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Fine Hierarchy and Definability in the Lindenbaum Algebra

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Abstract

The paper continues an earlier work on applications of a fine hierarchy defined by the author to the classification of index sets. We refine some results relating the fine hierarchy to Boolean terms, prove a useful completeness condition for the levels of the fine hierarchy, discuss a general scheme of the classification of definable index sets, and completely classify index sets of the predicates definable in the Lindenbaum algebra of sentences of any finite rich language.

1 Introduction

The paper continues an earlier work on the classification of index sets by means of hierarchies. In the development of this topic one can identify three stages. In the first stage people classified a lot of concrete index sets (which via a natural coding represent decision problems); most of them turned out to be m-complete in a level of the arithmetical hierarchy, see e.g. Ro67. The second stage consists of several examples (Er68, Hay69, Sc82, Se84, Le87) of the classification of some infinite sequences of index sets; this sometimes needs a refinement of the arithmetical hierarchy, e.g. the difference hierarchy. The third stage initiated by the author (Se83, Se89—Se92) tries to

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classify index sets of the predicates definable in some language; this needs a further refinement of the arithmetical hierarchy, namely the fine hierarchy (FH) $\{\Sigma_\alpha\}_{\alpha<\varepsilon_0}$ introduced in Se83.

The mentioned process of subsequent refinements of the arithmetical hierarchy is in some respects similar to the extensions of rationals in order to be able to measure natural geometric quantities. In this analogy the FH is similar to reals (or, maybe, to the algebraic reals): in Se83 it was shown that the sequence $\{\Sigma_\alpha\}$ has a natural closure property and contains finite levels of many hierarchies. The FH has several natural descriptions, the one from Se9? means that it is the finite version of the Wadge hierarchy of Borel sets. So one can hope to use the FH as a scale for the classification of (m-degrees of) naturally arising sets.

Recall that the degree structures were invented as instruments for the estimation of the undecidability of sets. The long (and not yet finished) investigation of these structures showed that they are extremely complicated. Using them one can not associate to a given set simple invariants (like ordinals) reflecting the unsolvability of the problem. But it remains the possibility to find simple substructures of the degree structures serving as scales for the estimation of most "naturally arising" sets. We believe that the sequence of m-degrees of Σ_α-complete sets ($\alpha < \varepsilon_0$) is such a scale of length $\varepsilon_0 = sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\}$ for the structure of m-degrees. The results of the cited papers show that this sequence picks up from the huge structure of m-degrees some really important elements.

The aim of this paper is two–fold. Firstly, we present a new clear description of the FH and of a general scheme of its application; we hope that this presentation is close to an optimal one. Secondly, we completely classify index sets of the predicates definable in the Lindenbaum algebra of sentences of any finite rich language; this is of some interest for the predicate logic and gives a good typical example of an application of the FH.

The paper is organized as follows: in Section 2 we formulate and discuss our main results, in Sections 3 and 4 we summarize some ground facts on the FH, and in Sections 5—7 we prove the main results. Many results of this paper strengthen or simplify earlier facts from Se91 and Se9?, so it would be good for the reader to have those papers at hand. Some facts from those papers are cited without proofs.

2 Statements and Discussion

Let T be the set of Boolean terms (i.e. terms in the language $\{\cup, \cap, \bar{}, 0, 1\}$ of Boolean algebras (BA)) with variables $v_k^n(k, n < \omega)$. We call $v_k^n(k < \omega)$ *variables of type n*. Relate to any term $t \in T$ the set $t[L]$ of all its values when the variables of type n range over the level Σ_{n+1}^0 of the arithmetical hierarchy; L denotes here the sequence $\{\Sigma_{n+1}^0\}_{n<\omega}$. In Se9? we stated the following relation of the introduced classes to the FH (as usual, Π_α denote the dual class for Σ_α, and $\Delta_\alpha = \Sigma_\alpha \cap \Pi_\alpha$):

2.1. Theorem. *The collections $\{t[L] : t \in T\}$ and $\{\Sigma_\alpha, \Pi_\alpha : \alpha < \varepsilon_0\}$ coincide.*

The result seems interesting for two reasons. Firstly, it gives a new natural description of the FH (for others see Se83, Se89 and the next section). Secondly, some its refinement implies useful completeness conditions for the levels of FH. To formulate the refinement we need some terminology.

Let F be a finite subset of $V = \{v_k^n | n, k \in \omega\}$ and T_F be the set of terms with variables in F. Relate to any $R \subseteq F$ the term $e_R = (\bigcap_{v \in R} v) \cap (\bigcap_{v \in F \setminus R} \bar{v})$ from T_F. By *a F-assignment* we mean a map A from F into the class $P(\omega)$ of subsets of ω such that $A_k^n = A(v_k^n) \in \Sigma_{n+1}^0$ for $v_k^n \in F$ (the assignment may be written as a sequence $\{A_k^n | v_k^n \in F\}$). For $t \in T_F$ let $t[A]$ denote the value of t on A (when v_k^n is interpreted as A_k^n).

Now let $\mathbf{F} = (F; \mathcal{F})$ be a pair with F as above and $\mathcal{F}$ a nonempty finite subset of $P(F)$. By *a $\mathbf{F}$-assignment* we mean a F-assignment A such that $e_R[A] = \emptyset$ for all $R \in P(F) \setminus \mathcal{F}$ (intuitively, $\mathcal{F}$ specifies the F-assignments satisfying all Boolean identities from $P(F) \setminus \mathcal{F}$). Let $t[L, \mathbf{F}]$ be the set of values of t on all $\mathbf{F}$-assignments. These sets are also closely related to the levels of the FH.

2.2. Theorem. *The collections $\{t[L, \mathbf{F}] : t \in T_F\}$ and $\{\Sigma_\alpha, \Pi_\alpha, \Delta_{1+\alpha} : \alpha < \varepsilon_0\}$ coincide.*

Relate to any F-assignment A the pair $\mathbf{F}_A = (F; \mathcal{F}_A)$, where $\mathcal{F}_A = \{R \subseteq F : e_R[A] \neq \emptyset\}$; A is clearly a $\mathbf{F}_A$-assignment. A F-assignment B is *m-reducible* to A (in symbols $B \leq_m A$), if there is a recursive function f such that $B_k^n = f^{-1}(A_k^n)$ for all $v_k^n \in F$. Note that if $B \leq_m A$ then B is an $\mathbf{F}_A$-assignment. We call A *a complete F-assignment* if any $\mathbf{F}_A$-assignment is m-reducible to A. This notion generalizes several similar notions in recursion theory, i.e. the notion of m-complete (or effectively inseparable) pair of disjoint r.e. sets.

Theorem 2.2 implies the following sufficient condition for a set to be m-complete in a level of the FH.

