta(\vec{x}) \in \mathrm{Form}_{\mathcal{L}_{\omega\omega}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then*

1. *β is $\mathcal{L}_{\omega\omega}$-definably continuous in p iff it is globally $\mathcal{L}_{\omega\omega}$-definably continuous in p, and*
2. *β is $\mathcal{L}_{\omega\omega}$-definably continuous in p with parameters iff it is globally $\mathcal{L}_{\omega\omega}$-definably continuous in p with parameters.*

Proof. 1. We only prove the *only-if* direction. Let $\mathfrak{A} \in \mathrm{Str}[\boldsymbol{\tau}]$. By continuity there is an s-free $\mathcal{L}_{\omega\omega}$-formula γ such that $\mathfrak{A} = (A, T, \ldots) \models \beta[\vec{u}]$ implies that

$$(A, \{\, \vec{t} : \mathfrak{A} \models \gamma(\vec{x}; \vec{y})[\vec{u}\vec{t}] \,\}, \ldots) \models \beta(\vec{x}) \wedge \forall \vec{y}\, (\gamma(\vec{x}; \vec{y}) \to p(\vec{y}))[\vec{u}], \tag{6}$$

and hence

$$(A, \{\, \vec{t} : \mathfrak{A} \models \gamma(\vec{x}; \vec{y})[\vec{u}\vec{t}] \,\}, \ldots) \models [\lambda \vec{y}.\, \gamma(\vec{x}; \vec{y})/p]\beta(\vec{x}) \wedge \forall \vec{y}\, (\gamma(\vec{x}; \vec{y}) \to p(\vec{y}))[\vec{u}].$$

Let $\beta'_{\mathfrak{A},\vec{u}}$ denote the latter formula. Then $\mathfrak{A} \models \beta \leftrightarrow \beta'_{\mathfrak{A},\vec{u}}[\vec{u}]$. So

$$\models \beta \leftrightarrow \bigvee_{\{\, (\mathfrak{A},\vec{u}) \,:\, \mathfrak{A} \models \beta[\vec{u}] \,\}} \beta'_{\mathfrak{A},\vec{u}}[\vec{u}].$$

By compactness the latter disjunction reduces to a finite one, that is, for some n we have

$$\models \beta \leftrightarrow \bigvee_{0 \leq i \leq n} \Big([\lambda \vec{y}.\, \gamma_i(\vec{x}; \vec{y})/p]\beta(\vec{x}) \wedge \forall \vec{y}\, (\gamma_i(\vec{x}; \vec{y}) \to p(\vec{y}))\Big),$$

where all γ_is are s-free $\mathcal{L}_{\omega\omega}$-formulas.

2. This is proved like 1. We replace (6) with

$$\Big(A, \{\, \vec{t} : \mathfrak{A} \models \gamma(\vec{x}; \vec{y}; \vec{z})[\vec{u}\vec{t}\vec{w}] \,\}, \ldots\Big) \models \beta(\vec{x}) \wedge \forall \vec{y}\, (\gamma(\vec{x}; \vec{y}; \vec{z}) \to p(\vec{y}))[\vec{u}\vec{w}], \tag{7}$$

where γ is the formula given by the continuity of β, and the $\vec{w}$ are additional parameters. Clearly, (7) implies that $(A, \{\, \vec{t} : \mathfrak{A} \models \gamma(\vec{x};\vec{y};\vec{z})[\vec{u}\vec{t}\vec{w}]\,\}, \ldots)$ satisfies

$$\exists \vec{z}\left([\lambda\vec{y}.\,\gamma(\vec{x};\vec{y};\vec{z})/p]\beta(\vec{x}) \land \forall\vec{y}\,(\gamma(\vec{x};\vec{y};\vec{z}) \to p(\vec{y}))\right)$$

at $\vec{u}$. Reasoning as before one derives that

$$\models \beta \leftrightarrow \bigvee_{0\le i\le n} \exists \vec{z}_i\left([\gamma_i(\vec{x};\vec{y};\vec{z}_i)/p]\beta(\vec{x}) \land \forall\vec{y}\,(\gamma(\vec{x};\vec{y};\vec{z}_i) \to p(\vec{y}))\right).$$

This implies that β is globally $\mathcal{L}_{\omega\omega}$-definably continuous with parameters. ⊣

4.23. LEMMA. *Let $\pi(\vec{x}) \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ be upward monotone in p.*

1. *Assume $\alpha(\vec{x}) \in \mathrm{Form}_{\mathcal{L}'}[\boldsymbol{\tau}]$ is globally $\mathcal{L}$-definably continuous in p. Then $\forall\vec{p}\,(\alpha(\vec{x}) \to \pi(\vec{x}))$ reduces to a p-free formula via suitable instantiations.*
2. *Assume $\alpha(\vec{x}) \in \mathrm{Form}_{\mathcal{L}'}[\boldsymbol{\tau}]$ is globally $\mathcal{L}$-definably continuous in p with additional parameters. Then $\forall\vec{p}\vec{z}\,(\alpha(\vec{x};\vec{z}) \to \pi(\vec{x};\vec{z}))$ reduces to a p-free formula via suitable instantiations.*

Proof. The instantiations needed are of the form $\lambda\vec{z}.\,\gamma(\vec{x};\vec{y})$, where γ is a p-free $\mathcal{L}$-formula, $\lambda\vec{z}.\,\vec{z} = \vec{z}$, and $\lambda\vec{z}.\,\vec{z} \ne \vec{z}$, depending on whether p occurs both in α and π, only in α or only in π. To see this, assume p occurs both in α and π (the other cases are as before). Let $\gamma_0(\vec{x};\vec{y})$, ..., $\gamma_n(\vec{x};\vec{y})$ be the p-free formulas given by the definable continuity of α. It suffices to show that

$$\models \left(\bigwedge_{0\le i\le n} [\lambda\vec{y}.\,\gamma_i(\vec{x};\vec{y})/p](\alpha(\vec{x}) \to \pi(\vec{x}))\right) \to \left(\forall p(\alpha(\vec{x}) \to \pi(\vec{x}))\right),$$

the converse direction being an instantiation. So assume that $\mathfrak{A} = (A, T, \ldots) \models \alpha[\vec{u}]$; then, for some i we have that $(A, X_{\gamma_i}, \ldots) \models \alpha[\vec{u}]$, where X_{γ_i} is the subset of T defined by γ_i (notation as in 4.20). This implies $(A, X_{\gamma_i}, \ldots) \models [\lambda\vec{y}.\,\gamma_i(\vec{x};\vec{y})/p]\alpha[\vec{u}]$, and by assumption, $(A, X_{\gamma_i}, \ldots) \models [\lambda\vec{y}.\,\gamma_i(\vec{x};\vec{y})/p]\pi[\vec{u}]$; by monotonicity this gives$(A, T, \ldots) \models \pi[\vec{u}]$.

Next assume α is globally definably continuous with parameters. The instantiations we need are of the form $\lambda\vec{z}.\,\gamma(\vec{x};\vec{y};\vec{z})$, where γ is a p-free $\mathcal{L}$-formula, $\lambda\vec{z}.\,\vec{z} = \vec{z}$, and $\lambda\vec{z}.\,\vec{z} \ne \vec{z}$. Assume that p occurs both in α and in π, and let $\gamma_0(\vec{x};\vec{y};\vec{z}_0)$, ..., $\gamma_n(\vec{x};\vec{y};\vec{z}_n)$ be p-free formulas witnessing the continuity of α. Reasoning as before we find

$$\models \left(\bigwedge_{0\le i\le n} \forall\vec{z}_i\,[\lambda\vec{y}.\,\gamma_i(\vec{x};\vec{y};\vec{z}_i)/p](\alpha(\vec{x}) \to \pi(\vec{x}))\right) \to \left(\forall\vec{p}\,(\alpha(\vec{x}) \to \pi(\vec{x}))\right).$$

This suffices. ⊣

EXAMPLE. As an example, consider the formula $\Box\Box p \to \Box p$ in $\mathcal{ML}(\Diamond)$.

- Higher-order equivalent:

$$\forall p\left(\forall y\,(Rxy \to \forall z\,(Ryz \to p(z))) \to \forall v\,(Rxv \to p(v))\right),$$

- after rewriting: $\forall p\,(\underline{\forall yz\,(Rxy \land Ryz \to p(z))} \to \forall v\,(Rxv \to p(v)))$, whose antecedent is definably continuous in p with $\lambda u.\,R^2xz$ as the p-free definition,
- substituting $\lambda u.\,R^2xu$ for p the formula reduces to $\forall v\,(Rxv \to R^2xu)$, i.e. R is dense.

EXAMPLE. In Blackburn & Spaan (1993)'s attribute value logic $\mathbf{L}^{KR[*]}$ with master modality $[*]$ one has models with a stock of binary relations R_l and $x \models [*]p$ iff for all y with $(x, y) \in (\bigcup_l R_l)^*$: $y \models \phi$. Consider the formula $\langle * \rangle [*]p \to p$.

