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NNIL, A Study in Intuitionistic Propositional Logic

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NNIL

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ABSTRACT: In this paper we study NNIL, the class of formulas of the Intuitionistic Propositional Calculus IPC, with no nestings of implications to the left. We show that the formulas of this class are precisely the formulas of the language of IPC that are preserved under taking submodels of Kripke models for IPC (for various notions of submodel). This makes NNIL an analogue of the purely universal formulas in Predicate Logic. We prove a number of interpolation properties for NNIL, and explore the extent to which these properties can be generalized to more complicated classes of formulas.

1 Introduction: In this paper we study a special class of formulas of the Intuitionistic Propositional Calculus IPC. This is the class of Π_1-formulas or NNIL-formulas. These formulas are the formulas without nestings of implications to the left. Examples of NNIL-fomulas are: p, $\top$, $(p\rightarrow(q\vee(r\rightarrow s)))\wedge((q\wedge t)\rightarrow((r\rightarrow p)\vee(s\rightarrow r)))$.

The usual Kripke semantics for IPC provides us with a translation of IPC-formulas to one-variable formulas of the Classical Predicate Calculus (CPdC). The NNIL-formulas are seen to translate to purely universal formulas or Π_1-formulas in the sense of CPdC. In fact our results imply that every Π_1-formula in the sense of CPdC, under certain further appropriate general conditions shared by all IPC-formulas, is provably equivalent to a NNIL-formula.

We will prove in this paper that the NNIL-formulas are precisely the IPC-formulas preserved under taking submodels. Moreover the NNIL formulas satisfy the interrelated properties of left and right approximation and (uniform) left and right interpolation. We define these properties below. Let $\mathfrak{L}$ be the language of IPC. $\mathfrak{L}(\mathbf{p})$ is the restriction of $\mathfrak{L}$ to the finite set of propositional variables $\mathbf{p}$. PV(A) is the set of propositional variables in A. Define:

- NNIL satisfies *left-interpolation* (IPL) if for every $A\in$ NNIL and $B\in\mathfrak{L}$, there is an I in NNIL such that $\vdash A\rightarrow I$ and $\vdash I\rightarrow B$ and $PV(I)\subseteq PV(A)\cap PV(B)$. I is called the NNIL left-interpolant.
- NNIL satisfies *right-interpolation* (IPR) if for every $A\in\mathfrak{L}$ and $B\in$ NNIL, there is an I in NNIL such that $\vdash A\rightarrow I$ and $\vdash I\rightarrow B$ and $PV(I)\subseteq PV(A)\cap PV(B)$. I is called

the NNIL left-interpolant.

- NNIL satisfies *uniform left-interpolation* (UIPL) if for every B and every **p**, there is a formula $B^*(\mathbf{p}) \in NNIL(\mathbf{p})$, such that for all $A \in NNIL$ satisfying $PV(A) \cap PV(B) \subseteq \mathbf{p}$ we have: $\vdash A \to B \Leftrightarrow \vdash A \to B^*(\mathbf{p})$. We call $B^*(\mathbf{p})$ the uniform NNIL left-interpolant of B.
- NNIL satisfies *uniform right-interpolation* (UIPR) if for every A and every **p**, there is a formula $A^\circ(\mathbf{p}) \in NNIL(\mathbf{p})$, such that for all $B \in NNIL$ satisfying $PV(A) \cap PV(B) \subseteq \mathbf{p}$ we have: $\vdash A \to B \Leftrightarrow \vdash A^\circ(\mathbf{p}) \to B$. We call $A^\circ(\mathbf{p})$ the uniform NNIL right-interpolant of A.
- NNIL satisfies *left-approximation* (APL) if for every B, there is a formula $B^* \in NNIL$, such that for all $A \in NNIL$: $\vdash A \to B \Leftrightarrow \vdash A \to B^*$.
- NNIL satisfies *right-approximation* (APR) if for every A, there is a formula $A^\circ \in NNIL$, such that for all $B \in NNIL$: $\vdash A \to B \Leftrightarrow \vdash A^\circ \to B$.

1.2 Sources of interest: The work reported in this paper is connected with with several lines of interest of its authors.

Our first reason are derived from the analogy with modal logic. Under a suitable translation, both modal propositional logic and IPC can be viewed as especially tractable fragments of full first-order logic. (In fact IPC can be viewed as a fragment of propositional modal logic.) The question is then to which extent these fragments inherit desirable meta-properties of first-order logic. Obvious cases of transfer are purely universal properties (such as Löwenheim-Skolem), but the matter is less straightforward with properties that involve existential quantification over first-order formulas, such as the Loś' Theorem or Interpolation. (Both are in fact Π_2 in their formula quantifiers.) Our results show for IPC that transfer works for Loś' Theorem, too, and moreover, our proofs provide methods for investigating such issues more generally. Both propositional modal logic and IPC have miniature versions of the classical hierarchies of formulas in predicate logic. In this paper we study (mainly) the lowest part of one such hierarchy. For more information about the programme of transposing classical results, see De Rijke[93].

A second concern is interpolation in fragments. A common definition of fragments is in terms of the formulas with connectives from a certain set of (primitive or defined) connectives. In classical propositional logic, interpolation holds in all such fragments (proved by F. Ville: see Kreisel & Krivine[71], Kreisel & Krivine[72], Ch. 1, Exercises). In intuitionistic logic, the situation is different. On the one hand, it is shown in Renardel[89], that all fragments based on some subset of $\{\wedge, \vee, \to, \neg\}$ satisfy interpolation (the only non-trivial cases are the fragments [$\to$], [$\to,\vee$] and [$\to,\vee,\neg$]; for [$\to$], interpolation was already proved by J. Zucker in Zucker[78]). On the other hand, there are many fragments for which interpolation fails, e.g. the fragment [$\wedge,\to,\neg,\delta$] (where $\delta(A,B,C) = (A\vee\neg A) \wedge (A\to B) \wedge (\neg A\to C)$) in Zucker[78]; see also Renardel[86] for more counterexamples. Other information on fragments based on a subset of $\{\wedge,\vee,\to,\neg,\leftrightarrow,\neg\neg\}$ such as the structure and size of finite fragments, can be found in DHR[88].

For intermediate (also called superintuitionistic) logics, there is the remarkable result by Maksimova (see: Maksimova[77]) that there are only five logics between classical and intuitionistic propositional logic for which interpolation holds for the full fragment; they are axiomatized by

- $\neg A \vee \neg\neg A$
- $A \vee (A \to (B\vee\neg B))$
- $A \vee (A \to (B\vee\neg B)), (A\to B) \vee (B\to A) \vee ((A\to\neg B) \wedge (B\to\neg A))$
- $A \vee (A \to (B\vee\neg B)), \neg A \vee \neg\neg A$
- $(A\to B) \vee (B\to A)$

respectively. The general situation is as for intuitionistic logic: see Porebska[85], where it is shown that every intermediate logic has many fragments for which interpolation fails.

About the situation in predicate logic not much is known. Renardel[81] gives many fragments of intuitionistic predicate logic for which interpolation fails, but the definition of fragment used there is rather awkward. However, it is easy to see that interpolation fails in [$\to,\forall$] and in any other fragment containing $\to,\forall$ where $\exists$ is not definable. To see this, observe that: $A(c) \vdash \forall y(A(y) \to B) \to B$, but any interpolant I can only contain the parameter A and is hence equivalent to $\exists xA(x)$, which is not in the fragment (in classical logic, $\forall y(A(y) \to \forall xA(x)) \to \forall xA(x)$ is equivalent to $\exists xA(x)$ and can serve as an interpolant).

NNIL can be viewed as a fragment in an extended sense, since it is generated like the language of IPC, restricting the formation rule for implications.

We turn to the third concern. Consider Heyting's Arithmetic (HA). Let X be set of sentences of $\mathfrak{L}_{HA}$, the language of Arithmetic. A formula A of $\mathfrak{L}(\mathbf{p})$, the language of IPC restricted to the finite set of propositional variables $\mathbf{p}$, is *HA,X-exactly provable* if there is a substitution $\mathfrak{f}$ of elements of X for the variables in $\mathbf{p}$, such that for any $B\in\mathfrak{L}(\mathbf{p})$:

$$HA\vdash B\mathfrak{f} \Leftrightarrow IPC\vdash A\to B.$$

The question *What are the HA,X-exactly provable formulas?* is connected to the question *What are the propositional derived rules of HA?* In his paper De Jongh[82], Dick de Jongh shows that the HA,$\mathfrak{L}_{HA}$-exactly provable formulas in one propositional variable p are precisely p, ¬p, ¬¬p, ¬¬p→p, ⊤. For two propositional variables the precise set of HA,$\mathfrak{L}_{HA}$-exactly provable formulas is unknown. In Visser[85], it is shown that the HA,Σ_1-exactly provable formulas (in any number of variables) are precisely the consistent NNIL-formulas with the Disjunction Property. Moreover for any finite Σ_1-substitution $\mathfrak{f}$ we can find an A in NNIL such that: $HA \vdash B\mathfrak{f} \Leftrightarrow IPC \vdash A \rightarrow B$. Using this result the propositional derived rules of HA for Σ_1-substitutions can be fully characterized. It also follows that the propositional derived rules of HA for $B(\Sigma_1)$-substitutions are the same as the propositional derived rules of IPC itself (see DV[93]). Thus from the point of the study of the logical properties of constructive theories like HA, NNIL turns out to be a significant class.

1.3 Genesis of the paper: In 1983-1984, A. Visser was studying the provability logic of Heyting's Arithmetic HA. As a subproblem he considered Σ_1-substitutions of propositional formulas. NNIL emerged from this work on HA. Two questions came up of a purely propositional character. The first was whether the NNIL formulas are precisely the ones that are preserved under taking submodels of Kripke models. This question was answered positively by Johan van Benthem (in correpondence with Visser) using a model-theoretical argument close to the argument presented in this paper and, independently, by Visser using different methods (see Visser[85]). Van Benthem's proof appeared (for the case of temporal logic) in his Van Benthem[91]. For the further development on the modal side the reader is referred to De Rijke[93] and the forthcoming paper ANV[?].

The other question was to prove NNIL-interpolation. This question was posed in Visser[85]. In that paper, an indirect proof for NNIL left-interpolation (see 1.1.4 in Visser[85]) was given. A direct proof both for left and for right interpolation is given in Renardel[86]. Renardel's proofs are in presented in section 4 of this paper.

In 1993 De Jongh and Visser decided to take up the project concerning IPC and HA again. The present paper brings together the results on NNIL for propositional logic.

1.4 Organization of the paper: Section 2 contains the syntactical preliminaries. In section 3 we consider properties like uniform interpolation from a mildly abstract point of view. Section 4 provides a proof of NNIL interpolation using cut-elimination. Section 5 gives the basics of Kripke models and section 6 adds the basics of subsimulations between models. In this section we prove the promised result that the NNIL formulas are precisely the ones preserved under submodels. Section 7 contains the proof of uniform NNIL interpolation by model-theoretical means. In section 8 we prove that (uniform) right interpolation also holds for Π_2, but that uniform left interpolation fails for Π_2 and that right interpolation fails for Π_3. Appendix A contains a characterization of IPC as a fragment of Predicate Logic. In appendix B we develop the notions of simulation appropriate for the model-theoretic characterization of arbitrary Π_n.

Each of the selections:

$\langle 1,2,3,4\rangle$, $\langle 1,2,3,5,6,7\rangle$, $\langle 1,2,3,5,6,7,8\rangle$, $\langle 1,2,5,6,A\rangle$, $\langle 1,2,5,B\rangle$,

can be read as a reasonably selfcontained paper, and, of course, any of their unions can.

2 Matters of syntax: Let $\mathfrak{L}$ be the language of Intuitonistic Propositional Logic IPC. We take as connectives: $\wedge,\vee,\rightarrow,\top$ and $\bot$. $\neg A$ is defined by $A\rightarrow\bot$. PV is the set of propositional variables, denoted by p,q,...; together with $\top,\bot$ we call them *atoms*. A,B,C,... are formulas; Γ,Δ,Γ',... are finite (possibly empty) sets of formulas. We write Γ,Δ for the union of Γ and Δ; Γ,A stands for $\Gamma,\{A\}$. Let $\mathfrak{P}$ be a set of propositional variables. We write $\mathfrak{L}(\mathfrak{P})$ for $\mathfrak{L}$ restricted to $\mathfrak{P}$. Similar notation will be used for other classes of formulas. **p**,**q**,**r**,.. will range over *finite* sets of propositional variables.

The substitution operator A[p:=B] ("substitute B for all occurrences of p in A") is defined as usual. PV(A) is the set of of propositional variables occurring in A.

We define a measure of complexity ρ, which counts the left-nesting of $\rightarrow$, as follows:

- $\rho(p) := \rho(\bot) := \rho(\top) := 0$
- $\rho(A\wedge B) := \rho(A\vee B) := \max(\rho(A),\rho(B))$
- $\rho(A\rightarrow B) := \max(\rho(A)+1,\rho(B))$.

We define $\Pi_n{:=}\{A\in\mathfrak{L} \mid \rho(A)\leq n\}$. We will also use Π_n is the context of Predicate Logic in its usual sense. This overloading of notations is explained in section 2.1.

We will at some points confuse propositional formulas with their equivalence classes modulo IPC-provable equivalence. Under this confusion IPC becomes $\mathfrak{H}_{IPC}$, the free Heyting Algebra on $\aleph_0$ generators.

2.1 Remark: connection with Predicate Logic: Consider the language $\mathfrak{L}\mathfrak{R}$ of Predicate Logic with constant b, relation symbols =, $\leq$ and with infinitely many unary predicate symbols P,Q,R,... . We define the Π_n- and Σ_n-formulas of our language as follows:

- $\Pi_0 := \Sigma_0 :=$ all Boolean combinations of atomic formulas,
- Π_{n+1} and Σ_{n+1} are the smallest classes such that:
  - $\Sigma_n \subseteq \Pi_{n+1}$,
  - $\Pi_n \subseteq \Sigma_{n+1}$,
  - Π_{n+1} is closed under $\wedge$, $\vee$ and $\forall$,
  - Σ_{n+1} is closed under $\wedge$, $\vee$ and $\exists$,
  - If $A \in \Sigma_{n+1}$ and $B \in \Pi_{n+1}$, then $A \to B \in \Pi_{n+1}$,
  - If $A \in \Pi_{n+1}$ and $B \in \Sigma_{n+1}$, then $A \to B \in \Sigma_{n+1}$.

Let T be the theory in $\mathfrak{LK}$ consisting of:

- Classical Predicate Logic,
- The theory of identity for =,
- The theory of partial orders for ≤ with bottom element b,
- The persistence property: $(Px \wedge x \leq y) \to Py$, for all unary predicate symbols P.

We translate formulas of IPC into $\mathfrak{LK}$. I(A,x), the Kripke translation of A at x, is defined by:

- $I(p,x) := Px$, $I(\bot,x) := \bot$, $I(\top,x) := \top$,
- $I(A \wedge B,x) := I(A,x) \wedge I(B,x)$
- $I(A \vee B,x) := I(A,x) \vee I(B,x)$
- $I(A \to B,x) := \forall y \geq x(I(A,y) \to I(B,y))$

We have by a simple induction on A: $I(A,x) \in \Pi_{\rho(A)}$.

Clearly every Kripke model (see section 5) can be considered as a model of T and vice versa. We have: $k \models_{\mathbb{K}} A \Leftrightarrow \mathbb{K} \models I(A,\underline{k})$, where the last satisfaction relation is the one of Predicate Logic. One can show in analogy to a result of Johan van Benthem for modal logic, that every one-variable formula A(x) of $\mathfrak{LK}$, that is (i) persistent and (ii) preserved under (partial) bisimulations, is CPdC-provably equivalent to a formula I(B,x) for some $B \in \mathfrak{L}$. For a precise formulation of the result and a proof, see appendix A.

As sketched above our measure of complexity corresponds via the Kripke translation with *depth of quantifier alternations.* In modal logic there is a similar correspondence: the relevant measure of complexity is there *depth of box alternations.* (The alternations are w.r.t. polarity.) A striking difference between the modal and the intuitionistic case is that the $\Pi_n(\mathbf{p})$ of modal logic are generally infinite modulo provable equivalence, where, as we will see, the $\Pi_n(\mathbf{p})$ of IPC are finite modulo provable equivalence.