2.3. Theorem. *Any Boolean combination of members of a complete F-assignment is m-complete in one of levels* $\Sigma_\alpha, \Pi_\alpha, \Delta_{\alpha+1}$ $(\alpha < \varepsilon_0)$*, and all the possibilities are realized.*

This result means that the FH is essential for the classification of Boolean combinations of sets m-complete in levels of the arithmetical hierarchy. An easy example of this is the following result from Se91a.

2.4. Corollary. *Let* P_n *(for any* $n < \omega$*) be a* Σ^0_{n+1}*-complete predicate on* ω*. Then for any* $\alpha < \varepsilon_0$ *there is a Boolean combination of the predicates* P_n *which is* Σ_α*-complete.*

Theorem 2.3 is the main ingredient in the classification of definable index sets. In Section 7 we prove a deep and technically difficult result of this type formulated as follows. Let Ω be a finite language containing a symbol of arity > 1 or at least two unary functional symbols (we call such a language *rich*). Let B be the Lindenbaum algebra of sentences of Ω and β be the numeration of B induced by the Gödel numeration of Ω-sentences (recall that B is the quotient of the structure $(S; \wedge, \vee, \neg)$ of Ω-sentences by the equivalence in the predicate calculus).

2.5. Theorem. *For any formula* $\varphi(v_0, \ldots, v_k)$ *in the language of BA's, the set* $\{\langle x_0, \ldots, x_k\rangle | B \models \varphi(\beta x_0, \ldots, \beta x_k)\}$ *is m-complete in one of the levels* $\Sigma_\alpha, \Pi_\alpha, \Delta_{\lambda+1}$ $(\alpha, \lambda < \varepsilon_0$*,* λ *is limit), and all the possibilities are realized.*

2.6. Remark. Theorem 2.5 was first announced in Se92 (that paper contains much additional information on the Lindenbaum algebra) but in an inaccurate formulation: the levels $\Delta_{\lambda+1}$ were omitted. Another drawback in Se92 is in the formulation of the Hanf–Peretiatkin theorem: it is true only for finite languages. We apologize for these inaccuracies.

Earlier we proved some other results similar to 2.5. The proofs are long and have some common parts, so it makes sense to try to formulate a general framework for such results. We conclude this section by a discussion of this topic for a typical particular case.

Let $\mathbf{A} = (A; \alpha, L)$ be an *arithmetic structure*, i.e. $(A; L)$ is a structure of a finite language L, and α is a map from ω onto A in which L-functions are representable by recursive functions and the L-predicates as well as the equality relation are arithmetic. By *α-index set* of a predicate $P(v_0, \ldots, v_k)$ on A we mean the set $\alpha^{-1}(P)$ of all codes $\langle x_0, \ldots, x_k\rangle$ of tuples of numbers for which $P(\alpha x_0, \ldots, \alpha x_k)$ is true. By *definable index sets* (in $\mathbf{A}$) we mean index sets of the predicates P_φ defined by L-formulas φ. So Theorem 2.5 classifies m-degrees of definable index sets in the Lindenbaum algebra. In general, we study the structure $(DI_{\mathbf{A}}; \leq_m)$ of m-degrees of $\mathbf{A}$-definable index sets.

Call a structure $\mathbf{A}$ *easy* if any $\mathbf{A}$-definable index set is m-complete in a level of the FH, and call $\mathbf{A}$ *hard* if the structure $\mathbf{N} = (\omega; +, \cdot)$ is elementarily definable in $\mathbf{A}$ without parameters (the definition see e.g. in Er80). These notions are in a sense opposite to one another: for $\mathbf{A}$ easy the structure $(DI_{\mathbf{A}}; \leq_m)$ is very simple (*almost well–ordered*, i.e. well–founded and for all $\mathbf{a}, \mathbf{b} \in DI_{\mathbf{A}}$ either $\mathbf{a} \leq_m \mathbf{b}$ or $\{\bar{B} | B \in \mathbf{b}\} \leq_m \mathbf{a}$), and for $\mathbf{A}$ hard it is extremely complicated, as it follows from the next evident fact in which $\leq_m^h$ denote the m-reducibility by the functions recursive in h, and $\equiv_m^h$ is the equivalence relation generated by $\leq_m^h$.

2.7. Proposition. *For any hard structure* $\mathbf{A}$ *there exists an arithmetical oracle* h *such that the quotient of* $(DI_{\mathbf{A}}; \leq_m)$ *modulo* $\equiv_m^h$ *coincides with the structure of all arithmetical* $\leq_m^h$*-degrees.*

Sketch of proof. It suffices to find an arithmetical oracle h such that any arithmetical set $B, B \notin \{\emptyset, \omega\}$, is $\leq_m^h$-equivalent to a definable index set. Let $\varphi_0, \ldots, \varphi_n$ be L-formulas defining $\mathbf{N}$ in $\mathbf{A}$, and h be any arithmetical set such that (index sets of) all the L-predicates, the equality relation and the predicates $P_{\varphi_0}, \ldots, P_{\varphi_n}$ are recursive in h. Let P be the predicate on A corresponding to the predicate "$x \in B$" in the chosen definition of $\mathbf{N}$ in $\mathbf{A}$. Then $\alpha^{-1}(P) \equiv_m^h B$ completing the proof.

In proving that a given structure $\mathbf{A}$ is easy one usually should completely understand the definable predicates. A full classification of definable index sets includes the following stages:

(i) find a sequence $\varphi_k (k < \omega)$ of "easy" formulas such that any L-formula is equivalent in $\mathbf{A}$ to a Boolean combination of these formulas;

(ii) prove that for any $m < \omega$ there is a complete F-assignment A such that any set $\alpha^{-1}(P_{\varphi_k}), k < m$, is a member of this assignment;

(iii) by an additional analysis of the assignments in (ii) exclude the levels of FH not containing m-complete $\mathbf{A}$-definable index sets;

(iv) prove that any level excepting those from (iii) contains a m-complete $\mathbf{A}$-definable index set.

Note that (i), (ii) and Theorem 2.3 imply that $\mathbf{A}$ is easy, and (iii) and (iv) describe the levels containing m-complete definable index sets. Note also that (i) is purely logical, (ii) is recursion–theoretic and (iii), (iv) are made (using the results of this paper) by some routine computations. So the whole problem is divided into independent parts.

The analysis (i)—(iv) usually proceeds in an effective way implying the existence of an algorithm computing from a given L-formula φ the level in which the set $\alpha^{-1}(P_\varphi)$ is m-complete (e.g. this is the case in the proof of 2.5 below). It is easy to see that this effectiveness implies the decidability of the elementary theory of **A**. The notion of an easy structure generalize the well–known notion of a decidable (or, in terminology of Er80, strongly constructive) structure: any decidable structure is easy, even in the effective sense described above. But there are easy structures which are even not recursively presentable; an example is the structure from 2.5.