- Higher-order equivalent:

$$\forall p \left(\exists y \left((x, y) \in (\textstyle\bigcup_l R_l)^* \wedge \forall z \left((y, z) \in (\bigcup_l R_l)^* \to p(z) \right) \right) \to p(x) \right),$$

- after rewriting:

$$\forall p \forall y \left((x, y) \in (\textstyle\bigcup_l R_l)^* \wedge \underline{\forall z \left((y, z) \in (\bigcup_l R_l)^* \to p(z) \right)} \to p(x) \right),$$

  where the underlined formula is definably continuous with $\lambda u. \left((y, u) \in (\bigcup_l R_l)^* \right)$ as the p-free definition,
- substituting $\lambda u. \left((y, u) \in (\bigcup_l R_l)^* \right)$ for p gives

$$\forall y \left((x, y) \in (\textstyle\bigcup_l R_l)^* \to (y, x) \in (\bigcup_l R_l)^* \right).$$

EXAMPLE. Shehtman (1993) uses a progressive operator Π in addition to the usual temporal operators F, P to approximate the meaning of the English progressive: $x \models \Pi p$ iff

$$\exists x' x'' \left(Rx'x \wedge Rxx'' \wedge \forall z \left(Rx'z \wedge Rzx'' \to z \models p \right) \right).$$

Consider the formula $\Pi p \to Fp$.

- Higher-order translation:

$$\forall p \left(\exists x' x'' \left(Rx'x \wedge Rxx'' \wedge \forall z \left(Rx'z \wedge Rzx'' \to p(z) \right) \right) \to \exists y \left(Rxy \wedge p(y) \right) \right),$$

- after rewriting:

$$\forall p \forall x' x'' \left(Rx'x \wedge Rxx'' \wedge \underline{\forall z \left(Rx'z \wedge Rzx'' \to p(z) \right)} \to \exists y \left(Rxy \wedge p(y) \right) \right),$$

  where the underlined formula is definably continuous with $\lambda u. \left(Rx'u \wedge Rux'' \right)$ as its p-free definition,
- substituting $\lambda u. \left(Rx'u \wedge Rux'' \right)$ for p reduces the formula to

$$\forall x' x'' \left(Rx'x \wedge Rxx'' \to \exists y \left(Rxy \wedge Rx'y \wedge Ryx'' \right) \right).$$

Despite the somewhat baroque definition of definably continuous formulas, for the definably continuous first-order formulas an explicit syntactic characterization can be given. As in earlier cases a form of distributivity is needed.

4.24. DEFINITION. Let $\beta \in \mathrm{Form}_{\mathcal{L}_{\omega\omega}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then β is called *type 3 distributive in p* if it is a disjunction of formulas of the form $\forall \vec{y} \left(\beta'(\vec{y}) \to p(\vec{y}) \right) \wedge \gamma$, where β' and γ are p-free formulas. Also, β is called *type 4 distributive in p* if it is a disjunction of formulas of the form $\exists \vec{z} \left(\gamma(\vec{x}; \vec{z}) \wedge \forall \vec{y} \left(\beta'(\vec{y}; \vec{z}) \to p(\vec{y}) \right) \right)$, with the same restrictions on β' and γ as before.

4.25. THEOREM. *Let $\beta \in \mathrm{Form}_{\mathcal{L}_{\omega\omega}}[\boldsymbol{\tau}]$, and let p be an s-variable. Then*

1. *β is definably $\mathcal{L}_{\omega\omega}$-continuous in p iff it is equivalent to a formula that is type 3 distributive in p, and*
2. *β is definably $\mathcal{L}_{\omega\omega}$-continuous in p with parameters iff it is equivalent to a formula that is type 4 distributive in p.*

Proof. Use the proof of 4.22. $\dashv$

Observe that if α is type 3 or type 4 distributive, reductions of the kind described in Lemma 4.23 take their substitutions from antecedents β of the Horn-like conditions $\forall\vec{y}\,(\beta \to p)$ occurring in α; the only (possible) difference between the two is that if α is type 4 distributive, β is allowed to contain additional parameters.

To conclude this section, Table 1 summarizes the main points.

semantic property	substitutions needed	syntactic form
upward monotonicity	$\lambda\vec{x}.\,\vec{x} \neq \vec{x}$	positive occurrences only (4.5)
downward monotonicity	$\lambda\vec{x}.\,\vec{x} = \vec{x}$	negative occurrences only (4.5)
continuity	$\lambda\vec{x}.\,\vec{x} = \vec{y}$, $\lambda\vec{x}.\,\vec{x} \neq \vec{x}$ and $\lambda\vec{x}.\,\vec{x} = \vec{x}$	distributive (4.9)
ω-continuity	$\lambda\vec{x}.\,\bigvee_i(\vec{x} = \vec{y}_i)$, $\lambda\vec{x}.\,\vec{x} \neq \vec{x}$ and $\lambda\vec{x}.\,\vec{x} = \vec{x}$	ω-distributive (4.17)
definable continuity	$\lambda\vec{x}.\,\gamma(\vec{x};\vec{y})$ ('s-free')	type 3 distributive (4.24)
definable continuity with parameters	$\lambda\vec{x}.\,\gamma(\vec{x};\vec{y};\vec{z})$ ('s-free')	type 4 distributive (4.24)

Table 1: Forms of continuity.

5 Reduction algorithms

We put our findings of §4 to work. Our input consists of s-universal formulas β, and the aim is to reduce such formulas β to (combinations of) formulas of the form

$$\beta' \equiv \forall\vec{p}\,(\alpha \to \pi),$$

where α satisfies one of the distributivity conditions of §4 for all of its s-variables, and π is positive in all of its s-variables. Given the syntactic form of α the instantiations yielding the required reduction to an s-free formula can then be read of from β'.

There are several ways of rewriting β to β'. The earliest approaches are due to Sahlqvist (1975) and Van Benthem (1976, 1983). A recent version can be found in (Sambin & Vaccaro 1989). These approaches all deal with the uni-modal language with a single diamond $\Diamond$ and box $\Box$ only. They describe a fragment of this language, and show that all formulas in this fragment reduce to first-order formulas. In addition, Sahlqvist (1975) and Sambin & Vaccaro (1989) show that whenever the basic modal logic **K** is extended with axioms taken from this fragment, the resulting system is axiomatically complete. Kracht (1993) obtains those reducibility and completeness results in one go as part of a unifying approach towards definability in modal logic. Venema (1993) obtains a similar double result for certain modal languages containing a difference operator D. Finally, Gabbay & Ohlbach (1992) and Simmons (1992) extend the Sahlqvist-van Benthem algorithm by using Skolem functions; in addition, the latter considers modal languages with arbitrary many unary modal operators instead of a single one.

In this section we first describe the Sahlqvist-van Benthem algorithm extended to arbitrary languages; then its limitations are pointed out, and general strategies for *disproving* reducibility are sketched. Finally we show how the Gabbay-Ohlbach-Simmons approach overcomes some but not all of these limitations.

THE SAHLQVIST-VAN BENTHEM ALGORITHM

This is our strategy: we define a class of formulas Σ and show that every $\beta \in \Sigma$ can be rewritten to a combination of formulas of the form $\forall\vec{p}\forall\vec{y}\,(\alpha(\vec{x};\vec{y}) \to \pi(\vec{x};\vec{y}))$, where α is type 4 distributive in $\vec{p}$, and π is positive in $\vec{p}$. We then apply our results from §4 to show that β must be reducible. For the remainder of this subsection we fix a classical vocabulary $\boldsymbol{\tau}$ and a sort s in $\boldsymbol{\tau}$.

5.1. DEFINITION. (Sahlqvist formulas) We say that $\beta \in \text{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$ is a *simple Sahlqvist formula for* s if it is an s-universal formula of the form $\forall\vec{p}\forall\vec{y}\,(\alpha(\vec{x};\vec{y}) \to \pi(\vec{x};\vec{y}))$, where α is type 4 distributive in all its s-variables, and π is positive in all its s-variables.

The Sahlqvist formulas for s are built up as follows. First, a formula β (not containing any quantifiers binding s-variables) is called an *s-block* if

- it is negative in all its s-variables, or
- it is type 4 distributive in all its s-variables, or
- it is s-free.

Next, *s-antecedents* are defined by the rule

$$\alpha ::= \beta \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \exists\vec{y}\,\alpha,$$

where β is an s-block. Finally, a *Sahlqvist formula* is an s-universal formula of the form $\forall\vec{p}\forall\vec{y}\,\gamma(\vec{x})$ where

$$\gamma ::= \forall\vec{u}\,(\alpha(\vec{u}) \to \pi(\vec{u})) \mid \forall\vec{u}\,(\delta(\vec{u}) \to \gamma(\vec{u})) \mid \gamma_1 \wedge \gamma_2 \mid \gamma_1 \vee \gamma_2, \tag{8}$$

where the formation of disjunctions is subject to the condition that γ_1, γ_2 share no s-variables and no individual variables except for $\vec{x}$, and where α is an s-antecedent, π is positive in all its s-variables, and δ is s-free.

