MacNeille completions and canonical extensions

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Abstract

Let V be a variety of monotone bounded lattice expansions, that is, lattices endowed with additional operations, each of which is order preserving or reversing in each coordinate. We prove that if V is closed under MacNeille completions, then it is also closed under canonical extensions. As a corollary we show that in the case of Boolean algebras with operators, any such variety V is generated by an elementary class of relational structures.

Our main technical construction reveals that the canonical extension of a monotone bounded lattice expansion can be embedded in the MacNeille completion of any sufficiently saturated elementary extension of the original structure.

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1 Introduction

Given a lattice L with additional operations, one often wishes to find a completion of L, and an extension of the additional operations to this completion, so that the resulting structure preserves some given set of identities satisfied by the original.

The first example of this is Dedekind's famous construction of the reals from the rationals. Not only did Dedekind (conditionally) complete the rationals to the reals, but he also extended the basic arithmetic operations on the rationals to the reals and showed these extensions preserved the usual identities of arithmetic.

MacNeille [24] generalized Dedekind's method to provide a method of completion for arbitrary posets. This completion will be called the MacNeille completion here, although it is also known as the normal completion, and the completion by cuts. The matter of extending additional operations to the MacNeille completion seems to have been addressed mostly in a piecemeal fashion with various authors noting that the varieties of Boolean algebras, Heyting algebras, closure algebras, and ortholattices [2, 23], among others, could naturally be considered to be closed under MacNeille completions via some natural extension of the operations. The only systematic studies of extensions of additional operations to MacNeille completions were done in the setting of Boolean algebras with additional operations by Monk [25], Givant & Venema [12], and Hirsch & Hodkinson [19].

The second type of completion of interest here, the canonical extension, arises from Stone's duality theorems. The canonical extension of a Boolean algebra B is the embedding of B into the power set of its Stone space. Canonical extensions for bounded distributive lattices can be realized in a similar way either through the Stone space, or the Priestley space, of the distributive lattice. It is probably fair to say that a systematic investigation of canonical extensions for general bounded lattices was initiated by Gehrke & Harding [8].

The matter of extending additional operations on a Boolean algebra to operations on its canonical extension was initiated by Jónsson & Tarski [20, 21]. They were originally motivated by their study of relational algebras. The later introduction of Kripke semantics for modal logics further increased activity in this area. The key fact is that the closure of a variety of modal algebras under canonical extensions implies the completeness of the corresponding logic with respect to its Kripke semantics.

These investigations culminated into a result, proved by Sahlqvist [27] for Boolean algebras with additional well-behaved operations, giving sufficient conditions for an identity to be preserved under canonical extensions. This theorem was later generalized to various kinds of lattice expansions; see for instance Gehrke, Nagahashi & Venema [10] for a result in the setting of distributive lattices. The work of Givant & Venema [12], again in the setting of Boolean algebras with additional well-behaved operations, gave a type of Sahlqvist theorem for preservation of identities under MacNeille completions.

The identities that Givant & Venema showed are preserved under MacNeille completions form a subset of those known to be preserved under canonical extensions. This naturally brings up the question, also raised by Goldblatt [16], whether the closure of a variety under MacNeille completions implies closure under canonical extensions. The purpose of this paper is to show this is indeed the case. Moreover, our results apply not only to the Boolean setting that motivated them, but to any variety of monotone lattice expansions, that is, bounded lattices with additional operations that uniformly preserve or reverse order in each coordinate.

Our method of proof is to show, for any bounded lattice L with additional operations, that the canonical extension of L can be embedded into the MacNeille completion of a saturated non-standard extension of L. That this is reminiscent of a well-known proof method and resulting theorem in modal logic (see Fine [6] and Goldblatt [14]) is not a coincidence — the result in modal logic essentially follows by applying ours to the special case of a complete atomic Boolean algebra with operations that are completely additive in each coordinate.

Another connection with the theory of Boolean algebras with operators (BAOs) concerns the relation between varieties of BAOs that are closed under taking canonical extensions on the one hand, and classes of relational structures that are closed under taking ultraproducts, on the other. Prime examples of the latter kind are provided by the elementary classes, that is, those classes of relational structures that are definable by a set of formulas in first order logic. It is well known that one can construct a BAO from the power set of a relational structure, the so-called complex algebra of the structure. Doing so for all algebras in a class of relational structures, we may subsequently consider the variety generated by all of these complex algebras. Now Goldblatt [14] has strengthened the original result of Fine [6] to show that if the original class of relational structures is closed under taking ultraproducts, then the generated variety of BAOs is canonical, that is, closed under taking canonical extensions.

For a long time it was an open problem whether the converse of these results holds as well, and in particular, whether every canonical variety of BAOs is generated by some elementary class of relational structures. A recent result of Goldblatt, Hodkinson & Venema [17] shows this is not the case in general. However, our results do imply that in the case of a variety of BAOs that is closed under taking MacNeille completions, the answer is positive: all such varieties are generated by elementary classes of structures. These results are discussed in detail in a separate section at the end of the paper.

Before closing this introduction we should briefly remark on a third method of completion that is often used, the ideal lattice completion. Roughly, the ideal lattice completion works well when the additional operations are order preserving, but is of limited use when order inverting operations are allowed. While our results show that canonical extensions are preferable to MacNeille completions in regards to preservation of identities, there is no such comparison possible between the ideal lattice completion and either the MacNeille or canonical completion. For example, the variety of Boolean algebras is closed under MacNeille completions but not under ideal lattice completions, and the variety of modular lattices is closed under ideal lattice completions, see Crawley & Dilworth [5], but not under canonical extensions (Harding [18]).

2 Preliminaries

Throughout, all lattices are bounded. In this section we review the two kinds of lattice completions, and for each completion, we explain the two natural ways in which additional, monotone operations on the lattice can be extended to these completions. We finish this section with a brief discussion of our approach to the notion of saturation.