From the last paragraph it is clear that easy structures are rare. It turns out that some naturally arising structures are indeed hard. E.g. such are the structures of r.e. m- and Turing degrees (for the first one it is shown in Ni9?, and for the second one—in a recent work by A.Nies, R.Shore and T.Slaman). For such structures it makes no sense to try to classify all definable index sets in the FH. But it does make sense to classify some particular index sets (say, index sets of the predicates definable by the existential formulas). This is closely related to the decidability of restricted theories of such structures.

3 Fine Hierarchy

Here we summarize some facts on an abstract version of the FH. Most of them are contained in Se89, Se91 and (in a complete systematized form) in Se9?, so many proofs are omitted.

Let $(B;\cup,\cap,\bar{},0,1)$ be a BA with $0\neq 1$. By *a base* (in B) we mean any sequence $L=\{L_n\}_{n<\omega}$ of sublattices of $(B;\cup,\cap,0,1)$ satisfying $L_n\cup\check{L}_n\subseteq L_{n+1}$ (we use the following notation: for any $A\subseteq B$, let $\check{A}=\{\bar{a}|a\in A\}$ and $\tilde{A}=A\cap\check{A}$; $a\cap b$ is sometimes abbreviated to ab). Define the operation *Bisep* on subsets of B by

$$Bisep(X,Y_0,Y_1,Y_2)=\{x_0y_0\cup x_1y_1\cup\bar{x}_0\bar{x}_1y_2|x_i\in X,y_j\in Y_j,x_0x_1y_0=x_0x_1y_1\}.$$

3.1. Definition. By *the fine hierarchy over* L we mean the sequence $\{S_\alpha\}_{\alpha<\varepsilon_0}$, where $S_\alpha=S^0_\alpha$ and the classes $S^n_\alpha(n<\omega)$ are defined by induction on α: $S^n_0=\{0\}$; $S^n_{\omega^\gamma}=S^{n+1}_\gamma$ for $\gamma>0$; $S^n_{\beta+1}=Bisep(L_n,S^n_\beta,\check{S}^n_\beta,S^n_0)$ and $S^n_{\delta+\omega^\gamma}=Bisep(L_n,S^n_\delta,\check{S}^n_\delta,S^n_{\omega^\gamma})$ for $\delta=\omega^\gamma\cdot\delta_1>0$, $\gamma>0$.

Definition uses some ordinal arithmetic as described e.g. in KM67. To see that this definition is correct note that every nonzero ordinal $\alpha < \varepsilon_0$ is uniquely representable in the form $\alpha = \omega^{\gamma_0} + \cdots + \omega^{\gamma_k}$ for a finite sequence $\gamma_0 \geq \cdots \geq \gamma_k$ of ordinals $< \alpha$. Applying 3.1 we subsequently get $S^n_{\omega^{\gamma_0}}, S^n_{\omega^{\gamma_0}+\omega^{\gamma_1}}, \ldots, S^n_\alpha$. The classes S^n_γ for $n > 0$ play a technical role, they are among the classes S_α. Note that by Stone Representation Theorem we may (and sometimes will) think of them as of classes of sets.

Most important for this paper is the FH over $L = \{\Sigma^0_{n+1}\}$; we call it simply FH and denote S_α as Σ_α. Other examples are considered later on.

Note that $\cup_\alpha S_\alpha = \cup_n L_n$, and any L_n is among the classes S_α. Some other properties are given in the next assertion.

3.2. Properties. (i) *If $\alpha < \beta < \varepsilon_0$ then $S_\alpha \subseteq \tilde{S}_\beta$.*

(ii) *For any limit ordinal $\lambda < \varepsilon_0$, the classes S^n_λ and $\check{S}^n_\lambda$ are closed under intersection with any element of $\tilde{L}_{n+1}$.*

(iii) *For any $\alpha < \varepsilon_0$, $S^n_{\alpha+2}$ is the class of intersections of elements of $\check{S}^n_{\alpha+1}$ with the elements of L_n.*

(iv) *For any $\alpha < \varepsilon_0$, the class S^n_α (resp. $\check{S}^n_{1+\alpha}$) is closed under intersection with any element of L_n (resp. of $\tilde{L}_n$).*

(v) *For any $\alpha < \varepsilon_0$, if $u_0, u_1 \in \tilde{L}_n$ and $au_0, au_1 \in S^n_\alpha$ (resp. $\check{S}^n_\alpha$), then $a(u_0 \cup u_1) \in S^n_\alpha$ (resp. $\check{S}^n_\alpha$).*

For brevity we call the classes $S_\alpha \setminus \check{S}_\alpha$, $\check{S}_\alpha \setminus S_\alpha$ and $\tilde{S}_\alpha \setminus \cup_{\beta<\alpha}(S_\beta \cup \check{S}_\beta)$ respectively the S_α-, $\check{S}_\alpha$- and $\tilde{S}_\alpha$-*constituents* of the FH over L. The term is justified by the following

3.3. Corollary. *The constituents of the FH over L are pairwise disjoint and exhaust the class $\cup_n L_n$.*

Proof. The disjointness follows immediately from 3.2.(i), so it suffices to show that any element a from $\cup_n L_n$ belongs to a constituent. For some $\alpha < \varepsilon_0$ we have $a \in S_\alpha \cup \check{S}_\alpha$. Take the least such α, then a is in one of the α-constituents above. This completes the proof.

We are interested in the cases when some of the $\tilde{S}_\alpha$-constituents are empty.

3.4. Definition. (i) A base L is *discrete* if 1 is join–irreducible in L_0 (i.e. $1 = u \cup v$ in L_0 imply that either $u = 1$ or $v = 1$).

(ii) L is *interpolable* if for all $n < \omega$ any two disjoint elements $a, b \in \check{L}_{n+1}$ are separable by an element c of the BA generated by L_n (i.e. $a \subseteq c \subseteq \bar{b}$).

(iii) L *perfect* if it is both discrete and interpolable.

(iv) L is *reducible* if any L_n has the reduction property (i.e. for all $a, b \in$

L_n there are disjoint $a^*, b^* \in L_n$ with $a^* \subseteq a$, $b^* \subseteq b$ and $a^* \cup b^* = a \cup b$).