What Definition 5.1 boils down to is that (modulo some 'extras') a Sahlqvist formula is a formula of the form $\forall\vec{p}\forall\vec{y}\,(\alpha \to \pi)$ where π is positive, and in α no $\exists$ or $\vee$ occurs in the scope of a $\forall$.

5.2. LEMMA. (Rewriting Lemma) *Assume that γ is a Sahlqvist formula of the form $\forall\vec{p}\forall\vec{y}\,(\alpha \to \pi)$ with α, π as in (8). Then γ is equivalent to a conjunction of simple Sahlqvist formulas.*

Proof. We first give an inductive recipe for rewriting conjunctions γ of Sahlqvist formulas of the form $\forall\vec{p}\forall\vec{y}\,(\alpha \to \pi)$ to conjunctions of the form

$$\bigwedge_i \forall\vec{p}_i\forall\vec{y}_i \Big(\bigwedge_{j_i} B_{j_i} \to \pi_i\Big), \tag{9}$$

where the B_i are s-blocks and the π_i are positive.

1. if $\forall\vec{p}\forall\vec{y}\,(\exists\vec{z}\,\alpha \to \pi)$ is a conjunct in γ, replace it with $\forall\vec{p}\forall\vec{y}\forall\vec{z}\,(\alpha \to \pi)$;
2. if $\forall\vec{p}\forall\vec{y}\,(\alpha_1 \vee \alpha_2 \to \pi)$ is a conjunct in γ, replace it with $\forall\vec{p}\forall\vec{y}\,(\alpha_1 \to \pi) \wedge \forall\vec{p}\forall\vec{y}\,(\alpha_2 \to \pi)$;

3. if $\forall\vec{p}\forall\vec{y}\,(\alpha_1 \wedge \exists\vec{z}\,\alpha_2 \to \pi)$ is a conjunct in γ, replace it with $\forall\vec{p}\forall\vec{y}\forall\vec{z}\,(\alpha_1 \wedge \alpha_2 \to \pi)$;
4. if $\forall\vec{p}\forall\vec{y}\,(\alpha_1 \wedge (\alpha_2 \vee \alpha_3) \to \pi)$ is a conjunct in γ, replace it with $\forall\vec{p}\forall\vec{y}\,((\alpha_1 \wedge \alpha_2) \vee (\alpha_1 \wedge \alpha_3) \to \pi)$.

Clearly, every conjunct in γ is equivalent to a formula of the form occurring in the antecedent of 1–4. It is also clear that the output of this rewriting recipe has the form described in (9).

Next we show how conjunctions γ of the form (9) can be rewritten to simple Sahlqvist formulas. Take any conjunct in γ; it may be assumed to have the form

$$\forall\vec{p}\forall\vec{y}\,(D \wedge N \wedge F \to \pi), \tag{10}$$

where D is a conjunction of type 4 distributive formulas, N is a conjunction of negative formulas, and F is a conjunction of s-free formulas. Now (10) is equivalent to

$$\forall\vec{p}\forall\vec{y}\,(D \to \pi \vee \neg N \vee \neg F). \tag{11}$$

This is a simple Sahlqvist formula, as $\pi \vee \neg N \vee \neg F$ is positive in all its s-variables. Repeating the procedure for all conjuncts in γ completes the proof. $\dashv$

5.3. THEOREM. (The Sahlqvist-van Benthem Algorithm) *Assume that $\beta(\vec{x})$ is (equivalent to) a Sahlqvist formula for s. Then $\beta(\vec{x})$ reduces to an s-free formula via suitable instantiations. Moreover, these instantiations can be effectively obtained from $\beta(\vec{x})$.*

Proof. We first prove the result for conjunctions of simple Sahlqvist formulas. Let $\forall\vec{p}\forall\vec{y}\,(\alpha \to \pi)$ be such a formula. It is equivalent to a conjunction of formulas of the form

$$\forall\vec{p}\forall\vec{y}\,(D \to \pi'), \tag{12}$$

where π' is positive, and D is type 4 distributive in all s-variables. By Lemma 4.23 (12) reduces to an s-free formula via substitutions that can be read of from D. By Lemma 3.6 the conjunction of reducible formulas is also reducible.

Next, if $\forall\vec{p}\forall\vec{y}\,\gamma$ is a Sahlqvist formula for s, reducibility is obtained by an inductive argument.

- First, Sahlqvist formulas of the form $\forall\vec{p}\forall\vec{y}\,(\alpha \to \pi)$ are equivalent to conjunctions of simple Sahlqvist formulas, by the Rewriting Lemma; hence it is reducible to an s-free formula by the first half of the proof.
- If $\forall\vec{p}\forall\vec{y}\,\gamma$ is of the form $\forall\vec{p}\forall\vec{y}\,(\gamma_1 \wedge \gamma_2)$, it rewrites to $\forall\vec{p}\forall\vec{y}\,\gamma_1 \wedge \forall\vec{p}\forall\vec{y}\,\gamma_2$; the latter reduces to an s-free formula whenever both conjuncts $\forall\vec{p}\forall\vec{y}\,\gamma_1$ and $\forall\vec{p}\forall\vec{y}\,\gamma_2$ do so (by Lemma 3.6).
- If $\forall\vec{p}\forall\vec{y}\,\gamma$ is of the form $\forall\vec{p}\forall\vec{y}\vec{z}\,(\gamma_1 \vee \gamma_2)$, it rewrites to $\forall\vec{p}\forall\vec{y}\,\gamma_1 \vee \forall\vec{p}\,\vec{z}\,\gamma_2$ as only formulas not sharing any bound variables are disjoined; the latter reduces to an s-free formula iff both disjuncts do (Lemma 3.6).
- If $\forall\vec{p}\forall\vec{y}\,\gamma_1$ is of the form $\forall\vec{p}\forall\vec{y}\,(\delta(\vec{x};\vec{y}) \to \pi)$; the latter reduces to an s-free formula iff $\forall\vec{p}\forall\vec{y}\,\gamma_1(\vec{y})$ does (Lemma 3.7). $\dashv$

5.4. REMARK. To recap, the strategy in Theorem 5.3 is to obtain reductions through instantiations. The instances are found by carefully rewriting Sahlqvist formulas into certain combinations of *simple* Sahlqvist formulas $\forall\vec{p}\forall\vec{y}\,(\alpha \to \pi)$, and then simply reading them of from the antecedents α. Detailed examples are provided in §6 below.

Theorem 5.3 takes type 4 distributive formulas as its basic building blocks supplying the required instantiations. Scaled-down analogues of the Sahlqvist Theorem may be obtained by taking one of the other syntactic forms occurring in Table 1, as the basic building blocks.

LIMITATIONS OF THE SAHLQVIST-VAN BENTHEM ALGORITHM

Formulas that are typically excluded from the set of Sahlqvist formulas have implications $\alpha \to \pi$ as their matrix with α containing a $\forall\exists$ or $\forall\,(\ldots\vee\ldots)$-combination. Van Benthem (1983) shows that these limitations occur even in the weakest language we consider here, $\mathcal{ML}(\Diamond)$. Below I will repeat one case (in $\mathcal{ML}(\Diamond)$) of non-reducibility due to a forbidden $\forall\exists$-combination. By way of examples I will show how this case may be used to obtain further *non*-reducibility results for arbitrary modal formulas with first-order definable truth definitions, that contain a forbidden quantification of the form $\forall\exists$.

5.5. PROPOSITION. (Van Benthem (1983)) *The (translation of the) McKinsey formula* $\Box\Diamond p \to \Diamond\Box p$ *does not reduce to a p-free formula over* R, $=$.

Proof. The higher-order translation of the McKinsey formula reads

$$\forall p\left(\forall y \exists z\,(Rxy \to (Ryz \wedge p(z))) \to \exists y' \forall z'\,(Rxy' \wedge (Ry'z' \to p(z')))\right). \tag{13}$$

Non-reducibility is proved by showing that it lacks a first-order equivalent over R, $=$. To this end we show that it does not enjoy the Löwenheim-Skolem property. Consider the frame $\mathfrak{F} = (W, R)$, where

- $W = \{ a, b_n, b_{n_i} : n \in \mathbb{N}, i \in \{0,1\} \} \cup \{ c_f : f : \mathbb{N} \to \{0,1\} \}$,
- $R = \{ (a, b_n), (b_n, b_{n_i}), (b_{n_i}, b_{n_i}) : n \in \mathbb{N}, i \in \{0,1\} \} \cup \{ (a, c_f), (c_f, b_{n_{f(n)}}) : f : \mathbb{N} \to \{0,1\} \}$.