Lattice completions

We find it convenient to work with abstract characterizations of the canonical and MacNeille completions. The characterization of the MacNeille completion below is due to Banaschewski [1] and independently to Schmidt [28], and that of the canonical extension is due to Gehrke and Harding [8].

Definition 2.1 *Let L be a lattice, C be a complete lattice, and $L \leq C$. We say*

1. *L is join-dense in C if each element of C is a join of elements of L.*
2. *L is meet-dense in C if each element of C is a meet of elements of L.*
3. *L is dense in C if each element of C is both a join of meets and a meet of joins of elements of L.*
4. *L is compact in C if for each $A, B \subseteq L$ with $\bigwedge A \leq \bigvee B$ there are finite $A' \subseteq A, B' \subseteq B$ with $\bigwedge A' \subseteq \bigvee B'$.*

Theorem 2.2 *For any lattice L there exists a complete lattice C such that*

1. *$L \leq C$.*
2. *L is both join-dense and meet-dense in C.*

Further, the lattice C is unique up to unique isomorphism fixing L. We call such C the MacNeille completion of L and denote it $\overline{L}$.

Theorem 2.3 *For any lattice L there exists a complete lattice C such that*

1. *$L \leq C$.*
2. *L is dense in C.*
3. *L is compact in C.*

Further, the lattice C is unique up to unique isomorphism fixing L. We call such C the canonical extension of L and denote it L^σ.

Definition 2.4 *Define the sets $K(L^\sigma)$ and $O(L^\sigma)$ of closed and open elements by*

1. *$K(L^\sigma)$ is all elements of L^σ that are meets of elements of L.*
2. *$O(L^\sigma)$ is all elements of L^σ that are joins of elements of L.*

Remark 2.5 Note that density of L in L^σ means that each element of L^σ is a join of closed elements and a meet of open elements. Also, L being compact in L^σ means that if a closed element x lies beneath an open element y, then there is some element $a \in L$ with $x \leq a \leq y$. In particular, the elements of L are exactly the clopen elements.

Remark 2.6 Almost all of our methods and results could be phrased in terms of various topologies that naturally arise in the context that we have just defined. We have decided to present the results in a topology-free environment. However, since we do believe the topological perspective to be very useful, we will occasionally point out a topological connection. For that purpose, we here introduce the three topological families that are involved.

To start with, there are the well-known lower, upper and interval topologies, which can in fact be defined on arbitrary partial orders. Given a poset $(P, \leq)$, obtain the lower topology $\iota^\downarrow$ and the upper topology $\iota^\uparrow$ on P by taking as a subbasis the collection consisting of the empty set together with the collection of complements $P \setminus (\uparrow p)$ of principal upsets and the collection of complements $P \setminus (\downarrow p)$ of principal downsets, respectively. The interval topology ι is defined as the join of $\iota^\downarrow$ and $\iota^\uparrow$; its name is explained by the fact that it is the smallest topology for which all intervals $[p, q] = \{x \in P \mid p \leq x \leq q\}$, where p and q are either elements of P or $+\infty$ or $-\infty$, are closed. In the case that P is a complete lattice satisfying some (infinitary) distributive laws, these three topologies have various other nice characterizations as well; we need not go into the details here, referring to Gierz et alii [11] instead.

Perhaps the most important contribution of Gehrke and Jónsson [9] is the topological perspective on canonical extensions. They introduce on L^σ the topologies σ, $\sigma^\uparrow$ and $\sigma^\downarrow$, having as their bases, respectively, the collections of sets of the form $[x, y]$, $\uparrow x$ and $\downarrow y$, with $x \in K(L^\sigma)$ and $y \in O(L^\sigma)$. Using the denseness and compactness of L in L^σ one can show that the set L is (topologically) dense in (L^σ, σ), and that the members of L are exactly the isolated points of (L^σ, σ).

Similarly, on the MacNeille completion $\overline{L}$ of a lattice L, one can define topologies ρ, $\rho^\uparrow$ and $\rho^\downarrow$ generated by, respectively, the collection of all sets of the form $[a, b]$, $\uparrow a$ and $\downarrow b$, with $a, b \in L$. It is now easy to prove that L is dense in $\overline{L}$, and that the members of L are exactly the isolated points of $(\overline{L}, \rho)$.

Extending monotone maps to lattice completions

We now explain the two natural ways in which additional, monotone operations on a lattice can be extended to the MacNeille completion and canonical extension. We start with maps that are order preserving.

Definition 2.7 *For $f : L \to M$ order preserving, define $\overline{f}, \widehat{f} : \overline{L} \to \overline{M}$ by*

$$\begin{aligned} \overline{f}(u) &= \bigvee\{f(a) | a \leq u \text{ and } a \in L\}, \\ \widehat{f}(u) &= \bigwedge\{f(a) | u \leq a \text{ and } a \in L\}. \end{aligned}$$

We call $\overline{f}$ and $\widehat{f}$ the lower and upper MacNeille extensions of f respectively.

In the case of a Boolean algebra B with completely additive operators, the expansion of B^σ with the lower extension of the operators is known in the literature as the Monk completion of the original structure, cf. Hirsch & Hodkinson [19].

Definition 2.8 *For $f : L \to M$ order preserving, define $f^\sigma, f^\pi : L^\sigma \to M^\sigma$ by*

$$\begin{aligned} f^\sigma(u) &= \bigvee\{\bigwedge\{f(a) | x \leq a \in L\} | u \geq x \in K(L^\sigma)\}, \\ f^\pi(u) &= \bigwedge\{\bigvee\{f(a) | y \geq a \in L\} | u \leq y \in O(L^\sigma)\}. \end{aligned}$$

We call f^σ and f^π the lower and upper canonical extensions of f respectively.