3.5. Proposition. *If L is discrete (interpolable, perfect) then the $\tilde{S}_\alpha$-constituents are empty for all successor (resp. limit, nonzero) ordinals α.*

Now we formulate a simpler description of the FH over a reducible base. Define a simplified version *bisep* of the operation *Bisep* as follows:

$$bisep(X, Y_0, Y_1, Y_2) = \{x_0y_0 \cup x_1y_1 \cup \bar{x}_0\bar{x}_1y_2 | x_i \in X, y_j \in Y_j, x_0x_1 = \emptyset\}.$$

3.6. Proposition. *Classes of the fine hierarchy over a reducible base coincide with the corresponding classes obtained by using the operation bisep in place of Bisep.*

Next we want to give an alternative, technically important description of the fine hierarchy. First some notation and terminology. Let $\omega^{<\omega}$ ($2^{<\omega}$) be the set of all finite strings of numbers (resp. of numbers < 2). By $\sigma \subseteq \tau$ we denote that the string σ is an initial segment of the string τ. By $|\sigma|$ we denote the length of a string σ, by $\sigma\tau$ (σk)—the concatenation of strings σ, τ (respectively of a string σ and a number k). For $k < 2$ let $\bar{k} = 1 - k$, and let $\bar{\sigma}$ be defined by $\bar{\emptyset} = \emptyset$ and $\overline{\sigma k} = \bar{\sigma}\bar{k}$. By 0^m we denote the string of m zeros.

Fix a string $\mu \in \omega^{<\omega}$. By *a μ-tree* (in B) we mean a sequence $\{a_\sigma\}_{\sigma \in 2^{<\omega}}$ of elements of B such that $a_\sigma = 0$ for $|\sigma| > |\mu|$, $a_{\sigma k} \in L_{\mu(|\sigma|)}$ for $|\sigma| < |\mu|$ and $a_\sigma \supseteq a_{\sigma k}$. A tree is called *reduced*, if $a_{\sigma 0}a_{\sigma 1} = 0$, and *special*, if $a_0 \cup a_1 = 1$. The elements $a^*_\tau = a_\tau \bar{a}_{\tau 0}\bar{a}_{\tau 1}$ are called *components* of a tree $\{a_\sigma\}$.

We say that an element $a \in B$ *is defined* by a tree $\{a_\sigma\}$, if $a \subseteq a_0 \cup a_1$, $aa_{\sigma 0} \subseteq a_{\sigma 00} \cup a_{\sigma 01}$ and $\bar{a}a_{\sigma 1} \subseteq a_{\sigma 10} \cup a_{\sigma 11}$. This notion does not depend on $a_\emptyset$; applying it we usually think that $a_\emptyset = 1$ (if not, just replace $a_\emptyset$ by 1). Let us cite from Se9? and Se91a some properties of the introduced notions.

3.7. Properties. (i) *$\{a_\sigma\}$ defines a iff $a = \cup_\sigma a^*_{\sigma 1}$ and $\bar{a} = a^*_\emptyset \cup (\cup_\sigma a^*_{\sigma 0})$.*

(ii) *$\{a_\sigma\}$ defines some element iff the components $a^*_\emptyset$, $a^*_{\sigma 0}$ are disjoint with the components $a^*_{\tau 1}$.*

(iii) *Any reduced tree has pairwise disjoint components, satisfies $\cup_\sigma a^*_\sigma = 1$ and defines some element.*

(iv) *If $\{a_\sigma\}$ defines a, then $\{a_\sigma x\}$ defines ax.*

(v) *If $\{a_\sigma\}$ defines ax, $\{b_\sigma\}$ defines ay, and $a_\sigma \subseteq x$, $b_\sigma \subseteq y$, then $\{a_\sigma \cup b_\sigma\}$ defines $a(x \cup y)$.*

(vi) *Let $\{a_\sigma\}$ defines a and let $b_\emptyset = 1$, $b_1 = a_1 \cup x$ and $b_\sigma = a_\sigma \bar{x}$ for $\sigma \neq \emptyset, 1$. Then $\{b_\sigma\}$ defines $a \cup x$.*

Let M_μ (M^s_μ, R_μ, R^s_μ) be the set of all elements defined by μ-trees (resp. by special, reduced and by special reduced μ-trees). The next result states some closure properties of these classes under the operation $Bisep$.

3.8. Lemma. *Let strings μ, ν and a number n be such that $n < \mu(i)$ for $i < |\mu|$ and $n \leq \nu(j)$ for $j < |\nu|$, and let $\xi = \mu n \nu$.*

(i) *Over any base, $Bisep(L_n, M_\nu, \check{M}_\nu, M_\mu) = M_\xi$ and in the case $\mu \neq \emptyset$ $Bisep(L_n, M_\nu, \check{M}_\nu, \tilde{M}_\mu) = \tilde{M}_\xi$.*

(ii) *Over any reducible base, the analog of (i) is true with bisep in place of Bisep and R in place of M.*

Proof. (i) The first equation was stated in the proof of Theorem 4.3 in Se9?, and it implies the inclusion from left to right in the second equation. So it remains to show that any $a \in \tilde{M}_\xi$ is in $Bisep(L_n, M_\nu, \check{M}_\nu, \tilde{M}_\mu)$. Let $u_i, v_i \in L_n, a_0, b_0, \bar{a}_1, \bar{b}_1 \in M_\nu$ and $a_2, b_2 \in M_\mu$ be some elements satisfying

$$\begin{aligned} a &= u_0 a_0 \cup u_1 a_1 \cup \bar{u}_0 \bar{u}_1 a_2, \quad u_0 u_1 a_0 = u_0 u_1 a_1, \\ \bar{a} &= v_0 b_0 \cup v_1 b_1 \cup \bar{v}_0 \bar{v}_1 b_2, \quad v_0 v_1 b_0 = v_0 v_1 b_1. \end{aligned} \tag{1}$$

For the elements $x_0 = u_0 \cup v_1$, $x_1 = u_1 \cup v_0$, $e_0 = a_0 u_0 \cup \bar{b}_1 v_1$, $\bar{e}_1 = \bar{a}_1 u_1 \cup b_0 v_0$ and $e_2 = a_2 \bar{x}_0 \bar{x}_1$ we have $a = x_0 e_0 \cup x_1 e_1 \cup \bar{x}_0 \bar{x}_1 e_2$ and $x_0 x_1 e_0 = x_0 x_1 e_1$ (because $a x_0 = e_0$ and $\bar{a} x_1 = \bar{e}_1$), so it remains to show that $e_0, \bar{e}_1 \in M_\nu$ and $e_2 \in \tilde{M}_\mu$.

First note that 3.7.(iv),(v) and (vi) imply that for all $d \in M_\nu, c \in M_\mu$ and $y, z \in L_n$ we have: $dy \in M_\nu$; if $dy, dz \in M_\nu$ then $d(y \cup z) \in M_\nu$; if $c \subseteq \bar{y}$ then $c \cup y \in M_\mu$ (the last fact only for $\mu \neq \emptyset$). Now, $e_0 = a u_0 \cup a v_1$ and $a u_0 = a_0 u_0 \in M_\nu, a v_1 = \bar{b}_1 v_1 \in M_\nu$, so $e_0 \in M_\nu$ and similarly $\bar{e}_1 \in M_\nu$. Finally, we have $e_2 = a_2 \bar{x}_0 \bar{x}_1 \in M_\mu$ (because $\bar{x}_0 \bar{x}_1 \in L_{n+1}$ and $n+1 \leq \mu(i)$ for $i < |\mu|$) and $\bar{e}_2 \bar{x}_0 \bar{x}_1 = \bar{a} \bar{x}_0 \bar{x}_1 = b_2 \bar{x}_0 \bar{x}_1 \in M_\mu$, so $\bar{e}_2 = \bar{e}_2 \bar{x}_0 \bar{x}_1 \cup x_0 \cup x_1 \in M_\mu$ and $e_2 \in \tilde{M}_\mu$.