It may be shown that $\mathfrak{F}, a \models (13)$. Take a countable elementary subframe $\mathfrak{F}'$ of $\mathfrak{F}$ containing a and all b_n, b_{n_i}. For some $f : \mathbb{N} \to \{0,1\}$, f is not in $\mathfrak{F}'$ (as $\mathfrak{F}'$ is countable). This f may be used to refute (13) at a in $\mathfrak{F}'$. Hence (13) lacks a first-order equivalent. $\dashv$

Now, the strategy for porting the above non-reducibility to arbitrary modal languages in which all operators have first-order definable patterns, is to code formulas with forbidden quantifier patterns up into the above example 5.5 using first-order means. Here is an example taken from unary interpretability logic (De Rijke 1992*d*). The latter extends provability logic with an operator $\mathbf{I}$ used to simulate the notion of relative interpretability over a given base theory. The semantics of $\mathbf{I}$ is based on a binary relation R and a ternary relation S as follows:

$$(W, R, S, V), x \models \mathbf{I}p \ \text{ iff } \ \forall y\,(Rxy \to \exists z\,(Sxyz \wedge z \models p)).$$

Consider the formula $\mathbf{I}p \to \neg\mathbf{I}\neg p$ whose classical equivalent on frames reads

$$\forall p\left(\forall y \exists z\,(Rxy \to Sxyz \wedge p(z)) \to \exists y' \forall z'\,(Rxy' \wedge (Sxy'z' \to p(z')))\right), \tag{14}$$

which is of the form $\forall p\,(\alpha \to \pi)$ with π positive (in p), and α containing a $\forall\exists$-combination.

5.6. PROPOSITION. *The formula (14) does not reduce to a p-free formula over* R, S.

Proof. Let $\mathfrak{G} = (W, R, S)$ where $\mathfrak{F} = (W, R)$ is as in the proof of Proposition 5.5, and S is defined by

- $\forall xyz\,(Sxyz \leftrightarrow (Rxy \wedge Ryz))$.

Then $\mathfrak{G}, a \models (14) \leftrightarrow (13)$. Hence $\mathfrak{G} \models (14)$. But for $\mathfrak{G}' = (\mathfrak{F}', S')$ with $\mathfrak{F}'$ as in the proof of Proposition 5.5, and S' defined like S above, we must have $\mathfrak{G}' \not\models (14)$, for otherwise $\mathfrak{F}' \models (13)$. ⊣

The same strategy shows non-reducibility results for (classical equivalents of) formulas involving $\forall\exists$-combinations in many other modal languages, with *Until*, *Since*-logic as an obvious example.

As to the second kind of forbidden combinations mentioned earlier, viz. configurations $\forall (\ldots \vee \ldots)$, Van Benthem (1983) gives a non-reducible formula in $\mathcal{ML}(\Diamond)$ whose higher-order equivalent $\forall p\, (\alpha \rightarrow \pi)$ contains such a combination in its antecedent α. Analogous to the above case of $\forall\exists$ this example may be used as a tool for establishing non-reducibility results for 'forbidden formulas' in arbitrary modal languages in which all patterns are first-order definable.

The Gabbay-Ohlbach-Simmons algorithm

Unlike the Sahlqvist-van Benthem algorithm the Gabbay-Ohlbach-Simmons algorithm is able to deal with some cases like (14). By Proposition 5.6 the Gabbay-Ohlbach-Simmons algorithm cannot reduce (14) to a first-order formula involving only R and S (assuming the algorithm is sound). To arrive at a p-free equivalent it uses quantification over Skolem functions. Here is an example. Consider (13) again:

$$\forall p \left(\forall y \exists z\, (Rxy \rightarrow Ryz \wedge p(z)) \rightarrow \exists y' \forall z'\, (Rxy' \wedge (Ry'z' \rightarrow p(z')))\right).$$

The antecedent of the matrix of (13), $\forall y \exists z\, (Rxy \rightarrow Ryz \wedge p(z))$ is equivalent to

$$\exists f \forall y\, (Rxy \rightarrow Ryf(x,y) \wedge p(f(x,y))).$$

Thus (13) is equivalent to

$$\forall p \forall f \Big(\forall y\, (Rxy \rightarrow Ryf(x,y) \wedge p(f(x,y))) \rightarrow \\ \exists y' \forall z'\, (Rxy' \wedge (Ry'z' \rightarrow p(z')))\Big).$$

Substituting $\lambda u.\, \exists z\, (Rxz \wedge u = f(x,z))$ for p in the above gives

$$\forall f \Big(\forall y\, (Rxy \rightarrow Ryf(x,y)) \rightarrow \\ \exists y' \forall z'\, (Rxy' \wedge (Ry'z' \rightarrow \exists v\, (Rxv \wedge z' = f(x,v))))\Big). \qquad (15)$$

A remark is in order: (15) replaces a quantification over unary predicates in (14) with quantification over functions — what has been gained? Besides revealing a link between different fragments of classical logic that may in itself be of logical interest, such replacements are computationally relevant, as is shown by Gabbay & Ohlbach (1992).

We now present the Gabbay-Ohlbach-Simmons algorithm in analogy with the Sahlqvist-van Benthem algorithm. First, we need a set of formulas for the algorithm to operate on.

5.7. Definition. (Extended Sahlqvist formulas) We assume that our vocabulary has function symbols. The type 4 distributive formulas over this vocabulary are defined as in Definition 4.24 — where the arguments of p may now involve function symbols. From these, simple Sahlqvist formulas and s-blocks are defined as in Definition 5.1.

To define extended s-antecedents α we consider an intermediate set of formulas α' generated by

$$\alpha' ::= \beta' \mid \alpha'_1 \wedge \alpha'_2 \mid \exists \vec{y}\, \alpha' \mid \forall \vec{y}\, \alpha',$$

with β' an s-block such that if β' is a type 4 distributive formula, then it should be of the form $\exists \vec{z}\, (\gamma \wedge \forall \vec{y}\, (\beta \to p))$. Then, the *extended s-antecedents* are generated by the rule

$$\alpha ::= \beta \mid \alpha' \mid \alpha_1 \wedge \alpha_2 \mid \alpha_1 \vee \alpha_2 \mid \exists \vec{y}\, \alpha,$$

where β is an s-block. The important restriction here is that no $\forall$ governs a $\vee$. Finally, *extended Sahlqvist formulas* are generated using extended s-antecedents analogous to (8).

For the poly-modal language $\mathcal{ML}(\langle a \rangle : a \in A)$ the above definition specifies the same fragment as the one given by (Simmons 1992). The proof of this claim would require a lengthy and boring induction, and is therefore omitted.

The Gabbay-Ohlbach-Simmons algorithm extends the Sahlqvist-van Benthem algorithm. First there is an Extended Rewriting Lemma.

5.8. LEMMA. (Extended Rewriting Lemma) *Let $\beta \equiv \forall \vec{p} \forall \vec{f} \forall \vec{y}\, (\alpha \to \pi)$ be an extended Sahlqvist formula with α an extended s-antecedent and π positive. Then β is equivalent to a conjunction of (almost simple) Sahlqvist formulas of the form*

$$\forall \vec{p} \forall \vec{f} \forall \vec{y}\, (\bigwedge_i D_i \to \pi), \tag{16}$$

with $\bigwedge_i D_i$ a conjunction of type 4 distributive formulas.

Proof. This is similar to the proof of Lemma 5.2. First we rewrite to a formula as in (16), but with a conjunction of s-blocks in antecedent position rather than distributive formulas. The following rewrite instructions need to be added to the stock in 5.2; their purpose is to move quantifications over functions to the prefix, and to push occurrences of $\forall$ inside as far as possible until they 'reach' a distributive formula that doesn't start with a $\exists$-prefix — without breaking down negative formulas or s-free formulas.

5. if $\forall \vec{p} \forall \vec{f} \forall \vec{y}\, (\ldots \forall \vec{z}\, (\alpha_1 \wedge \alpha_2) \ldots \to \pi)$ is a conjunct in γ, replace it with

$$\forall \vec{p} \forall \vec{f} \forall \vec{y} \left(\ldots \forall \vec{z}\, \alpha_1 \wedge \forall \vec{z}\, \alpha_2 \ldots \to \pi \right);$$

6. if $\forall \vec{p} \forall \vec{f} \forall \vec{y}\, (\ldots \forall \vec{z} \exists u_1 \ldots u_n\, \alpha \ldots \to \pi)$ is a conjunct in γ, replace it with

$$\forall \vec{p} \forall \vec{f} \forall \vec{y} \left(\ldots \exists f_1 \ldots f_n \forall \vec{z} [f_1(\vec{y}, \vec{z})/u_1] \ldots [f_n(\vec{y}, \vec{z})/u_n]\, \alpha \ldots \to \pi \right),$$

for fresh function symbols $f_1, \ldots, f_n$;