Note, finally, that there are alternative hierarchies both for modal logic and for IPC corresponding simply to depth of quantifiers. In the IPC one counts *depth of implications*; in modal logic one counts *depth of boxes*. For this notion the complexity classes restricted to **p** are finite modulo provable equivalence both in modal logic and in IPC. ❍

The subject in this paper is the class NNIL := Π_1.

As is easily seen every formula in Π_{n+1} is provably equivalent to a formula resulting from substituting Π_n-formulas in a NNIL-formula.

Consider a NNIL-fomula. Conjunctions and disjunctions in front of implications can clearly be removed using:

- $\vdash ((A \vee B) \to C) \leftrightarrow ((A \to C) \wedge (B \to C))$,
- $\vdash ((A \wedge B) \to C) \leftrightarrow (A \to (B \to C))$.

So NNIL coincides modulo provable equivalence with $NNIL_0$, the smallest class containing the propositional atoms, closed under conjunction and disjunction and:

- if $A \in NNIL_0$, then $(p \to A) \in NNIL_0$.

2.2 **Fact:** NNIL(**p**) is finite (modulo provable equivalence).

Proof: Each element of NNIL(**p**) can be rewritten as a conjunction of disjunctions of atoms and elements of the form: $p \to A$, where A is in NNIL(**p**/{p}). So the result follows immediately with induction of the cardinality of **p**. ❑

By our earlier observation that every $\Pi_{n+1}(\mathbf{p})$-formula can be obtained by substituting $\Pi_n(\mathbf{p})$-formulas in a NNIL-formula, it follows by induction on n, that:

2.3 **Fact:** $\Pi_n(\mathbf{p})$ is finite (modulo provable equivalence). ❍

Since $\Pi_n(\mathbf{p})$ is finite, it is a finite distributive lattice under $\wedge$ and $\vee$. Hence it is also a Heyting Algebra, with implication, say: $\to_{n,\mathbf{p}}$. Note that $\to$ need not be $\to_{n,\mathbf{p}}$. We *do* have (for $A,B \in \Pi_n(\mathbf{p})$): $\vdash (A \to_{n,\mathbf{p}} B) \to (A \to B)$.

3 Closure and Coclosure Operations on the Heyting Algebra of IPC

Many important classes of formulas can be represented (modulo provable equivalence) as sets of fixed points of (co)closure operations on IPC considered as the free Heyting Algebra $\mathfrak{H}_{IPC}$ on $\aleph_0$ generators.

Note that if A in $\mathfrak{H}_{IPC}$ is generated by **p** and generated by **q**, then it is also generated by

$\mathbf{p \cap q}$. Thus the notation PV(A) makes sense: it denotes the minimum set of generators of A. In this section we will confuse formulas and their equivalence classes (or their standard interpretations in $\mathfrak{H}_{IPC}$). We should, however, remember that PV(A) for A qua formula is not necessarily identical to PV(A) for A qua equivalence class.

$\Phi:\mathfrak{H}_{IPC}\rightarrow\mathfrak{H}_{IPC}$ is a *closure operation* if:

- $A \leq \Phi(A)$ (Φ is inductive)
- $A \leq B \Rightarrow \Phi(A) \leq \Phi(B)$ (Φ is monotonic)
- $\Phi(A) = \Phi(\Phi(A))$ (Φ is idempotent)

Ψ is a *coclosure operation* if Ψ is monotonic, idempotent and has the following property:

- $\Psi(A) \leq A$ (Ψ is coinductive)

Note that if a set X is the image of a closure operation Φ at all, then Φ is completely determined by X, since: $\Phi(A) = \mathrm{Min}(\{B \in X \mid A \leq B\})$. Thus $\Phi(A)$ is the *smallest upper X-approximation* of A. Similarly for coclosure operations Ψ we have: $\Psi(A) = \mathrm{Max}(\{B \in X \mid B \leq A\})$, *the greatest lower X-approximation* of A. We will sometimes write Φ_X and X_Φ to express the dependence of X and a closure operation Φ. We will use Ψ_X for a corresponding coclosure operation.

3.1 Fact: Suppose X is finite (modulo provable equivalence) and closed under conjunction (disjunction), then Φ_X (Ψ_X) exists.

Proof: Trivial. ❑

We call Θ *Propositional Variable Preserving* or *PVP* if:

- $PV(\Theta(A)) \subseteq PV(A)$.

We state a simple sufficient condition for (co)closure operations to be PVP.

3.2 Fact: Suppose that Θ is a (co)closure operation and that X_Θ is closed under permutations of propositional variables. Then Θ is PVP.

Proof: As is easily seen it follows that for any permutation σ of PV: $\Theta(\sigma A)=\sigma\Theta(A)$. Now suppose $\vdash \Theta(A) \leftrightarrow B(q)$ for some q not in A. Choose r outside of $PV(A) \cup PV(\Theta(A))$. Suppose σ interchanges precisely q and r. Then:

$$\vdash B(q) \leftrightarrow \Theta(A) \leftrightarrow \Theta(\sigma A) \leftrightarrow \sigma\Theta(A) \leftrightarrow B(r).$$

Hence: $\vdash B(q) \leftrightarrow B(\top)$. So q does not occur essentially in $\Theta(A)$. ❑

X is called *PV-finite* if for each $\mathbf{p}$ $X(\mathbf{p})$ is finite (modulo provable equivalence). A (co)closure operation is called PV-finite if its image is.

A well known example of a PV-finite, PVP closure operation is double negation $\neg\neg$. $X_{\neg\neg}$ is the class of stable formulas.

A major discovery on IPC is the result by A. Pitts (Pitts' Uniform Interpolation Theorem, see Pitts[92]), that $\mathfrak{L}(\mathbf{p})$ is the image both of a PVP closure operation, say, $\mathfrak{E}\mathbf{p}$ and a PVP coclosure operation, say, $\mathfrak{A}\mathbf{p}$. Equivalently Pitts' result can be viewed as providing a PVP closure operation $\exists\mathbf{p}$ and a PVP coclosure operation $\forall\mathbf{p}$ with $\mathfrak{L}(PV/\mathbf{p})$ as image. (We leave the simple proof of the equivalence to the reader.)

It is well known that if Φ is a closure operation, then X_Φ is closed under $\wedge$. Similarly if Ψ is a coclosure operation, then X_Ψ is closed under $\vee$.

3.3 **Fact:** If Φ is a PVP closure operation, then X_Φ is closed under $\mathfrak{A}\mathbf{p}$. Similarly if Φ is a PVP coclosure operation, then X_Φ is closed under $\mathfrak{E}\mathbf{p}$.

Proof: Let Φ be a PVP closure operation and $X:=X_\Phi$. Suppose $A\in X$. We have: $\mathfrak{A}\mathbf{p}A\leq A$ and hence $\Phi(\mathfrak{A}\mathbf{p}A)\leq\Phi(A)=A$. Since Φ is PVP: $\Phi(\mathfrak{A}\mathbf{p}A)\in\mathfrak{L}(\mathbf{p})$. Ergo: $\Phi(\mathfrak{A}\mathbf{p}A) = \mathfrak{A}\mathbf{p}\Phi(\mathfrak{A}\mathbf{p}A) \leq \mathfrak{A}\mathbf{p}A$. On the other hand: $\mathfrak{A}\mathbf{p}A\leq\Phi(\mathfrak{A}\mathbf{p}A)$. Hence $\mathfrak{A}\mathbf{p}A = \Phi(\mathfrak{A}\mathbf{p}A)$. ❑

It follows for example that $\mathfrak{A}\mathbf{p}\neg\neg A$ is stable.

3.4 **Fact:** Consider any set subset X of $\mathfrak{H}_{IPC}$. Then X is the image of a PVP closure operator iff each $X(\mathbf{p})$ is the image of a PVP closure operator. Similarly for coclosure.

Proof: Suppose X is the image of the PVP closure operation Φ. It is easily seen that $\Phi\circ\mathfrak{E}\mathbf{p}$ is a PVP-closure operation with image $X(\mathbf{p})$. E.g. by the fact that Φ is PVP: $\Phi\circ\mathfrak{E}\mathbf{p}\circ\Phi\circ\mathfrak{E}\mathbf{p} = \Phi\circ\Phi\circ\mathfrak{E}\mathbf{p} = \Phi\circ\mathfrak{E}\mathbf{p}$.

Conversely suppose that for each $\mathbf{p}$ $\Phi_\mathbf{p}$ is a PVP closure operation with image $X(\mathbf{p})$. Note that $\Phi_\mathbf{q}(A)\in X(\mathbf{q}\cap PV(A))$. Consider any A and any $\mathbf{q}\supseteq PV(A)$. We claim: $\Phi_\mathbf{q}(A) = \Phi_{PV(A)}(A)$. First $\Phi_\mathbf{q}(A)\in X(PV(A))$ and $A\leq\Phi_\mathbf{q}(A)$, hence:

$$\Phi_{PV(A)}(A) \leq \Phi_{PV(A)}(\Phi_\mathbf{q}(A)) = \Phi_\mathbf{q}(A).$$

Conversely $PV(\Phi_{PV(A)}(A)) \subseteq PV(A) \subseteq \mathbf{q}$ and $A \leq \Phi_{PV(A)}(A)$, hence: $\Phi_{\mathbf{q}}(A) \leq \Phi_{PV(A)}(A)$.

Set: $\Phi(A) := \Phi_{PV(A)}(A)$. Clearly $A \in X \Leftrightarrow A \in X(PV(A)) \Leftrightarrow \Phi_{PV(A)}(A) = A$. We claim that Φ is a PVP closure operator. We have:

i) $A \leq \Phi_{PV(A)}(A)$

ii) Let $\mathbf{p} \supseteq PV(A) \cup PV(B)$, then: $A \leq B \Rightarrow \Phi_{\mathbf{p}}(A) \leq \Phi_{\mathbf{p}}(B) \Rightarrow \Phi_{PV(A)}(A) \leq \Phi_{PV(B)}(B)$

iii) Clearly $PV(\Phi_{PV(A)}(A)) \subseteq PV(A)$. We have:
$\Phi_{PV(A)}(A) = \Phi_{PV(A)}(\Phi_{PV(A)}(A)) = \Phi(\Phi_{PV(A)}(A))$. ❑

It follows that if X is PV-finite and closed under conjunction, then X is the image of a PVP-closure operation if each $\Phi_{X(\mathbf{p})}$ is PVP.

3.5 Fact: Suppose X is the image of a PVP closure operation Φ and X is closed under $\mathfrak{E}\mathbf{p}$, then $\Phi \circ \mathfrak{E}\mathbf{p} = \mathfrak{E}\mathbf{p} \circ \Phi$.

Proof: Left to the reader. ❑

We introduce some concepts and give their connections with (co)closure operations.

Define:

- X satisfies *left-interpolation* (IPL) if for every $A \in X$ and $B \in \mathfrak{L}$, there is an I in X such that $\vdash A \to I$ and $\vdash I \to B$ and $PV(I) \subseteq PV(A) \cap PV(B)$. I is called the X left-interpolant.
- X satisfies *right-interpolation* (IPR) if for every $A \in \mathfrak{L}$ and $B \in X$, there is an I in X such that $\vdash A \to I$ and $\vdash I \to B$ and $PV(I) \subseteq PV(A) \cap PV(B)$. I is called the X left-interpolant.
- X satisfies *uniform left-interpolation* (UIPL) if for every B and every **p**, there is a formula $B^*(\mathbf{p}) \in X(\mathbf{p})$, such that for all $A \in X$ satisfying $PV(A) \cap PV(B) \subseteq \mathbf{p}$ we have: $\vdash A \to B \Leftrightarrow \vdash A \to B^*(\mathbf{p})$. We call $B^*(\mathbf{p})$ the uniform X left-interpolant of B.
- X satisfies *uniform right-interpolation* (UIPR) if for every A and every **p**, there is a formula $A^\circ(\mathbf{p}) \in X(\mathbf{p})$, such that for all $B \in X$ satisfying $PV(A) \cap PV(B) \subseteq \mathbf{p}$ we have: $\vdash A \to B \Leftrightarrow \vdash A^\circ(\mathbf{p}) \to B$. We call $A^\circ(\mathbf{p})$ the uniform X right-interpolant of A.
- X satisfies *left-approximation* (APL) if for every B, there is a formula $B^* \in X$, such that for all $A \in X$: $\vdash A \to B \Leftrightarrow \vdash A \to B^*$.
- X satisfies *right-approximation* (APR) if for every A, there is a formula $A^\circ \in X$, such that for all $B \in X$: $\vdash A \to B \Leftrightarrow \vdash A^\circ \to B$.

3.6 Theorem: Suppose X satisfies IPR, X is PV-finite and X is closed under conjunction, then X is the image of a PVP-closure operation. Similarly for IPL, disjunction and coclosure.

Proof: Consider A in X. Consider A and let $\mathbf{p}$:=PV_0(A) and $\mathbf{p}\subseteq\mathbf{q}$. We have:

$$A_0 := \Phi_{X(\mathbf{p})}(A) = \bigwedge\{B\in X(\mathbf{p}) \mid \vdash A\rightarrow B\},$$

Clearly $A_0\in X(\mathbf{p})$. Now consider any $C\in X$ with $\vdash A\rightarrow C$. By right-interpolation there is a $B\in X(\mathbf{p})$, such that $\vdash A\rightarrow B$ and $\vdash B\rightarrow C$. Hence $\vdash A_0\rightarrow B$ and so $\vdash A_0\rightarrow C$. So we can put: $\Phi_X(A) := \Phi_{X(\mathbf{p})}(A)$ for $\mathbf{p}$:=PV(A). ❑

3.7 Theorem: X satisfies APR(L) iff X is the image of a (co)closure operation.

Proof: Trivial. ❑

3.8 Theorem: The following are equivalent:

(i) X is the image of a PVP (co)closure operation,

(ii) X satisfies UIPR(L).

(iii) X satisfies IPR(L) and APR(L)

Proof

(i)⇒(ii). This is theorem 3.3.

(ii)⇒(iii). Clearly UIPR(L) implies IPR(L). To get e.g. APR from UIPR, take: A°:=A°(PV(A)).

(iii)⇒(i). Suppose e.g. IPR and APR. Define $\Phi(A)$:=A°. Clearly Φ is a closure operation. We have IPC$\vdash A\rightarrow A^\circ$ and $A^\circ\in X$. So by IPR there is a $B\in X(PV(A))$ with IPC$\vdash A\rightarrow B$ and IPC$\vdash B\rightarrow A^\circ$. It follows that IPC$\vdash B\leftrightarrow A^\circ$. Hence Φ is PVP. ❑

4 Interpolation, the proof theoretic approach: In this section we prove that NNIL satisfies both left- and right-interpolation by a proof-theoretic argument. The proof consists in constructing the interpolant I from a (cut-free) proof of A$\vdash$B in a sequent calculus system. By the results of section 3 it follows that NNIL also satisfies uniform left- and right interpolation, since NNIL is PV-finite and closed under disjunction and conjunction.

4.1 Positive and negative occurrences: We define $PV^+(A)$ [$PV^-(A)$], the set of all positively [negatively] occurring propositional variables in A, by:

- $PV^+(\top) = PV^-(\top) = PV^+(\bot) = PV^-(\bot) = \emptyset$
- $PV^+(p) = \{p\}$, $PV^-(p) = \emptyset$
- $PV^+(A\wedge B) = PV^+(A\vee B) = PV^+(A) \cup PV^+(B)$
- $PV^-(A\wedge B) = PV^-(A\vee B) = PV^-(A) \cup PV^-(B)$
- $PV^+(A\to B) = PV^-(B\to A) = PV^-(A) \cup PV^+(B)$

Clearly $PV(A) = PV^+(A) \cup PV^-(A)$.