(ii) is also proved by the argument above, because we can take the elements u_i, v_i above pairwise disjoint (replacing them on pairwise disjoint elements u_i^*, v_i^* so that $u_i^* \subseteq u_i, v_i^* \subseteq v_i$ and $u_0^* \cup u_1^* \cup v_0^* \cup v_1^* = u_0 \cup u_1 \cup v_0 \cup v_1$ existing by the reduction property). This completes the proof of the lemma.

3.9. Lemma. *Over any base, $\tilde{M}_\xi = M^s_\xi$. Over any reducible base, $\tilde{R}_\xi = R^s_\xi$.*

Proof. If a (reduced) special ξ-tree $\{a_\sigma\}$ defines a then $\{a_{\bar{\sigma}}\}$ is a (reduced) special ξ-tree defining $\bar{a}$. This proves the inclusions from right to left. The reverse inclusion is checked by induction on $|\xi|$. The case $\xi = \emptyset$ is trivial, so let ξ be nonempty and $a \in \tilde{M}_\xi$. Represent ξ as above, $\xi = \mu n \nu$, and consider first the case when μ is nonempty. Represent a as in the preceding proof: $a = x_0 e_0 \cup x_1 e_1 \cup \bar{x}_0 \bar{x}_1 e_2$, so in particular $e_2 \in \tilde{M}_\mu$. By induction hypothesis, $e_2 \in M^s_\mu$. Choose ν-trees $\{a_\sigma\}, \{b_\sigma\}$ and a special μ-tree $\{c_\tau\}$ defining respectively $e_0, \bar{e}_1$ and e_2. By 3.7.(iv), the ν-trees $\{a_\sigma x_0\}, \{b_\sigma x_1\}$ and the μ-tree $\{c_\tau \bar{x}_0 \bar{x}_1\}$ define respectively the elements $a x_0 = x_0 e_0, \bar{a} x_1 = x_1 \bar{e}_1$ and $a \bar{x}_0 \bar{x}_1 = \bar{x}_0 \bar{x}_1 e_2$. Define a ξ-tree $\{d_\rho\}$ as follows:

$d_\emptyset = 1$; $d_{0^l} = c_{0^l} \bar{x}_0 \bar{x}_1 \cup x_0 \cup x_1$ for $0 < l \leq |\mu|$; $d_\rho = c_\rho \bar{x}_0 \bar{x}_1$ for $|\rho| \leq |\mu|, \rho \not\subseteq 0^{|\mu|}$; $d_{0^{|\mu|}k} = x_k$ for $k < 2$; $d_{0^{|\mu|}0\sigma k} = a_{\sigma k} x_0$, $d_{0^{|\mu|}1\sigma k} = \overline{b_{\sigma k}} x_1$ for $k < 2$, $|\sigma| < |\nu|$; $d_\rho = 0$ for $|\rho| > |\mu|, \rho \not\supseteq 0^{|\mu|}$.

By cases it is not difficult to check that $\{d_\rho\}$ defines a. But $\{d_\rho\}$ is special, because $d_0 \cup d_1 = (c_0 \bar{x}_0 \bar{x}_1 \cup x_0 \cup x_1) \cup c_1 \bar{x}_0 \bar{x}_1 = 1$. So $a \in M^s_\xi$, as desired.

For the case of empty string μ represent $a \in \tilde{M}_\xi$ as in (1) with a_2, b_2 being empty. Let $e_i, x_i (i < 2)$ be the elements from the proof of 3.8, then in particular $a = e_0 x_0 \cup e_1 x_1$ and $x_0 \cup x_1 = 1$. Define $\{d_\rho\}$ as above (for the trivial μ-tree $\{c_\tau\}$). Then $d_i = x_i$, so $\{d_\rho\}$ is a special ξ-tree defining a and *a fortiori* $a \in M^s_\xi$.

For the reducible case the proof is the same with all trees above being reduced and x_0, x_1 disjoint. This completes the proof of the lemma.

Now we can describe the FH in terms of trees. Define strings $\mu^n_\alpha (n < \omega)$ by induction on α as follows: $\mu^n_0 = \emptyset$, $\mu^n_{\alpha+1} = n \mu^n_\alpha$, $\mu^n_{\omega^\gamma} = \mu^{n+1}_\gamma$ for $\gamma > 0$, $\mu^n_{\delta+\omega^\gamma} = \mu^n_{\omega^\gamma} n \mu^n_\delta$ for $\delta = \omega^\gamma \cdot \delta_1 > 0$, $\gamma > 0$. Let $\mu_\alpha = \mu^0_\alpha$.

3.10. Proposition. *Over any base, $S_\alpha = M_{\mu_\alpha}$ and $\tilde{S}_\alpha = M^s_{\mu_\alpha}$. Over any reducible base, $S_\alpha = R_{\mu_\alpha}$ and $\tilde{S}_\alpha = R^s_{\mu_\alpha}$.*

Proof. By induction on α we check that $S^n_\alpha = M_{\mu^n_\alpha}$ (and $S^n_\alpha = R_{\mu^n_\alpha}$ in the reducible case) for all n. For $\alpha = 0, \omega^\gamma$ this is evident. For $\alpha = \beta + 1$ we have by induction hypothesis and by 3.9

$$S^n_\alpha = Bisep(L_n, S^n_\beta, \check{S}^n_\beta, S^n_0) = Bisep(L_n, M_{\mu^n_\beta}, \check{M}_{\mu^n_\beta}, M_\emptyset) = W_{n \mu^n_\beta} = M_{\mu^n_\alpha}.$$

The case $\alpha = \delta + \omega^\gamma$ is considered in the same way.

Now, by 3.9 we have $\tilde{S}_\alpha = \tilde{M}_{\mu_\alpha} = M^s_{\mu_\alpha}$, and similarly in the reducible case. This completes the proof.

We conclude this section by a result relating the fine hierarchies over different bases. By *a morphism* of a base L into a base L' (in B') we mean a BA-homomorphism $g : \cup_n L_n \to \cup_n L'_n$ sending any L_n into L'_n. The notion of an isomorphism on bases looks similarly. In the next result the assertion (i) is immediate by definition, and (ii) by induction on α.