7. if $\forall \vec{p} \forall \vec{f} \forall \vec{y}\, (\alpha_1 \wedge \exists g\, \alpha_2 \to \pi)$ is a conjunct in γ, replace it with $\forall \vec{p} \forall \vec{f} g \forall \vec{y}\, (\alpha_1 \wedge \alpha_2 \to \pi)$;
8. if $\forall \vec{p} \forall \vec{f} \forall \vec{y}\, (\ldots \forall \vec{z}\, (\delta \to \theta \wedge p) \ldots \to \pi)$ is a conjunct in γ, replace it with

$$\forall \vec{p} \forall \vec{f} \forall \vec{y} \left(\ldots \forall \vec{z}\, (\delta \to \theta) \wedge \forall \vec{z}\, (\delta \to p) \ldots \to \pi \right).$$

The second half of the proof is similar to the second half of 5.2. ⊣

5.9. THEOREM. (The Gabbay-Ohlbach-Simmons algorithm) *Let $\boldsymbol{\tau}$ be a vocabulary with sufficiently many function symbols, and s a sort in $\boldsymbol{\tau}$. Let $\beta(\vec{x})$ be (equivalent to) an extended Sahlqvist formula for s. Then $\beta(\vec{x})$ reduces to an s-free formula, possibly involving additional function symbols, via suitable instantiations. These instantiations can be effectively obtained from β.*

Proof. This is almost the same as the proof of 5.3; the substitutions arising from distributive formulas involving function symbols of the form $\forall\vec{y}\,(\gamma(\vec{x};\vec{y}) \to p(f(\vec{x};\vec{y})))$ are $\lambda\vec{u}.\,\exists\vec{y}\,(\gamma(\vec{x};\vec{y}) \wedge \vec{u} = f(\vec{x};\vec{y}))$. $\dashv$

Limitations of the Gabbay-Ohlbach-Simmons algorithm

The main gain of extended Sahlqvist formulas over Sahlqvist formulas is that the former allow $\forall\exists$-combinations. However the extended Sahlqvist formulas still suffer from the restriction on $\forall(\ldots \vee \ldots)$-combinations. The importance of the restriction is best explained by an example. Consider Löb's formula in $\mathcal{ML}(\Diamond)$ $\Box(\Box p \to p) \to \Box p$, which translates into

$$\forall p \left(\forall y\,(Rxy \to \exists z\,(Rxz \wedge \neg p(z)) \vee p(y)) \to \forall u\,(Rxu \to p(u))\right)$$

on frames. After Skolemization and rewriting this gives

$$\forall p \forall f \exists y \left((Rxy \to (Rxf(x,y) \wedge \neg p(f(x,y))) \vee p(y)) \to \forall u\,(Rxu \to p(u))\right). \tag{17}$$

At this point we need to define a substitution to achieve a reduction to a p-free formula. However, there is no obvious candidate — because of the disjunction occurring in the antecedent of (17). It seems that to be able to handle cases such as the Löb formula, higher-order functions are needed, ones that take infinite sequences, or even whole 'R-trees' as arguments. On the other hand, it may be that the Löb formula is not expressible in the Gabbay-Ohlbach-Simmons fragment. I will leave this for further study.

6 Applying the algorithms

Section 5 presented the general Sahlqvist-van Benthem and Gabbay-Ohlbach-Simmons algorithms for obtaining reducibility results. To actually apply them to individual modal languages requires a further detailed analysis of those languages to locate the Sahlqvist fragments. Below we illustrate this by examining the languages of standard modal logic, D-logic, *Since*, *Until*-logic, as well as the language of Peirce algebras, and infinitary modal languages. Finally, applications are given to areas other than modal logic, including circumscription.

Standard modal logic

Formulas of the standard modal language $\mathcal{ML}(\Diamond)$ translate into a strict subset of the language of monadic second-order logic. Its Sahlqvist fragment is a strict subset of the general Sahlqvist fragment of the latter (5.1). To be precise, let the set of Sahlqvist formulas $\mathcal{SF}(\Diamond) \subseteq \mathcal{ML}(\Diamond)$ be defined by putting $\chi \in \mathcal{SF}(\Diamond)$ iff it is produced by the following rules

- $\phi ::= \Box^i p \mid \nu \mid \delta$, where ν is negative in all proposition letters occurring in it, and δ is p-free,
- $\psi ::= \phi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \Diamond\psi$,

– $\chi ::= \psi \to \pi \mid \chi_1 \wedge \chi_2 \mid \chi_1 \vee \chi_2 \mid \Box\chi$, where π is positive in all its proposition letters, and $\vee$ is applied only to formulas χ_1, χ_2 that don't share proposition letters.

When interpreted on frames every $\chi \in \mathcal{SF}(\Diamond)$ translates into a Sahlqvist formula over a vocabulary with a single binary relation symbol, and unary predicate variables corresponding to the proposition letters in $\mathcal{ML}(\Diamond)$. By Theorem 5.3 every element of the Sahlqvist fragment $\mathcal{SF}(\Diamond)$ reduces to an s-free formula.

The set of instances needed to reduce every formula in $\mathcal{SF}(\Diamond)$ is an atomic join semi-lattice with partial operators, the atoms being the terms denoting singletons, and the operators correspond to necessitation and are defined on $\vee$-free terms only.

Now that we are considering individual modal languages, much more fine-grained issues become visible than in our general analysis of §§4, 5. As an example, given the Sahlqvist fragment $\mathcal{SF}(\Diamond)$ one may strive for an explicit syntactic description.

6.1. DEFINITION. (Kracht 1993) An individual variable v is called *inherently universal in* α if either it is free in α, or α is of the form $\forall x\,(Rxy \to \beta)$ and v is inherently universal in β. *Inherently existential* is defined similarly. A first-order formula α is *restricted* if it is built using only restricted quantifiers $\forall v\,(Rxv \to \ldots)$ and $\exists v\,(Rxv \wedge \ldots)$.

A *Sahlqvist reduct* is a first-order formula over a binary relation symbol R and $=$ that is equivalent to a positive, restricted formula in which every subformula $R^i yz$ contains at least one inherently universal variable.

6.2. THEOREM. *A first-order formula is definable by means of a Sahlqvist formula in the standard modal language* $\mathcal{ML}(\Diamond)$ *iff it is a Sahlqvist reduct.*

Proof. One direction follows from Theorem 5.3. The other one involves a simple but long case analysis which is too lengthy to be included here. Instead we give an example. Consider the formula

$$\exists z\,(Rxz \wedge \forall y(R^2zy \to Rxy) \wedge Rzx). \tag{18}$$

The idea is to view (18) as being the result of certain substitutions into the translation of a positive modal formula π, to extract those substitutions from (18), and to prefix their modal counterparts as a Sahlqvist antecedent to π. Here we go:

1. the restricted quantification $\exists z\,(Rxz \wedge \ldots$ stems from a diamond $\Diamond$: $\Diamond(\ldots$;
2. the conjunct Rzx refers back to x, thus calling for a proposition letter p to be true at x, and z 'seeing' p: $p \to \Diamond(\Diamond p \wedge \ldots$;
3. finally, in $\forall y\,(R^2zy \to Rxy)$ the antecedent calls for 2 boxes $\Box$, and the consequent refers to 'being a successor of x' which calls for a boxed proposition letter being true at x: $p \wedge \Box q \to \Diamond(\Diamond p \wedge \Box\Box q)$. ⊣

Two short remarks: a similar syntactic analysis can be given for the *extended* Sahlqvist formulas as well (Definition 5.7); and recently Hans-Joachim Ohlbach has announced general results on associating modal equivalents to first-order formulas.

Next, as an application of the Gabbay-Ohlbach-Simmons algorithm in the standard modal language, we show that any modal reduction principle reduces to a p-free formula. First, a *modal reduction principle* in $\mathcal{ML}(\Diamond)$ (mrp) is a modal formula of the form $¥p \to \$p$, where ¥, \$ are (possibly empty) sequences of modal operators $\Diamond$ and $\Box$.

6.3. THEOREM. *The Gabbay-Ohlbach-Simmons algorithm reduces every modal reduction principle* $¥p \to \$p$ *to a* p*-free formula.*

Proof. Van Benthem (1983, Theorem 10.8) fully classifies the mrp's that reduce to a p-free formula by means of the Sahlqvist-van Benthem algorithm. From this result it follows that the use of additional function symbols (as in the Gabbay-Ohlbach-Simmons algorithm) is essential.

To prove the theorem it suffices to observe that every mrp translates into an extended Sahlqvist formula over R, $=$. To get some feel as to how an arbitrary mrp is reduced to a p-free formula, it may be instructive to go over the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$ and its higher-order translation (13) again. $\dashv$

D-LOGIC

We describe the Sahlqvist fragment $\mathcal{SF}(\Diamond, D)$ of the modal language $\mathcal{ML}(\Diamond, D)$ studied in (De Rijke 1992*b*). Put $\chi \in \mathcal{SF}(\Diamond, D)$ if it is produced by the following rules:

- $\phi ::= \overline{\#}_1 \ldots \overline{\#}_n p \mid \nu \mid \delta$, where $\overline{\#}_i \in \{\Box, \overline{D}\}$, ν is negative in all its proposition letters, and δ is p-free,
- $\psi ::= \phi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \#\phi$, where $\# \in \{\Diamond, D\}$,
- $\chi ::= \psi \to \pi \mid \chi_1 \wedge \chi_2 \mid \chi_1 \vee \chi_2 \mid \overline{\#}\chi$, where π is positive in all its proposition letters, $\vee$ is applied only to formulas χ_1, χ_2 having no proposition letters in common, and $\overline{\#} \in \{\Box, \overline{D}\}$.