4.2 The derivation system: We use the following sequent calculus:

(p) $\quad \Gamma,p \vdash p$

($\top$) $\quad \Gamma \vdash \top$

($\bot$) $\quad \Gamma,\bot \vdash C$

($\wedge$R) $$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A\wedge B}$$

($\wedge$L) $$\frac{\Gamma,A,B \vdash C}{\Gamma,A\wedge B \vdash C}$$

($\vee$R) $$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1\vee A_2} \quad (i=1,2)$$

($\vee$L) $$\frac{\Gamma,A \vdash C \qquad \Gamma,B \vdash C}{\Gamma,A\vee B \vdash C}$$

($\to$R) $$\frac{\Gamma,A \vdash B}{\Gamma \vdash A\to B}$$

($\to$L) $$\frac{\Gamma \vdash A \qquad \Gamma,B \vdash C}{\Gamma,A\to B \vdash C}$$

Related systems can be found in Schütte[62], Takeuti[75]. All these systems are equivalent in the sense that they yield the same class of derivable formulas and that they satisfy the following properties, to be used later:

CE cut elimination: if $\Gamma \vdash A$ and $\Gamma,A \vdash B$ then $\Gamma \vdash B$

W weakening: if $\Gamma \vdash A$ then $\Gamma,\Delta \vdash A$

S substitution: if $\Gamma \vdash A$ then $\Gamma[p:=B] \vdash A[p:=B]$

PS positive substitution: if $p \notin PV^-(A)$ then $A,(p\to B) \vdash A[p:=B]$

The proofs are standard.

4.3 Schütte's interpolation method: Schütte gives in Schütte[62] a method to build an interpolant from a derivation of $A\vdash B$. This method yields for every derivable sequent $\Gamma,\Delta\vdash C$ an interpolant I satisfying:

$\Gamma\vdash I$, $\Delta,I\vdash C$ and $PV(I)\subseteq PV(\Gamma)\cap PV(\Delta,C)$.

Using the shorthand $\Gamma[I]\Delta \vdash C$ for ($\Gamma \vdash I$ and $\Delta, I \vdash C$), Schütte's method can be rendered as follows:

(ip1) $\Gamma[\top]\Delta, p \vdash p$ (ip2) $\Gamma, p\ [p]\ \Delta \vdash p$

(i$\top$) $\Gamma[\top]\Delta \vdash \top$

(i$\bot$1) $\Gamma[\top]\Delta, \bot \vdash C$ (i$\bot$2) $\Gamma, \bot\ [\bot]\ \Delta \vdash C$

(i$\wedge$R) $$\frac{\Gamma[I_1]\Delta \vdash A \qquad \Gamma[I_2]\Delta \vdash B}{\Gamma[I_1 \wedge I_2]\Delta \vdash A \wedge B}$$

(i$\vee$L1) $$\frac{\Gamma\ [I_1]\ A, \Delta \vdash C \qquad \Gamma\ [I_2]\ B, \Delta \vdash C}{\Gamma\ [I_1 \wedge I_2]\ A \vee B, \Delta \vdash C}$$

(i$\vee$L2) $$\frac{\Gamma, A\ [I_1]\ \Delta \vdash C \qquad \Gamma, B\ [I_2]\ \Delta \vdash C}{\Gamma, A \vee B\ [I_1 \vee I_2]\ \Delta \vdash C}$$

(i$\to$L1) $$\frac{\Gamma[I_1]\Delta \vdash A \qquad \Gamma\ [I_2]\ B, \Delta \vdash C}{\Gamma\ [I_1 \wedge I_2]\ A \to B, \Delta \vdash C}$$

(i$\to$L2) $$\frac{\Delta\ [I_1]\ \Gamma \vdash A \qquad \Gamma, B\ [I_2]\ \Delta \vdash C}{\Gamma, A \to B\ [I_1 \to I_2]\ \Delta \vdash C}$$

We explain this notation with an example. (i$\wedge$R) means:

if $\Gamma \vdash I_1$ and $I_1, \Delta \vdash A$ and $\Gamma \vdash I_2$ and $I_2, \Delta \vdash B$,

then $\Gamma \vdash I_1 \wedge I_2$ and $I_1 \wedge I_2, \Delta \vdash A \wedge B$.

So (i$\wedge$R) indicates how an interpolant for $\Gamma, \Delta \vdash A \wedge B$ can be obtained from interpolants for $\Gamma, \Delta \vdash A$ and $\Gamma, \Delta \vdash B$. For rules not mentioned here (($\wedge$L), ($\vee$R), ($\to$R), ($\neg$R)), the interpolant for the conclusion is the same as for the premise.

Now the Interpolation theorem is proved as follows. Assume $A \vdash B$, then there is a derivation of $A \vdash B$ in the sequent calculus defined in 4.2. With induction over the length of the derivation it is shown (using Schütte's method) that any partition $\Gamma, \Delta \vdash C$ of a sequent in the derivation has an interpolant I, i.e.:

$\Gamma \vdash I$, $I, \Delta \vdash C$ and $PV(I) \subseteq PV(\Gamma) \cap PV(\Delta, A)$.

Hence $A \vdash B$ has an interpolant.

Applying Schütte's method to derivations of $A \vdash B$ with $A \in$ NNIL does not always yield an $I \in$ NNIL:

$$\dfrac{\dfrac{\dfrac{\dfrac{p[p]\vdash p \qquad q[q]p\vdash q}{p\to q\,[p\to q]\,p\vdash q}\,(\to L) \qquad p,r\,[R]\,p\to q\vdash r}{p,\,q\to r\,[(p\to q)\to r]\,p\to q\vdash r}\,(\to L)}{p\wedge(q\to r)\,[(p\to q)\to r]\,p\to q\vdash r}\,(\wedge L)}{p\wedge(q\to r)\,[(p\to q)\to r]\vdash (p\to q)\to r}\,(\to R)$$

It turns out that (i→L2), the only place where an → is added to I, has to be modified. This will be done in the next section.

4.4 **The proof**: We first prove NNIL left-interpolation and then NNIL right-interpolation.

4.4.1 Lemma:. Assume $\Gamma,\Delta \vdash C$ and $\Gamma \subseteq \mathrm{NNIL}_0$. Then there is an I with:

i) $\Gamma \vdash I$ and $I,\Delta \vdash C$,

ii) $PV(I) \subseteq PV(\Gamma) \cap PV(\Delta, C)$,

iii) $I \in$ NNIL;

iv) $\{C\} \cap PV^-(I) \subseteq PV(\Delta)$ (i.e. if $C \in PV^-(I)$, then $C \in PV(\Delta)$).

Proof: Induction over the length of a derivation of $\Gamma,\Delta \vdash C$.

$\Gamma,\Delta \vdash C$ is an axiom or the conclusion of a rule different from (→L): apply Schütte's method (4.3). (i)-(iv) follow directly by induction.

$\Gamma,\Delta \vdash C$ is the conclusion of (→L). Let $A \to B$ be the 'new' formula in the conclusion. We distinguish two cases: $A \to B \in \Gamma$ or $A \to B \in \Delta$.

Case 1: $A \to B \in \Gamma$. Then $A \to B \in$ NNIL, so A is a propositional variable, p say. By the induction hypothesis, we have a Γ' with $\Gamma' \cup \{p \to B\} = \Gamma$ and I_1, I_2 with

a) $\Gamma' \vdash I_1$; $I_1,\Delta \vdash p$; $\Gamma',B \vdash I_2$; $I_2,\Delta \vdash C$

b) $PV(I_1) \subseteq PV(\Gamma') \cap PV(\Delta,p)$; $PV(I_2) \subseteq PV(\Gamma',B) \cap PV(\Delta,C)$

c) $I_1,I_2 \in$ NNIL

d) $p \in PV^-(I_1) \Rightarrow p \in PV(\Delta)$; $\{C\} \cap PV^-(I_2) \subseteq PV(\Delta)$.

Now we must find an I and show that (i)-(iv) hold. We consider three subcases.

Subcase IA: $C = p$. Put $I := I_1$.
i) $\Gamma \vdash I$ follows from $\Gamma' \vdash I_1$ (by (a)), $\Gamma' \subseteq \Gamma$ and (W). We get.(ii), (iii) and (iv): directly from (b), (c) and (d).

Subcase 1B: $C \neq p$, $p \in PV(\Delta)$. Put $I := I_1 \wedge (p \rightarrow I_2)$.

i)

$$\dfrac{\Gamma', p \rightarrow B \vdash I_1 \ (a,W) \qquad \dfrac{\dfrac{\Gamma', p \vdash p \qquad \Gamma', p, B \vdash I_2 \ (a,W)}{\Gamma', p \rightarrow B, p \vdash I_2}}{\Gamma', p \rightarrow B \vdash p \rightarrow I_2}}{\Gamma', p \rightarrow B \vdash I_1 \wedge (p \rightarrow I_2) \quad \text{i.e. } \Gamma \vdash I;}$$

$$\dfrac{\dfrac{I_1, \Delta \vdash p \ (a) \qquad I_1, I_2, \Delta \vdash C \ (a,W)}{I_1, p \rightarrow I_2, \Delta \vdash C}}{I_1 \wedge (p \rightarrow I_2), \Delta \vdash C \quad \text{i.e. } I, \Delta \vdash C}$$

ii) $PV(I_1, I_2) \subseteq PV(\Gamma) \cap PV(\Delta, C)$ is easy; as $\Gamma = \Gamma' \cup \{p \rightarrow B\}$ and $p \in PV(\Delta)$ we also have $p \in PV(\Gamma) \cap PV(\Delta, C)$, so $PV(I) = PV(I_1) \cup PV(I_2) \cup \{p\} \subseteq PV(\Gamma) \cap PV(\Delta, C)$.

iii) $I_1, I_2 \in NNIL$ (by (c)), so $I = I_1 \wedge (p \rightarrow I_2) \in NNIL$ by definition of NNIL.

iv) Assume $C \in PV^-(I)$, then $C \in PV^-(I_1) \cup PV^-(I_2)$, for $C \neq p$. Now if $C \in PV^-(I_1)$ then $C \in PV(\Delta) \cup \{p\}$ by (b), so $C \in PV(\Delta)$ (for $C \neq p$); and if $C \in PV^-(I_2)$ then $C \in PV(\Delta)$ by (d). Conclusion: $C \in PV(\Delta)$ and (iv) is proved.

Subcase 1C: $C \neq p$, $p \notin PV(\Delta)$. Put $I := I_1[p := I_2]$.

i) As in Subcase 1B, we have $\Gamma', p \rightarrow B \vdash I_1 \wedge (p \rightarrow I_2)$; by (d) and $p \notin PV(\Delta)$ we have $p \notin PV^-(I_1)$, so $I_1, (p \rightarrow I_2) \vdash I$ by (PS); now apply (CE) and we get $\Gamma', p \rightarrow B \vdash I$, i.e. $\Gamma \vdash I$. Furthermore:

$$\dfrac{\dfrac{I_1, \Delta \vdash p}{I_1[p := I_2], \Delta \vdash I_2 \ \ (p \notin PV(\Delta))}\,(S) \qquad I_2, \Delta \vdash C}{I_1[p := I_2], \Delta \vdash C \text{ i.e. } I, \Delta \vdash C.}\,(CE)$$

ii)

$$\begin{aligned} PV(I) &= (PV(I_1)/\{p\}) \cup PV(I_2) \\ &\subseteq (PV(\Gamma') \cap PV(\Delta, p)/\{p\}) \cup (PV(\Gamma', B) \cap PV(\Delta, C)) \\ &= PV(\Gamma', B) \cap PV(\Delta, C) \subseteq PV(\Gamma) \cap PV(\Delta, C), \end{aligned}$$

using (b) and $p \notin PV(\Delta)$.

iii) $I \in NNIL$ follows from (c) and $p \notin PV^-(I_1)$, a consequence of (d) and $p \notin PV(\Delta)$.

iv) Assume $C \in PV$ and $C \in PV^-(I)$, then $C \in PV^-(I_1) \cup PV^-(I_2)$ (for $p \notin PV^-(I_1)$, see (i)). Now continue as for (iv) under 1B.

Case 2: $A \rightarrow B \in \Delta$. Apply ($i{\rightarrow}L1$) of Schütte's method: it yields an interpolant $I = I_1 \wedge I_2$ and (i) - (iv) follow directly. ❑

As a corollary, we immediately have NNIL left-interpolation (using that NNIL and $NNIL_0$ coincide modulo IPC-provable equivalence). Right-interpolation is somewhat easier. We prove it now.

4.4.2 Lemma: If $\Gamma,\Delta \vdash C$, then there is an I with

i) $\Gamma \vdash I$; $I,\Delta \vdash C$;

ii) $PV(I) \subseteq PV(\Gamma) \cap PV(\Delta,C)$;

iii) if $C \in NNIL$ and $\Delta \subseteq PV$ then $I \in NNIL$;

iv) if $\Gamma \subseteq PV$ then $I = p_1 \wedge \ldots \wedge p_n$ for some $p_1,\ldots,p_n \in PV$.

Proof: Induction over the length of a derivation of $\Gamma,\Delta \vdash C$.

If $\Gamma,\Delta \vdash C$ is an axiom or the conclusion of a rule different from ($\rightarrow$L), then apply Schütte's method. If $\Gamma,\Delta \vdash C$ is the conclusion of ($\rightarrow$L), then we distinguish three cases.

Case 1: $A \rightarrow B \in \Gamma$, $C \in NNIL$, $\Delta \subseteq PV$. Here ($i{\rightarrow}L2$) of Schütte's method prescribes the interpolant $I_1 \rightarrow I_2$. This interpolant satisfies (i) and (ii), but in general not (iii) (only if $I_1 \in PV$). However, I_1 is the interpolant of $\Delta,\Gamma' \vdash A$ (with Γ' such that $\Gamma = \Gamma' \cup \{A \rightarrow B\}$) and $\Delta \subseteq PV$, so (by (iv) of the induction hypothesis) $I = p_1 \wedge \ldots \wedge p_n$. Now put $I := p_1 \rightarrow (\ldots(p_n \rightarrow I_2)\ldots)$, then $I \equiv I_1 \rightarrow I_2$ so I satisfies (i) and (ii); also $I \in NNIL$ for $I_2 \in NNIL$ (by induction hypothesis). (iv) is trivially satisfied.

Case 2: $A \rightarrow B \in \Gamma$, ($C \notin NNIL$ or $\Delta/PV \neq \emptyset$). Now follow ($I{\rightarrow}L2$) of Schütte's method, then (i), (ii) are satisfied, (iii) and (iv) are trivially true.

Case 3: $A \rightarrow B \notin \Gamma$. Then $A \rightarrow B \in \Delta$. Now follow ($i{\rightarrow}L1$) of Schütte's method: this yields an interpolant $I = I_1 \wedge I_2$ for which (i), (ii) hold. (iii) is trivially true (for $A \rightarrow B \in \Delta$) and (iv) follows by the induction hypothesis. ❑

As a corollary we have NNIL right-interpolation.

4.4.3 Positive and negative occurrence: Schütte's method yields an interpolant I for $A \vdash B$ with:

(±) $PV^+(I) \subseteq PV^+(A) \cap PV^+(B)$, $PV^-(I) \subseteq PV^-(A) \cap PV^-(B)$.

However, our adaptation of Schütte's method used in 4.3.1 does not respect (±): e.g., in subcase 1B we have $p \in PV^-(I)$, but $p \in PV^+(\Delta) \cup PV^-(C)$ is not excluded. We therefore state the following open problem: does NNIL interpolation hold if (±) is added? ❍

By the results of section 3 we may conclude that NNIL satisfies also uniform right- and left-interpolation and right- and left-approximation.

In the next two sections we will introduce the notions and prove the lemmas necessary for the following results. We give a Kripke model proof of the fact that NNIL satisfies uniform interpolation and we show that a formula is in NNIL if and only if it is robust in some appropriate sense.

5 Kripke models: We suppose that the reader is familiar with Kripke models for IPC (see TV[88a], or Smoryński[73]). To fix notations: a Kripke model is a structure $\mathbb{K} = \langle K, b, \leq, \mathfrak{P}, \models \rangle$, where K is a non-empty set of nodes; $\leq$ is a partial ordering; $b \in K$ is the bottom element w.r.t. $\leq$; $\mathfrak{P}$ is a set of propositional variables; $\models$ is the atomic forcing relation on $\mathfrak{P}$: it is a relation between nodes and propositional variables in $\mathfrak{P}$, satisfying:

$k \leq k'$ and $k \models p \Rightarrow k' \models p$ (persistence).

$\models$ is extended to $\mathfrak{L}(\mathfrak{P})$ in the standard way. The resulting relation is again persistent. We will say that $\mathbb{K}$ is a *$\mathfrak{P}$-model* if its set of propositional variables is $\mathfrak{P}$. A model is *finite* if all its components are finite.

