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CONNECTIFICATION FOR n-CONTRACTION

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Abstract

In this paper, we introduce connectification operators for intuitionistic and classical linear algebras corresponding to linear logic and to some of its extensions with n-contraction. In particular, n-contraction ($n \geq 2$) is a version of the contraction rule, where $n + 1$ occurrences of a formula may be contracted to n occurrences. Since cut cannot be eliminated from the systems with n-contraction considered most of the standard proof-theoretic techniques to investigate meta-properties of those systems are useless. However, by means of connectification we establish the disjunction property for both intuitionistic and classical affine linear logics with n-contraction.

1 Introduction

The idea of connectification has often been used in the literature to give a model-theoretic proof of the disjunction property (DP) and of the existence property (EP) for various intuitionistic theories. Besides the well-known connectification operators for Ω models (see Troelstra, van Dalen [10]), Smorynski in [8] used connectification for Kripke models to prove (DP) and (EP) and many other closure properties for Heyting arithmetic. Further, there are well-known generalizations of Smorynski's method, corresponding to intuitionistic higher order theories, the Freyd-cover of a topos (see Moerdijk [5], Scedrov, Scott [7]) and an alternative to the Freyd-cover introduced by Moerdijk [4]. These so-called glueing techniques, i.e. connectification methods, also have corresponding syntactic counter-parts known as "slashing-relations", such as the Aczel slash (see Troelstra, van Dalen [10], Smorynski [8]) and the Friedman slash (see Scedrov, Scott [7]).

In this paper, we shall introduce a suitable connectification operator for intuitionistic and classical algebras corresponding to linear logic and some of its extensions with n-contraction ($n \geq 2$), i.e. affine case: $\mathbf{IPL}_n^a$, $\mathbf{CPL}_n^a$, and non-affine case: $\mathbf{IPL}_n$, $\mathbf{CPL}_n$. To be specific, n-contraction ($n \geq 2$) is a version of the contraction rule, where $(n+1)$ occurrences of a formula may be contracted to n occurrences. A variant of $\mathbf{CPL}_n^a$ was first considered in Prijatelj [6], while $\mathbf{IPL}_n$ appeared, soon after that, in a slightly more general guise in Hari, Ono and Schellinx [2]. Moreover, we shall introduce a connectification operator for a non-commutative version of $\mathbf{IPL}_n^a$-algebras, corresponding to extended directional Lambek calculi, $\mathbf{L}_n^a$ (for a comparison, see Kanazawa [3]).

However, it will become clear later on that only the connectification operators for affine linear algebras are useful to prove the disjunction property for the underlying intuitionistic and classical systems. The crucial reason is that the presence of weakening in the systems enforces the top element of a lattice to coincide with the unit of a monoid in the corresponding algebras. The shortcoming of the connectification operators for the non-affine algebras to handle (DP) will be discussed in the last section.

Since none of the extensions of linear logic considered here, enjoys cut-elimination (witness counter-examples in Hari, Ono and Schellinx [2] and in Prijatelj [6]), it is difficult to establish almost any of their meta-properties syntactically. Thus, we shall in what follows focus on a model-theoretic proof of (DP) for the systems $\mathbf{IPL}_n^a$ and $\mathbf{CPL}_n^a$ by means of connectification.

In the last section we shall introduce connectification operators for the rest of linear algebras discussed above. We shall show that for a particular subclass of $\mathbf{IPL}_n$-models the valuation of any $\perp$-free $\mathbf{IPL}_n$-formula is preserved under the connectification considered. Moreover, this distinguished class of $\mathbf{IPL}_n$-models is complete for the $\perp$-free $\mathbf{IPL}_n$ system, as pointed out in the sequel.

Let us finally mention that Troelstra's notation for the operators of linear logic will be used in this paper (see Troelstra [9]).

2 Intuitionistic Systems with n-Contraction and Weakening

For any $n \geq 2$, an intuitionistic system of affine propositional logic with n-contraction, $\mathbf{IPL}_n^a$, is given by the following axioms and rules. Throughout the below, Λ denotes the empty multiset, Φ denotes either an occurrence of an $\mathbf{IPL}_n^a$-formula or the empty multiset, and Γ, Γ_1, Γ_2 stand for finite multisets of $\mathbf{IPL}_n^a$-formulas.

Axioms

$$A \Rightarrow A \qquad 0 \Rightarrow \Lambda \qquad \Lambda \Rightarrow 1$$

Logical rules

$$L\star \ \frac{\Gamma, A, B \Rightarrow \Phi}{\Gamma, A \star B \Rightarrow \Phi} \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \star B} \ R\star$$

$$L \multimap \ \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2, B \Rightarrow \Phi}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Phi} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B} \ R \multimap$$

$$L\sqcap \ \frac{\Gamma, A_i \Rightarrow \Phi}{\Gamma, A_1 \sqcap A_2 \Rightarrow \Phi} \ (i = 1, 2) \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B} \ R\sqcap$$

$$L\sqcup \ \frac{\Gamma, A \Rightarrow \Phi \qquad \Gamma, B \Rightarrow \Phi}{\Gamma, A \sqcup B \Rightarrow \Phi} \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \sqcup A_2} \ (i = 1, 2) \ R\sqcup$$

Structural rules

$$LW \ \frac{\Gamma \Rightarrow \Phi}{\Gamma, A \Rightarrow \Phi} \qquad \frac{\Gamma \Rightarrow \Lambda}{\Gamma \Rightarrow A} \ RW$$

$$LC_n \ \frac{\Gamma, A^{(n+1)} \Rightarrow \Phi}{\Gamma, A^{(n)} \Rightarrow \Phi}$$

where $A^{(k)} \equiv A, A, \ldots, A$, i.e. k copies of formula A.

$$CUT \ \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2, A \Rightarrow \Phi}{\Gamma_1, \Gamma_2 \Rightarrow \Phi}$$

Remark: A non-involutive negation can be defined by $\sim A = A \multimap 0$. Note that the respective left and right rules

$$L\sim \ \frac{\Gamma \Rightarrow A}{\Gamma, \sim A \Rightarrow \Lambda} \qquad\qquad \frac{\Gamma, A \Rightarrow \Lambda}{\Gamma \Rightarrow \ \sim A}\ R\sim$$

are derivable in $\mathbf{IPL}_n^a$.

3 Algebraic Models for $\mathbf{IPL}_n^a$

Definition 3.1 $\mathbf{X} = \langle X, \star, \multimap, \sqcap, \sqcup, 0, 1\rangle$ *is an* $\mathbf{IPL}_n^a$ *algebra, if:*

(1) $\langle X, \star, 1\rangle$ *is a commutative monoid with unit* 1*;*

(2) $\langle X, \sqcap, \sqcup, 0, 1\rangle$ *is a lattice with bottom* 0 *and top* 1*;*

(3) $\star$ *is monotone with respect to the lattice order* $\leq$*, i.e. for all* $x, y, z \in X$*, if* $x \leq y$*, then* $x \star z \leq y \star z$*;*

(4) for all $x, y, z \in X$*,* $x \star y \leq z$ *if and only if* $x \leq y \multimap z$*;*

(5) for all $x \in X$*,* $x^n \leq x^{n+1}$*, where* $x^k = x \star \cdots \star x$ *with* k *copies of* x*.*

Remark: Note that, an $\mathbf{IPL}_n^a$-algebra is in fact an intuitionistic linear algebra with zero (see Troelstra [9]), satisfying in addition:

- $\bot = 0$ and $x \leq 1$ for all $x \in X$ (corresponding to weakening);
- clause (5) (corresponding to n-contraction).

However, for the notational perspicuity in the next two sections we shall here adopt the above given formulation of an $\mathbf{IPL}_n^a$-algebra.