3.11. Proposition. (i) *For any m, $\{S^m_\alpha\}$ is the FH over $\{L_k\}_{k\geq m}$.*

(ii) *Any morphism (isomorphism) g from L into (resp. onto) L' sends any class S^n_α into (resp. bijectively onto) $S^{n\prime}_\alpha$.*

4 Syntactic Fine Hierarchy

Here we study technically useful fine hierarchies over some bases defined in terms of the typed variables $v^n_k \in V$ from Section 2. The word "syntactic" is not very informative, it just stresses the specific nature of these hierarchies.

Fix a pair $\mathbf{F} = (F; \mathcal{F})$ formed by a finite set $F \subseteq V$ and a nonempty family $\mathcal{F} \subseteq P(F)$. Define preorderings $\leq_n$ $(n < \omega)$ on $P(F)$ as follows: $R \leq_n S$, if $R^{<n} = S^{<n}$ and $R^n \subseteq S^n$, where $R^n(R^{<n})$ denote the set of variables of type n (resp. of type $< n$) in R. Note that for sufficiently large n the relation $\leq_n$ is the equality, and that $R \leq_{n+1} S$ imply $R \equiv_n S$ ($\equiv_n$ is the equivalence relation generated by $\leq_n$). Let $\mathcal{L}_n$ be the class of all subsets of $\mathcal{F}$ closed upwards under $\leq_n$, and $\mathcal{L} = \mathcal{L}^{\mathbf{F}} = \{\mathcal{L}_n\}$. The next result is clear.

4.1. Lemma. *For any $\mathbf{F}$, $\mathcal{L}^{\mathbf{F}}$ is a base satisfying $\cup_n \mathcal{L}_n = P(\mathcal{F})$.*

We will study the FH $\{\mathcal{S}_\alpha\}$ over $\mathcal{L}$ called also the (syntactic) FH over $\mathbf{F}$. The next result provides invariants for all constituents of this hierarchy.

For a given $\mu \in \omega^{<\omega}$, by *a μ-alternating tree* for $\mathcal{X}$ we mean a sequence $\{R_\sigma : \sigma \in 2^{<\omega}, |\sigma| \leq |\mu|\}$ of elements of $\mathcal{F}$ such that $R_\emptyset \notin \mathcal{X}$ and $R_{\sigma 0} \notin \mathcal{X}, R_{\sigma 1} \in \mathcal{X}, R_\sigma \leq_{\mu_\alpha(|\sigma|)} R_{\sigma k}$ for $|\sigma| < |\mu|$ and $k < 2$. By *a special μ-alternating tree* for $\mathcal{X}$ we mean a sequence $\{R_\sigma : 0 < |\sigma| \leq |\mu|\}$ of elements of $\mathcal{F}$ satisfying the properties above (with the exception $R_\emptyset$) and the additional property $R_0 \equiv_k R_1$ for $k = \mu(0) - 1$ (for $k = -1$ we think that $R \equiv_k S$ for all $R, S \in \mathcal{F}$). For a nonempty μ such a tree may be considered as the "pair" of trees $\{R_{0\sigma}, R_{1\sigma}\}_{|\sigma|<|\mu|}$. Let μ_α be the string from Proposition 3.10.

4.2. Proposition. (i) *For all $\mathcal{X} \subseteq \mathcal{F}$ and $\alpha < \varepsilon_0$, $\mathcal{X} \notin \check{\mathcal{S}}_\alpha$ iff there is a μ_α-alternating tree for $\mathcal{X}$.*

(ii) *For all $\mathcal{X} \subseteq \mathcal{F}$ and nonzero $\alpha < \varepsilon_0$, $\mathcal{X} \notin \cup_{\beta<\alpha}(\mathcal{S}_\beta \cup \check{\mathcal{S}}_\beta)$ iff there is a special μ_α-alternating tree for $\mathcal{X}$.*

Proof. (i) One direction was stated in the proof of Theorem 5.1 in Se9? (see also Se91), the other is similar to the proof of Theorem 2 in Se91a.

(ii) Let first $\alpha = \beta + 1$. By (i), $\mathcal{X} \notin \mathcal{S}_\beta \cup \check{\mathcal{S}}_\beta$ iff there are μ_β-alternating trees $\{R_\sigma\}$ and $\{S_\sigma\}$ for $\mathcal{X}$ and $\bar{\mathcal{X}}$ respectively. This is equivalent to the existence of a a special μ_α-alternating tree for $\mathcal{X}$ (consider the pair of trees $\{R_\sigma, S_{\bar\sigma}\}$ and note that $\mu_\alpha(0) = 0$ and $R_\emptyset \equiv_{-1} S_\emptyset$).

The case of a limit ordinal α is more tedious. First let us relate to any such ordinal the ordinals $\alpha_k = g(\alpha, k)$ by induction: $g(\omega^{\beta+1}, k) = \omega^\beta(k+1)$, $g(\omega^\lambda, k) = \omega^{g(\lambda,k)}$ for a limit λ, $g(\delta + \omega^\gamma, k) = \delta + g(\omega^\gamma, k)$ for $\delta = \omega^\gamma \cdot \delta_1 > 0$ and $\gamma > 0$. It is clear that for $\alpha = \omega$ and $\alpha = \delta + \omega$ all α_k are successors, in other cases all α_k are limit ordinals, and in any case $\alpha_0 < \alpha_1 < \cdots$ and $\alpha = sup\{\alpha_k | k < \omega\}$.

It suffices to show that $\mathcal{X} \notin \cup_k \mathcal{S}_{\alpha_k}$ (i.e. $\mathcal{X}$ has a μ_{α_k}-alternating tree for any k) iff $\mathcal{X}$ has a special μ_α-alternating tree. By induction on α we prove the following more general assertion:

$(*)$ *for all $n < \omega, m \leq n$ and $\xi \in \omega^{<\omega}$, $\mathcal{X}$ has a $\mu^n_{\alpha_k} m\xi$-alternating tree for any k iff $\mathcal{X}$ has a special $\mu^n_\alpha m\xi$-alternating tree.*

We shorten $\mu^n_\alpha m\xi$ and $\mu^n_{\alpha_k} m\xi$ respectively to ν and ν_k and consider the following alternatives for the ordinal α: ω, $\omega^{\gamma+1}$ for a nonzero γ, ω^λ for a limit λ, $\delta + \omega$, $\delta + \omega^{\gamma+1}$, $\delta + \omega^\lambda$. In the last three cases $\gamma > 0$, λ is limit and δ satisfy the usual condition, as above.