Remark 2.9 There is in fact a very natural way to generalize the above definitions to arbitrary maps between lattices, as has been shown by Gehrke & Jónsson [9] in the case of canonical extensions. For instance, to define the lower extension f^σ for an arbitrary map $f : L \to M$, one can take

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) | a \in [x, y]_L\} | x \leq u \leq y,\ x \in K(L^\sigma),\ y \in O(L^\sigma)\},$$

where $[x, y]_L$ denotes the set $\{a \in L \mid x \leq a \leq y\}$.

From the perspective of the topologies introduced in Remark 2.6, this definition can be seen as

$$f^\sigma \ = \ \underline{\lim}_\sigma f,$$

where, for a dense subset D of a topological space (X, τ), a complete lattice C and a map $g : D \to C$, one puts

$$\underline{\lim}_\tau g(u) \ = \ \bigvee\{\bigwedge g(U \cap D) \mid u \in U \in \tau\}.$$

The fact that every element of L is an isolated point of (L^σ, σ) then implies that f^σ is indeed an extension of f. Connecting the topological families, $\{\sigma^\uparrow, \sigma^\downarrow, \sigma\}$ on L^σ and $\{\iota^\uparrow, \iota^\downarrow, \iota\}$ on M^σ, one can show that the maps f^σ and f^π are the largest $(\sigma, \iota^\downarrow)$-continuous and the smallest $(\sigma, \iota^\uparrow)$-continuous extension of f, respectively. Likewise, involving the ρ topologies on the MacNeille completion of the domain, it holds that $\overline{f}$ and $\widehat{f}$ are the largest $(\rho, \iota^\downarrow)$-continuous and the smallest $(\rho, \iota^\uparrow)$-continuous extension of f, respectively.

Many useful properties of maps between lattices can be expressed in terms of these topologies. To mention a couple of examples, it follows almost immediately from the facts mentioned above that the map $f : L \to M$ is *smooth* (meaning that its lower and upper extension coincide: $f^\sigma = f^\pi$) if and only if f has a (σ, ι)-continuous extension $g : L^\sigma \to M^\sigma$. Other useful consequences of the topological characterizations concern the fact that in some cases, the operation of extending maps commutes with that of composing them. For example, if g^σ happens to be (σ, σ)-continuous then $f^\sigma g^\sigma \leq (fg)^\sigma$. This latter statement is actually of the kind involved in our proofs.

For more details concerning this approach to canonical extensions the reader is referred to Gehrke & Jónsson [9].

Here we confine ourselves to maps that are monotone, that is, maps of the form $f : L^n \to M$ that preserve or reverse the order in each coordinate. It will be convenient to introduce some terminology.

Definition 2.10 *An element $\epsilon \in \{1, d\}^n$ will be called an order type or monotonicity type (of length n). Given a lattice L, we let L^ϵ denote the lattice product $\prod_{i=1}^n L^{\epsilon_i}$ where L^{ϵ_i} is L if $\epsilon_i = 1$, and L^{ϵ_i} is the order dual L^d of L if $\epsilon_i = d$.*

Note that L^n and L^ϵ are based on the same carrier set, so that a map $f : L^n \to M$ can (set-theoretically) be viewed as a map from $L^\epsilon \to M$ as well.

Definition 2.11 (Monotone lattice expansions) *Let ϵ be a monotonicity type. An operation $f : L^n \to M$ is ϵ-monotone if it is order preserving when viewed as a map from $L^\epsilon \to M$. An operation $f : L^n \to M$ is called monotone if it is ϵ-monotone for some monotonicity type ϵ.*

A monotone lattice expansion is a bounded lattice expanded with a family of monotone operations.

Definition 2.12 *Let $f : L^n \to L$ be a monotone operation of type ϵ. Viewing f as an order preserving map from $L^\epsilon \to L$, we let $\overline{f}$ and $\widehat{f}$, respectively, denote the lower and upper MacNeille extension of f, as in Definition 2.7. Similarly, we define the lower and upper canonical extensions f^σ and f^π of f, as in Definition 2.8.*

The reader may worry that the above definition depends on the monotonicity type of f. So what if there would not be a *unique* $\epsilon \in \{1, d\}^n$ such that f is ϵ-monotone? Fortunately, this could only happen if f is both order preserving and reversing in some coordinates — but then f does not depend on the argument in those coordinates. This shows that $\overline{f}, \widehat{f}, f^\sigma$ and f^π have been well defined.

Using the facts that $\overline{(L^\epsilon)} = (\overline{L})^\epsilon$ and $(L^\epsilon)^\sigma = (L^\sigma)^\epsilon$, it is easy to find more direct definitions of the extensions of monotone maps.

Example 2.13 A Heyting algebra H is a bounded distributive lattice with an additional binary operation f (often written $\to$) satisfying certain identities. The operation f is order reversing in the first coordinate and order preserving in the second. Then

$$\overline{f}(u, v) = \bigvee \{f(a, b) | u \leq a, b \leq v\},$$

$$\widehat{f}(u,v) = \bigwedge\{f(a,b)|a \leq u, v \leq b\}.$$

And as $K((L^d \times L)^\sigma) = O(L^\sigma) \times K(L^\sigma)$ and $O((L^d \times L)^\sigma) = K(L^\sigma) \times O(L^\sigma)$

$$f^\sigma(u,v) \bigvee\{\bigwedge\{f(a,b)|x \geq a, y \leq b\}|u \leq x \in O(L^\sigma), v \geq y \in K(L^\sigma)\},$$

$$f^\pi(u,v) \bigwedge\{\bigvee\{f(a,b)|x \leq a, b \leq y\}|u \geq x \in K(L^\sigma), v \leq y \in O(L^\sigma)\}.$$

It is well known that the lower completions $(\overline{H}, \overline{f})$ and (H^σ, f^σ) need not be Heyting algebras, while the upper completions $(\overline{H}, \widehat{f})$ and (H^σ, f^π) are Heyting algebras.