Lemma 3.2 *In any* $\mathbf{IPL}_n^a$*-algebra* $\mathbf{X} = \langle X, \star, \multimap, \sqcap, \sqcup, 0, 1\rangle$*, the following are satisfied for all* $x, x', y, y', z \in X$*:*

(a) $x \star (y \sqcup z) = (x \star y) \sqcup (x \star z)$*, and moreover,*
$x \star \bigsqcup_{i\in I} y_i = \bigsqcup_{i \in I}(x \star y_i)$ *, provided* $\bigsqcup_{i\in I} y_i$ *exists;*

(b) if $x \leq x'$ *and* $y \leq y'$*, then* $x' \multimap y \leq x \multimap y'$*;*

(c) $y \multimap z = \max\{x \in X \mid x \star y \leq z\}$*;*

(d) $x \star y \leq x$;

(e) $x \star 0 = 0$;

(f) $x^n = x^{n+1}$.

Proof: The proof of (a)-(c) is standard (see Troelstra [9]), (d) and (f) are straightforward consequences of the affine character of an $\mathbf{IPL}_n^a$-algebra. $\diamond$

The following corresponds to a fact well-known from the theory of Heyting algebras.

Fact 3.3 *Let the clauses (1)-(3) of the definition of an* $\mathbf{IPL}_n^a$*-algebra be satisfied for some* $\langle X, \star, \sqcap, \sqcup, 0, 1\rangle$. *If* $\multimap$ *is well-defined on* $X \times X$ *by*

$$y \multimap z = \max\{x \in X \mid x \star y \leq z\},$$

then also (4) of definition 3.1 is fulfilled for $\langle X, \star, \multimap, \sqcap, \sqcup, 0, 1\rangle$.

Definition 3.4 $\mathbf{M} = \langle \mathbf{X}, [\![\,.\,]\!]\rangle$ *is an* $\mathbf{IPL}_n^a$*-model, if*

(1) $\mathbf{X}$ *is an* $\mathbf{IPL}_n^a$*-algebra;*

(2) $[\![\,.\,]\!]$ *is a valuation satisfying:*

(i) $[\![P]\!] \in X$, *for every propositional variable* P;

(ii) $[\![0]\!] = 0$, $[\![1]\!] = 1$;

$[\![\,.\,]\!]$ *is extended to arbitrary* $\mathbf{IPL}_n^a$*-formula inductively, as follows:*

$$[\![A \bullet B]\!] = [\![A]\!] \bullet [\![B]\!], \quad \textit{with} \quad \bullet \in \{\star, \multimap, \sqcap, \sqcup\}.$$

Moreover, $[\![\,.\,]\!]$ *is extended to multisets by:*

$$[\![\Lambda]\!] = 1 \ \textit{ and } \ [\![\Gamma, \Delta]\!] = [\![\Gamma]\!] \star [\![\Delta]\!].$$

A sequent $\Gamma \Rightarrow A$ *is valid in* $\mathbf{M}$, *denoted by* $\models_{\mathbf{M}} \Gamma \Rightarrow A$, *if and only if* $[\![\Gamma]\!] \leq [\![A]\!]$. *Moreover, we stipulate that a sequent of the form* $\Gamma \Rightarrow \Lambda$ *is valid in* $\mathbf{M}$ *if and only if a sequent* $\Gamma \Rightarrow 0$ *is valid in* $\mathbf{M}$, *i.e. if and only if* $[\![\Gamma]\!] \leq 0$.

Remark: A sequent of the form $\Lambda \Rightarrow A$ is valid in $\mathbf{M}$ if and only if $[\![A]\!] = 1$, since 1 is the top element in X.

Proposition 3.5 (Soundness) *Given an* $\mathbf{IPL}_n^a$*-sequent* $\Gamma \Rightarrow \Phi$,

$$\text{if } \mathbf{IPL}_n^a \vdash \Gamma \Rightarrow \Phi, \quad \text{then} \quad \models_{\mathbf{M}} \Gamma \Rightarrow \Phi,$$

for every $\mathbf{IPL}_n^a$*-model* $\mathbf{M}$.

Proof: By induction on the length of a derivation of $\Gamma \Rightarrow \Phi$. $\diamond$

Proposition 3.6 (Completeness) *There exists an* $\mathbf{IPL}_n^a$*- model* $\mathbf{M}_L$*, such that*

$$\text{if } \models_{\mathbf{M}_L} \Gamma \Rightarrow A, \quad \text{then} \quad \mathbf{IPL}_n^a \vdash \Gamma \Rightarrow A.$$

Proof: Observe that the Lindenbaum algebra of $\mathbf{IPL}_n^a$ is an $\mathbf{IPL}_n^a$-algebra. The rest of the proof is standard. $\diamond$

4 Connectification with a new top element

Definition 4.1 *Let* $\mathbf{X} = \langle X, \star, \multimap, \sqcap, \sqcup, 0, 1\rangle$ *be an* IPL_n^a*-algebra. The connectification of* $\mathbf{X}$ *with a new top element* $1_c \notin X$ *is the* $\mathbf{IPL}_n^a$*-algebra* $\mathbf{X}_c = \langle X \cup \{1_c\}, \star_c, \multimap_c, \sqcap_c, \sqcup_c, 0\rangle$*, given by:*

(1) $\star_c$ *is the extension of* $\star$ *on* $X \cup \{1_c\}$ *, defined by:*

$$x \star_c 1_c = 1_c \star_c x = x \text{ for all } x \in X \cup \{1_c\}.$$

(2) $\leq_c$ *is the extension of the lattice order* $\leq$ *on* $X \cup \{1_c\}$*, given by:*

$$\text{for all } x \in X \cup \{1_c\} : \quad x \leq_c 1_c$$

(3) $\multimap_c$ *is defined on* $(X \cup \{1_c\}) \times (X \cup \{1_c\})$ *by:*

$$y \multimap_c z = \max\{x \in X \cup \{1_c\} | \, x \star_c y \leq_c z\}.$$

Remark: Note that, explicitly:

$$y \multimap_c z = \begin{cases} 1_c & \text{if } y \leq_c z \\ z & \text{if } y = 1_c \\ y \multimap z & \text{otherwise} \end{cases}$$

Hence, $\multimap_c$ is in fact well-defined on $(X \cup \{1_c\}) \times (X \cup \{1_c\})$.

Next we verify that $\mathbf{X}_c = \langle X \cup \{1_c\}, \star_c, \multimap_c, \sqcap_c, \sqcup_c, 0\rangle$ is an $\mathbf{IPL}_n^a$-algebra. By (1) and (2) above, clauses (1) and (2) of the definition of an $\mathbf{IPL}_n^a$-algebra are satisfied. To justify monotonicity of $\star_c$ with respect to $\leq_c$, clause (d) of lemma 3.2 is to be used. So far, clauses (1)-(3) of the definition of an $\mathbf{IPL}_n^a$-algebra are satisfied for $\langle X \cup \{1_c\}, \star_c, \sqcap, \sqcup, 0\rangle$. Moreover, $\multimap_c$ is well-defined on $(X \cup \{1_c\}) \times X \cup \{1_c\})$, and therefore, by fact 3.3, clause (4) of definition of an $\mathbf{IPL}_n^a$-algebra is fulfilled for $\mathbf{X}_c$ as well. Since 1_c is the unit for $\star_c$ in $X \cup \{1_c\}$, clause (5) of definition 3.1 is satisfied for $\mathbf{X}_c$ too.

Next, we shall introduce the connectification of an $\mathbf{IPL}_n^a$-model with a new top element.

Definition 4.2 *Let $\mathbf{M} = \langle \mathbf{X}, [\![\,.\,]\!]\rangle$ be an $\mathbf{IPL}_n^a$-model. The connectification of $\mathbf{M}$ with a new top element 1_c is the $\mathbf{IPL}_n^a$-model $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\,.\,]\!]_c\rangle$, given by:*

(1) $\mathbf{X}_c$ is the connectification of $\mathbf{X}$ with a new top element 1_c;

(2) $[\![\,.\,]\!]_c$ is the valuation, defined by: $[\![P]\!]_c = [\![P]\!]$, for any propositional variable P.

5 Disjunction Property for $\mathbf{IPL}_n^a$

We are now ready to prove a useful

Lemma 5.1 *Let $\mathbf{M} = \langle \mathbf{X}, [\![\,.\,]\!]\rangle$ be an $\mathbf{IPL}_n$-model and $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\,.\,]\!]_c\rangle$ the connectification of $\mathbf{M}$ with a new top element 1_c. Then the following holds true for any $\mathbf{IPL}_n^a$-formula A:*

(i) if $[\![A]\!]_c = 1_c$, then $[\![A]\!] = 1$;

(ii) if $[\![A]\!]_c <_c 1_c$, then $[\![A]\!]_c = [\![A]\!]$.