Our Kripke models are what is usually called *rooted Kripke models*. In many contexts it is more natural to omit the root. However, for the purposes of the present paper it is more convenient to have all our models rooted.

We write $\mathbb{K} \models A$ for: $b \models A$ (or equivalently: $\forall k \in K\ k \models A$).

Let **p** be a finite set of propositional variables. We will write $\mathbb{K}(\mathbf{p})$ for the result of restricting the atomic forcing of $\mathbb{K}$ to **p**.

For any $k \in K$ $\mathbb{K}[k]$ is the model $\langle K', k, \leq', \mathfrak{P}, \models' \rangle$, where $\mathbb{K}' := \{k' | k \leq k'\}$ and where $\leq'$ and $\models'$ are the restrictions of $\leq$ respectively $\models$ to K'. (We will often simply write $\leq$ and $\models$ for $\leq'$ and $\models'$.)

We write $\mathrm{Th}(\mathbb{K}) := \{A\in\mathfrak{L} \mid \mathbb{K}\models A\}$, $\mathrm{Th}_X(\mathbb{K}) := \{A\in X \mid \mathbb{K}\models A\}$, where X is a set of formulas. We will often write $\mathrm{Th}_X(k)$ for $\mathrm{Th}_X(\mathbb{K}[k])$.

The central result connecting structures and language is:

5.1 Kripke Model Completeness Theorem: We have:
$\mathrm{IPC}\vdash A \Leftrightarrow$ For all (finite) PV(A)-models $\mathbb{K}$ $\mathbb{K}\models A$.

6 Subsimulations: We start by repeating some usual notions about relations. Define:

- $x(R\circ S)y :\Leftrightarrow \exists z\, xRzSy$,
- $xR^{\wedge}y :\Leftrightarrow yRx$,
- $\mathrm{ID}_X\subseteq X\times X$ and $x\mathrm{ID}_X y :\Leftrightarrow x=y$, i.e. $\mathrm{ID}_X=\{\langle x,x\rangle \mid x\in X\}$.

Let $\mathbb{K}$ an $\mathbb{M}$ be $\mathfrak{P}$-models. A relation R on K×M has *the zig-property (w.r.t $\mathbb{K}$ and $\mathbb{M}$)* if:

- $kRm \Rightarrow \forall p\in\mathfrak{P}\, (k\models p \Leftrightarrow m\models p)$,
- $k'\geq kRm \Rightarrow \exists m'\, k'Rm'\geq m$, i.e. $\geq\circ R \subseteq R\circ\geq$.

We will say that R is *zig*. We do *not* require that R preserves roots, i.e. $b_{\mathbb{K}}Rb_{\mathbb{M}}$. We will also call R a *subsimulation of $\mathbb{K}$ in $\mathbb{M}$.*. The 'sub' witnesses that roots are not necessarily preserved. Note that the empty relation is a subsimulation between any two models.

R is total if $\forall k\in K\, \exists m\in M\, kRm$. If R is zig and root-preserving we say that R is *+-zig*. We will say that a +-zig R is a *simulation.* Note that simulations are in our context automatically total.

R is *zag* if $R^{\wedge}$ is zig, etcetera.

Define:

- $R:\mathbb{K}\preccurlyeq\mathbb{M} :\Leftrightarrow$ R is a total subsimulation of $\mathbb{K}$ in $\mathbb{M}$,
- $\mathbb{K}\preccurlyeq\mathbb{M} :\Leftrightarrow \exists R\, R:\mathbb{K}\preccurlyeq\mathbb{M}$,
- $R:\mathbb{K}\preccurlyeq^{+}\mathbb{M} :\Leftrightarrow$ R is a simulation of $\mathbb{K}$ in $\mathbb{M}$,
- $\mathbb{K}\preccurlyeq^{+}\mathbb{M} :\Leftrightarrow \exists R\, R:\mathbb{K}\preccurlyeq^{+}\mathbb{M}$.

The existence of simulations and the existence of total subsimulations is related in essentially simple ways. It is good to have total subsimulations since they are 'definable' in a way that simulations are not (see section 7). It is good to have simulations, since they are better for constructions on models. We give two connections between the two notions. First note that :

$\mathbb{K}\preccurlyeq\mathbb{M} \Rightarrow$ for all $k\in K$ there is an $m\in M$: $\mathbb{K}[k]\preccurlyeq^{+}\mathbb{M}[m]$.

Moreover:

$\mathbb{K}\preccurlyeq\mathbb{M} \Leftrightarrow$ for some $m\in M$: $\mathbb{K}\preccurlyeq^{+}\mathbb{M}[m]$.

Secondly take $\mathbb{K}^{+}$ to be the result of adding a new root $\mathfrak{b}$ to $\mathbb{K}$ with:

- $\mathfrak{b}\models p :\Leftrightarrow \mathbb{K}\models p$ and $\mathbb{M}\models p$.

We have:

$\mathbb{K}\preccurlyeq\mathbb{M} \Leftrightarrow \mathbb{K}^{+}\preccurlyeq^{+}\mathbb{M}$.

We leave the simple verifications to the reader.

We list the basic facts about (sub)simulations.

6.1 Fact: Let **P** be a set of subsimulations. Then $\bigcup$**P** is a subsimulation. It follows that the set of subsimulations has a maximum. Moreover if one of the elements of **P** is total (root-preserving), then $\bigcup$**P** is total (root-preserving).

Proof: trivial. ❑

Note that R defined by: $kRm :\Leftrightarrow \mathbb{K}[k]\preccurlyeq^{+}\mathbb{M}[m]$, is the maximum subsimulation.

6.2 Fact

$R{:}\mathbb{K}\preccurlyeq^{(+)}\mathbb{M}$ and $S{:}\mathbb{M}\preccurlyeq^{(+)}\mathbb{N} \Rightarrow R\circ S{:}\mathbb{K}\preccurlyeq^{(+)}\mathbb{N}$,

$ID_K{:}\mathbb{K}\preccurlyeq^{+}\mathbb{K}$.

Proof: Trivial. ❑

6.2 tells us that $\preccurlyeq^{(+)}$ is a preorder. We write $\equiv^{(+)}$ for the induced equivalence relation.

Note that if $\mathbb{K}\equiv\mathbb{M}$ and $R{:}\mathbb{K}\preccurlyeq\mathbb{M}$, then $R\cup\{\langle b_{\mathbb{K}},b_{\mathbb{M}}\rangle\}$ is a simulation. It follows that $\mathbb{K}\equiv^{+}\mathbb{M}$. So $\equiv$ and $\equiv^{+}$ coincide.

A relation R between $\mathbb{K}$ and $\mathbb{M}$ is a *bisimulation* if R and R^ are both subsimulations. If a bisimulation is total and surjective, we can always extend it to preserve roots. We write:

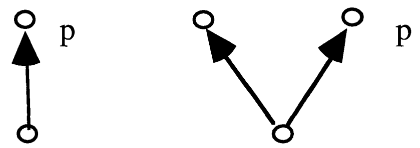
- $R{:}\mathbb{K}\simeq\mathbb{M}$ for: R is a total, surjective bisimulation between $\mathbb{K}$ and $\mathbb{M}$,
- $\mathbb{K}\simeq\mathbb{M} :\Leftrightarrow \exists R\; R{:}\mathbb{K}\simeq\mathbb{M}$.

It is easy to see that $\simeq$ is an equivalence relation and that: $\mathbb{K}\simeq\mathbb{M} \Rightarrow \mathbb{K}\equiv\mathbb{M}$. Bisimulations are closed under unions, so the set of bisimulations between $\mathbb{K}$ and $\mathbb{M}$ has a maximum. Note that R with:

- kRm $:\Leftrightarrow \mathbb{K}[k]\simeq\mathbb{M}[m]$,

is the maximal bisimulation between $\mathbb{K}$ and $\mathbb{M}$.

6.3 **Example:** The following is an example of two models $\mathbb{U}$ and $\mathbb{V}$ with $\mathbb{U}\equiv\mathbb{V}$ but not $\mathbb{U}\simeq\mathbb{V}$.



$\mathbb{U}$ $\mathbb{V}$

(If an atom is not displayed at a node, then it is not forced.)

In section 7 we will see that if we restrict ourselves to **p**-models, then the number of $\equiv$-equivalence classes is finite (in contrast to the number of $\simeq$-equivalence classes).

In the next two facts we relate total subsimulations and simulations with the behaviour of models on the formulas of IPC. A converse of 6.4 (for **p**-models) will be proved in section 7. (6.5 has a number of somewhat weakened converses, but we won't prove them in this paper.)

6.4 **Fact:** Let R be zig. Then: kRm $\Rightarrow Th_{NNIL}(m) \subseteq Th_{NNIL}(k)$.

It follows immediately that: $\mathbb{K}\preccurlyeq\mathbb{M} \Rightarrow Th_{NNIL}(\mathbb{M}) \subseteq Th_{NNIL}(\mathbb{K})$.

Proof: By induction on $NNIL_0$. E.g. suppose $kRm\models(p\rightarrow A)$ for $A\in NNIL_0$ and $k\leq k'\models p$. Then for some m': $k'Rm'\geq m$ and hence $m'\models p$. Since $m'\geq m\models(p\rightarrow A)$, it follows that $m'\models A$ and hence by the Induction Hypothesis: $k'\models A$. We may conclude that $k\models(p\rightarrow A)$. ❑

6.5 **Fact:** $\mathbb{K}\simeq\mathbb{M} \Rightarrow Th(\mathbb{K})=Th(\mathbb{M})$.

Proof: By induction on $\mathfrak{L}$. ❑

Define:

- $\mathbb{K}\preccurlyeq_1\mathbb{M}$:⇔ ∃F (F:$\mathbb{K}\preccurlyeq\mathbb{M}$ and F is a function)
- $\mathbb{K}\subseteq\mathbb{M}$:⇔ K⊆M, $\leq_{\mathbb{K}}\subseteq\leq_{\mathbb{M}}$ and $\models_{\mathbb{K}} = \models_{\mathbb{M}}\restriction K$,
- $\mathbb{K}\subseteq_{full}\mathbb{M}$:⇔ $\mathbb{K}\subseteq\mathbb{M}$, $\leq_{\mathbb{K}} = \leq_{\mathbb{M}} \cap(K\times K)$,
- $\mathbb{K}\subseteq_{ini}\mathbb{M}$:⇔ $\mathbb{K}\subseteq_{full}\mathbb{M}$ and for all $m\leq_{\mathbb{M}}m'\in K$: $m\in K$ or $m=b_{\mathbb{M}}$,

For all these notions we have the obvious +-versions, e.g. for $\mathbb{K}\subseteq^{+}\mathbb{M}$ we demand that $b_{\mathbb{K}}=b_{\mathbb{M}}$. Note:

- $\mathbb{K}\subseteq^{+}_{ini}\mathbb{M}$ ⇔ $\mathbb{K}\subseteq^{+}_{full}\mathbb{M}$ and for all $m\leq_{\mathbb{M}}m'\in K$: $m\in K$.

Note that: $\mathbb{K}\subseteq_{ini}\mathbb{M} \Rightarrow \mathbb{K}\subseteq_{full}\mathbb{M} \Rightarrow \mathbb{K}\subseteq\mathbb{M} \Rightarrow \mathbb{K}\preccurlyeq_1\mathbb{M} \Rightarrow \mathbb{K}\preccurlyeq\mathbb{M}$.

We will now prove the central result of the present section. It will give us (in section 7) both the desired results on robustness and on uniform interpolation.

6.7 Lifting Theorem: Let **q**, **p** and **r** be disjoint sets of variables. Let $\mathbb{K}$ be a **q**,**p**-model and let $\mathbb{M}$ be a **p**,**r**-model. Suppose $\mathbb{K}(\mathbf{p})\preccurlyeq^{(+)}\mathbb{M}(\mathbf{p})$. Then there are **q**,**p**,**r**-models $\mathbb{K}'$,$\mathbb{M}'$ such that: $\mathbb{K}\simeq\mathbb{K}'(\mathbf{q},\mathbf{p})$, $\mathbb{K}'\subseteq^{(+)}_{full}\mathbb{M}'$ and $\mathbb{M}\simeq\mathbb{M}'(\mathbf{p},\mathbf{r})$.

Proof: We give the proof for the +-case. (To drop the +, first extend $\mathbb{K}$ to $\mathbb{K}^{+}$ taking the forcing at the new bottom for the elements of **q** arbitrary within the bounds dictated by persistence. After the construction drop the extra bottom of $\mathbb{K}^{+}{}'(\mathbf{q},\mathbf{p})$ to obtain $\mathbb{K}'(\mathbf{q},\mathbf{p})$.)

Assume R:$\mathbb{K}(\mathbf{p})\preccurlyeq^{+}\mathbb{M}(\mathbf{p})$. We construct the promised new models. We first specify $\mathbb{K}'$. (We index the various relations to keep track where we are. Later we will omit these indices.) Define:

- K' := {⟨k,m⟩ | kRm}. (So K' is just R viewed as a set of pairs.),
- $\langle k,m\rangle \leq_{\mathbb{K}'} \langle k',m'\rangle$:⇔ $k\leq_{\mathbb{K}}k'$ and $m\leq_{\mathbb{M}}m'$,
- $b_{\mathbb{K}'} := \langle b_{\mathbb{K}},b_{\mathbb{M}}\rangle$,
- $\mathfrak{P}_{\mathbb{K}'}$:= **q**,**p**,**r**,
- $\langle k,m\rangle \models_{\mathbb{K}'} s$:⇔ $k\models_{\mathbb{K}}s$ or $m\models_{\mathbb{M}}s$.

Note that:

$s\in \mathbf{q},\mathbf{p} \Rightarrow (\langle k,m\rangle \models_{\mathbb{K}'} s \Leftrightarrow k\models_{\mathbb{K}}s)$,

$s\in \mathbf{p},\mathbf{r} \Rightarrow (\langle k,m\rangle \models_{\mathbb{K}'} s \Leftrightarrow m\models_{\mathbb{K}}s)$.

Define B by: kB⟨k',m⟩ :⇔ k=k'. It is immediate that B:$\mathbb{K}\simeq\mathbb{K}'(\mathbf{q},\mathbf{p})$.

We specify $\mathbb{M}'$.

- $M' := \{\langle k,m\rangle \mid \exists m'\in M\ kRm'\leq_{\mathbb{M}}m\}$,
- $\langle k,m\rangle \leq_{\mathbb{M}'} \langle k',m'\rangle :\Leftrightarrow k\leq_{\mathbb{K}}k'$ and $m\leq_{\mathbb{M}}m'$,
- $b_{\mathbb{M}'} := \langle b_{\mathbb{K}},b_{\mathbb{M}}\rangle$,
- $\mathfrak{P}_{\mathbb{M}'} := \mathbf{q},\mathbf{p},\mathbf{r}$,
- $\langle k,m\rangle \vDash_{\mathbb{M}'} s :\Leftrightarrow k\vDash_{\mathbb{K}}s$ or $m\vDash_{\mathbb{M}}s$.

Note that:

$q\in \mathbf{q} \Rightarrow (\langle k,m\rangle \vDash_{\mathbb{M}'} q \Leftrightarrow k\vDash_{\mathbb{K}}q)$,

$p\in \mathbf{p} \Rightarrow (k\vDash_{\mathbb{K}}p \Rightarrow \langle k,m\rangle \vDash_{\mathbb{M}'}p)$,

$s\in \mathbf{p},\mathbf{r} \Rightarrow (\langle k,m\rangle \vDash_{\mathbb{M}'} s \Leftrightarrow m\vDash_{\mathbb{M}}s)$,

$\langle k,m\rangle\in K'$ and $m\leq_{\mathbb{M}}m' \Rightarrow \langle k,m'\rangle\in K'$ and $\langle k,m\rangle\leq_{\mathbb{M}'}\langle k,m'\rangle$.

Define C by: $mC\langle k,m'\rangle :\Leftrightarrow m=m'$. It is easily seen that $C:\mathbb{M}\simeq\mathbb{M}'(\mathbf{p}.\mathbf{r})$.

Finally it is completely trivial that: $\mathbb{K}'\subseteq^{+}_{\text{full}}\mathbb{M}'$. ❑

Note that 6.7 goes through even in case some of **q**, **p** and **r** are infinite.