Saturated Extensions

Finally, we shall use extensively results from non-standard analysis. The reader should consult Gehrke [7] for an introduction to the use of non-standard analysis in lattice theory in particular, and Stroyan and Luxemburg [29] for details on non-standard analysis in general.

In brief, a binary relation R is concurrent on a subset X of its domain if for any finite number n of elements $x_1, \ldots, x_n$ in X there is some y with $x_1Ry, \ldots, x_nRy$. A non-standard extension L^* of L is called κ-saturated if for any internal binary relation R of L^* that is concurrent on a subset X of its domain with cardinality less than κ, there is a y with xRy for all $x \in X$. We remark that R need not be a relation over the individuals of L^*, but could, for instance, be an internal binary relation over the n-fold power of the individuals of L^*. This fact will be of importance in later considerations of n-ary operations.

The crucial point is that for any structure L and any cardinal κ, there exists a κ-saturated non-standard extension of L. In fact, such a κ-saturated extension of L can be chosen to be an ultrapower of L. For a complete account, see [29].

For readers who are more familiar with the concept of saturation from classical model theory, rather than non-standard analysis, it will not be difficult to replace the details of our proofs with an argumentation based on notions from Chang & Keisler [4]. At the end of section 3 we provide a sketch of this alternative approach.

3 The Main Theorem

Throughout L is a monotone lattice expansion of cardinality κ and L^* is a κ^+-saturated non-standard extension of L. Here κ^+ is the successor cardinal to κ. We first show that there is a complete lattice embedding of the canonical extension of L into the MacNeille completion of L^*.

Definition 3.1 *For $i : L \to \overline{L^*}$ the identity map define $\varphi : L^\sigma \to \overline{L^*}$ by*

$$\varphi(u) = \bigvee\{\bigwedge\{i(a)|x \leq a \in L\}|u \geq x \in K(L^\sigma)\}.$$

Theorem 3.2 *The map φ is the unique complete lattice embedding of L^σ into $\overline{L^*}$ that extends the identical embedding.*

Proof. It is not difficult to see that φ is an order preserving extension of the identity embedding i. In order to prove the other properties of φ, we need the following technical fact:

$$\text{For all } p \in L^* \text{ and } y \in O(L^\sigma), \text{ if } p \leq \varphi(y) \text{ then there is some } a \in L \text{ with } a \leq y \text{ and } p \leq i(a). \tag{3.1}$$

The condition (3.1) can be understood both lattice theoretically and topologically. Lattice theoretically, one can show that (3.1) is equivalent to saying that any ideal I of L generates a normal ideal of L^*. For the topological interpretation, one can show that the condition as stated is equivalent to the following condition:

$$\text{For all } p \in L^* \text{ and } u \in L^\sigma, \text{ if } p \leq \varphi(u) \text{ then there is some } x \in K(L^\sigma) \text{ with } x \leq u \text{ and } p \leq \varphi(x),$$

which one may easily verify to spell out the condition of φ being $(\sigma^\uparrow, \rho^\uparrow)$-continuous.

To prove (3.1), pick $p \in L^*$ and $y \in O(L^\sigma)$ with $p \leq \varphi(y)$, and set $I = \{a \in L | a \leq y\}$. Suppose $p \not\leq a$ for each $a \in I$. Define a relation R by setting $uRv \Leftrightarrow u \leq v$ and $p \not\leq v$. For any $a_1, \ldots, a_n \in I$ we have $a = a_1 \vee \cdots \vee a_n$ belongs to I, so by assumption $p \not\leq a$, therefore $a_1 Ra, \ldots, a_n Ra$. So the internal binary relation R is concurrent on the set I of cardinality less than κ^+, and therefore there is $z \in L^*$ with aRz for all $a \in I$. Thus $p \not\leq z$ and $i(a) = a \leq z$ for all $a \in I$. By the compactness of L in L^σ we have $\varphi(y) = \bigvee\{i(a)|y \geq a \in L\}$, so $\varphi(y) \leq z$, contrary to the assumption that $p \leq \varphi(y)$. This proves (3.1).

We next turn to the proof that φ is a complete lattice homomorphism, that is, preserves arbitrary meets and joins. We first show that φ preserves joins (in L^σ) of elements of L. Suppose a_i ($i \in I$) is a family of elements of L. As φ is order preserving we have $\bigvee_I \varphi(a_i) \leq \varphi(\bigvee_I a_i)$. Setting $y = \bigvee_I a_i$, the compactness of L in L^σ gives $\varphi(y) = \bigvee\{i(b)|y \geq b \in L\}$. But if $b \leq y = \bigvee_I a_i$, then by compactness $b \leq a_{i_1} \vee \cdots \vee a_{i_n}$ for some $i_1, \ldots, i_n$. It follows that $\varphi(\bigvee_I a_i) \leq \bigvee_I \varphi(a_i)$.

We next show that φ preserves arbitrary meets. Note that as each element of L^σ is a meet of open elements, it is enough to show $\varphi(\bigwedge_I y_i) = \bigwedge_I \varphi(y_i)$ whenever y_i ($i \in I$) is a family of open elements. That $\varphi(\bigwedge_I y_i) \leq \bigwedge_I \varphi(y_i)$ follows as φ is order preserving. As L^* is join dense in $\overline{L^*}$, for the other inequality it is enough to show each $p \in L^*$ under $\bigwedge_I \varphi(y_i)$ is also under $\varphi(\bigwedge_I y_i)$. By (3.1), as $p \leq \varphi(y_i)$ there is $a_i \leq y_i$ with $a_i \in L$ and $p \leq \varphi(a_i)$. Set $x = \bigwedge_I a_i$. Then $x \leq \bigwedge_I y_i$ and $\varphi(x) = \bigwedge\{i(c)|x \leq c \in L\}$. But if $x = \bigwedge_I a_i \leq c \in L$, we have by compactness $a_{i_1} \wedge \cdots \wedge a_{i_n} \leq c$ for some $i_1, \ldots, i_n$, hence $p \leq c$, so $p \leq \varphi(x) \leq \varphi(\bigwedge_I y_i)$ as required.