Proof: By induction on the complexity of A.

To illustrate the proof, we will here work out the only tricky case. Assume $A = B \multimap D$.

(i) If $[\![B \multimap D]\!]_c = 1_c$, then the following two possibilities are to be distinguished:

(1) $[\![D]\!]_c = 1_c$, then by induction hypothesis we get: $[\![D]\!] = 1$. Thus, due to lemma 3.2(c) and the fact, that 1 is the top element in X, we get: $[\![B \multimap D]\!] = [\![B]\!] \multimap [\![D]\!] = [\![B]\!] \multimap 1 = 1$, what was to be shown.

(2) $[\![D]\!]_c <_c 1_c$. Then, due to assumption (i) and definition of $\multimap_c$, we know: $[\![B]\!]_c \leq_c [\![D]\!]_c$, yielding $[\![B]\!]_c <_c 1_c$. Using now the induction hypothesis, we get: $[\![D]\!]_c = [\![D]\!]$ and $[\![B]\!]_c = [\![B]\!]$. And therefore, $[\![B]\!] \leq [\![D]\!]$. Hence, $[\![B \multimap D]\!] = [\![B]\!] \multimap [\![D]\!] = 1$, due to lemma 3.2(c) and the fact, that 1 is the top element and the unit for $\star$ in X.

(ii) If $[\![B \multimap D]\!]_c <_c 1_c$, then the following two possibilities are to be distinguished:

(1) $[\![B]\!]_c = 1_c$, then by definition of $\multimap_c$, assumption (ii) and the fact that 1_c is the top element in $X \cup \{1_c\}$, we know: $[\![D]\!]_c <_c 1_c$. Now we can use induction hypotheses yielding: $[\![B]\!] = 1$ and $[\![D]\!]_c = [\![D]\!]$. And hence, $[\![B \multimap D]\!] = [\![B]\!] \multimap [\![D]\!] = 1 \multimap [\![D]\!] = [\![D]\!]$, due to lemma 3.2(c) and the fact, that 1 is the unit for $\star$ in X. But, $[\![D]\!] = [\![D]\!]_c = 1_c \multimap_c [\![D]\!]_c = [\![B]\!]_c \multimap_c [\![D]\!]_c = [\![B \multimap D]\!]_c$, what was to be shown.

(2) $[\![B]\!]_c <_c 1_c$, then by induction hypothesis we have: $[\![B]\!]_c = [\![B]\!]$. Moreover, due to assumption (ii), and definition of $\multimap_c$, we know that $[\![B]\!]_c \not\leq_c [\![D]\!]_c$, yielding $[\![D]\!]_c <_c 1_c$. Using induction hypothesis once again gives: $[\![D]\!]_c = [\![D]\!]$. From that and definition of $\multimap_c$, we get: $[\![B \multimap D]\!]_c = [\![B]\!]_c \multimap_c [\![D]\!]_c = [\![B]\!]_c \multimap [\![D]\!]_c = [\![B]\!] \multimap [\![D]\!] = [\![B \multimap D]\!]$. And we are done. $\diamond$

Remark: The above lemma shows that the class of all $\mathbf{M}_c$-valid formulas is a subclass of $\mathbf{M}$-valid formulas.

Lemma 5.2 *Let* $\mathbf{M} = \langle \mathbf{X}, [\![\ .\ ]\!] \rangle$ *be an* $\mathbf{IPL}^a_n$*-model and* $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\ .\ ]\!]_c \rangle$ *the connectification of* $\mathbf{M}$ *with a new top element* 1_c.

$$\text{If } [\![A]\!] < 1 \text{ and } [\![B]\!] < 1, \text{ then } [\![A \sqcup B]\!]_c \leq_c 1,$$

where A *and* B *are* $\mathbf{IPL}^a_n$*-formulas.*

Proof: Suppose $\mathbf{M} = \langle \mathbf{X}, [\![\,.\,]\!] \rangle$ is an $\mathbf{IPL}_n^a$-model, such that for some $\mathbf{IPL}_n^a$-formulas A and B, $[\![A]\!] < 1$ and $[\![B]\!] < 1$. Then, by contraposition of the statement (i) of lemma 5.1, we get $[\![A]\!]_c \leq_c 1$ and $[\![B]\!]_c \leq_c 1$, since 1 and 1_c are the top elements in X and in X_c respectively. Hence, $[\![A \sqcup B]\!]_c = [\![A]\!]_c \sqcup_c [\![B]\!]_c \leq_c 1$, and we are done. $\diamond$

Remark: The above lemma can be rewritten as follows. Given an $\mathbf{IPL}_n^a$-model $\mathbf{M}$ such that $\not\models_{\mathbf{M}} \Lambda \Rightarrow A$ and $\not\models_{\mathbf{M}} \Lambda \Rightarrow B$, then $\not\models_{\mathbf{M}_c} \Lambda \Rightarrow A \sqcup B$, where $\mathbf{M}_c$ is the connectification of $\mathbf{M}$ with a new top element.

Lemma 5.3 *The product of* $\mathbf{IPL}_n^a$*-models,* $\mathbf{M}_1 = \langle \mathbf{X}_1, [\![\,.\,]\!]_1 \rangle$ *and* $\mathbf{M}_2 = \langle \mathbf{X}_2, [\![\,.\,]\!]_2 \rangle$*, is the* $\mathbf{IPL}_n^a$*-model* $\mathbf{M}_1 \times \mathbf{M}_2 = \langle \mathbf{X}_1 \times \mathbf{X}_2, ([\![\,.\,]\!]_1, [\![\,.\,]\!]_2) \rangle$*, with the operations in* $\mathbf{X}_1 \times \mathbf{X}_2$ *defined component-wise.*

Proposition 5.4 *The system* $\mathbf{IPL}_n^a$ *enjoys the following disjunction property:*

$$\textit{if}\ \ \mathbf{IPL}_n^a \vdash \Lambda \Rightarrow A \sqcup B, \quad \textit{then}\ \ \mathbf{IPL}_n^a \vdash \Lambda \Rightarrow A \quad \textit{or}\ \ \mathbf{IPL}_n^a \vdash \Lambda \Rightarrow B.$$

Proof: Suppose that $\mathbf{IPL}_n^a \not\vdash \Lambda \Rightarrow A$ and $\mathbf{IPL}_n^a \not\vdash \Lambda \Rightarrow B$. Then, due to completeness, see proposition 3.6, there are $\mathbf{IPL}_n^a$-models $\mathbf{M}_1 = \langle \mathbf{X}_1, [\![\,.\,]\!]_1 \rangle$ and $\mathbf{M}_2 = \langle \mathbf{X}_2, [\![\,.\,]\!]_2 \rangle$, such that:

$$\not\models_{\mathbf{M}_1} \Lambda \Rightarrow A \ \text{ and } \ \not\models_{\mathbf{M}_2} \Lambda \Rightarrow B.$$

This means, that

$$[\![A]\!]_1 <_1 1_1 \quad \text{and} \quad [\![B]\!]_2 <_2 1_2,$$

where 1_1 and 1_2 are the top elements in the corresponding $\mathbf{IPL}_n^a$-models respectively. But, by lemma 5.3, $\mathbf{M}_1 \times \mathbf{M}_2 = \langle \mathbf{X}_1 \times \mathbf{X}_2, ([\![\,.\,]\!]_1, [\![\,.\,]\!]_2) \rangle$ is again an $\mathbf{IPL}_n^a$-model. Moreover, we know that

$$([\![A]\!]_1, [\![A]\!]_2) < (1_1, 1_2) \ \text{ and } \ ([\![B]\!]_1, [\![B]\!]_2) < (1_1, 1_2),$$

with $(1_1, 1_2)$ being the top element of the model $\mathbf{M}_1 \times \mathbf{M}_2$. And hence, using lemma 5.2, we may conclude that

$$[\![A \sqcup B]\!]_c \leq_c (1_1, 1_2) <_c 1_c,$$

where $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\,.\,]\!]_c \rangle$ is the connectification of the $\mathbf{IPL}_n^a$-model $\mathbf{M}_1 \times \mathbf{M}_2$ with a new top element 1_c. Thus, $\not\models_{\mathbf{M}_c} \Lambda \Rightarrow A \sqcup B$. Hence, due to soundness, see proposition 3.5, $\mathbf{IPL}_n^a \not\vdash \Lambda \Rightarrow A \sqcup B$ and we are done. $\diamond$