An immediate consequence of the proof is that for any **q**-model $\mathbb{K}$ and for any **r**-model $\mathbb{M}$ with **q** and **r** disjoint, there is a **q**,**r**-model $\mathbb{N}$ such that $\mathbb{K}\simeq\mathbb{N}(\mathbf{q})$ and $\mathbb{M}\simeq\mathbb{N}(\mathbf{r})$. (Take R in the proof the universal relation between K and M.)

6.8 **Corollary**: Suppose $\mathbb{K}$ and $\mathbb{M}$ are **p**-models. Then:

$\mathbb{K}\preccurlyeq^{(+)}\mathbb{M} \Leftrightarrow \exists\mathbb{K}',\mathbb{M}'\ \mathbb{K}\simeq\mathbb{K}'\subseteq^{(+)}_{\text{full}}\mathbb{M}'\simeq\mathbb{M}$.

Proof: "$\Rightarrow$" Is immediate from 6.7, taking $\mathbf{q}=\mathbf{r}=\emptyset$. "$\Leftarrow$" Suppose:

$\mathbb{K}\simeq\mathbb{K}'\subseteq^{(+)}_{\text{full}}\mathbb{M}'\simeq\mathbb{M}$.

It follows that: $\mathbb{K}\preccurlyeq^{(+)}\mathbb{K}'\preccurlyeq^{(+)}\mathbb{M}'\preccurlyeq^{(+)}\mathbb{M}$. Hence: $\mathbb{K}\preccurlyeq^{(+)}\mathbb{M}$. ❑

6.9 **Corollary**: Suppose $\mathbb{K}$ and $\mathbb{M}$ are **p**-models. Then:

$\mathbb{K}\preccurlyeq^{(+)}\mathbb{M} \Leftrightarrow \exists\mathbb{K}'\ \mathbb{K}\simeq\mathbb{K}'\preccurlyeq_{1}^{(+)}\mathbb{M}$.

Proof: Note, by inspecting the proof of 6.7, that in 6.8 the total, surjective bisimulation C between $\mathbb{M}'$ and $\mathbb{M}$ is in fact a function (and thus a p-morphism). ❑

We can improve our result to obtain models embedded via $\subseteq^{(+)}_{\text{ini}}$.

6.10 Strenghened Lifting Theorem: Let **q**, **p** and **r** be disjoint sets of variables. Let $\mathbb{K}$ be a **q**,**p**-model and let $\mathbb{M}$ be a **p**,**r**-model. Suppose $\mathbb{K}(\mathbf{p})\preccurlyeq^{(+)}\mathbb{M}(\mathbf{p})$. Then there are **q**,**p**,**r**-models $\mathbb{K}',\mathbb{M}'$ such that: $\mathbb{K}\approx\mathbb{K}'(\mathbf{q},\mathbf{p})$, $\mathbb{K}'\sqsubseteq^{(+)}_{ini}\mathbb{M}'$ and $\mathbb{M}\approx\mathbb{M}'(\mathbf{p},\mathbf{r})$.

Proof: We just specify the new models for the the +-case and leave the routine verification to the reader. $\mathbb{K}'$ is given as follows:

- $K' := \{\langle k,m,m\rangle \mid kRm\}$,
- $\langle k,m,m\rangle \leq_{\mathbb{K}'} \langle k',m',m'\rangle :\Leftrightarrow k\leq_{\mathbb{K}}k'$ and $m\leq_{\mathbb{M}}m'$,
- $b_{\mathbb{K}'} := \langle b_{\mathbb{K}},b_{\mathbb{M}},b_{\mathbb{M}}\rangle$,
- $\mathfrak{P}_{\mathbb{K}'} :=$ **q**,**p**,**r**,
- $\langle k,m,m\rangle \vDash_{\mathbb{K}'} s :\Leftrightarrow k\vDash_{\mathbb{K}}s$ or $m\vDash_{\mathbb{M}}s$.

$\mathbb{M}'$ is the following model:

- $M' := \{\langle k,m,m'\rangle \mid kRm\leq_{\mathbb{M}}m'\}$,
- $\langle k_0,m_0,m_0'\rangle \leq_{\mathbb{M}'} \langle k_1,m_1,m_1'\rangle :\Leftrightarrow$ ($m_0=m_0'$ and $k_0\leq_{\mathbb{K}}k_1$ and $m_0\leq_{\mathbb{M}}m_1$) or ($k_0=k_1$ and $m_0=m_1$ and $m_0'\leq_{\mathbb{M}}m_1'$),
- $b_{\mathbb{M}'} := \langle b_{\mathbb{K}},b_{\mathbb{M}},b_{\mathbb{M}}\rangle$,
- $\mathfrak{P}_{\mathbb{M}'} :=$ **q**,**p**,**r**,
- $\langle k,m,m'\rangle \vDash_{\mathbb{M}'} s :\Leftrightarrow k\vDash_{\mathbb{K}}s$ or $m'\vDash_{\mathbb{M}}s$. ❑

6.11 Corollary: Suppose $\mathbb{K}$ and $\mathbb{M}$ are **p**-models. Then:

$$\mathbb{K}\preccurlyeq^{(+)}\mathbb{M} \Leftrightarrow \exists\mathbb{K}',\mathbb{M}'\ \mathbb{K}\approx\mathbb{K}'\sqsubseteq^{(+)}_{ini}\mathbb{M}'\approx\mathbb{M}.$$

Proof: Like the proof of 6.8. ❑

Let $\lhd$ be any relation between models. A formula A is *$\lhd$-robust* if

for all $\mathbb{M}$ ($\mathbb{M}\vDash A \Rightarrow$ for all $\mathbb{N}\lhd\mathbb{M}$: $\mathbb{N}\vDash A$).

6.12 Fact: The following are equivalent:

i) A is $\preccurlyeq$-robust,

ii) A is $\preccurlyeq_1$-robust,

iii) A is $\sqsubseteq$-robust,

iv) A is $\sqsubseteq_{full}$-robust,

v) A is $\sqsubseteq_{ini}$-robust.

Proof: Trivially (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). We prove (v) $\Rightarrow$ (i). Suppose A is $\sqsubseteq_{ini}$-robust and that $\mathbb{K}\preccurlyeq\mathbb{M}\vDash A$. By 6.10 there are $\mathbb{K}',\mathbb{M}'$ with $\mathbb{K}\approx\mathbb{K}'\sqsubseteq_{ini}\mathbb{M}'\approx\mathbb{M}$. By bisimulation $\mathbb{M}'\vDash A$. By robustness $\mathbb{K}'\vDash A$. By bisimulation again: $\mathbb{K}\vDash A$. ❑

7 NNIL and subsimulations: To each model $\mathbb{K}$ and finite set of atoms $\mathbf{p}$, we assign NNIL($\mathbf{p}$)-formulas $\nu_{\mathbb{K}}(\mathbf{p})$ and $\eta_{\mathbb{K}}(\mathbf{p})$ as follows:

- $\nu_{\mathbb{K}}(\mathbf{p}) := \bigvee\{A \in \mathrm{NNIL}(\mathbf{p}) \mid \mathbb{K} \nvDash A\}$,
- $\eta_{\mathbb{K}}(\mathbf{p}) := \bigwedge\{A \in \mathrm{NNIL}(\mathbf{p}) \mid \mathbb{K} \vDash A\}$.

Define also:

- $\rho_{\mathbb{K}}(\mathbf{p}) := \bigvee\{p \in \mathbf{p} \mid \mathbb{K} \nvDash p\}$,
- $\pi_{\mathbb{K}}(\mathbf{p}) := \bigwedge\{p \in \mathbf{p} \mid \mathbb{K} \vDash p\}$.

If from the context it is clear that we are considering a model $\mathbb{K}$ we write $\nu_k(\mathbf{p})$ for $\nu_{\mathbb{K}[k]}(\mathbf{p})$ and similarly for η, ρ and π.

7.1 Fact: $\mathbb{K} \nvDash \nu_{\mathbb{K}}(\mathbf{p})$ and $\mathbb{K} \vDash \eta_{\mathbb{K}}(\mathbf{p})$.

Proof: Obvious. ❑

7.2 Theorem: Consider two $\mathbf{p}$-models $\mathbb{K}$ and $\mathbb{M}$. Then:

$$\mathbb{K} \preccurlyeq \mathbb{M} \Leftrightarrow \mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(\mathbb{M}) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(\mathbb{K}).$$

Proof: "⇒" is immediate from 6.4. "⇐" Suppose $\mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(\mathbb{M}) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(\mathbb{K})$. Define R by:

- $k R m :\Leftrightarrow \mathrm{Th}_{\mathbf{p}}(k) \subseteq \mathrm{Th}_{\mathbf{p}}(m)$ and $\mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(m) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(k)$.

Then R is total zig. To show that R is total consider any k in K. Clearly $\mathbb{K} \nvDash \pi_k(\mathbf{p}) \rightarrow \nu_k(\mathbf{p})$ and hence by assumption (since $(\pi_k(\mathbf{p}) \rightarrow \nu_k(\mathbf{p})) \in \mathrm{NNIL}(\mathbf{p})$): $\mathbb{M} \nvDash \pi_k(\mathbf{p}) \rightarrow \nu_k(\mathbf{p})$. It follows that for some m: $m \vDash \pi_k(\mathbf{p})$ and $m \nvDash \nu_k(\mathbf{p})$. Ergo: kRm.

To prove that R is zig suppose kRm. It follows that:

$$\mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(\mathbb{M}[m]) \subseteq \mathrm{Th}_{\mathrm{NNIL}(\mathbf{p})}(\mathbb{K}[k]).$$

Hence by the previous argument R restricted to the domains of $\mathbb{K}[k]$ and $\mathbb{M}[m]$ is total. But this gives us precisely the zig-property. ❑

7.3 Theorem: Let $\mathbb{K}$ and $\mathbb{M}$ be $\mathbf{p}$-models. Then:

$$\mathbb{K} \preccurlyeq \mathbb{M} \Leftrightarrow \mathbb{M} \nvDash \nu_{\mathbb{K}}(\mathbf{p}) \Leftrightarrow \mathbb{K} \vDash \eta_{\mathbb{M}}(\mathbf{p}) .$$

Proof: immediate from 7.2. ❑

7.4 Fact: Let $\mathbb{K}$ and $\mathbb{M}$ be $\mathbf{p}$-models. Then:

$$\mathbb{K} \preccurlyeq \mathbb{M} \Leftrightarrow \vdash \nu_{\mathbb{K}}(\mathbf{p}) \rightarrow \nu_{\mathbb{M}}(\mathbf{p}) \Leftrightarrow \vdash \eta_{\mathbb{K}}(\mathbf{p}) \rightarrow \eta_{\mathbb{M}}(\mathbf{p}).$$

Proof: Suppose $\mathbb{K}\preccurlyeq\mathbb{M}$. If $\mathbb{N}\nvDash\nu_{\mathbb{M}}(\mathbf{p})$, then $\mathbb{M}\preccurlyeq\mathbb{N}$ and hence $\mathbb{K}\preccurlyeq\mathbb{N}$, so $\mathbb{N}\nvDash\nu_{\mathbb{K}}(\mathbf{p})$. By the Completeness Theorem, we may conclude: $\vdash\nu_{\mathbb{K}}(\mathbf{p})\rightarrow\nu_{\mathbb{M}}(\mathbf{p})$. For the converse assume $\vdash\nu_{\mathbb{K}}(\mathbf{p})\rightarrow\nu_{\mathbb{M}}(\mathbf{p})$. Since $\mathbb{M}\nvDash\nu_{\mathbb{M}}(\mathbf{p})$, it follows that $\mathbb{M}\nvDash\nu_{\mathbb{K}}(\mathbf{p})$. Hence by 7.3: $\mathbb{K}\preccurlyeq\mathbb{M}$.

Suppose $\mathbb{K}\preccurlyeq\mathbb{M}$. If $\mathbb{N}\vDash\eta_{\mathbb{K}}(\mathbf{p})$, then $\mathbb{N}\preccurlyeq\mathbb{K}$ and hence $\mathbb{N}\preccurlyeq\mathbb{M}$. It follows that $\mathbb{N}\vDash\eta_{\mathbb{M}}(\mathbf{p})$. By the completeness Theorem: $\vdash \eta_{\mathbb{K}}(\mathbf{p}) \rightarrow\eta_{\mathbb{M}}(\mathbf{p})$. For the converse suppose $\vdash \eta_{\mathbb{K}}(\mathbf{p}) \rightarrow\eta_{\mathbb{M}}(\mathbf{p})$. Since $\mathbb{K}\vDash\eta_{\mathbb{K}}(\mathbf{p})$, it follows that $\mathbb{K}\vDash\eta_{\mathbb{M}}(\mathbf{p})$, and hence that $\mathbb{K}\preccurlyeq\mathbb{M}$. ❑

7.5 Fact: The number of $\equiv$-equivalence classes of $\mathbf{p}$-models is finite.

Proof: By 7.4 for $\mathbf{p}$-models $\mathbb{K}$ and $\mathbb{M}$: $\mathbb{K}\equiv\mathbb{M} \Leftrightarrow \vdash \nu_{\mathbb{K}}(\mathbf{p}) \leftrightarrow \nu_{\mathbb{M}}(\mathbf{p})$. But there only finitely many NNIL($\mathbf{p}$)-formulas modulo provable equivalence. ❑

We want to prove the uniform interpolation theorem. Clearly if the uniform left-interpolant $A^*(\mathbf{p})$ exists, then it is equivalent to $\bigvee\{B\in \text{NNIL}(\mathbf{p}) \mid \vdash B\rightarrow A\}$. Par abus de langage we now define:

- $A^*(\mathbf{p}) := \bigvee\{B\in \text{NNIL}(\mathbf{p}) \mid \vdash B\rightarrow A\}$,

and prove that this formula is the uniform left-interpolant. Similarly we define:

- $A^\circ(\mathbf{p}) := \bigwedge\{B\in \text{NNIL}(\mathbf{p}) \mid \vdash A\rightarrow B\}$,

and prove that it is the uniform right-interpolant.

7.6 Uniform Interpolation Theorem: NNIL satisfies uniform interpolation.

Proof: Consider formulas A and B and a finite set of propositional variables $\mathbf{p}$. Let $PV(A)\subseteq\mathbf{p},\mathbf{r}$ and $PV(B)\subseteq\mathbf{q},\mathbf{p}$ for $\mathbf{q},\mathbf{p},\mathbf{r}$ disjoint. Suppose $\vdash A\rightarrow B$.

We show that $B^*(\mathbf{p})$ is the uniform NNIL left-interpolant of B. Clearly $\vdash B^*(\mathbf{p})\rightarrow B$, so it is sufficient to show:

if $A\in$ NNIL, then $\vdash A\rightarrow B^*(\mathbf{p})$.

Suppose $A\in$ NNIL and $\nvdash A\rightarrow B^*(\mathbf{p})$. Let $\mathbb{M}$ be a $\mathbf{p},\mathbf{r}$-model such that $\mathbb{M}\vDash A$ and $\mathbb{M}\nvDash B^*(\mathbf{p})$. Suppose $\eta_{\mathbb{M}}(\mathbf{p})\vdash B$, then $\eta_{\mathbb{M}}(\mathbf{p})\vdash B^*(\mathbf{p})$ by the definition of $B^*(\mathbf{p})$. But then $\mathbb{M}\vDash B^*(\mathbf{p})$. Quod non. Ergo $\eta_{\mathbb{M}}(\mathbf{p})\nvdash B$. By the Completeness Theorem there is a $\mathbf{q},\mathbf{p}$-model $\mathbb{K}$ such that $\mathbb{K}\vDash\eta_{\mathbb{M}}(\mathbf{p})$ and $\mathbb{K}\nvDash B$. By 7.3: $\mathbb{K}(\mathbf{p})\preccurlyeq\mathbb{M}(\mathbf{p})$. We apply the Lifting Theorem 6.7 to obtain $\mathbf{q},\mathbf{p},\mathbf{r}$-models $\mathbb{K}',\mathbb{M}'$ such that: $\mathbb{K}\simeq\mathbb{K}'(\mathbf{q},\mathbf{p})$, $\mathbb{K}'\preccurlyeq\mathbb{M}'$ and $\mathbb{M}\simeq\mathbb{M}'(\mathbf{p},\mathbf{r})$. We find $\mathbb{K}'\nvDash B$ and $\mathbb{M}'\vDash A$. Since $\mathbb{K}'\preccurlyeq\mathbb{M}'$ and $A\in$ NNIL we get: $\mathbb{K}'\vDash A$. Ergo $\nvdash A\rightarrow B$. A contradiction. So $\vdash A\rightarrow B^*(\mathbf{p})$.