Here are examples of Sahlqvist formulas in $\mathcal{ML}(\Diamond, D)$ and $\mathcal{ML}(F, P, D)$ plus their reductions to first-order conditions.

EXAMPLE. Consider the $\mathcal{ML}(\Diamond, D)$-formula $\Diamond p \to Dp$.

- Second-order translation: $\forall p\, (\exists y\, (Rxy \wedge p(y)) \to \exists z\, (x \neq z \wedge p(z)))$,
- after rewriting: $\forall p \forall y\, (Rxy \wedge \underline{p(y)} \to \exists z\, (x \neq z \wedge p(z)))$,
- substituting $\lambda u.\, u = y$ for p reduces this to $\forall y\, (Rxy \to \exists z\, (x \neq z \wedge z = y))$, or $\forall y\, (Rxy \to x \neq y)$, or $\neg Rxx$.

EXAMPLE. A slightly more complex example: $p \wedge \neg Dp \to A \neg \Diamond p$, or equivalently, $p \wedge E\Diamond p \to Dp$.

- Second-order translation:

$$\forall p\, \Big(p(x) \wedge \exists yz\, (Ryz \wedge p(z)) \to \exists v\, (v \neq x \wedge p(v))\Big),$$

- after rewriting: $\forall p \forall yz\, (\underline{p(x)} \wedge Ryz \wedge \underline{p(z)} \to \exists v\, (v \neq x \wedge p(v)))$,
- substituting $\lambda u.\, (u = x \vee u = z)$ for p reduces this to

$$\forall yz (Ryz \to \exists v\, (v \neq x \wedge (v = x \vee v = z))),$$

  or $\neg \exists y\, (Ryx)$.

EXAMPLE. As a final example in $\mathcal{ML}(F, P, D)$, consider $Gp \vee Hp \to \overline{D}p$.

- Higher-order equivalent:

$$\forall p\, \Big(\underline{\forall y\, (Rxy \to p(y))} \wedge \underline{\forall y\, (Ryx \to p(y))} \to \forall y\, (y \neq x \to p(y))\Big),$$

- substituting $\lambda u.\, (Rxu \vee Rux)$ for p reduces this to $\forall y\, (x \neq y \to Rxy \vee Ryx)$.

Until, *Since*-LOGIC

The above definition of the Sahlqvist fragment $\mathcal{SF}(\Diamond)$ of $\mathcal{ML}(\Diamond)$ can easily be extended to the language $\mathcal{ML}(F,P)$ of temporal logic with the operators F and P. But the more powerful binary modal operators *Until* (whose pattern reads: $\lambda pq.\, \exists y\, (Rxy \wedge p(y) \wedge \forall z\, (Rxz \wedge Rzy \rightarrow q(z))))$ and *Since* $(\lambda pq.\, \exists y\, (Ryx \wedge p(y) \wedge \forall z\, (Ryz \wedge Rzx \rightarrow q(z))))$ can also be accommodated. To define a Sahlqvist fragment $\mathcal{SF}(\mathit{Until}, \mathit{Since})$ of the modal language with *Until*, *Since*, recall that both F and P are definable using *Until*, *Since*. Let $\#$ range over F, P, and $\overline{\#}$ over G, H. Put $\chi \in \mathcal{SF}(\mathit{Until}, \mathit{Since})$ if it is produced by the following rules

- $\phi ::= \overline{\#}_1 \ldots \overline{\#}_n p \mid \nu \mid \delta$, where ν is negative in all its proposition letters, and δ is p-free,
- $\psi ::= \phi \mid \psi_1 \wedge \psi_2 \mid \psi_1 \vee \psi_2 \mid \#\psi \mid \mathit{Until}(\#_1 \ldots \#_n \psi, \overline{\#}_1 \ldots \overline{\#}_m p) \mid \mathit{Since}(\#_1 \ldots \#_n \psi, \overline{\#}_1 \ldots \overline{\#}_m p)$,
- $\chi ::= \psi \rightarrow \pi \mid \chi_1 \wedge \chi_2 \mid \chi_1 \vee \chi_2 \mid \overline{\#}\chi$, where π is positive in all its proposition letters, and $\vee$ is applied only to formulas χ_1, χ_2 having no proposition letters in common.

All formulas in $\mathcal{SF}(\mathit{Until}, \mathit{Since})$ translate into Sahlqvist formulas over R and $=$; in particular, the 'between-ness' property $\exists y\, (Rxy \wedge p(y) \wedge \forall z\, (Rxz \wedge Rzy \rightarrow q(z)))$ itself is distributive in p and type 4 distributive in q. Thus, by Theorem 5.3, every formula in $\mathcal{SF}(\mathit{Until}, \mathit{Since})$ reduces to a first-order formula.

EXAMPLE. Consider the formula $Fp \rightarrow \mathit{Until}(p,q)$.

- Higher-order equivalent:

$$\forall pq \Big(\exists y\, (Rxy \wedge p(y)) \rightarrow \exists y'\, (Rxy' \wedge p(y') \wedge \forall z'\, (Rxz' \wedge Rz'y' \rightarrow q(z')))\Big),$$

- after rewriting:

$$\forall pq \forall y \Big(Rxy \wedge \underline{p(y)} \rightarrow \exists y'\, (Rxy' \wedge p(y') \wedge \forall z'\, (Rxz' \wedge Rz'y' \rightarrow q(z')))\Big),$$

- substituting $\lambda u.\, u = y$ for p, and $\lambda u.\, u \neq u$ for q gives $\forall y\, (Rxy \rightarrow \neg \exists z\, (Rxz \wedge Rzy))$.

THE LOGIC OF PEIRCE ALGEBRAS

De Rijke (1993*b*) uses a two-sorted modal language $\mathcal{ML}_2$ to axiomatize the representable Peirce algebras (Brink, Britz & Schmidt 1993). $\mathcal{ML}_2$ has an operator $\langle\cdot\rangle\cdot$, interpreted using a ternary relation P, that takes a relation and a set, and returns a set (the Peirce product), and an operator $\circ$, interpreted using a ternary relation C, which is the modal counterpart of relation composition. One of the axioms is $\langle a \circ b\rangle p \rightarrow \langle a\rangle\langle b\rangle p$, where a, b range over relations, and p ranges over sets (or propositions).

- Higher-order translation of $\langle a \circ b\rangle p \rightarrow \langle a\rangle\langle b\rangle p$:

$$\forall ab \forall p \Big(\exists y_r y'_r y''_r z_s \Big(Px_s y_r z_s \wedge Cy_r y'_r y''_r \wedge a(y'_r) \wedge b(y''_r) \wedge p(z_s)\Big) \rightarrow \exists v_r v'_r v''_r z_s \Big(Px_s v_r z'_s \wedge Pv_r v'_r v''_r \wedge a(v'_r) \wedge b(v''_r) \wedge p(z'_s)\Big)\Big),$$

- after rewriting:

$$\forall ab \forall p \forall y_r y'_r y''_r z_s \Big(Px_s y_r z_s \wedge Cy_r y'_r y''_r \wedge \underline{a(y'_r)} \wedge \underline{b(y''_r)} \wedge \underline{p(z_s)} \rightarrow \exists v_r v'_r v''_r z_s\, (Px_s v_r z'_s \wedge Pv_r v'_r v''_r \wedge a(v'_r) \wedge b(v''_r) \wedge p(z'_s))\Big),$$

– substituting $\lambda u_r.\, u_r = y'_r$ for a, $\lambda u_r.\, u_r = y''_r$ for b, and $\lambda u_s.\, u_s = z_s$ for p reduces this to (CP2):

$$\forall y_r y'_r y''_r z_s \left(Px_s y_r z_s \wedge Cy_r y'_r y''_r \to \exists z'_s \left(Px_s y'_s z'_s \wedge Pz'_s y''_r z_s\right)\right).$$

Infinitary modal logic

So far we have applied our methods mainly to modal logics whose operators have first-order patterns. But they can be applied equally well beyond the first-order realm. For instance, they are easily extended to infinitary modal languages such as **PDL**, where one has multiple diamonds $\langle a\rangle$ as well as composition $\langle a; b\rangle$, union $\langle a \cup b\rangle$ and iteration $\langle a^*\rangle$. Because of the Kleene star * **PDL** translates into a fragment of $\mathcal{L}_{\omega_1\omega}$, and on frames into Π^1_1-conditions over $\mathcal{L}_{\omega_1\omega}$. A Sahlqvist fragment for **PDL** is easily defined, resulting in a set of **PDL**-formulas whose $\Pi^1_1(\mathcal{L}_{\omega_1\omega})$-equivalent reduces to a $\mathcal{L}_{\omega_1\omega}$-formula over $R_a, \ldots$ and $=$. Here is an example: $[a^*](\langle b\rangle p \to \langle a^*\rangle p)$.