A natural question arising at this point is whether the disjunction property can be generalized to some suitable class of $\mathbf{IPL}_n^a$-formulas that may occur in the antecedent of the sequents considered. For that purpose, consider the following modification of the connectification of an $\mathbf{IPL}_n^a$-model. Let clause (2) of definition 4.2 be replaced by:
(2') for any propositional variable P,

$$[\![P]\!]_c = \begin{cases} 1_c & \text{if } [\![P]\!] = 1 \\ [\![P]\!] & \text{otherwise} \end{cases}$$

It is easy to see that also for this version of connectification lemma 5.1 holds and so does the rest of the proof establishing (DP) of $\mathbf{IPL}_n^a$. Moreover, the following preservation result can be obtained.

Proposition 5.5 *Let A be any $\sqcup$-free $\mathbf{IPL}_n^a$-formula. Given an $\mathbf{IPL}_n^a$-model $\mathbf{M}$ and its connectification $\mathbf{M}_c$, in the sense above,*

$$[\![A]\!]_c = \begin{cases} 1_c & \textit{if } [\![A]\!] = 1 \\ [\![A]\!] & \textit{otherwise} \end{cases}$$

Proof: By induction on the complexity of A. $\diamond$

Remark: Clearly given an $\mathbf{IPL}_n^a$-model $\mathbf{M}$ the validity of any $\sqcup$-free formula A is preserved under the connectification just introduced, i.e. $\models_{\mathbf{M}} \Lambda \Rightarrow A$ if and only if $\models_{\mathbf{M}_c} \Lambda \Rightarrow A$.

Definition 5.6 *Let $\mathcal{I}$ be the class of those $\sqcup$-free $\mathbf{IPL}_n^a$-formulas D with the property:*
for every $\mathbf{IPL}_n^a$-formula F, such that $\mathbf{IPL}_n^a \not\vdash D \Rightarrow F$ there is an $\mathbf{IPL}_n^a$-model $\mathbf{M}$ satisfying $\models_{\mathbf{M}} \Lambda \Rightarrow D$ and $\not\models_{\mathbf{M}} \Lambda \Rightarrow F$.

Remark: First, observe that $\mathcal{I}$ is not empty, since every $\sqcup$-free $\mathbf{IPL}_n^a$-formula provably equivalent to 1 is an element of $\mathcal{I}$. Further, note that $\mathcal{I}$ is not a subclass of $\mathbf{IPL}_n^a$-formulas provably equivalent to 1, since any $\sqcup$-free $\mathbf{IPL}_n^a$-formula provably equivalent to 0 also belongs to $\mathcal{I}$. Further, we are going to show that $\mathcal{I}$ is just a proper subclass of all $\sqcup$-free $\mathbf{IPL}_n^a$-formulas. For that purpose, we proceed, as follows. Given a propositional variable P, we know that $\mathbf{IPL}_n^a \not\vdash P \Rightarrow P \star P$, for $n \geq 2$ (see Prijatelj [6] for a counter-model). However, for any $\mathbf{IPL}_n^a$-model $\mathbf{M}$ the following holds: if $\models_{\mathbf{M}} \Lambda \Rightarrow P$ (i.e. $[\![P]\!] = 1$), then $\models_{\mathbf{M}} \Lambda \Rightarrow P \star P$ (i.e. $[\![P]\!] \star [\![P]\!] = 1$). Hence,

P does not belong to $\mathcal{I}$.

Proposition 5.7 *Given $D \in \mathcal{I}$ the following holds for any $\mathbf{IPL}_n^a$-formulas A and B:*

$$\textit{if}\ \ \mathbf{IPL}_n^a \vdash D \Rightarrow A \sqcup B, \quad \textit{then}\ \ \mathbf{IPL}_n^a \vdash D \Rightarrow A \quad \textit{or}\ \ \mathbf{IPL}_n^a \vdash D \Rightarrow B.$$

Proof: Suppose that $\mathbf{IPL}_n^a \nvdash D \Rightarrow A$ and $\mathbf{IPL}_n^a \nvdash D \Rightarrow B$. Since by assumption $D \in \mathcal{I}$, we know that there are $\mathbf{IPL}_n^a$-models $\mathbf{M}_1$ and $\mathbf{M}_2$ satisfying:
$\models_{\mathbf{M}_1} \Lambda \Rightarrow D$, $\not\models_{\mathbf{M}_1} \Lambda \Rightarrow A$, and $\models_{\mathbf{M}_2} \Lambda \Rightarrow D$, $\not\models_{\mathbf{M}_2} \Lambda \Rightarrow B$. Using the same arguments as in the proof of proposition 5.4 one can show that $[\![A \sqcup B]\!]_c <_c 1_c$, where $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\,.\,]\!]_c \rangle$ is the connectification of the $\mathbf{IPL}_n^a$-model $\mathbf{M}_1 \times \mathbf{M}_2$ with a new top element 1_c. Moreover, by the preservation proposition 5.5 we get $[\![D]\!]_c = 1_c$. Thus clearly, $[\![D]\!]_c \not\leq_c [\![A \sqcup B]\!]_c$, i.e. $\not\models_{\mathbf{M}_c} D \Rightarrow A \sqcup B$. Hence, $\mathbf{IPL}_n^a \nvdash D \Rightarrow A \sqcup B$. And we are done. $\diamond$

Summing up the results of this section, i.e. proposition 5.4 and a straightforward generalization of proposition 5.7, we obtain

Proposition 5.8 *The system $\mathbf{IPL}_n^a$ enjoys the following disjunction property:*

$$\textit{if}\ \ \mathbf{IPL}_n^a \vdash \Gamma \Rightarrow A \sqcup B, \quad \textit{then}\ \ \mathbf{IPL}_n^a \vdash \Gamma \Rightarrow A \quad \textit{or}\ \ \mathbf{IPL}_n^a \vdash \Gamma \Rightarrow B,$$

provided that either Γ is the empty multiset or the $\star$-product of all the formulas in Γ is provably equivalent to some element of $\mathcal{I}$, in particular to 0 or 1.

6 Classical Systems with n-Contraction and Weakening

For any $n \geq 2$, a classical system of affine propositional logic with n-contraction, $\mathbf{CPL}_n^a$, is given by the following axioms and rules. Throughout the below, Λ denotes the empty multiset and $\Gamma, \Gamma_1, \Gamma_2, \Delta, \Delta_1, \Delta_2$ stand for finite multisets of $\mathbf{CPL}_n^a$-formulas.

Axioms

$$A \Rightarrow A \qquad 0 \Rightarrow \Lambda \qquad \Lambda \Rightarrow 1$$

Logical rules

$$L{\sim}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, {\sim}A \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow\ {\sim}A, \Delta}\ R{\sim}$$

$$L\star\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \star B \Rightarrow \Delta} \qquad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \star B, \Delta_1, \Delta_2}\ R\star$$

$$L+\ \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A + B \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A + B, \Delta}\ R+$$

$$L\sqcap\ \frac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_1 \sqcap A_2 \Rightarrow \Delta}\ (i = 1, 2) \qquad \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \sqcap B, \Delta}\ R\sqcap$$

$$L\sqcup\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \sqcup B \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A_i, \Delta}{\Gamma \Rightarrow A_1 \sqcup A_2, \Delta}\ (i = 1, 2)\ R\sqcup$$

Structural rules

$$LW\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}\ RW$$

$$LC_n\ \frac{\Gamma, A^{(n+1)} \Rightarrow \Delta}{\Gamma, A^{(n)} \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow A^{(n+1)}, \Delta}{\Gamma \Rightarrow A^{(n)}, \Delta}\ RC_n$$

where $A^{(k)} \equiv A, A, \ldots, A$, i.e. k copies of formula A.

$$CUT \quad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, A \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Remark: A linear implication can be defined by $A \multimap B =\sim A + B$. Note that, the respective left and right rules

$$L \multimap \quad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Delta_1, \Delta_2} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta} \quad R \multimap$$

are derivable in $\mathbf{CPL}^a_n$.