By duality the map $\psi : L^\sigma \to \overline{L^*}$ defined by

$$\psi(u) = \bigwedge\{\bigvee\{i(a)|y \geq a \in L\}|u \leq y \in O(L^\sigma)\}$$

is an extension of the identity that preserves arbitrary joins. But for $u \in L^\sigma$ we have $u = \bigwedge\{\bigvee\{a|y \geq a \in L\}|u \leq y \in O(L^\sigma)\}$. Then as φ preserves arbitrary meets and joins of elements of L, we have $\varphi(u) = \psi(u)$. Thus φ preserves arbitrary joins and meets.

In order to prove the injectivity of φ, we claim that it suffices to show the following:

$$\text{If } \varphi(x) \leq \varphi(y) \text{ for } x \in K(L^\sigma) \text{ and } y \in O(L^\sigma), \text{ then } x \leq y. \tag{3.2}$$

Suppose (3.2) holds and $u, v \in L^\sigma$ are such that $\varphi(u) \leq \varphi(v)$. Then by complete additivity of φ and the density of L in L^σ it follows that $\varphi(x) \leq \varphi(y)$ for all closed $x \leq u$ and all open $y \geq v$. Then by (3.2) $x \leq y$ for each closed $x \leq u$ and open $y \geq v$, hence by density $u \leq v$. Thus φ is an order embedding, and therefore is injective.

For the proof of (3.2), set $F = \{a \in L | x \leq a\}$ and define a relation R by setting $uRv \Leftrightarrow u \geq v$ and $v \in F^*$. This relation is not only internal but actually standard. Also, for any $a_1, \ldots, a_n \in F$ we have $a = a_1 \wedge \cdots \wedge a_n$ belongs to F, so $a_1 Ra, \ldots, a_n Ra$. Because R is standard, $\kappa-$saturation is not even needed to conclude that there is $p \in F^*$ with $p \leq a = i(a)$ for all $a \in F$ as this is true in any non-standard extension, e.g. see [7]. Now, as $\varphi(x) = \bigwedge\{i(a) | x \leq a \in L\}$ we have $p \leq \varphi(x)$, hence as $\varphi(x) \leq \varphi(y)$, we have $p \leq \varphi(y)$. Condition (3.1) then yields $a \in L$ with $p \leq a$ and $a \leq y$. As F is a filter of L, then F^* is a filter of L^*. Then as $p \in F^*$ and $p \leq a$, we then have $a \in F^*$, and as a is standard, $a \in F$. Therefore $x \leq a \leq y$.

Finally, uniqueness of φ follows as L^σ is completely generated by L. QED

We now show that the embedding φ is also a homomorphism with respect to the additional operations.

Proposition 3.3 *Suppose f is a monotone n-ary operation on L. Then*

$$\begin{aligned} \overline{f^*}(\varphi(u_1), \ldots, \varphi(u_n)) &= \varphi f^\sigma(u_1, \ldots, u_n), \\ \widehat{f^*}(\varphi(u_1), \ldots, \varphi(u_n)) &= \varphi f^\pi(u_1, \ldots, u_n). \end{aligned}$$

Proof. To avoid introducing cumbersome notation we prove the result in detail for the lower extension $\overline{f^*}$ of an order preserving unary operation and indicate the changes necessary for a binary operation that preserves order in the first coordinate and reverses order in the second coordinate. The argument for the lower extension of a general monotone operation is merely a formalization of this, and the argument for upper extensions follows by duality as φ is the unique extension of the identity map to a complete homomorphism.

Suppose $f : L \to L$ is order preserving. We need the following fact:

For all $p \in L^*$ and $x \in K(L^\sigma)$, if $p \geq \overline{f^*}\varphi(x)$ then there is some $a \in L$ with $a \geq x$ and $p \geq i(f(a))$. (3.3)

Condition (3.3) is fairly transparently a continuity condition, namely

For all $p \in L^*$ and $x \in K(L^\sigma)$, if $p \geq \overline{f^*}\varphi(x)$ then there is $y \in O(L^\sigma)$ with $y \geq x$ and $p \geq \overline{f^*}\varphi(y)$.

Condition (3.3) is obtained from this latter condition by applying compactness to the inequality $y \geq x$ and then applying the definitions of the various functions. But this latter condition is exactly the condition that $\overline{f^*}\varphi$ is $(\sigma^\downarrow, \rho^\downarrow)$-continuous on $K(L^\sigma)$. Now one may wonder whether, just as for condition (3.1), we have that this in turn is equivalent to:

For all $p \in L^*$ and $u \in L^\sigma$, if $p \geq \overline{f^*}\varphi(u)$ then there is some $y \in O(L^\sigma)$ with $y \geq u$ and $p \geq \overline{f^*}\varphi(y)$.

which is the condition that $\overline{f^*}\varphi$ is $(\sigma^{\downarrow}, \rho^{\downarrow})$-continuous on all of L^σ. However, this is not the case at this level of generality. But, if f is an operator (modulo the appropriate flipping of coordinates in the n-ary case), then it is true.

For the proof of (3.3), take $x \in K(L^\sigma)$, $p \in L^*$, and let $F = \{a \in L | x \leq a\}$. Assume $f(a) \not\leq p$ for each $a \in F$. We must show $\overline{f^*}\varphi(x) \not\leq p$. Define a relation R by setting $uRv \Leftrightarrow u \geq v$ and $f^*(v) \not\leq p$. For any $a_1, \ldots, a_n \in F$ if we set $a = a_1 \wedge \cdots \wedge a_n$ then $a_1 Ra, \ldots, a_n Ra$ since $f(a) \not\leq p$ for each $a \in F$. So the internal binary relation R is concurrent on the set F of cardinality less than κ^+, thus there is $z \in L^*$ with aRz for all $a \in F$. This means that $z \leq a$ for all $a \in F$, hence $z \leq \varphi(x)$, and $f^*(z) \not\leq p$. It then follows by the monotonicity of $\overline{f^*}$ that $\overline{f^*}\varphi(x) \not\leq p$. This proves (3.3).