– Higher-order translation:

$$\forall p \forall y \left(\bigvee_n (R^n_a xy) \to \left(\exists z\, (R_b yz \wedge p(z) \to \exists v \bigvee_n (R^n_a yv) \wedge p(v))\right)\right),$$

– after rewriting:

$$\forall p \forall yz \left(\bigvee_n (R^n_a xy) \wedge R_b yz \wedge \underline{p(z)} \to \exists v \bigvee_n (R^n_a yv) \wedge p(v)\right),$$

– substituting $\lambda u.\, u = z$ for p reduces this to

$$\forall yz \left(\bigvee_n (R^n_a xy) \wedge R_b yz \to \bigvee_n (R^n_a yz)\right).$$

Our methods apply equally well to modal languages with more explicitly infinitary constructs, such as arbitrary disjunctions and conjunctions, as in the infinitary basic modal languages $\mathcal{BML}(\boldsymbol{\tau})$ of (De Rijke 1993*c*). I invite the reader to think up examples for himself.

Beyond modal logic

Applications of 5.3 outside the field of modal logic are easily found. Here are some examples.

First, 5.3 provides us with a scheme for reducing a large class of Π^{n+1}_1-formulas to n-th order formulas. To see this, assume that we are working in a fragment without $(n+1)$-st order constant symbols, let s be a sort containing *all* $(n+1)$-st order variables, and let X be any set of s-free formulas. Then, if χ is an $(n+1)$-st order formulas that is in fact a Sahlqvist formula for s, χ reduces to an s-free formula, i.e. to an n-th order formula.

Second, the Sahlqvist machinery may be used to remove *sorts* from a many-sorted (first-order) theory Δ. Let s be a sort in the language of Δ. If all of Δs axioms are Sahlqvist formulas for s, then Δ has an axiomatization using s-free formulas only — by Theorem 5.3.

Third, recall that *circumscription* is the minimization of predicates subject to restrictions expressed by first-order formulas that is proposed for the purpose of formalizing non-monotonic aspects of common sense reasoning (Lifschitz 1985). The general definition of circumscription involves second-order quantification: circumscription of P with respect to $\alpha(P)$ is

$$Circ(P, \alpha(P)) = \alpha(P) \wedge \forall p \left(\alpha(p) \wedge \forall y\, (p(y) \to P(y)) \to \forall y\, (P(y) \to p(y))\right),$$

or

$$\alpha(P) \wedge \forall p \left(\alpha(p) \rightarrow \forall y\,(P(y) \rightarrow p(y)) \vee \exists y\,(p(x) \wedge \neg P(y))\right). \tag{19}$$

The consequent of the matrix of $\forall p\,(\ldots)$ in (19) is positive in p, so by Theorem 5.3 (19) reduces to a first-order formula whenever $\alpha(p)$ is a p-antecedent (Definition 5.1). As an example, consider $\alpha \equiv \exists x\, Px$. $Circ(P, \exists x\, Px)$ asserts that the extension of P is a minimal non-empty set, that is, a singleton.

- $Circ(P, \exists x\, Px)$: $\exists x\, Px \wedge \forall p\,(\exists x\, p(x) \rightarrow \forall y\,(Py \rightarrow p(y)) \vee \exists y\,(p(y) \wedge \neg Py))$,
- after rewriting: $\exists x\, Px \wedge \forall p \forall x\,(\underline{p(x)} \rightarrow \forall y\,(Py \rightarrow p(y)) \vee \exists y\,(p(y) \wedge \neg Py))$,
- substituting $\lambda u.\, u = x$ for p reduces this to $\exists x Px \wedge \forall x\,(Px \rightarrow \forall y \rightarrow y = x)$.

Lifschitz (1985) presents a 'small' Sahlqvist Theorem. He describes a large class of first-order formulas whose circumscription is first-order; all formulas he gives are Sahlqvist formulas. In effect, the way Lifschitz show his circumscribed formulas to be equivalent to first-order conditions is by means of appropriate substitutions.

7 Another perspective: global restrictions

In previous sections we obtained reducibility results by isolating 'reducible' fragments of a given modal language. We end this paper by considering certain extreme cases of reducibility where full languages become reducible, and where our algorithmic approach of earlier sections no longer work. Below we consider certain *global* restrictions that yield reducibility results. Natural candidates include

- restrictions on the possible values of the variables that are up for reduction,
- restrictions on the vocabulary in which those variables live.
- constraints on the structure of models.

We discuss the first two options. The third option is known as *relative correspondence theory*; Van Benthem (1983) gives a worked-out example in the standard modal language $\mathcal{ML}(\Diamond)$, with the constraint being that the relation R in structures for $\mathcal{ML}(\Diamond)$ should be transitive.

Restricting values

Nominal Tense Logic (Blackburn 1993*b*) extends tense logic with the addition of a new sort of atomic symbols called *nominals*, whose distinguishing feature is that they are true at exactly one point in a model. Here we briefly consider the language $\mathcal{ML}_n^-(\Diamond)$ with the standard diamond, a collection N of nominals, and *no* ordinary proposition letters. The standard translation for $\mathcal{ML}_n^-(\Diamond)$ is as usual for $\neg$, $\wedge$, $\Diamond$, while a nominal i has $ST(i) = (x_i = x)$, where x_i is an individual variable, and x represents the point of evaluation as usual. For $\mathfrak{F} = (W, R)$ a frame of $\mathcal{ML}_n^-(\Diamond)$, we have that $\mathfrak{F}, w \models \phi$ iff $\mathfrak{F}, w \models \forall x_{i_1} \ldots \forall x_{i_n}\, ST(\phi)$, for all $\phi \in \mathcal{ML}_n^-(\Diamond)$; that is: both on frames and on models $\mathcal{ML}_n^-(\Diamond)$-formulas end up as first-order formulas over R.

This observation can be generalized to include sorts of propositional symbols whose truth depends on sets of at most a *fixed* finite number of objects — on frames formulas of such sorted modal languages will all reduce to first-order conditions.

Obviously, at this point many options are available for further analysis. For a modal language whose patterns and connectives all live in a classical logic $\mathcal{L}$, these options are covered by the following restriction:

for all atomic symbols p: $V(p)$ is definable in $\mathcal{L}$.

The result is that in any modal language $\mathcal{ML}$ satisfying this restriction all formulas reduce to 'p-free' $\mathcal{L}$-conditions when interpreted on frames.

The link between the above observations and our results in §§4–5 is best explained by means of a rather bulky definition.

7.1. DEFINITION. Let $\boldsymbol{\tau}$ be a classical vocabulary, s a sort in $\boldsymbol{\tau}$. $M^{def}_{\mathcal{L}(\tau)}(s)$ is the set of all s-universal formulas $\forall\vec{p}\,\alpha$, $\alpha \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, satisfying the following implication

$$\mathfrak{M} \models (\forall \text{ definable } p)\,\alpha \Rightarrow \mathfrak{M} \models \forall p\,\alpha.$$

More precisely, for $\mathfrak{A}$ a $\boldsymbol{\tau}$-structure, let $\mathcal{W}$ consist of all subsets of the (appropriate) domain parametrically definable by means of an s-free $\beta \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$, i.e. $\mathcal{W} = \{\, \{\, u : \mathfrak{A} \models \beta[uv_1 \dots v_n]\,\} : \beta \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}],\ s\text{-free}\,\}$. Then $(\forall\vec{p}\,\alpha) \in M^{def}_{\mathcal{L}(\tau)}(s)$ iff for all $\mathfrak{A}$

$$\mathfrak{A} \models \forall\vec{p} \in \mathcal{W}\,\alpha[\vec{u}] \text{ implies } \mathfrak{A} \models \forall\vec{p}\,\alpha[\vec{u}].$$

Informally, $M^{def}_{\mathcal{L}(\tau)}(s)$ contains all s-universal formulas whose truth depends on $\mathcal{L}$-definable parts of models only. Definition 7.1 generalizes (Van Benthem 1983, Definition 9.14), where a class M^{def}_1 is defined as the set of formulas in $\mathcal{ML}(\Diamond)$ preserved in passing from a general frame $(\mathfrak{F}, \mathcal{W})$ with $\mathcal{W}$ containing all subsets of the domain parametrically definable by means of a first-order formula over R, to the underlying frame $\mathfrak{F}$.

By an easy argument, if $\alpha(p)$ is type 4 distributive in all s-variables, then $\forall\vec{p}\,\alpha$ is in $M^{def}_{\mathcal{L}(\tau)}(s)$. Conversely, assuming $\mathcal{L}$ is compact, if $\forall\vec{p}\,\alpha$ is in $M^{def}_{\mathcal{L}(\tau)}(s)$, it must be equivalent to an (s-free) finite conjunction of formulas of the form $[\beta/p]\alpha$, for β s-free. It is an open question whether this implies that α is equivalent to a type 4 distributive formula for s.