7 Algebraic Models for $\mathbf{CPL}^a_n$

Definition 7.1 $\mathbf{X} = \langle X, \sim, \star, +, \sqcap, \sqcup, 0, 1\rangle$ *is a* $\mathbf{CPL}^a_n$*-algebra, if:*

(1) $\langle X, \star, 1\rangle$ *and* $\langle X, +, 0\rangle$ *are commutative monoids with units* 1 *and* 0 *respectively;*

(2) $\langle X, \sqcap, \sqcup, 0, 1\rangle$ *is a lattice with bottom* 0 *and top* 1*;*

(3) $\sim$ *is involution, i.e.* $\sim\sim x = x$ *for all* $x \in X$*;*

(4) $\star$ *and* $+$ *are monotone with respect to the lattice order* $\leq$*, i.e. for all* $x, y, z \in X$*, if* $x \leq y$*, then* $x \star z \leq y \star z$ *and* $x + z \leq y + z$*;*

(5) for all $x, y, z \in X$*,* $x \star y \leq z$ *if and only if* $x \leq \sim y + z$*;*

(6) for all $x \in X$*,* $x^n \leq x^{n+1}$ *and* $(n+1)x \leq nx$*, where* $x^k = x \star \cdots \star x$ *and* $kx = x + \cdots + x$ *with* k *copies of* x *respectively.*

Remark: Note that, a $\mathbf{CPL}^a_n$-algebra is just a classical linear algebra (provided$\multimap$ is taken as primitive while $\top$, $\sim$ and $+$ are defined in a usual way, see Troelstra [9]), satisfying in addition:

- $\perp = 0$ and $\top = 1$ (corresponding to weakening);
- clause (5) (corresponding to n-contraction).

However, we will here choose the fully symmetric formulation of $\mathbf{CPL}^a_n$-algebra establishing a proper intuition for defining, later on, the connectification operator for the classical algebras considered.

Lemma 7.2 *In any* $\mathbf{CPL}_n^a$*-algebra* $\mathbf{X} = \langle X, \sim, \star, +, \sqcap, \sqcup, 0, 1\rangle$*, the following are satisfied for all* $x, y, z \in X$*:*

(a) $x\star \sim x = 0$ *and* $x+ \sim x = 1$*;*

(b) $\sim 0 = 1$ *and* $\sim 1 = 0$*;*

(c) *for all* $x, y, z \in X$, $x \star (y + z) \leq (x \star y) + z$*;*

(d) *De Morgan laws expressing that the following pairs of operators are dual to each other:* $(\sim, \sim)$, $(\star, +)$, $(\sqcap, \sqcup)$, $(0, 1)$*;*

(e) $\sim$ *is anti-monotone with respect to* $\leq$*, i.e.* $x \leq y$ *iff* $\sim y \leq \sim x$*;*

(f) *distributivity of* $\star$ *and of* $+$ *over* $\sqcup$ *and over* $\sqcap$ *respectively; and moreover,*
$x \star \bigsqcup_{i\in I} y_i = \bigsqcup_{i\in I}(x \star y_i)$ *, provided* $\bigsqcup_{i\in I} y_i$ *exists;*
$x + \sqcap_{i\in I} y_i = \sqcap_{i\in I}(x + y_i)$ *, provided* $\sqcap_{i\in I} y_i$ *exists;*

(g) $y + z = \max\{x \in X \mid x\star \sim y \leq z\}$*;*

(h) $x \star y \leq x$ *and* $x \leq x + y$*;*

(i) $x \star 0 = 0$ *and* $x + 1 = 1$*;*

(j) $x^n = x^{n+1}$ *and* $(n + 1)x = nx$*.*

Proof: Straightforward. $\diamond$

Fact 7.3 *Let the clauses (1)-(4) of definition 7.1 be satisfied for some* $\langle X, \sim, \star, \sqcap, \sqcup, 0, 1\rangle$*. If* $+$ *is well-defined on* $X \times X$ *by*

$$y + z = max\{x \in X \mid x\star \sim y \leq z\},$$

then, also, (5) of definition 7.1 is fulfilled for $\langle X, \sim, \star, +, \sqcap, \sqcup, 0, 1\rangle$*.*

Definition 7.4 $\mathbf{M} = \langle \mathbf{X}, [\![\,.\,]\!]\rangle$ *is a* $\mathbf{CPL}_n^a$*-model, if:*

(1) $\mathbf{X}$ *is a* $\mathbf{CPL}_n^a$*-algebra;*

(2) $[\![\,.\,]\!]$ *is a valuation satisfying the same conditions as an* $\mathbf{IPL}_n^a$*-model;*

$[\![\ .\ ]\!]$ is extended to $\mathbf{CPL}_n^a$-formulas inductively, by: $[\![\sim A]\!] = \sim [\![A]\!]$ and $[\![A \bullet B]\!] = [\![A]\!] \bullet [\![B]\!]$, with $\bullet \in \{\star, +, \sqcap, \sqcup\}$;
Moreover, a $\mathbf{CPL}_n^a$-sequent $A_1, \dots, A_k \Rightarrow B_1, \dots, B_m$ (where k, m may not both be zero) is valid in $\mathbf{M}$ if and only if $[\![A_1]\!] \star \cdots \star [\![A_k]\!] \leq [\![B_1]\!] + \cdots + [\![B_m]\!]$.

Finally, the soundness and the completeness theorem for the classical case considered can be established in an analogous way to the previously discussed intuitionistic case.

8 Connectification with new top and bottom elements

Definition 8.1 *Let $\mathbf{X} = \langle X, \sim, \star, +, \sqcap, \sqcup, 0, 1\rangle$ be a $\mathbf{CPL}_n^a$-algebra. The connectification of $\mathbf{X}$ with a new top element $1_c \notin X$ and with a new bottom element $0_c \notin X$ is the $\mathbf{CPL}_n^a$-algebra $\mathbf{X}_c = \langle X \cup \{0_c, 1_c\}, \sim_c, \star_c, +_c, \sqcap_c, \sqcup_c\rangle$, given by:*

(1) $\sim_c$ is the extension of $\sim$ on $X \cup \{0_c, 1_c\}$, defined by:

$$\sim_c 0_c = 1_c \ \ \textit{and} \ \ \sim_c 1_c = 0_c.$$

(2) $\leq_c$ is the extension of the lattice order $\leq$ on $X \cup \{0_c, 1_c\}$, given by:

$$0_c \leq_c x \leq_c 1_c, \ \ \textit{for all} \ \ x \in X \cup \{0_c, 1_c\}.$$

(3) $\star_c$ is defined on $(X \cup \{0_c, 1_c\}) \times (X \cup \{0_c, 1_c\})$ by:

$$y \star_c z = \begin{cases} 0_c & \textit{if } y \leq_c \sim_c z \\ y & \textit{if } z = 1_c \\ z & \textit{if } y = 1_c \\ y \star z & \textit{otherwise} \end{cases}$$

(4) $+_c$ is defined on $(X \cup \{0_c, 1_c\}) \times (X \cup \{0_c, 1_c\})$ by:

$$y +_c z = \begin{cases} 1_c & \textit{if } \sim_c y \leq_c z \\ y & \textit{if } z = 0_c \\ z & \textit{if } y = 0_c \\ y + z & \textit{otherwise} \end{cases}$$

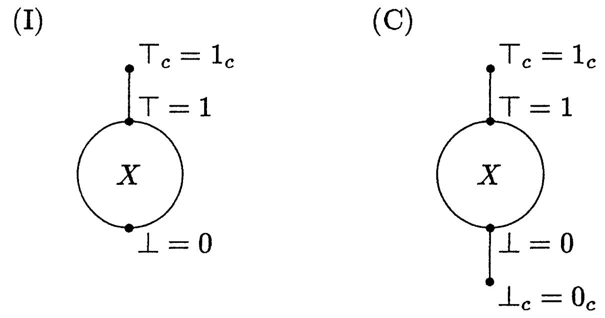

Figure 1: The connectification of affine linear algebras: (I) intuitionistic case, (C) classical case.