Let A be -again- an arbitrary formula. We show that $A^\circ(\mathbf{p})$ is the uniform NNIL right-interpolant of A. Clearly $\vdash A \to A^\circ(\mathbf{p})$, so it is sufficient to show:

If $B \in$ NNIL, then $\vdash A^\circ(\mathbf{p}) \to B$.

Suppose $A \in$ NNIL and $\nvdash A^\circ(\mathbf{p}) \to B$. Let $\mathbb{K}$ be a $\mathbf{q},\mathbf{p}$-model such that $\mathbb{K} \models A^\circ(\mathbf{p})$ and $\mathbb{K} \not\models B$. Suppose $A \vdash \nu_{\mathbb{K}}(\mathbf{p})$, then $A^\circ(\mathbf{p}) \vdash \nu_{\mathbb{K}}(\mathbf{p})$ by the definition of $A^\circ(\mathbf{p})$. But then $\mathbb{K} \models \nu_{\mathbb{K}}(\mathbf{p})$. Quod non. Ergo $A \nvdash \nu_{\mathbb{K}}(\mathbf{p})$. By the Completeness Theorem there is a $\mathbf{p},\mathbf{r}$-model $\mathbb{M}$ such that $\mathbb{M} \models A$ and $\mathbb{M} \not\models \nu_{\mathbb{K}}(\mathbf{p})$. By 7.3: $\mathbb{K}(\mathbf{p}) \preccurlyeq \mathbb{M}(\mathbf{p})$. We apply the Lifting Theorem 6.7 to obtain $\mathbf{q},\mathbf{p},\mathbf{r}$-models $\mathbb{K}',\mathbb{M}'$ such that: $\mathbb{K} \simeq \mathbb{K}'(\mathbf{q},\mathbf{p})$, $\mathbb{K}' \preccurlyeq \mathbb{M}'$ and $\mathbb{M} \simeq \mathbb{M}'(\mathbf{p},\mathbf{r})$. We find $\mathbb{K}' \not\models B$ and $\mathbb{M}' \models A$. Since $\mathbb{K}' \preccurlyeq \mathbb{M}'$ we get: $\mathbb{M}' \not\models B$. Ergo $\nvdash A \to B$. A contradiction. So $\vdash B \to A^*(\mathbf{p})$. ❑

Note that by the results of section 3 it also follows that the NNIL approximants A^* and A° exist. Moreover if $PV(A) \subseteq \mathbf{p}$, then:

$\vdash A^* \leftrightarrow A^*(\mathbf{p})$ and $\vdash A^\circ \leftrightarrow A^\circ(\mathbf{p})$

We give an alternative characterization of $A^*(\mathbf{p})$ and $A^\circ(\mathbf{p})$.

7.7 Theorem: Consider $A \in \mathfrak{L}(\mathbf{p},\mathbf{q})$. We have:

i) $\vdash A^*(\mathbf{p}) \leftrightarrow \bigwedge\{\nu_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \not\models A\}$,

ii) $\vdash A^\circ(\mathbf{p}) \leftrightarrow \bigvee\{\eta_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \models A\}$.

Proof: (i) "$\to$" Suppose $\mathbb{K} \not\models A$, then $\mathbb{K} \not\models A^*(\mathbf{p})$ and hence by the definition of $\nu_{\mathbb{K}}(\mathbf{p})$: $\vdash A^*(\mathbf{p}) \to \nu_{\mathbb{K}}(\mathbf{p})$. Ergo: $\vdash A^*(\mathbf{p}) \to \bigwedge\{\nu_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \not\models A\}$,

"$\leftarrow$" Suppose that $\nvdash \bigwedge\{\nu_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \not\models A\} \to A$. Then for some $\mathbb{M}$:

$\mathbb{M} \models \bigwedge\{\nu_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \not\models A\}$ and $\mathbb{M} \not\models A$.

But then $\mathbb{M} \models \nu_{\mathbb{M}}(\mathbf{p})$, quod non. Hence $\vdash \bigwedge\{\nu_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \not\models A\} \to A$ and so by the definition of $A^*(\mathbf{p})$: $\vdash \bigwedge\{\nu_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \not\models A\} \to A^*(\mathbf{p})$.

(ii) "$\leftarrow$" Suppose $\mathbb{K} \models A$, then $\mathbb{K} \models A^\circ(\mathbf{p})$ and hence by the definition of $\eta_{\mathbb{K}}(\mathbf{p})$: $\vdash \eta_{\mathbb{K}}(\mathbf{p}) \to A^\circ(\mathbf{p})$. Ergo: $\vdash \bigvee\{\eta_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \models A\} \to A^\circ(\mathbf{p})$.

"$\to$" Suppose that $\nvdash A \to \bigvee\{\eta_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \models A\}$. Then for some $\mathbb{M}$:

$\mathbb{M} \models A$ and $\mathbb{M} \not\models \bigvee\{\eta_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \models A\}$.

But then $\mathbb{M} \not\models \eta_{\mathbb{M}}(\mathbf{p})$, quod non. Hence $\vdash A \to \bigvee\{\eta_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \models A\}$ and so by the definition of $A^\circ(\mathbf{p})$: $\vdash A^\circ(\mathbf{p}) \to \bigvee\{\eta_{\mathbb{K}}(\mathbf{p}) \mid \mathbb{K} \models A\}$. ❑

Define:

- $\mathbb{M}\vDash\bigcirc A \quad :\Leftrightarrow \forall\mathbb{K}\preccurlyeq\mathbb{M}\ \mathbb{K}\vDash A$,
- $\mathbb{M}\vDash\bigcirc_1 A \quad :\Leftrightarrow \forall\mathbb{K}\preccurlyeq_1\mathbb{M}\ \mathbb{K}\vDash A$,
- $\mathbb{M}\vDash\Diamond A \quad :\Leftrightarrow \exists\mathbb{K}\succcurlyeq\mathbb{M}\ \mathbb{K}\vDash A$.

7.8 **Theorem:** Consider $\mathbb{M}$ and $A\in\mathfrak{L}(\mathbf{p})$. Then:

i) $\mathbb{M}\vDash\bigcirc_1 A \Leftrightarrow \mathbb{M}\vDash\bigcirc A \Leftrightarrow \mathbb{M}\vDash A^*$.

ii) $\mathbb{M}\vDash\Diamond A \Leftrightarrow \mathbb{M}\vDash A^\circ(\mathbf{p})$.

Proof: (i) The first equivalence is immediate by 6.9. We prove the second equivalence. "$\Leftarrow$" By 6.4. "$\Rightarrow$" Suppose $\mathbb{M}\nvDash A^*$. We have $\mathbb{M}\nvDash A^*(\mathbf{p})$. Then by 7.7 for some $\mathbb{K}$ with $\mathbb{K}\nvDash A$: $\mathbb{M}\nvDash \nu_{\mathbb{K}}(\mathbf{p})$. Clearly we may assume that $\mathbb{K}$ is a $\mathbf{p}$-model. It follows that $\mathbb{K}\preccurlyeq\mathbb{M}(\mathbf{p})$. For some (possibly infinite) $\mathbf{r}$, disjoint from $\mathbf{p}$, $\mathbb{M}$ is a $\mathbf{p},\mathbf{r}$-model. By the Lifting Theorem 6.7 we can find a $\mathbf{p},\mathbf{r}$-model $\mathbb{K}'$ such that: $\mathbb{K}\simeq\mathbb{K}'(\mathbf{p})$ and $\mathbb{K}'\preccurlyeq\mathbb{M}$. Hence there is a $\mathbb{K}'$ with $\mathbb{K}'\nvDash A$ and $\mathbb{K}'\preccurlyeq\mathbb{M}$. Ergo $\mathbb{M}\nvDash\bigcirc A$.

ii) "$\Rightarrow$"Suppose $\exists\mathbb{K}\succcurlyeq\mathbb{M}\ \mathbb{K}\vDash A$. Then $\mathbb{K}\vDash A^\circ(\mathbf{p})$ and hence $\mathbb{M}\vDash A^\circ(\mathbf{p})$. In other words: $\mathbb{M}\vDash A^\circ$. "$\Leftarrow$" Suppose $\mathbb{M}\vDash A^\circ(\mathbf{p})$. By 7.6 for some $\mathbb{K}$ with $\mathbb{K}\vDash A$: $\mathbb{M}\vDash\eta_{\mathbb{K}}(\mathbf{p})$. We may assume that $\mathbb{K}$ is a $\mathbf{p}$-model. We find $\mathbb{M}(\mathbf{p})\preccurlyeq\mathbb{K}$. For some (possibly infinite) $\mathbf{q}$, disjoint from $\mathbf{p}$, $\mathbb{M}$ is a $\mathbf{q},\mathbf{p}$-model. By the Lifting Theorem 6.7 we can find a $\mathbf{q},\mathbf{p}$-model $\mathbb{K}'$ such that: $\mathbb{K}\simeq\mathbb{K}'(\mathbf{p})$ and $\mathbb{M}\preccurlyeq\mathbb{K}'$. Hence there is a $\mathbb{K}'$ with $\mathbb{K}'\vDash A$ and $\mathbb{M}\preccurlyeq\mathbb{K}'$. Ergo $\mathbb{M}\vDash\bigcirc A$. ❑

7.9 **Question:** Is there a reasonable, complete set of inference rules for $\bigcirc$? The same question for $\Diamond$ and for $\bigcirc$ and $\Diamond$ together. ❍

7.10 **Example:** Define $\mathbb{M}\vDash\triangle A :\Leftrightarrow \forall\mathbb{K}\subseteq\mathbb{M}\ \mathbb{K}\vDash A$. Consider the model $\mathbb{U}$ of 6.3. Clearly $\mathbb{U}\vDash\triangle((\neg\neg p\rightarrow p)\rightarrow(p\vee\neg p))$, but $\mathbb{U}\nvDash(p\vee\neg p)$. Moreover:

$$((\neg\neg p\rightarrow p)\rightarrow(p\vee\neg p))^* = (p\vee\neg p).$$

So 7.8 does not generalize to $\triangle$. ❍

7.12 **Theorem:** A is $\preccurlyeq$-robust $\Leftrightarrow$ A is in NNIL.

Proof: "$\Leftarrow$" is immediate by 6.4. "$\Rightarrow$" Say A is in $\mathfrak{L}(\mathbf{p})$. Suppose A is $\preccurlyeq$-robust. Suppose $\mathbb{K}\vDash A$, then $\mathbb{K}\vDash\bigcirc A$ and hence by 7.8: $\mathbb{K}\vDash A^*$. Ergo $\vdash A\leftrightarrow A^*$. So A is in NNIL. ❑

Note that by 6.11 we also have:

A is $\preccurlyeq_1$-robust $\Leftrightarrow$ A is in NNIL,

A is $\subseteq$-robust $\Leftrightarrow$ A is in NNIL,

A is $\subseteq_{full}$-robust $\Leftrightarrow$ A is in NNIL,

A is $\subseteq_{ini}$-robust $\Leftrightarrow$ A is in NNIL.

7.13 Corollary: We use the notions introduced in 2.1. We have:

A is a NNIL formula $\Leftrightarrow$ I(A,x) is T-provably equivalent to a Π_1-formula of predicate logic.

Proof: "$\Rightarrow$" By 2.1. "$\Leftarrow$" Suppose I(A,x) is T-provably equivalent to a Π_1-formula of predicate logic. Then I(A,x) is preserved under taking submodels of T. But this implies means that A is preserved under taking sub Kripke models. By 7.12 A is in NNIL. ❑

In appendix A we show how 7.13 can be improved.

We close the paper by giving an alternative characterization of $\nu_{\mathbb{K}}(\mathbf{p})$.

7.14 Excursion: ν and anti-model-descriptions: Let $\mathbb{K}$ be a model. There is a pleasant way of characterizing $\nu_{\mathbb{K}}(\mathbf{p})$ as a formula giving an 'anti-description'. Remember:

- $\rho_{\mathbb{K}}(\mathbf{p}) := \bigvee\{p\in\mathbf{p} \mid \mathbb{K}\nvDash p\}$,
- $\pi_{\mathbb{K}}(\mathbf{p}) := \bigwedge\{p\in\mathbf{p} \mid \mathbb{K}\vDash p\}$.

Define:

- $\alpha_{\mathbb{K}}(\mathbf{p}) := \pi_{\mathbb{K}}(\mathbf{p}) \to (\rho_{\mathbb{K}}(\mathbf{p}) \vee \bigvee\{\alpha_{\mathbb{K}[k]}(\mathbf{p}) \mid b_{\mathbb{K}}<k\})$.

We again employ the convention of the empty conjunction being $\top$ and the empty disjunction being $\bot$. Modulo provable equivalence $\alpha_{\mathbb{K}}(\mathbf{p})$ can be written more efficiently, by restricting ourselves to the immediate strict <-successors of $b_{\mathbb{K}}$ in the last disjunction of the definition. Clearly $\alpha_{\mathbb{K}}(\mathbf{p})\in$ NNIL($\mathbf{p}$).

7.14.1 Fact: $\mathbb{K}\nvDash\alpha_{\mathbb{K}}(\mathbf{p})$.

Proof: We prove by induction on the depth of k in $\mathbb{K}$ that $k\nvDash\alpha_k(\mathbf{p})$. Suppose $k\vDash\alpha_k(\mathbf{p})$. Since clearly $k\vDash\pi_k(\mathbf{p})$ and $k\nvDash\nu_k(\mathbf{p})$, it follows that $k\vDash\bigvee\{\alpha_{k'}(\mathbf{p}) \mid k<k'\}$ and hence for some k'>k $k\vDash\alpha_{k'}(\mathbf{p})$. This contradicts the induction hypothesis. ❑

7.14.2 Fact: $\vdash \alpha_{\mathbb{K}}(\mathbf{p}) \leftrightarrow \nu_{\mathbb{K}}(\mathbf{p})$.

Proof: "→" Is immediate by 7.14.1 and the definition of $\nu_{\mathbb{K}}(\mathbf{p})$."←" We prove by induction on de depth of k in $\mathbb{K}$ that for all $A \in NNIL(\mathbf{p})$: $k \nvDash A \Rightarrow \vdash A \to \alpha_k(\mathbf{p})$. Consider k. The proof proceeds by a subinduction on A in $NNIL_0(\mathbf{p})$. The cases of atoms, conjunction and disjunction are straightforward. Suppose A is of the form $p \to B$. In case $k \vDash p$ we find: $k \nvDash B$. Hence by our subinduction hypothesis: $\vdash B \to \alpha_k(\mathbf{p})$. Since p is a conjunct of $\pi_k(\mathbf{p})$, we find: $\vdash (p \to B) \to \alpha_k(\mathbf{p})$. In case $k \nvDash p$ there is a $k' > k$ such that $k' \vDash p$ and $k' \nvDash B$. It follows by our main induction hypothesis that $\vdash (p \to B) \to \alpha_{k'}(\mathbf{p})$. Hence by the definition of $\alpha_k(\mathbf{p})$: $\vdash (p \to B) \to \alpha_k(\mathbf{p})$. ❑

7.14.3 Exercise: The reader may amuse himself by proving $\top^* = \top$ and $\bot^* = \bot$ using 7.6 ❍

7.14.4 Example: Consider the characterization of A* of 7.7. Note that by 7.4 we can restrict the conjunction to $\nu_{\mathbb{K}}(\mathbf{p})$ for $\mathbb{K}$ a representative of a $\preccurlyeq$-minimal $\equiv$-equivalence class X such that $\mathbb{M} \nvDash A$ for some $\mathbb{M} \in X$. This insight allows us to use 7.14.2 for actual computation of $A^*(\mathbf{p})$. We compute e.g.:

$$A := ((\neg\neg p \to p) \to (p \vee \neg p))^*.$$

Consider the models $\mathbb{U}$ and $\mathbb{V}$ of example 6.3. It is easy to see that any model $\mathbb{K}$ such that $\mathbb{K} \nvDash A$ has $\mathbb{V}$ as a submodel. Moreover $\mathbb{V} \nvDash A$ and $\mathbb{U} \equiv \mathbb{V}$. So:

$$A^* = \alpha_{\mathbb{U}} = \bigwedge\emptyset \to (\bigvee\{p\} \vee \bigvee\{(\bigwedge\{p\} \to (\bigvee\emptyset \vee \bigvee\emptyset))\}) = (p \vee \neg p).$$ ❍

8 Beyond NNIL: Can we extend our results to the higher complexity classes? It turns out that the characterization of the complexity classes in terms of an appropriate notion of simulation extends in an immediate way. How to do this is sketched in appendix B. In this section we show that *yes* we can get uniform right interpolation for Π_2, but *no* we cannot get uniform left interpolation for Π_2, and *no* we cannot get uniform left interpolation for Π_3. Note that these classes are locally finite and closed under disjunction and conjunction. So by the results of section 3, it follows that Π_2 does not have IPL and that Π_3 does not have IPR.