We first show that $\overline{f^*}\varphi(x) = \varphi f^\sigma(x)$ for x closed. As L^* is meet-dense in $\overline{L^*}$ condition (3.3) provides $\overline{f^*}\varphi(x) \geq \bigwedge\{\varphi f(a) | x \leq a \in L\}$ and monotonicity provides the other inequality. Then as φ is complete $\overline{f^*}\varphi(x) = \varphi f^\sigma(x)$.

For arbitrary $u \in L^\sigma$ we have $\varphi f^\sigma(u) = \varphi(\bigvee\{f^\sigma(x) | u \geq x \in K(L^\sigma)\})$, and using the fact that φ is complete in conjunction with the above result for closed elements, $\varphi f^\sigma(u) = \bigvee\{\overline{f^*}\varphi(x) | x \leq u\}$. Monotonicity then gives $\varphi f^\sigma(u) \leq \overline{f^*}\varphi(u)$. For the other inequality note $\overline{f^*}\varphi(u) = \bigvee\{f^*(p) | \varphi(u) \geq p \in L^*\}$ by definition of $\overline{f^*}$. Suppose $p \leq \varphi(u)$ where $p \in L^*$. Expressing u as a meet of open elements $u = \bigwedge_I y_i$ we have $p \leq \varphi(y_i)$ for each $i \in I$. Then by (3.1), $p \leq \varphi(a_i)$ for some $a_i \leq y_i$. Setting $x = \bigwedge_I a_i$ we have that x is closed, $x \leq u$, and as φ is complete $p \leq \varphi(x)$. It follows that $f^*(p) \leq \overline{f^*}\varphi(x) = \varphi f^\sigma(x)$, and so, by monotonicity, $f^*(p) \leq \varphi f^\sigma(u)$. Therefore $\overline{f^*}\varphi(u) \leq \varphi f^\sigma(u)$.

The argument for a binary operation preserving order in the first coordinate and reversing order in the second requires the statement of (3.3) to be modified to apply to closed elements (x, y) of $(L \times L^d)^\sigma$. Specifically, for $x \in K(L^\sigma)$ and $y \in O(L^\sigma)$ with $\overline{f^*}(\varphi(x), \varphi(y)) \leq p$ we must show there are $a, b \in L$ with $x \leq a$, $b \leq y$ and $f(a, b) \leq p$. This is accomplished by considering the binary relation R on $L^* \times L^*$ defined by $(u_1, u_2)R(v_1, v_2) \Leftrightarrow u_1 \geq v_1$, $u_2 \leq v_2$ and $f^*(v_1, v_2) \not\leq p$, then noting that R is concurrent on the set $F = \{(a, b) \in L \times L | x \leq a, b \leq y\}$. The argument then follows the lines above showing first that $\overline{f^*}(\varphi(x), \varphi(y)) = \varphi f^\sigma(x, y)$ for x closed and y open, then using this result and (3.1) and its dual to show equality for arbitrary $u, v \in L^\sigma$. QED

We now have all the ingredients for our main technical result, viz., Theorem 3.5 below, which extends Theorem 3.2 to monotone lattice expansions. In the most general formulation, we may be dealing with lattices endowed with more than one additional operation; depending on whether we take the lower or upper extension for each of these, we arrive at a different extension of the algebra. We need some terminology for this.

Definition 3.4 *Suppose L is a lattice with a family of additional operations indexed by the set I. A completion type β for L is a specification of which operations f_i are to be extended with the lower extension, and which are to be extended with the upper extension. The β-MacNeille and β-canonical completions of L are then defined in the obvious ways.*

Theorem 3.5 *Suppose L is a lattice of cardinality κ expanded with a family of monotone operations. Then for any κ^+-saturated non-standard extension L^* and any completion type β of L, the β-canonical extension of L can be embedded into the β-MacNeille completion of L^* via an embedding that preserves all joins and meets.*

Proof. Theorem 3.2 shows φ is a complete embedding of the lattice L^σ into $\overline{L^*}$. The previous proposition shows that φ is compatible with the extensions of the operation f_i provided both are lower extensions or both are upper extensions. QED

Theorem 3.5 paves the way for our main result which states that varieties that are closed under MacNeille completions are also closed under canonical extensions. In fact, we can prove something slightly stronger.

We recall a formula Φ is a universal formula if all the quantifiers of its prenex normal form are universal quantifiers. A class K is a universal class if it can be axiomatized by universal formulas. It is well-known (see for instance Burris & Sankappanavar [3, pg. 215]) that a class K is universal iff it is closed under isomorphism, substructures and ultraproducts.

Theorem 3.6 *Let K be a universal class (for instance, a variety) of monotone lattice expansions. If K is closed under β-MacNeille completions, then K is closed under β-canonical extensions.*

Proof. Suppose L belongs to K and L has cardinality κ. Let L^* be a κ^+-saturated ultrapower of L. Then L^* also belongs to K, and as the β-canonical extension of L is isomorphic to a subalgebra of the β-MacNeille completion of L^*, the result follows. QED

Remark 3.7 It would be wrong to conclude from Theorem 3.6 that under all circumstances, canonical extensions are 'better' completions than MacNeille completions. For instance, consider the class of Boolean algebras extended with two operations $\blacklozenge$ and $\lozenge$ that preserve arbitrary existing joins. Abbreviate $\lozenge^0 x = x$ and $\lozenge^{n+1}x = \lozenge\lozenge^n x$, for all $n < \omega$. Now let C be the class of algebras that satisfy the following constraint:

$$\blacklozenge x \;=\; \bigvee_{n<\omega} \lozenge^n x. \tag{3.4}$$

(That is, for all x, the join on the right hand side exists, and it is equal to the element on the left hand side.) This condition plays an important role in the theory of dynamic algebras under the name of $*$-continuity, see Pratt [26].