For the case $\alpha = \omega$ we have $\mu^n_\alpha = (n+1)$, $\alpha_k = k+1$ and $\mu^n_{\alpha_k} = (n, \ldots, n)$, the last string is of length $k+1$. Suppose for a contradiction that the direct implication in $(*)$ is false, and choose k bigger than the cardinality of $\mathcal{F}$. Then $\mathcal{X}$ has no special ν-alternating tree but has a ν_k-aternating tree $\{R_\sigma\}$. By definition, $R_\sigma \leq_n R_{\sigma i}$ for $|\sigma| \leq k$ and $i < 2$. We claim that $R_{\sigma 0} \not\equiv_n R_{\sigma 1}$ (otherwise let $S_i = R_{\sigma i}$ and $S_{i\tau} = R_{\sigma i i^p \tau}$ for $i < 2, 0 < |\tau| \leq |\xi| + 1$ and $|\sigma i^p| = k$; from $m \leq n$ it follows that $\{S_{0\tau}, S_{1\tau}\}_{|\tau| < |\nu|}$ is a special ν-alternating tree for $\mathcal{X}$ which is a contradiction). So for any σ of length $\leq k$ there is $i < 2$ satisfying $R_\sigma <_n R_{\sigma i}$. But then there is a $<_n$-chain of k elements of $\mathcal{F}$ contradicting to the choice of k.

In the opposite direction, from a special ν-alternating tree $\{S_{i\tau}\}$ for $\mathcal{X}$ one constructs a ν_k-alternating tree (for any k) $\{R_\sigma\}$ for $\mathcal{X}$ as follows: $R_\sigma = S_0$ for $|\sigma| \leq k$, and $R_{\sigma i \tau} = S_{i\tau}$ for $|\sigma| = k$, $i < 2$ and $|\tau| < |\nu|$.

For the case $\alpha = \omega^{\gamma+1}$ we have $\mu^n_\alpha = \mu^{n+1}_{\gamma+1} = (n+1)\rho$ (where $\rho = \mu^{n+1}_\gamma$), $\alpha_k = \omega^\gamma(k+1)$ and $\mu^n_{\alpha_k} = \rho n \rho n \cdots \rho$ (n occurs k times). Let $l_j = (j+1)|\rho| + j$, so $\nu_k(l_k) = m, \nu_k(l_j) = n$ and $\nu(l_j + a) > n$ for all $j < k$ and $a < |\rho|$.

Suppose that the direct implication in $(*)$ is false and choose k and $\{R_\sigma\}$ as in the case $\alpha = \omega$. Define a function f on $2^{<\omega}$ by $f(\emptyset) = \emptyset$ and $f(\sigma i) = f(\sigma) 0^{|\rho|} i$. Then $|f(\sigma i)| = l_{|\sigma|} + 1$, so $R_{f(\sigma)} \leq_n R_{f(\sigma i)}$ for $|\sigma| < k$ and $i < 2$.

As above, it suffices to show that $R_{f(\sigma 0)} \not\equiv_n R_{f(\sigma 1)}$ for $|\sigma| < k$. Suppose not and set $p = l_k - l_{|\sigma|+1}$, $S_{i\eta} = R_{f(\sigma i)\eta}$ for $|\eta| \leq |\rho|$ and $S_{i\eta\tau} = R_{f(\sigma i)\eta i p_\tau}$ for $i < 2, 0 < |\tau| \leq |\xi| + 1$ and $|\eta| = |\rho|$. From $m \leq n$ and properties of l_j it follows that $\{S_{0\tau}, S_{1\tau}\}_{|\tau|<|\nu|}$ is a ν-alternating tree for $\mathcal{X}$ which is a contradiction.

The opposite implication is considered similarly to the previous case.

In the case $\alpha = \omega^\lambda$ we have $\mu^n_\alpha = \mu^{n+1}_\lambda$ and $\mu^n_{\alpha_k} = \mu^{n+1}_{\lambda_k}$. By induction hypothesis $(*)$ is true for λ, so it is true also for α. In the case $\alpha = \delta + \omega$ we have $\mu^n_\alpha = \mu^n_\omega n\delta$ and $\mu^n_{\alpha_k} = n \dots n\delta$, so the assertion $(*)$ immediately follows from the case $\alpha = \omega$. The last two cases follow respectively from the cases $\alpha = \omega^{\gamma+1}$ and $\alpha = \omega^\lambda$ in the same way. This completes the proof.

Now we state conditions on $\mathbf{F}$ implying the emptiness of some constituents. For $R \in \mathcal{F}$ and $n < \omega$, let $[R] = [R]_{n+1}$ denote the set of all $S \in \mathcal{F}$ with $S \leq_{n+1} R$. We say that $\mathbf{F}$ is: *discrete*, if $(\mathcal{F}; \leq_0)$ has a least element; *interpolable*, if for all $n < \omega$ any two disjoint sets of the form $[R], [S]$ defined above are separable by an element of the BA $(\mathcal{L}_n)$ generated by $\mathcal{L}_n$; *perfect*, if it is both discrete and interpolable.

The next evident fact gives examples illustrating the introduced notions for the pairs $\mathcal{G}_n = (\{v^n_0\}; \{\emptyset, \{v^n_0\}\})$ and $\mathcal{H}_n = (\{v^n_0, v^n_1\}; \{\{v^n_0\}, \{v^n_1\}\})$.

4.3. Lemma. (i) *The pairs $(\emptyset; \{\emptyset\})$ and $\mathcal{G}_n$ (for any $n < \omega$) are perfect.*

(ii) *The pair $\mathcal{H}_0$ is interpolable but not discrete.*

(iii) *For any n, the pair $\mathcal{H}_{n+1}$ is discrete but not interpolable.*

The next result relates the introduced notions to the emptiness of the constituents.

4.4. Proposition. *If $\mathbf{F}$ is discrete (interpolable, perfect) then the $\tilde{S}_\alpha$-constituents are empty for all successor (resp. limit, nonzero) ordinals α.*

Proof. By Proposition 3.5, it suffices to show that $\mathbf{F}$ is discrete (interpolable, perfect) iff the base $\mathcal{L}^{\mathbf{F}}$ has the same property. For the case of discreteness this is clear.

Let $\mathbf{F}$ be interpolable and disjoint $\mathcal{X}, \mathcal{Y} \in \check{\mathcal{L}}_{n+1}$ be given; we have to separate $\mathcal{X}$ from $\mathcal{Y}$ by a class from $(\mathcal{L}_n)$. By definition of $\mathcal{L}_{n+1}$, $\mathcal{X} = \cup_i [R_i]$ and $\mathcal{Y} = \cup_j [S_j]$ for some finite number of $R_i, S_j \in \mathcal{F}$. Let $\mathcal{Z}_{ij} \in (\mathcal{L}_n)$ separates $[R_i]$ from $[S_j]$, i.e. $[R_i] \subseteq \mathcal{Z}_{ij} \subseteq \overline{[S_j]}$. Then $\mathcal{X} \subseteq \mathcal{Z} \subseteq \overline{\mathcal{Y}}$ for $\mathcal{Z} = \cap_j \cup_i \mathcal{Z}_{ij} \in (\mathcal{L}_n)$, so $\mathcal{Z}$ separates $\mathcal{X}$ from $\mathcal{Y}$. The opposite direction (that $\mathbf{F}$ is interpolable if $\mathcal{L}^{\mathbf{F}}$ is) is trivial.