Let $\mathbb{K}$ and $\mathbb{M}$ be **p**-models. A pair of relations R,S is a *2-subsimulation* between $\mathbb{K}$ and $\mathbb{M}$, if R is a subsimulation between $\mathbb{K}$ and $\mathbb{M}$, S is a subsimulation between $\mathbb{M}$ and $\mathbb{K}$ and $R \subseteq S^\wedge$. We write:

$R,S:\mathbb{K}\preccurlyeq_2\mathbb{M} :\Leftrightarrow R,S$ is a 2-subsimulation between $\mathbb{K}$ and $\mathbb{M}$ and R is total

$R,S:\mathbb{K}\preccurlyeq_2{}^+\mathbb{M} :\Leftrightarrow R,S:\mathbb{K}\preccurlyeq_2\mathbb{M}$, and R is root preserving,

$\mathbb{K}\preccurlyeq_2\mathbb{M} :\Leftrightarrow \exists R,S:\mathbb{K}\preccurlyeq_2\mathbb{M}$,

$\mathbb{K}\preccurlyeq_2{}^+\mathbb{M} :\Leftrightarrow \exists R,S:\mathbb{K}\preccurlyeq_2{}^+\mathbb{M}$.

Note that we have: $\mathbb{K}\preccurlyeq_2\mathbb{M} \Leftrightarrow \exists m\in M:\mathbb{K}\preccurlyeq_2{}^+\mathbb{M}[m]$. Note that if R is rootpreserving, then S is also rootpreserving.

8.1 Theorem: Suppose $\mathbb{K}$ and $\mathbb{M}$ are **p**-models. Then:
$\mathbb{K}\preccurlyeq_2\mathbb{M} \Leftrightarrow Th_{\Pi_2(\mathbf{p})}(\mathbb{M})\subseteq Th_{\Pi_2(\mathbf{p})}(\mathbb{K})$.

Proof: We take:

$kRm :\Leftrightarrow Th_{\Pi_2(\mathbf{p})}(m)\subseteq Th_{\Pi_2(\mathbf{p})}(k)$ and $Th_{\Pi_1(\mathbf{p})}(k)\subseteq Th_{\Pi_1(\mathbf{p})}(m)$,

$mSk :\Leftrightarrow Th_{\Pi_1(\mathbf{p})}(k)\subseteq Th_{\Pi_1(\mathbf{p})}(m)$ and $Th_{\mathbf{p}}(m)\subseteq Th_{\mathbf{p}}(k)$.

The further verification is left to the reader (or: see appendix B). ❑

Define $A^{\circ\circ}(\mathbf{p}) := \bigwedge\{B\in\Pi_2(\mathbf{p}) \mid A\vdash B\}$. $A^{\circ\circ}:=A^{\circ\circ}(PV(A))$.

8.2 Theorem: Let $B\in\Pi_2$. Suppose $A\vdash B$, then $A^{\circ\circ}\vdash B$.

Proof: Let $\mathbf{q}\supseteq PV(A)\cup PV(B)$, $\mathbf{p}:=PV(A)$. Suppose $A^{\circ\circ}\nvdash B$. Let $\mathbb{K}$ be a **q**-model such that $\mathbb{K}\models A^{\circ\circ}$ and $\mathbb{K}\not\models B$. We claim that:

$A,\bigwedge\{C\in\Pi_1(\mathbf{p}) \mid b_{\mathbb{K}}\models C\}\nvdash\bigvee\{D\in\Pi_2(\mathbf{p}) \mid b_{\mathbb{K}}\not\models D\}$.

If it did, we would have:

$A^{\circ\circ} \vdash \bigwedge\{C\in\Pi_1(\mathbf{p}) \mid b_{\mathbb{K}}\models C\} \rightarrow \bigvee\{D\in\Pi_2(\mathbf{p}) \mid b_{\mathbb{K}}\not\models D\}$.

But this contradicts the fact that $b_{\mathbb{K}}\models A^{\circ\circ}(\mathbf{p})$. Let $\mathbb{M}$ be a **p**-model such that:

$\mathbb{M}\models A$, $\mathbb{M}\models \bigwedge\{C\in\Pi_1(\mathbf{p}) \mid b_{\mathbb{K}}\models C\}$ and $\mathbb{M}\not\models \bigvee\{D\in\Pi_2(\mathbf{p}) \mid b_{\mathbb{K}}\not\models D\}$.

It follows that for some R,S: $R,S:\mathbb{K}(\mathbf{p})\preccurlyeq_2{}^+\mathbb{M}\models A$. We will construct a **q**-model $\mathbb{N}$ such that: (a) $\mathbb{M}\simeq\mathbb{N}(\mathbf{p})$, (b) $\mathbb{K}\preccurlyeq_2\mathbb{N}$. It immediately follows that $\mathbb{N}\models A$ and hence $\mathbb{N}\models B$. But then $\mathbb{K}\models B$. A contradiction. Ergo $A^{\circ\circ} \vdash B$.

We take:

- $N := \{\langle m,k\rangle \mid mSk\}$,
- $\langle m,k\rangle \leq_{\mathbb{N}} \langle m',k'\rangle :\Leftrightarrow m\leq_{\mathbb{M}}m'$ and $k\leq_{\mathbb{K}}k'$.
- $b_{\mathbb{N}} := \langle b_{\mathbb{M}},b_{\mathbb{K}}\rangle$,
- $\mathfrak{P}_{\mathbb{N}} := \mathbf{q}$,
- $\langle m,k\rangle \models_{\mathbb{N}} q :\Leftrightarrow k\models_{\mathbb{K}}q$

Since S is total, zig from $\mathbb{M}$ to $\mathbb{K}(\mathbf{p})$, it follows that $\mathbb{M}\simeq\mathbb{N}(\mathbf{p})$. Define S' from $\mathbb{N}$ to $\mathbb{K}$

by:

- $\langle m,k\rangle$ S' k' :⇔ k=k'.

By definition S' is total, zig. Finally R' from $\mathbb{K}$ to $\mathbb{N}$ by:

- k R' $\langle m,k'\rangle$:⇔ k=k'Rm

Since R⊆S^, also R'⊆S'^. Also it is easy to see that R' is total, zig. ❑

It follows that $A^{\circ\circ}$ is the $\prod_2$ right-approximant of A. Moreover $(.)^{\circ\circ}$ is PVP. Hence, by 3.7, 3.8 we have uniform $\prod_2$ right-interpolation. By 3.3, it follows that $\prod_2$ is closed under Pitts' universal quantification.

We show that $\prod_2$ does not satisfy (uniform) left interpolation and that $\prod_3$ does not satisfy (uniform) right interpolation. By 3.3 this is immediate from 8.3. Theorem 8.3 illustrates that in general the growth of implicational complexity in constructing the Pitts' interpolants is necessary.

8.3 Theorem: $\exists\prod_2 = \forall\prod_3 = \mathfrak{L}$.

Proof: Suppose $A\in\mathfrak{L}(\mathbf{p})$. Let **q** be a set of variables disjoint from **p**, that is in 1-1 correspondence with the subformulas of the form (B→C) of A. let the correspondence be $\mathfrak{q}$. We define $\mathfrak{T}$:Sub(A)→Sub(A) as follows:

- $\mathfrak{T}$ commutes with atoms, conjunction and disjunction,
- $\mathfrak{T}(B\to C) := \mathfrak{q}(B\to C)$.

Define:

- $EQ := \bigwedge\{\ \mathfrak{q}(B\to C) \leftrightarrow (\mathfrak{T}(B)\to\mathfrak{T}(C)) \mid (B\to C)\in Sub(A)\ \}$.

Note that EQ is $\prod_2$. Finally we put:

- $A^{\#} := \exists\mathbf{q}\ (\ EQ \wedge \mathfrak{T}(A)\ )$,
- $A^{\$} := \forall\mathbf{q}\ (\ EQ \to \mathfrak{T}(A)\ )$.

Note that $A^{\#}\in\exists\prod_2$ and $A^{\$}\in\forall\prod_3$. By elementary reasoning in second order propositional logic we find: $\vdash A\leftrightarrow A^{\#}$ and $\vdash A\leftrightarrow A^{\$}$. ❑

A Appendix: IPC as a fragment of predicate logic: In this appendix we follow the notations of 2.1. Let A(x) be any $\mathfrak{RL}$-formula in one variable.

We say that A(x) is *persistent* if for any T-model (or equivalently: IPC-model) $\mathbb{K}$ and for any k, k' in $\mathbb{K}$: k≤k' and $\mathbb{K}\models A(\underline{k}) \Rightarrow \mathbb{K}\models A(\underline{k'})$. Note that:

A(x) is persistent iff $T\vdash (x\leq y \wedge A(x)) \to A(y)$.

We say that A(x) is *preserved under bisimulations* if for all T-models $\mathbb{K}$ and $\mathbb{M}$ and for all bisimulations R between $\mathbb{K}$ and $\mathbb{M}$ we have:

suppose kRm and $\mathbb{K}\models A(\underline{k})$, then $\mathbb{M}\models A(\underline{m})$.

We say that A(x) is *upwards preserved under (.)[.]* if for all $\mathbb{K}$ and all k in $\mathbb{K}$:

if $\mathbb{K}\models A(\underline{k})$, then $\mathbb{K}[k]\models A(\underline{k})$.

We say that A(x) is *downwards preserved under (.)[.]* if for all $\mathbb{K}$ and all k in $\mathbb{K}$:

if $\mathbb{K}[k]\models A(\underline{k})$, then $\mathbb{K}\models A(\underline{k})$.

A1 Theorem: Suppose A(x) is (1) persistent and (2) preserved under bisimulations, then there is a $B\in\mathfrak{L}$, such that CPdC $\vdash$ $A(x) \leftrightarrow I(B,x)$.

Before giving the proof, let's first look somewhat closer at (1) and (2). Note that (2) is equivalent to:

(2a) A(x) is preserved under total, surjective bisimulations,

(2b) A(x) is downwards preserved under [.](.),

(2c) A(x) is upwards preserved under [.](.).

We give separating examples for the four conditions (1),...,(2c), Each example is designated by the one condition it doesn't satisfy.

¬(1) $\neg P(x)$,

¬(2a) $\forall y\ (x\leq y \rightarrow y\leq x)$,

¬(2b) $\forall y\ P(y)$,

¬(2c) $\exists y\ \neg P(y)$.

Proof of A1: Suppose A(x) satisfies (1) and (2). Let $\Delta(x):=\{I(B,x)|\ CPdC\vdash A(x)\rightarrow I(B,x)\}$. If $T,\Delta(x)\vdash_{CPdC}A(x)$, we are easly done by compactness. If $T,\Delta(x)\nvdash_{CPdC}A(x)$, then, by results of CK[77] (alternatively: see De Rijke[93]) , there is an ω-saturated model $\mathbb{K}$ and an element k of $\mathbb{K}$, such that $\mathbb{K}\models\Delta(\underline{k})$ and $\mathbb{K}\models\neg A(\underline{k})$.

Let:

- $\Theta(x) := \{A(x)\} \cup \{I(B,x)|k\models_{\mathbb{K}}B\} \cup \{\neg I(C,x)|k\not\models_{\mathbb{K}}C\}$.

We claim that $\Theta(x)$ is consistent. If not, there are finite sets of IPC-formulas X and Y such that for every B in X: $k\models_{\mathbb{K}}B$ and for every C in Y: $k\not\models_{\mathbb{K}}C$ and such that:

$$T\vdash \neg\,(A(x)\wedge\bigwedge\{I(B,x)|B\in X\}\wedge\bigwedge\{\neg I(C,x)|C\in Y\}).$$

Hence:

$$T\vdash (A(x)\rightarrow(\bigwedge\{I(B,x)|B\in X\}\rightarrow\bigvee\{I(C,x)|C\in Y\})).$$

By predicate logic, the persistence of A(x) and the definition of I, it follows that:

$T \vdash (A(x) \to I(\bigwedge X \to \bigvee Y, x))$.

But then one the one hand $(\bigwedge X \to \bigvee Y) \in \Delta(x)$, but $\mathbb{K} \nVdash I(\bigwedge X \to \bigvee Y, \underline{k})$. A contradiction.

Let $\mathbb{M}$ be an ω-saturated model of Θ(x), say $\mathbb{M} \models \Theta(\underline{m})$. We define R between the nodes of $\mathbb{K}$ and $\mathbb{M}$ as follows:

- $k'Rm' :\Leftrightarrow \forall B \in \mathfrak{L}\ (k' \models_{\mathbb{K}} B \Leftrightarrow m' \models_{\mathbb{M}} B)$.

We claim that R is a bisimulation. Suppose e.g. $k'' \geq k'Rm'$. Take:

- $\Gamma(x) := \{x \geq \underline{m'}\} \cup \{I(B,x) \mid k'' \models_{\mathbb{K}} B\} \cup \{\neg I(C,x) \mid k'' \nvDash_{\mathbb{K}} C\}$.

We claim that Γ(x) is finitely satisfiable in $\mathbb{M}$. By ω-saturatedness it will follow that there is an $m'' \geq m$ satisfying Γ(x). Hence $k''Rm''$.

Consider any finite X and Y such that for all B in X: $k'' \models_{\mathbb{K}} B$ and for all C in Y: $k'' \nvDash_{\mathbb{K}} C$. Then evidently $k' \nvDash_{\mathbb{K}} (\bigwedge X \to \bigvee Y)$. since k'Rm', it follows that: $m' \nvDash_{\mathbb{M}} (\bigwedge X \to \bigvee Y)$. So for some $m'' \geq m'$: $m'' \models_{\mathbb{M}} \bigwedge X$ and $m'' \nvDash_{\mathbb{M}} \bigvee Y$. Evidently m'' satisfies: $\{x \geq \underline{m}\} \cup \{I(B,x) \mid B \in X\} \cup \{\neg I(C,x) \mid C \in Y\}$.

Since kRm and $\mathbb{M} \models A(\underline{m})$, we have $\mathbb{K} \models A(\underline{k})$. A contradiction. So $T, \Delta(x) \vdash_{CPdC} A(x)$. ❑

Note that it follows that if A(x) satisfies (1) and (2) and is Π_1 in CPdC, then A(x) is CPdC-provably equivalent to I(B,x). Moreover B will be preserved under taking submodels and hence IPC-provably equivalent to a NNIL-formula C. Ergo A(x) is CPdC-provably equivalent to I(C,x) with C in NNIL.

B Appendix: $\mathfrak{z}\mathfrak{z}$-simulations: In this appendix we describe the notion of simulation appropriate for the Π-hierarchy of IPC.

B.1 Extended numbers: It is pleasant, but not strictly necessary, to have some rules for calculating with infinity at hand. The rules I prefer make our structure *with the inverse ordering* into a residuation lattice. Let $\omega^+ := \omega \cup \{\infty\}$. We give ω^+ the obvious ordering. We let α,β,... range over ω^+ and we let m,n,... range over ω. Define:

- + has its usual meaning on ω,
- $\infty + \alpha := \alpha + \infty := \infty$,
- $\alpha \dot{-} \beta := 0$ if $\alpha \leq \beta$, $m \dot{-} n := m - n$ if $n < m$, $\infty \dot{-} n := \infty$.

Note that:

✪ $\alpha \leq \beta + \gamma \Leftrightarrow \alpha \dot{-} \beta \leq \gamma$.

It follows e.g. that for $X \subseteq \omega^+$:

$\sup(X) \dot{-} \alpha = \sup(\{\beta \dot{-} \alpha \mid \beta \in X\})$.