Remark: To verify that $\mathbf{X}_c$ is indeed a $\mathbf{CPL}_n^a$-algebra, observe the following facts. Note that $\star_c$ and $+_c$ are duals of each other and moreover, that they are commutative operations in $X \cup \{0_c, 1_c\}$. Observe also that (4) is just spelling out the effects of

$$y +_c z = \max\{x \in X \cup \{0_c, 1_c\} |\; x \star_c \sim_c y \leq_c z\}.$$

By fact 7.3 this yields clause (5) of the definition of an $\mathbf{CPL}_n^a$-algebra. Now, the verification is a trivial matter.

The connectification of a $\mathbf{CPL}_n^a$-model with new top and bottom elements is introduced, as follows.

Definition 8.2 *Let $\mathbf{M} = \langle \mathbf{X}, [\![\,.\,]\!] \rangle$ be a $\mathbf{CPL}_n^a$-model. The connectification of $\mathbf{M}$ with a new top element 1_c and with a new bottom element 0_c is the $\mathbf{CPL}_n^a$-model $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\,.\,]\!]_c \rangle$, given by:*

(1) $\mathbf{X}_c$ is the connectification of $\mathbf{X}$ with 1_c and 0_c;

(2) $[\![\,.\,]\!]_c$ is the valuation, defined by: for every propositional variable P,

$$[\![P]\!]_c = \begin{cases} 1_c & \textit{if } [\![P]\!] = 1 \\ 0_c & \textit{if } [\![P]\!] = 0 \\ [\![P]\!] & \textit{otherwise} \end{cases}$$

9 Disjunction Property for $\mathbf{CPL}_n^a$

In this section, we are going to show that also classical systems of affine linear logic with n-contraction ($n \geq 2$) enjoy the disjunction property. For that purpose, we shall first elaborate the necessary prerequisites.
To start with the central

Lemma 9.1 *Let* $\mathbf{M} = \langle \mathbf{X}, [\![\, . \,]\!] \rangle$ *be a* $\mathbf{CPL}_n^a$*-model and* $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\, . \,]\!]_c \rangle$ *the connectification of* $\mathbf{M}$ *with a new top element* 1_c *and with a new bottom element* 0_c*. Then, the following holds true for any* $\mathbf{CPL}_n^a$*-formula* A*:*

(i) if $[\![A]\!]_c = 1_c$*, then* $[\![A]\!] = 1$*;*

(ii) if $0_c <_c [\![A]\!]_c <_c 1_c$*, then* $[\![A]\!]_c = [\![A]\!]$*;*

(iii) if $[\![A]\!]_c = 0_c$*, then* $[\![A]\!] = 0$*.*

Proof: By induction on the complexity of A.
To illustrate the proof, we will here consider only the case for the connective $\sim$. Assume $A = \sim B$. Suppose that $0_c <_c [\![\sim B]\!]_c <_c 1_c$. Then, by definition $0_c <_c \sim_c [\![B]\!]_c <_c 1_c$, yielding $0_c <_c [\![B]\!]_c <_c 1_c$, due to the fact that $\sim_c$ is involution, anti-monotone with respect to $\leq_c$ (see lemma 7.2(e)), and that $\sim_c 0_c = 1_c$ and $\sim_c 1_c = 0_c$. We can now use induction hypothesis and get $[\![B]\!]_c = [\![B]\!]$. Hence, $[\![\sim_c B]\!]_c = \sim_c [\![B]\!]_c = \sim_c [\![B]\!]$. But $\sim_c$ restricted to X is just $\sim$. Therefore $\sim_c [\![B]\!] = \sim [\![B]\!] = [\![\sim B]\!]$. This verifies clause (ii) of the lemma while (i) and (iii) are left to the reader as well as the rest of the proof. $\diamond$

Next, the following preservation result can be established.

Proposition 9.2 *Let* A *be any* $\sqcap$*-*$\sqcup$*-free* $\mathbf{CPL}_n^a$*-formula. Given a* $\mathbf{CPL}_n^a$*-model* $\mathbf{M}$ *and its connectification* $\mathbf{M}_c$*,*

$$[\![A]\!]_c = \begin{cases} 1_c & \textit{if } [\![A]\!] = 1 \\ 0_c & \textit{if } [\![A]\!] = 0 \\ [\![A]\!] & \textit{otherwise} \end{cases}$$

Proof: By induction on the complexity of A. $\diamond$

Remark: Clearly given a $\mathbf{CPL}_n^a$-model $\mathbf{M}$ the validity of any $\sqcap$-$\sqcup$-free formula A is preserved under the connectification, i.e. $\models_{\mathbf{M}} \Lambda \Rightarrow A$ if and only if $\models_{\mathbf{M}_c} \Lambda \Rightarrow A$.
To continue with

Definition 9.3 *Let $\mathcal{C}$ be the class of those $\sqcap$-$\sqcup$-free* $\mathbf{CPL}_n^a$*-formulas D with the property:*
for every $\mathbf{CPL}_n^a$*-formula F, such that* $\mathbf{CPL}_n^a \not\vdash D \Rightarrow F$ *there is an* $\mathbf{CPL}_n^a$*-model* $\mathbf{M}$ *satisfying* $\models_{\mathbf{M}} \Lambda \Rightarrow D$ *and* $\not\models_{\mathbf{M}} \Lambda \Rightarrow F$.

Along the lines analogous to those for the intuitionistic case the following result can now be established:

Proposition 9.4 *The system* $\mathbf{CPL}_n^a$ *enjoys the following disjunction property:*

$$\textit{if}\ \ \mathbf{CPL}_n^a \vdash \Gamma \Rightarrow A \sqcup B, \quad \textit{then}\ \ \mathbf{CPL}_n^a \vdash \Gamma \Rightarrow A \quad \textit{or}\ \ \mathbf{CPL}_n^a \vdash \Gamma \Rightarrow B,$$

provided that either Γ is the empty multiset or the $\star$-product of all the formulas in Γ is provably equivalent to some element in $\mathcal{C}$, in particular to 0 *or* 1.

10 Some Variations of the Connectification Operator

In this section, we shall briefly introduce the connectification operators for:

(1) intuitionistic algebras corresponding to linear logic, **ILZ**-algebra, and to its extension with n-contraction, $\mathbf{IPL}_n$-algebra;

(2) classical algebras corresponding to linear logic, **CL**-algebra, and to its extension with n-contraction, $\mathbf{CPL}_n$-algebra.

(3) non-commutative versions of $\mathbf{IPL}_n^a$-algebras, corresponding to the directional Lambek calculi extended by additional operators, weakenning and n-contraction, $\mathbf{L}_n^a$-algebra.

Ad (1): The underlying systems $\mathbf{IPL}_n$ (**IPL**) are obtained from $\mathbf{IPL}_n^a$ by restricting weakenig only to 0 and 1 in the succedent and in the antecedent respectively,(omitting also n-contraction), and by adding the axiom for $\perp$. Moreover, an $\mathbf{IPL}_n$-algebra is an **ILZ**-algebra with the additional clause corresponding to n-contraction.