RESTRICTING THE LANGUAGE

We now show by way of example how restricting one's vocabulary may help in boosting reducibility. Here too there are many options. We restrict ourselves to examining what effect the exclusion of relation symbols (other than $=$) of arity ≥ 2 has.

For the time being, let $\mathcal{L}$ denote first-order logic, and let $\boldsymbol{\tau}$ contain only unary predicate symbols. Our aim is to show that for any $\alpha \in \mathrm{Form}[\boldsymbol{\tau}]$, $\forall\vec{p}\,\alpha$ reduces to a p-free (i.e. first-order) formula over $=$ (assuming it is p-universal). The result is not new — it was probably first proved by Ackerman (1954), but I believe the proof is.

Fix $\alpha \in \mathrm{Form}_{\mathcal{L}}[\boldsymbol{\tau}]$; let $p_0, \dots, p_{k-1}$ be the predicate symbols occurring in α, and let $\boldsymbol{\tau}_k$ be the restriction of $\boldsymbol{\tau}$ to these symbols. Let n be the quantifier rank of α.

Let $\mathfrak{M} = (W, P_0, \dots, P_{k-1})$ be a $\boldsymbol{\tau}_k$-structure. For $X \subseteq W$, $X^0 = X$, $X^1 = W \setminus X$. For $s \in 2^k$ the *s-slot* is

$$W^{\mathfrak{M}}_s = P^{s(0)}_0 \cap \dots \cap P^{s(k-1)}_{k-1}.$$

Let $\mathfrak{M} = (W, P_0, \dots, P_{k-1})$, $\mathfrak{M}' = (W', P'_0, \dots P'_{k-1})$ be $\boldsymbol{\tau}_k$-structures. We write $\mathfrak{M} \equiv_n \mathfrak{M}'$ if $\mathfrak{M}$ and $\mathfrak{M}'$ satisfy the same first-order sentences over $\boldsymbol{\tau}_k$ of quantifier rank at most n. For two sets X, Y we write $X \approx_n Y$ iff $|X| = |Y| < n$ or $|X|, |Y| \geq n$; by extension we put $\mathfrak{M} \approx_n \mathfrak{M}'$ iff for all $s \in 2^k$, $W^{\mathfrak{M}}_s \approx_n W^{\mathfrak{M}'}_s$. The important fact is that for any two $\boldsymbol{\tau}_k$-structures $\mathfrak{M}$, $\mathfrak{M}'$ we have $\mathfrak{M} \equiv_n \mathfrak{M}'$ iff $\mathfrak{M} \approx_n \mathfrak{M}'$ (cf. for example (Westerståhl 1989, Section 1.7)).

7.2. THEOREM. *Let $\boldsymbol{\tau}$ be a vocabulary containing only unary predicate letters. Let $\alpha \in \mathrm{Sent}_{\mathcal{L}}[\boldsymbol{\tau}]$, where $\mathcal{L}$ denotes first-order logic. Then $\forall p_1 \dots \forall p_k\,\alpha$ reduces to a first-order formula over $=$ (provided it is p-universal).*

Proof. By a routine argument $\approx_{k \cdot n}$ has finitely many equivalence classes, say $\mathsf{M} = \{\mathfrak{M}_1, \ldots, \mathfrak{M}_m\}$ contains a representative of every class. For every $\mathfrak{M} \in \mathsf{M}$, define a pure identity formula $\beta_{\mathfrak{M}}$ by

$$\beta_{\mathfrak{M}} = \begin{cases} \exists!|\mathfrak{M}| \text{ objects}, & \text{if } |\mathfrak{M}| < k \cdot n, \\ \exists \geq n \text{ objects}, & \text{otherwise.} \end{cases}$$

Define $\gamma = \bigwedge_{\mathfrak{M} \not\models \alpha} \neg\beta_{\mathfrak{M}}$, where $\mathfrak{M} \in \mathsf{M}$. Then $\models \forall\vec{p}\,\alpha \leftrightarrow \gamma$. To see this, assume $\mathfrak{A} \not\models \forall\vec{p}\,\alpha$, i.e. $\mathfrak{A}' = (\mathfrak{A}, P_0, \ldots, P_{k-1}) \models \neg\alpha$. Choose $\mathfrak{M} \in \mathsf{M}$ with $\mathfrak{M} \approx_n \mathfrak{A}'$. Then $\mathfrak{A}' \models \beta_{\mathfrak{M}}$, so $\mathfrak{A} \models \beta_{\mathfrak{M}}$ and $\mathfrak{A} \not\models \gamma$. And conversely, if $\mathfrak{A} \not\models \gamma$, say $\mathfrak{A} \models \beta_{\mathfrak{M}}$, then $\mathfrak{A}$ is 'large enough' so that we can define extensions of the predicates p_i in $\mathfrak{A}$ in a way that yields $\mathfrak{A}' = (\mathfrak{A}, P_0, \ldots, P_{k-1}) \approx_{k \cdot n} \mathfrak{M}$. By definition $\mathfrak{M} \not\models \alpha$, hence $\mathfrak{A}' \not\models \alpha$, and $\mathfrak{A} \not\models \forall\vec{p}\,\alpha$. ⊣

As a consequence of Theorem 7.2, in any modal language whose patterns and connectives are first-order definable over $=$, all formulas reduce to pure identity formulas when interpreted on frames. Examples of modal languages where this applies include

- $\mathcal{ML}(D)$, the language of D-logic studied in (De Rijke 1992*b*),
- $\mathcal{ML}(A)$, the language of the universal modality studied by Goranko & Passy (1992),
- the language of (certain versions of) graded modal logic (Van der Hoek & De Rijke 1992), and other modal languages with modal operators corresponding to first-order definable generalized quantifiers.

8 Concluding remarks

In this paper I have analyzed both the Sahlqvist-van Benthem and Gabbay-Ohlbach-Simmons algorithms for eliminating certain variables. Semantic and syntactic descriptions were given of formulas suitable as input for the algorithms. The algorithms themselves were described in quite general terms, and it was shown how their applications give rise to more fine-grained issues. Finally, we approached the issue of reducibility from a somewhat different angle by considering general restrictions that yield reducibility of *all* formulas of our example languages.

Despite the length of this paper many things had to be left out. What we have achieved, though, is an exposition of the mathematical core of the Sahlqvist-van Benthem and Gabbay-Ohlbach-Simmons algorithms, as well as ample demonstration of their methodology and use.

To conclude here are open questions and suggestions for further work.

1. The Gabbay-Ohlbach-Simmons algorithm was unable to deal with Löb's formula $\Box(\Box p \to p) \to \Box p$, despite the fact that it does have a p-free equivalent (namely well-foundedness). What further functions need to be assumed present to make an extension of the algorithm find this equivalent?
2. In the case of the standard modal language $\mathcal{ML}(\Diamond)$ can one characterize the Sahlqvist reducts (Definition 6.1) semantically? It is easy to see that they must be invariant under generated subframes, disjoint unions, p-morphisms and ultrafilter extensions — but what else, if anything, is needed to fully characterize the Sahlqvist reducts?
3. It can be shown that for *restricted* first-order formulas α, $\forall\vec{p}\,\alpha$ is reducible to an s-free formula iff its is preserved under ultrapowers. What about a result of a more general nature, at the level of abstraction pursued in this paper?
4. What is the complexity of reducibility? Van Benthem (1983, Theorem 17.10) shows that the class of first-order formulas in full Π^1_1-logic is not arithmetically definable. And Chagrova (1991) shows that the question whether a standard modal formula is first-order definable, is undecidable. By a simple argument the set of standard modal formulas

which are first-order definable as a result of the Sahlqvist-van Benthem algorithm, or the Gabbay-Ohlbach-Simmons algorithm is RE — but is it decidable?

5. Finally, a point that has to do with the fine-structure of correspondence theory. What can we say *constructively* about the complexity and shape of the reduced equivalents of a reducible formula? To be more specific, consider $\mathcal{ML}(\Diamond)$. Whereas on models two individual variables suffice to define the standard translation of any formula as was first observed by Dov Gabbay, on frames more variables are needed. As an example, transitivity — modally defined by $\Diamond\Diamond p \to \Diamond p$ — needs essentially 3 variables. What, then, is the connection between the shape of an $\mathcal{SF}(\Diamond)$-formula and the number of individual variables its first-order equivalent on frames needs? Likewise, one may wonder whether it is the case that if ϕ has modal depth n and a first-order equivalent α, then α must be definable with quantifier rank at most n; but this is false; $\Diamond\Box p \to \Box\Diamond p$ has depth 2, while its first-order equivalent is the Church-Rosser property, which has quantifier rank 3. Is there a reasonable function linking the two notions?

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