Another important principle -easily verified- is:

☆ If $\min(\beta,\gamma)<\infty$, then $(\alpha+\beta)\dot{-}\gamma = (\alpha\dot{-}(\gamma\dot{-}\beta)) + (\beta\dot{-}\gamma)$.
Immediate consequences are:
☆1 if $\min(\beta,\gamma)<\infty$ and $\gamma\leq\beta$, then $(\alpha+\beta)\dot{-}\gamma = \alpha + (\beta\dot{-}\gamma)$;
☆2 $(\alpha+n)\dot{-}n = \alpha$.
Finally we have:
✣ if $\alpha\leq\beta$, then $\alpha + (\beta\dot{-}\alpha) = \beta$.

B.2 Basics of $\mathfrak{z}\mathfrak{z}$-simulations: A *zigzag simulation* or *$\mathfrak{z}\mathfrak{z}$-simulation* R between $\mathfrak{P}$-models $\mathbb{K}$ and $\mathbb{M}$ is a quaternary relation between K, {zig,zag}, ω^+ and M, satisfying the conditions below. We will consider R also as a {zig,zag}$\times\omega^+$-indexed set of binary relations between K and M writing $kR_{\mathfrak{z},\alpha}m$ for $\langle k,\mathfrak{z},\alpha,m\rangle \in R$. We put: zig^:=zag and zag^:=zig. We give the conditions:

i) $kR_{zig,\alpha}m \Rightarrow Th_{\mathfrak{P}}(k)\supseteq Th_{\mathfrak{P}}(m)$, $kR_{zag,\alpha}m \Rightarrow Th_{\mathfrak{P}}(k)\subseteq Th_{\mathfrak{P}}(m)$,
ii) $\alpha>0$ and $k'\geq kR_{zig,\alpha}m \Rightarrow$ there is an m' with $k'R_{zig,\alpha}m'\geq m$,
iii) $\alpha>0$ and $kR_{zag,\alpha}m\leq m' \Rightarrow$ there is a k' with $k\leq k'R_{zag,\alpha}m'$,
iv) $\alpha>0$ and $kR_{\mathfrak{z},\alpha}m \Rightarrow kR_{\mathfrak{z}^\wedge,\alpha\dot{-}1}m$.

We call (ii) the *zig-property* and (iii) the *zag-property.*

If we set $kTPm :\Leftrightarrow Th_{\mathfrak{P}}(k)\supseteq Th_{\mathfrak{P}}(m)$, we can also formulate our conditions as:

i) $R_{zig,\alpha}\subseteq TP$, $R_{zag,\alpha}\subseteq TP^\wedge$,
ii) $\alpha>0 \Rightarrow \geq\circ R_{zig,\alpha} \subseteq R_{zig,\alpha}\circ\geq$,
iii) $\alpha>0 \Rightarrow R_{zag,\alpha}\circ\leq \subseteq \leq\circ R_{zag,\alpha}$,
iv) $\alpha>0 \Rightarrow R_{\mathfrak{z},\alpha} \subseteq R_{\mathfrak{z}^\wedge,\alpha\dot{-}1}$.

We write:

- $kR_\alpha m :\Leftrightarrow kR_{zig,\alpha}m$ and $kR_{zag,\alpha}m$,
- $kRm :\Leftrightarrow kR_\infty m$.

Note that by (iv) it follows that: $kR_{\mathfrak{z},\infty}m \Leftrightarrow kR_\infty m$.

A binary relation R between $\mathbb{K}$ and $\mathbb{M}$ is a *bisimulation* between $\mathbb{K}$ and $\mathbb{M}$ iff $R^+:=\{\langle k,\mathfrak{z},\infty,m\rangle \mid kRm\}$ is a $\mathfrak{z}\mathfrak{z}$-simulation. We will simply confuse bisimulations R with the corresponding $\mathfrak{z}\mathfrak{z}$-simulations R^+.

Suppose R is an $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{K}$ and $\mathbb{M}$ and that S is an $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{M}$ and $\mathbb{N}$. The composition $R\circ S$ is given by: $(R\circ S)_{\mathfrak{z},\alpha}:=R_{\mathfrak{z},\alpha}\circ S_{\mathfrak{z},\alpha}$. It is easily seen that $\mathfrak{z}\mathfrak{z}$-simulations are closed under composition.

Suppose $\mathfrak{R}$ is a set of $\mathfrak{z}\mathfrak{z}$-simulations between $\mathbb{K}$ and $\mathbb{M}$. It is easy to verify that $\bigcup\mathfrak{R}$ is again a $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{K}$ and $\mathbb{M}$. It follows that there is always a maximal $\mathfrak{z}\mathfrak{z}$-simulation between two models.

Suppose R is a $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{K}$ and $\mathbb{M}$. The inverse R^ is given by: $(R^\wedge)_{\mathfrak{z},\alpha} := (R_{\mathfrak{z}^\wedge,\alpha})^\wedge$, where $(.)^\wedge$ is the usual inverse of binary relations. Clearly $\mathfrak{z}\mathfrak{z}$-simulations are closed under $(.)^\wedge$.

Consider a $\mathfrak{z}\mathfrak{z}$-simulation R between $\mathbb{K}$ and $\mathbb{M}$. Define $R[\alpha]$ by:

- $kR[\alpha]_{\mathfrak{z},\beta}m :\Leftrightarrow kR_{\mathfrak{z},\alpha+\beta}m$.

We say that R is *downwards closed* if for all $\alpha\leq\beta$: $R_{\mathfrak{z},\beta}\subseteq R_{\mathfrak{z},\alpha}$. The *downwards closure* $R\!\downarrow$ of a $\mathfrak{z}\mathfrak{z}$-simulation R is the smallest downwards closed relation extending it.

Note that if R is downwards closed we automatically have for $\beta>0$:

for all $\alpha\leq\beta\dot{-}1$: $R_{\mathfrak{z},\beta}\subseteq R_{\mathfrak{z}^\wedge,\alpha}$.

B.2.1 Fact: let R be a $\mathfrak{z}\mathfrak{z}$-simulation. We have:

i) $R[\alpha]$ is an $\mathfrak{z}\mathfrak{z}$-simulation.

ii) The downwards closure of R is a $\mathfrak{z}\mathfrak{z}$-simulation.

Proof: (i) We verify the zig-property of $R[\alpha]$. Suppose $\beta>0$ and $k'\geq kR[\alpha]_{zig,\beta}m$. It follows that $k'\geq kR_{zig,\alpha+\beta}m$. Hence there is an $m'\geq m$ with $k'R_{zig,\alpha+\beta}m'$. Hence $k'R[\alpha]_{zig,\beta}m'$. The zag-property is analogous. Finally we have for $\beta>0$:

$$\begin{aligned} kR[\alpha]_{\mathfrak{z},\beta}m &\Rightarrow kR_{\mathfrak{z},\alpha+\beta}m \\ &\Rightarrow kR_{\mathfrak{z}^\wedge,(\alpha+\beta)\dot{-}1}m \\ &\Rightarrow kR_{\mathfrak{z}^\wedge,\alpha+(\beta\dot{-}1)}m \\ &\Rightarrow kR[\alpha]_{\mathfrak{z}^\wedge,\beta\dot{-}1}m. \end{aligned}$$

ii) It is sufficient to show that: $R\!\downarrow=\bigcup\{R[\alpha] \mid \alpha\in\omega^+\}$. This is immediate from:

$$\begin{aligned} \exists\gamma\ \beta\leq\gamma \text{ and } kR_{\mathfrak{z},\gamma}m &\Leftrightarrow \exists\gamma\ \beta\leq\gamma \text{ and } kR_{\mathfrak{z},\beta+(\gamma\dot{-}\beta)}m \\ &\Leftrightarrow \exists\gamma\ \beta\leq\gamma \text{ and } kR[\gamma\dot{-}\beta]_{\mathfrak{z},\beta}m \\ &\Leftrightarrow \exists\alpha\ kR[\alpha]_{\mathfrak{z},\beta}m \end{aligned}$$

The first equivalence is by ✣. To prove the "$\Leftarrow$"-direction of the third equivalence, we need to show that for all α there is a $\gamma\geq\beta$, such that $\beta+\alpha=\beta+(\gamma\dot{-}\beta)$. In case $\beta<\infty$, we can take $\gamma:=\alpha+\beta$ (by ☆2). If $\beta=\infty$, take e.g. $\gamma:=\beta$. ❑

Suppose R is a $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{K}$ and $\mathbb{M}$. Let $k\in K$ and $m\in M$. Then $R|(\uparrow k\times\uparrow m)$, the restriction of R to $\uparrow k\times\uparrow m$, is a $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{K}[k]$ and $\mathbb{M}[m]$.

B.3 Preorders based on $\mathfrak{z}\mathfrak{z}$-simulations: Let $\mathbb{K}$ and $\mathbb{M}$ be *rooted* $\mathfrak{P}$-models. Define:

- $R:\mathbb{K}\preccurlyeq_\alpha\mathbb{M} \Leftrightarrow R$ is a $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{K}$ and $\mathbb{M}$ and $\exists m\in M\ b_{\mathbb{K}}R_{zig,\alpha}m$.
- $\mathbb{K}\preccurlyeq_\alpha\mathbb{M} \Leftrightarrow \exists R\colon R:\mathbb{K}\preccurlyeq_\alpha\mathbb{M}$.
- $\mathbb{K}\approx_\alpha\mathbb{M} :\Leftrightarrow \mathbb{K}\preccurlyeq_\alpha\mathbb{M}$ and $\mathbb{M}\preccurlyeq_\alpha\mathbb{K}$.

B.3.1 Fact: $\preccurlyeq_\alpha$ is a partial preordering.

Proof: Clearly $ID:\mathbb{K}\preccurlyeq_\alpha\mathbb{K}$, where:

$ID := \{\langle k,\mathfrak{z},\alpha,k\rangle \mid \alpha\in\omega^+, \mathfrak{z}\in\{zig,zag\}, k\in K\}$.

Moreover if $R:\mathbb{K}\preccurlyeq_\alpha\mathbb{M}$ and $S:\mathbb{M}\preccurlyeq_\alpha\mathbb{N}$, then $R\circ S:\mathbb{K}\preccurlyeq_\alpha\mathbb{N}$. ❑

B.3.2 Fact

$\mathbb{K}\approx_\alpha\mathbb{M} \Leftrightarrow \exists R\ R$ is a $\mathfrak{z}\mathfrak{z}$-simulation between $\mathbb{K}$ and $\mathbb{M}$ and $b_{\mathbb{K}}R_\alpha b_{\mathbb{M}}$.

Proof: "$\Leftarrow$" Trivial. "$\Rightarrow$" Suppose $S:\mathbb{K}\preccurlyeq_\alpha\mathbb{M}$ and $T:\mathbb{M}\preccurlyeq_\alpha\mathbb{K}$. Take:

$R := \{\langle b_{\mathbb{K}},zig,\alpha,b_{\mathbb{M}}\rangle,\langle b_{\mathbb{K}},zag,\alpha,b_{\mathbb{M}}\rangle\} \cup S \cup T^\wedge$.

Suppose $b_{\mathbb{K}}S_\alpha m$. If $b_{\mathbb{K}}\nVdash p$, then $m\nVdash p$ and hence $b_{\mathbb{M}}\nVdash p$. Similarly in the other direction. It follows that $Th_{\mathfrak{P}}(b_{\mathbb{K}})=Th_{\mathfrak{P}}(b_{\mathbb{M}})$. We leave the rest of the verification to the avid reader. ❑

B.3.3 Fact: $\mathbb{K}\preccurlyeq_\infty\mathbb{M} \Leftrightarrow \exists m\in M\ \mathbb{K}\approx_\infty\mathbb{M}[m]$.

Proof: left to the reader. ❑

B.4 Fact: Suppose that R is an $\mathfrak{z}\mathfrak{z}$-simulation between the $\mathfrak{P}$-models $\mathbb{K}$ and $\mathbb{M}$. Then:

$kR_{zig,\alpha}m \Rightarrow Th_{\Pi_\alpha(\mathfrak{P})}(k) \supseteq Th_{\Pi_\alpha(\mathfrak{P})}(m)$,

$kR_{zag,\alpha}m \Rightarrow Th_{\Pi_\alpha(\mathfrak{P})}(k) \subseteq Th_{\Pi_\alpha(\mathfrak{P})}(m)$.

Proof: By induction on A in $\mathfrak{L}(\mathfrak{P})$, simultaneous for all k, m, α, zig and zag. The cases of atoms, conjunction and disjunction are trivial. Suppose e.g. $kR_{zig,\alpha}m$, $(B\rightarrow C)\in\Pi_\alpha$ and $k\nVdash(B\rightarrow C)$. Then for some $k'\geq k$ $k'\models B$ and $k'\nVdash C$. By the zig-property there is an $m'\geq m$, such that $k'R_{zig,\alpha}m'$ and hence by the induction hypothesis $m'\nVdash C$. Moreover B will be in $\Pi_{\alpha\dot{-}1}(\mathfrak{P})$, so by the Induction Hypothesis applied for $\alpha\dot{-}1$, noting that $k'R_{zag,\alpha\dot{-}1}m'$, we find: $m'\models B$. Ergo $m\nVdash(B\rightarrow C)$. ❑

B.5 Fact: Suppose $\mathbb{K}$ and $\mathbb{M}$ are $\mathfrak{P}$-models. Then:

$\mathbb{K}\preccurlyeq_\alpha\mathbb{M} \Rightarrow Th_{\Pi_\alpha(\mathfrak{P})}(\mathbb{M}) \subseteq Th_{\Pi_\alpha(\mathfrak{P})}(\mathbb{K})$.

Proof: Left to the reader. ❑

B.6 **Fact:** Suppose $\mathbb{K}$ and $\mathbb{M}$ are **p**-models. Then:

$Th_{\Pi_n(\mathbf{p})}(\mathbb{M}) \subseteq Th_{\Pi_n(\mathbf{p})}(\mathbb{K}) \Rightarrow \mathbb{K} \preccurlyeq_n \mathbb{M}$.

Proof: Suppose $\mathbb{K}$ and $\mathbb{M}$ are **p**-models and $Th_{\Pi_n(\mathbf{p})}(\mathbb{M}) \subseteq Th_{\Pi_n(\mathbf{p})}(\mathbb{K})$. We want to prove: $\mathbb{K} \preccurlyeq_n \mathbb{M}$. Define:

- $kR_{zig,i}m :\Leftrightarrow Th_{\Pi_i(\mathbf{p})}(k) \supseteq Th_{\Pi_i(\mathbf{p})}(m)$, and $i>0 \Rightarrow Th_{\Pi_{i-1}(\mathbf{p})}(k) \subseteq Th_{\Pi_{i-1}(\mathbf{p})}(m)$.
- $kR_{zag,i}m :\Leftrightarrow Th_{\Pi_i(\mathbf{p})}(k) \subseteq Th_{\Pi_i(\mathbf{p})}(m)$ and $i>0 \Rightarrow Th_{\Pi_{i-1}(\mathbf{p})}(k) \supseteq Th_{\Pi_{i-1}(\mathbf{p})}(m)$.

We check that R is an $\mathfrak{z}\mathfrak{z}$-simulation.

Suppose e.g. $i>0$ and $kR_{zig,i}m$. We verify the zig property. Suppose $k \leq k'$. Let:

$\eta_i(k') := (\bigwedge\{B \in \Pi_{i-1}(\mathbf{p}) \mid k' \models B\} \rightarrow \bigvee\{C \in \Pi_i(\mathbf{p}) \mid k' \not\models C\})$.

Clearly $k \not\models \eta_i(k')$ and $\eta_i(k') \in \Pi_i(\mathbf{p})$. Ergo $m \not\models \eta_i(k')$. But then for some $m' \geq m$:

$m' \models \bigwedge\{B \in \Pi_{i-1}(\mathbf{p}) \mid k' \models B\}$ and $m' \not\models \bigvee\{C \in \Pi_i(\mathbf{p}) \mid k' \not\models C\}$.

It follows that $k'R_{zig,i}m'$.

We leave the rest of the verification to the reader. Since $b_{\mathbb{M}} \not\models \eta_n(b_{\mathbb{K}})$,we can find an m such that: $b_{\mathbb{K}}R_{zig,i}m$. ❑

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