Definition 10.1 *Let* $\mathbf{X} = \langle X, \star, \multimap, \sqcap, \sqcup, 0, 1, \perp \rangle$ *be an* **ILZ** ($\mathbf{IPL}_n$)*-algebra. The connectification of* $\mathbf{X}$ *with a new top* $\top_c \notin X$ *is the* **ILZ** ($\mathbf{IPL}_n$)*-algebra* $\mathbf{X}_c = \langle X \cup \{\top_c\}, \star_c, \multimap_c, \sqcap_c, \sqcup_c, 0, 1, \perp \rangle$, *given by:*

(1) $\star_c$ *is the extension of* $\star$ *on* $X \cup \{\top_c\}$*, defined by:*
for all $x \in X \cup \{\top_c\}$*:*

$$\top_c \star_c x = x \star_c \top_c = \begin{cases} \top_c & \textit{otherwise} \\ \bot & \textit{if } x = \bot \end{cases}$$

(2) $\leq_c$ *is the extension of* $\leq$ *on* $X \cup \{\top_c\}$*, given by:*

$$\textit{for all } x \in X \cup \{\top_c\}: \ x \leq_c \top_c;$$

(3) $\multimap_c$ *is defined on* $(X \cup \{\top_c\}) \times (X \cup \{\top_c\})$ *by:*

$$y \multimap_c z = \begin{cases} \top_c & \textit{if } y = \bot \textit{ or } z = \top_c \\ \bot & \textit{if } y = \top_c \textit{ and } z \in X \\ y \multimap z & \textit{if } y \in X \setminus \{\bot\} \textit{ and } z \in X \end{cases}$$

Ad (2): The underlying systems $\mathbf{CPL}_n$ ($\mathbf{CPL}$) are obtained from $\mathbf{CPL}^a_n$, by restricting weakening as in (1) above, (by omitting n-contraction), and adding the corresponding axioms for $\bot$ and $\top$.
Moreover, a $\mathbf{CPL}_n$-algebra ($\mathbf{CPL}$-algebra) is obtained from a $\mathbf{CPL}^a_n$-algebra by replacing 0 and 1 with $\bot$ and $\top$ in the lattice respectively (ommiting also the clause corresponding to n-contraction.)

Definition 10.2 *Let* $\mathbf{X} = \langle X, \sim, \star, +, \sqcap, \sqcup, 0, 1, \bot, \top \rangle$ *be a* **CL** *(*$\mathbf{CPL}_n$*)-algebra. The connectification of* $\mathbf{X}$ *with a new top* $\top_c \notin X$ *and with a new bottom* $\bot_c \notin X$ *is the* **CL** *(*$\mathbf{CPL}_n$*)-algebra* $\mathbf{X}_c = \langle X \cup \{\bot_c, \top_c\}$, $\sim_c, \star_c, +_c, \sqcap_c, \sqcup_c, 0, 1\rangle$*, given by:*

(1) $\sim_c$ *is the extension of* $\sim$ *on* $X \cup \{\bot_c, \top_c\}$*, defined by:*

$$\sim_c \bot_c = \top_c \ \textit{and} \ \sim_c \top_c = \bot_c.$$

(2) $\leq_c$ *is the extension of the lattice order* $\leq$ *on* $X \cup \{\bot_c, \top_c\}$*, given by:*

$$\bot_c \leq_c x \leq_c \top_c, \ \textit{for all } x \in X \cup \{\bot_c, \top_c\}.$$

(3) $\star_c$ *is defined on* $(X \cup \{\bot_c, \top_c\}) \times (X \cup \{\bot_c, \top_c\})$ *by:*

$$y \star_c z = \begin{cases} \bot_c & \textit{if } y = \bot_c \textit{ or } z = \bot_c \\ \top_c & \textit{if } (y = \top_c \textit{ and } z \neq \bot_c) \textit{ or } (z = \top_c \textit{ and } y \neq \bot_c) \\ y \star z & \textit{if } y, z \in X \end{cases}$$

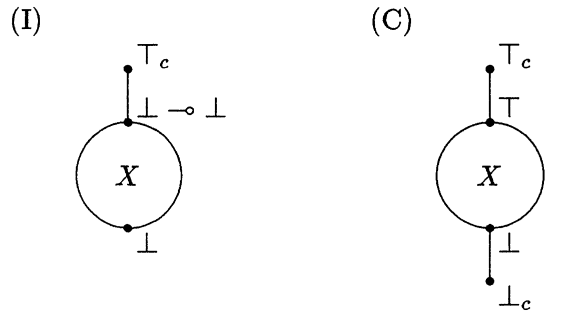

Figure 2: The connectification of non-affine linear algebras: (I) intuitionistic case, (C) classical case.

(4) $+_c$ *is defined on* $(X \cup \{\bot_c, \top_c\}) \times (X \cup \{\bot_c, \top_c\})$ *by:*

$$y +_c z = \begin{cases} \top_c & \textit{if } y = \top_c \textit{ or } z = \top_c \\ \bot_c & \textit{if } (y = \bot_c \textit{ and } z \neq \top_c) \textit{ or } (z = \bot_c \textit{ and } y \neq \top_c) \\ y + z & \textit{if } y, z \in X \end{cases}$$

Ad (3): For any $n \geq 2$ an extended directional Lambek calculus, $\mathbf{L}^a_n$, is a version of $\mathbf{IPL}^a_n$ based on sequences, with the linear implication being split into the left slash and into the right slash. The corresponding left and right introduction rules are, as follows:

$$L\backslash \; \frac{\Gamma \Rightarrow A \qquad \Delta_1, B, \Delta_2 \Rightarrow \Phi}{\Delta_1, \Gamma, A\backslash B, \Delta_2 \Rightarrow \Phi} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B} \; R\backslash$$

$$L/ \; \frac{\Gamma \Rightarrow A \qquad \Delta_1, B, \Delta_2 \Rightarrow \Phi}{\Delta_1, B/A, \Gamma, \Delta_2 \Rightarrow \Phi} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A} \; R/$$

We also emphasize that weakening is now built into the axiom schemes in order to prevent derivability of empty antecedent and of empty succedent sequents in the underlying systems.

Definition 10.3 $\mathbf{X} = \langle X, \star, \backslash, /, \sqcap, \sqcup, 0, 1 \rangle$ *is an* $\mathbf{L}^a_n$*-algebra, if:*

(1) $\langle X, \star, 1 \rangle$ *is a monoid with unit* 1*;*

(2) $\langle X, \sqcap, \sqcup, 0, 1\rangle$ *is a lattice with bottom* 0 *and top* 1*;*

(3) $\star$ *is left and right monotone with respect to the lattice order* $\leq$*, i.e. for all* $x, y, z \in X$*, if* $x \leq y$*, then* $z \star x \leq z \star y$ *and* $x \star z \leq y \star z$*;*

(4) for all $x, y, z \in X$*,* $x \star y \leq z$ *iff* $y \leq x \backslash z$ *iff* $x \leq z/y$*;*

(5) for all $x \in X$*,* $x^n = x^{n+1}$*, where* $x^k = x \star \cdots \star x$ *with* k *copies of* x*.*

Definition 10.4 *Let* $\mathbf{X} = \langle X, \star, \backslash, /, \sqcap, \sqcup, 0, 1\rangle$ *be an* L_n^a*-algebra. The connectification of* $\mathbf{X}$ *with a new top element* $1_c \notin X$ *is the* $\mathbf{L}_n^a$*-algebra* $\mathbf{X}_c = \langle X \cup \{1_c\}, \star_c, \backslash_c, /_c, \sqcap_c, \sqcup_c, 0\rangle$*, with clauses (1) and (2) identical to those in the commutative counter-part, but with the clause (3) being split into:*

(3') $\backslash_c$ *is defined on* $(X \cup \{1_c\}) \times (X \cup \{1_c\})$ *by:*

$$y \backslash_c z = \max\{x \in X \cup \{1_c\} |\, y \star_c x \leq_c z\}.$$

(3") $/_c$ *is defined on* $(X \cup \{1_c\}) \times (X \cup \{1_c\})$ *by:*

$$z /_c y = \max\{x \in X \cup \{1_c\} |\, x \star_c y \leq_c z\}.$$

Remark: Note that, explicitly:

$$y \backslash_c z = \begin{cases} 1_c & \text{if } y \leq_c z \\ z & \text{if } y = 1_c \\ y \backslash z & \text{otherwise} \end{cases}$$

$$z /_c y = \begin{cases} 1_c & \text{if } y \leq_c z \\ z & \text{if } y = 1_c \\ y / z & \text{otherwise} \end{cases}$$

At this point, the reader himself should be able to verify that each of the connectification operators introduced above is indeed well-defined. Moreover, by analogy with the previous cases of affine logics with n-contraction (DP) can also be established for the system $\mathbf{L}_n^a$. Let, in this case, $\mathcal{L}$ be the class of $\sqcup$-free $\mathbf{L}_n^a$-formulas, defined in an analogous way to the class $\mathcal{I}$ (see definition 5.6). We shall here write down only the main

Proposition 10.5 *The system* $\mathbf{L}_n^a$ *enjoys the following disjunction property:*

$$\text{if } L_n^a \vdash \Gamma \Rightarrow A \sqcup B, \quad \text{then } L_n^a \vdash \Gamma \Rightarrow A \quad \text{or} \quad L_n^a \vdash \Gamma \Rightarrow B,$$

provided the $\star$*-product (respecting the order) of all the formulas in* Γ *is provably equivalent to some element of* $\mathcal{L}$*, in particular to* 0 *or* 1.