It is fairly easy to see that the property (3.4) is preserved under taking lower MacNeille completions, but not under taking lower canonical extensions.

Remark 3.8 For readers familiar with the notion of saturation as defined in for instance Chang & Keisler [4], we briefly indicate how to use it to prove the results in this section.

For our fixed lattice L, let $\Sigma(L)$ be the first order signature extending the algebraic language of lattices with constants $\underline{a}$ for every element a of L, and predicate symbols $\underline{F}, \underline{I}$ for every filter (ideal, respectively) of L. Expand L to a $\Sigma(L)$-structure in the obvious way. It follows from the Theorems 6.1.4 and 6.1.8 in Chang & Keisler [4] that with respect to this language, L has κ-saturated elementary extensions for each cardinal κ; and that one can obtain these extensions as ultrapowers of the original structure.

In this approach, L^* denotes an arbitrary ω-saturated elementary extension of (the $\Sigma(L)$-expansion of) L. Then for any filter F or ideal I, respectively, F^* and I^* denote the interpretation in L^* of the monadic symbols $\underline{F}$ and $\underline{I}$. The same holds for a, $\underline{a}$ and a^* but we will write $i(a)$, or simply a, for a^*.

Let us now, as an example, see how to prove (3.1). Pick $p \in L^*$ and $y \in O(L^\sigma)$ with $p \leq \varphi(y)$, and set $I = \{a \in L | a \leq y\}$. Suppose for contradiction that $p \not\leq a$ for each $a \in I$. Define the $\Sigma(L)$-type

$$\Gamma(v) \ := \ \{\underline{p} \not\leq v\} \cup \{\underline{a} \leq v \mid a \in I\}.$$

For an arbitrary finite subset $\{a_1, \ldots, a_n\}$ of I we find that each a_i is below $a_1 \vee \cdots \vee a_n$, while by assumption, $p \not\leq a_1 \vee \cdots \vee a_n$. From this it easily follows that $\Gamma(v)$ is finitely satisfiable in L^*, whence by 2-saturation there is a $z \in L^*$ such that $p \not\leq z$ and $a \leq z$ for all $a \in I$. By the compactness of L in L^σ we have $\varphi(y) = \bigvee\{i(a) | y \geq a \in L\}$, so $\varphi(y) \leq z$, contrary to the assumption that $p \leq \varphi(y)$. This proves (3.1).

Note that in the above proof, 2-saturation suffices because all the formulas $\underline{a} \leq v$ with $a \in I$ belong to the language $\Sigma(L)$. In order to prove the other statements involving saturation, it suffices to take L^* to be ω-saturated; the price that we have to pay for this 'low' kind of saturation is a high cardinality of our language $\Sigma(L)$, which will be generally be κ^+.

4 Boolean algebras with operators

Before concluding, we turn to a very special situation, viz., that of modal logic and its algebraic counterpart, Boolean algebras with operators (BAOs). We will prove two corollaries to our results, one known, and one new. Both corollaries are related to a well-known result in modal logic due to Fine [6]:

$$\text{The modal logic of any elementary class of Kripke frames is canonical.} \qquad (4.1)$$

Before discussing the algebraic meaning of this result, we turn to Fine's proof, which used, respectively introduced, first order and modal notions of saturation. From this proof, Goldblatt [13] extracted a result that we can see as a manifestation of Theorem 3.5 in the context of Kripke frames: it states that the so-called ultrafilter extension of a Kripke frame S can be obtained as a certain 'bounded morphic image' of some elementary extension of S. Goldblatt [14] later generalized this result to the context of arbitrary relational structures, and showed that for the elementary extension one can actually take an ultrapower of S.

In order to avoid the introduction of such notions as ultrafilter extensions and bounded morphisms, we give here a more or less algebraic presentation. We do need the notion of a complex algebra.

Definition 4.1 *A relational structure $S = (P, (R_i)_{i\in I})$ consists of a set P and a collection $(R_i)_{i\in I}$ of relations R_i on P. A Kripke frame is a relational structure of the form (P, R) with R a single, binary, relation on P.*

Given an $n+1$-ary relation R on P, define the associated n-ary map $\langle R\rangle$ on the power set of P by

$$\langle R\rangle(Y_1, \ldots, Y_n) \ = \ \{x \in P \mid Rxy_1 \cdots y_n \text{ for some } y_1 \in Y_1, \ldots, y_n \in Y_n\}.$$

Adding these maps to the power set algebra of P, we obtain a Boolean algebra with additional operations that we call the complex algebra of S and denote by S^+.

Note that such complex algebras are rather special lattice expansions — they are complete, atomic Boolean algebras endowed with operations that are completely additive.

Goldblatt's result can now be formulated as the following corollary to Theorem 3.5.

Theorem 4.2 *For a relational structure S there is a non-standard extension S^* and an embedding φ of the canonical extension of S^+ in $(S^*)^+$ which preserves all meets and joins.*

Proof. We confine ourselves to the case that $S = (P, R)$ is a Kripke frame, and apply our results to the Boolean algebra with operators S^+. Let κ be the cardinality of the power set of P, and let $(S^+)^*$ be a κ^+-saturated non-standard extension of S^+. It follows immediately from Theorem 3.5 that the map φ as defined in Definition 3.1 is a complete embedding of $(S^+)^\sigma$ in the MacNeille completion of $(S^+)^*$:

$$(S^+)^\sigma \ \hookrightarrow^c \ \overline{(S^+)^*}.$$

Here we use the terms canonical extension and MacNeille completion to mean each operation is extended via its lower extension. As $(S^+)^*$ is a κ^+-saturated extension of S^+ it must also be atomic, and its atom structure will form a κ^+-saturated extension S^* of S. Also, since the operation $\langle R \rangle$ of S^+ is completely additive, it is residuated; that is, there is a residual map $g : S^+ \to S^+$ such that for all $X, Y \subset P$:

$$\langle R \rangle X \subseteq Y \ \text{ iff } \ X \subseteq g(Y).$$

From this it follows that $\langle R \rangle^*$ is residuated as well (by g^*), whence it preserves all existing joins. But then the lower MacNeille extension $\overline{\langle R \rangle^*}$ is completely additive, so that the MacNeille completion of $(S^+)^*$ is isomorphic to the complex algebra of its atom structure:

$$\overline{(S^+)^*} \ \cong \ (S^*)^+.$$

From the first and last displayed facts the result is immediate. QED

Remark 4.3 The reader may worry that this is not exactly the theorem from modal logic since the cited modal logicians were able to obtain an extension that is ω-saturated rather than the higher level of saturation required by our results. The difference can be explained by the fact that their notion of saturation is with respect to a first order language of a high cardinality — it contains a unary predicate symbol for every subset of the Kripke frame S. Had we chosen to work along similar lines (as sketched in Remark 3.8), then we would have obtained a version of our results referring to ω-saturated extensions as well.

Remark 4.4 Instead of viewing Theorem 4.2 as an application of our results, one may think of Theorem 3.5 as a version of the modal result which is applicable in a much more general setting. This may be useful when treating logics, such as substructural logics of various kinds, whose algebraic counterparts are not classes of BAOs but classes of possibly non-distributive lattice expansions.

The second application of our results concerns the converse of Fine's theorem, rather than its proof. To explain the meaning of (4.1) in more algebraic terms it is convenient to introduce some terminology.

Definition 4.5 *Given a class K of relational structures, $V(K)$ denotes the variety of algebras generated by the complex algebras of structures in K. That is, $V(K) = \mathsf{HSPCm}(K)$, where Cm denotes the class operation of taking complex algebras.*

Definition 4.6 *A variety of* BAO*s is said to be canonical if it is closed under lower canonical extensions.*

Fine's result (4.1) then translates as follows: An elementary class of Kripke frames generates a canonical variety of modal algebras. We remind the reader that a Kripke frame is a relational structure with a single binary relation (see definition 4.1). In subsequent years this theorem has been strengthened and generalized; for instance, Goldblatt [14] proves the following:

An Up-closed class of relational structures generates a canonical varieties of BAOs. (4.2)

Here a class of structures is called Up-closed if it is closed under taking ultraproducts. We remark that (4.2) can also be generalized to the setting of lattice expansions, see Gehrke & Jónsson [9] and Gehrke & Harding [8] for further discussion.

It has long been an open problem in the theory of BAOs whether the converse of this statement holds as well; that is, whether every canonical variety of BAOs is generated by some class of relational structures that is closed under ultraproducts; or, equivalently (as shown by Goldblatt [15]), whether every canonical variety of BAOs is generated by some elementary class of relational structures.

Goldblatt, Hodkinson & Venema [17] have recently shown this is not the case, however, as they point out, it remains an outstanding problem to characterize those varieties of BAOs that are generated by such a class of structures. There is a list of positive answers to this question in [17]; one of these results involves the notion of singleton persistence, also known as di-persistence. We need the following definition.

Definition 4.7 *Let S be a relational structure; its singleton algebra S° is the subalgebra of S^+ generated by the atoms of S^+ (i.e., by the singletons of S).*

Given a class C of Boolean algebras with operators, S is called a structure for C if S^+ belongs to C, and a weak structure for C if S° belongs to C. The class of structures and of weak structures for C are denoted as $\mathsf{Str}(C)$ and $\mathsf{Wst}(C)$, respectively.

The relevance of these notions is based on the following two facts. First, it is well-known and easy to see that any canonical variety is generated by its structures: for such V it even holds that $V = \mathsf{SCmStr}(V)$. Second, it has been shown by Venema [30] that for any universal class C of BAOs, the class $\mathsf{Wst}(C)$ is elementary; in fact, first order formulas defining $\mathsf{Wst}(C)$ can be effectively obtained from the universal formulas characterizing C. Put together, these facts explain the interest in cases where the classes $\mathsf{Wst}(V)$ and $\mathsf{Str}(V)$ coincide.

Definition 4.8 *A variety V of* BAO*s is singleton persistent if $\mathsf{Wst}(V) = \mathsf{Str}(V)$.*

Goldblatt [16] proves that every variety which is both singleton persistent and canonical is generated by an elementary class of relational structures. Our contribution here is that both conditions, that is, canonicity and singleton persistence, follow from the variety being closed under taking MacNeille completions.

Theorem 4.9 *Let V be a variety of Boolean algebras with operators which is closed under lower MacNeille completions. Then V is singleton persistent and canonical. Hence V is generated by an elementary class of relational structures, namely* $\mathsf{Str}(V)$.

Proof. One easily shows that the complex algebra of any relational structure S is the lower MacNeille completion of its singleton algebra: $S^+ = \overline{S^\circ}$. From this observation and the assumption on V it follows that $\mathsf{Wst}(V) \subseteq \mathsf{Str}(V)$. Since the opposite inclusion holds for any class of algebras that is closed under taking subalgebras, we find that $\mathsf{Wst}(V) = \mathsf{Str}(V)$. That is, V is singleton persistent. (This is essentially Theorem 5.1(a) of Goldblatt [16]; our contribution here has been to remove the restriction to complete additivity.) Theorem 3.6 gives directly that V is canonical. The last part of the theorem then follows from Corollary 3.4 of Goldblatt [16] — or from the two facts given above Definition 4.8. QED

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