We could prolong the story by specifying suitable connectification operators for the non-commutative algebras corresponding to the non-affine systems $\mathbf{L}$ and $\mathbf{L}_n$. However, we believe that the reader has got the sufficient routine to accomplish this task on his own. Instead, we are going to state a preservation result for $\perp$-free $\mathbf{IPL}_n$-formulas with respect to a certain subclass of $\mathbf{IPL}_n$ models and their connectification with a new top. For that purpose we proceed, as follows.
First, an $\mathbf{IPL}_n$-model $\mathbf{M} = \langle \mathbf{X}, [\![\ .\ ]\!] \rangle$, as well as the connectification of $\mathbf{M}$ with $\top_c$, $\mathbf{M}_c = \langle \mathbf{X}_c, [\![\ .\ ]\!]_c \rangle$, are defined in a usual way, with $[\![P]\!]_c = [\![P]\!]$, for every propositional variable P.
Next, let $\mathcal{M}_{>\perp}$ be the class of $\mathbf{IPL}_n$-models $\mathbf{M} = \langle \mathbf{X}, [\![\ .\ ]\!] \rangle$ satisfying $\perp < [\![A]\!]$ for any $\perp$-free formula A.

Proposition 10.6 *Let* A *be any* $\perp$*-free* $\mathbf{IPL}_n$*-formula. Given* $\mathbf{M} \in \mathcal{M}_{>\perp}$ *and its connectification* $\mathbf{M}_c$, $[\![A]\!]_c = [\![A]\!]$.

Proof: By induction on the complexity of A. $\diamond$

Remark: Clearly, by the theorem above, $1 \leq [\![A]\!]$ yields $1 \leq_c [\![A]\!]_c$, and vise versa, resulting in the following

Corollary 10.7 *For any* $\perp$*-free formula* A, $\models_{\mathbf{M}} \Lambda \Rightarrow A$ *iff* $\models_{\mathbf{M}_c} \Lambda \Rightarrow A$, *with* $\mathbf{M} \in \mathcal{M}_{>\perp}$.

We are now going to show that, in fact, $\mathcal{M}_{>\perp}$ is complete with respect to $\perp$-free $\mathbf{IPL}_n$ system (based on the language without the constant $\perp$). We shall work out the completeness proof by means of a suitable connectification of the Lindenbaum model of $\perp$-free $\mathbf{IPL}_n$.
First note that the Lindenbaum algebra, $\mathbf{X}_L$ of $\perp$-free $\mathbf{IPL}_n$ is an $\mathbf{IPL}_n$-algebra without bottom (top). Moreover, by standard arguments, one can prove that the Lindenbaum model, $\mathbf{M}_L$, is complete for $\perp$-free $\mathbf{IPL}_n$.
Next we introduce the connectification of $\mathbf{X}_L$, $\mathbf{X}_{L_c}$, with a bottom, $\perp$, and top, $\top$, as follows.

(1) for all $x, y \in X \cup \{\bot, \top\}$:

$$x \star_c y = \begin{cases} \bot & \text{if } x = \bot \text{ } or \text{ } y = \bot \\ \top & \text{if } (x = \top \text{ } and \text{ } y \neq \bot) \text{ } or \text{ } (y = \top \text{ } and \text{ } x \neq \bot) \\ x \star y & \text{otherwise} \end{cases}$$

(2) $\leq_c$ is the extension of $\leq$ on $X \cup \{\bot, \top\}$, given by:

$$\text{for all } x \in X \cup \{\bot, \top\}: \ \bot \leq_c x \leq_c \top;$$

(3) for all $y, z \in X \cup \{\bot, \top\}$,

$$y \multimap_c z = \begin{cases} \top & \text{if } y = \bot \text{ } or \text{ } z = \top \\ \bot & \text{if } (y = \top \text{ } and \text{ } z \neq \top) \text{ } or \text{ } (z = \bot \text{ } and \text{ } y \neq \bot) \\ y \multimap z & \text{otherwise} \end{cases}$$

Remark: Observe that $\mathbf{X}_{L_c}$ is an $\mathbf{IPL}_n$-algebra.
Further the connectification of $\mathbf{M}_L$ with $\bot$ and $\top$, $\mathbf{M}_{L_c}$, is defined in a usual way, by putting $[\![P]\!]_{L_c} = [\![P]\!]_L$.
We continue with a useful preservation result concerning $\mathbf{M}_L$ and $\mathbf{M}_{L_c}$.

Proposition 10.8 *Given a $\bot$-free $\mathbf{IPL}_n$-formula A, $[\![A]\!]_{L_c} = [\![A]\!]_L$.*

Proof: By induction on the complexity of A. $\diamond$

Due to the theorem above the completeness of $\mathcal{M}_{>\bot}$ with respect to $\bot$-free $\mathbf{IPL}_n$ system is now established by

Corollary 10.9 $\mathbf{M}_{L_c}$ *is complete for $\bot$-free $\mathbf{IPL}_n$ system and* $\mathbf{M}_{L_c} \in \mathcal{M}_{>\bot}$.

We conclude the paper with some general remarks. First, we shall point out why in the affine case the connectification operator is useful to prove the disjunction property as opposed to the non-affine case. For the affine case, a formula F, i.e. a sequent of the form $\Lambda \Rightarrow F$ is valid in a model if and only if $[\![F]\!] = 1$, since $1 = \top$. Thus, the connectification of any such a model with a new top (i.e. unit) 1_c (witness Figure 1) yields the following conclusion, essential to establish (DP):
for any formulas A and B, if $[\![A \sqcup B]\!]_c = 1_c$, then $[\![A]\!]_c = 1_c$ or $[\![B]\!]_c = 1_c$.
For the non-affine case, however, the validity condition in the corresponding models amounts to $1 \leq [\![F]\!]$. And therefore, the connectification of such a model with a new top (see Figure 2) does not generally permit the conclusion

below:
for any formulas A and B, if $1 \leq_c [\![A \sqcup B]\!]_c$, then $1 \leq_c [\![A]\!]_c$ or $1 \leq_c [\![B]\!]_c$ (note that 1 is preserved by the connectification operator).
Let us finally emphasize that for any of the systems considered in this paper omitting the cut rule (DP) can easily be established by a purely syntactic reasoning.

Proposition 10.10 *Let* T *denote the cut-free system of* **ILZ**, $\mathbf{IPL}_n$, $\mathbf{IPL}_n^a$, **CL**, $\mathbf{CPL}_n$, $\mathbf{CPL}_n^a$, **L**, $\mathbf{L}_n$, $\mathbf{L}_n^a$.
If $\mathbf{T} \vdash \Gamma \Rightarrow A \sqcup B$, *then* $\mathbf{T} \vdash \Gamma \Rightarrow A$ *or* $\mathbf{T} \vdash \Gamma \Rightarrow B$, *where no formula in* Γ *contains a strictly positive part of* $\sqcup$.

Proof: By induction on the length of a derivation of $\Gamma \Rightarrow A \sqcup B$. $\diamond$

Remark: Note that in the presence of cut such a syntactic reasoning must be given up. Namely, whatever restriction is imposed on Γ, following a derivation of a sequent $\Gamma \Rightarrow A \sqcup B$ bottom up, the cut rule may introduce a formula which violates the restriction in question.

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