The case of perfectness immediately follows from the preceding cases. This completes the proof.

Call pairs $\mathbf{F} = (F;\mathcal{F})$ and $\mathbf{G} = (G;\mathcal{G})$ *syntactically isomorphic* if there is a bijection $g : F \to G$ respecting the types of variables and satisfying $\mathcal{G} = \{g(R)|R \in \mathcal{F}\}$. Any such g induces an isomorphism of the bases $\mathcal{L}^{\mathbf{F}}, \mathcal{L}^{\mathbf{G}}$ and (by 3.11.(ii)) of the corresponding fine hierarchies.

Relate to any pairs $\mathbf{F} = (F;\mathcal{F})$ and $\mathbf{G} = (G;\mathcal{G})$ with disjoint F, G the pair $\mathbf{H} = (H;\mathcal{H})$ as follows: $H = F \cup G$ and $\mathcal{H} = \{R \subseteq H|RF \in \mathcal{F}, RG \in \mathcal{G}\}$ (the disjointness above is not essential because there is always a pair $\mathbf{G}'$ syntactically isomorphic to $\mathbf{G}$ and satisfying $F \cap G' = \emptyset$). Let us state some properties of the introduced operation.

4.5. Lemma. *If the pairs* $\mathbf{F} = (F;\mathcal{F})$ *and* $\mathbf{G} = (G;\mathcal{G})$ *are discrete (interpolable, perfect) then so is also the pair* $\mathbf{H} = \mathbf{F} \sqcup \mathbf{G}$.

Proof. The map $R \mapsto (RF, RG)$ is clearly an isomorphism of $(\mathcal{H};\leq_n)$ onto the Cartesian product $(\mathcal{F};\leq_n) \times (\mathcal{G};\leq_n)$ for any $n < \omega$. This fact and Proposition 4.4 immediately imply our assertion for the case of discreteness. This implies also that for all $R, S \in \mathcal{H}$ and $n < \omega$

$$[R][S] \neq \emptyset \ \ iff \ \ [RF][SF] \neq \emptyset \ \ and \ \ [RG][SG] \neq \emptyset, \qquad (2)$$

where $[R]$ is as above, and that the map $\mathcal{X} \mapsto \mathcal{X}^* = \{R \subseteq H|RF \in \mathcal{X}\}$ is a morphism from $\mathcal{L}^{\mathbf{F}}$ into $\mathcal{L}^{\mathbf{H}}$ (and similarly for G).

It remains to prove the lemma for the case of interpolability (for the perfectness it would then follow). Let n and disjoint elements $[R], [S]$ of $\check{\mathcal{L}}^{\mathbf{H}}_{n+1}$ be given; we have to separate the elements by a class from $(\mathcal{L}^{\mathbf{H}}_n)$. By (2), either the classes $[RF], [SF]$ or the classes $[RG], [SG]$ are disjoint. Consider e.g. the first case, then $[RF] \subseteq \mathcal{X} \subseteq \overline{[SF]}$ for some $\mathcal{X} \in (\mathcal{L}^{\mathbf{F}}_n)$. By the preceding paragraph, $[RF]^* \subseteq \mathcal{X}^* \subseteq \overline{[SF]}^*$ and $\mathcal{X}^* \in (\mathcal{L}^{\mathbf{H}}_n)$. But $[R] \subseteq [RF]^*$ and $[S] \subseteq [SF]^*$, so $\mathcal{X}^*$ separates $[R]$ from $[S]$ completing the proof.

We conclude this section by a result showing that Proposition 4.4 is optimal in the sense that all the constituents not excluded by this proposition can be nonempty.

4.6. Proposition. (i) *For any* $\alpha < \varepsilon_0$ *there exists a perfect pair with the nonempty* $\mathcal{S}_\alpha$*-constituent.*

(ii) *For any* $\alpha < \varepsilon_0$ *there exists an interpolable pair with the nonempty* $\tilde{\mathcal{S}}_{\alpha+1}$*-constituent.*

(iii) *For any limit* $\alpha < \varepsilon_0$ *there exists a discrete pair with the nonempty* $\tilde{\mathcal{S}}_\alpha$*-constituent.*

Proof. (i) It suffices to construct by induction on α a perfect pair $\mathbf{F}^n_\alpha (n < \omega)$ and a class $\mathcal{X}^n_\alpha \in \mathcal{S}^n_\alpha \setminus \check{\mathcal{S}}^n_\alpha$ (in the FH over $\mathbf{F}^n_\alpha$). We use the objects from

4.3 and 4.5 and sometimes omit the superscript n. Consider the following alternatives for the ordinal α: 0,1, ω^γ for a nonzero γ, $\beta+1$ for a successor β, $\beta+1$ for a limit β, $\beta+\omega^\gamma$ for $\beta=\omega^\gamma\cdot\beta_1>0$ and $\gamma>0$.

In the first two cases the objects $\mathbf{F}_0=(\emptyset;\{\emptyset\})$, $\mathbf{F}_1=\mathbf{G}_n$, $\mathcal{X}_0=\emptyset$ and $\mathcal{X}_1=\{\{v_0^n\}\}$ clearly work. The case $\alpha=\omega^\gamma$ is immediate by induction: let $\mathbf{F}_\alpha^n=\mathbf{F}_\gamma^{n+1}$ and $\mathcal{X}_\alpha^n=\mathcal{X}_\gamma^{n+1}$. For $\alpha=\beta+1$ and β successor, let $\mathbf{F}_\alpha=\mathbf{F}_\beta\sqcup\mathbf{G}_n$ and $\mathcal{X}_\alpha=\bar{\mathcal{X}}_\beta^*\mathcal{X}_1^*$. For $\alpha=\beta+1$ and β limit, let $\mathbf{F}_\alpha=\mathbf{F}_\beta\sqcup\mathbf{G}_n\sqcup\mathbf{G}_n$ and $\mathcal{X}_\alpha=\mathcal{Y}_0\mathcal{U}_0\cup\mathcal{Y}_1\mathcal{U}_1$, where $\mathcal{U}_i=\{\{u_i\}\}^*$ (u_0,u_1 are the variables from $\mathbf{G}_n\sqcup\mathbf{G}_n$) and $\mathcal{Y}_0=\mathcal{X}_\beta^*\bar{\mathcal{U}}_1$, $\mathcal{Y}_1=\bar{\mathcal{X}}_\beta^*\bar{\mathcal{U}}_0$. Finally, for $\alpha=\beta+\omega^\gamma$ let $\mathbf{F}_\alpha=\mathbf{F}_\beta\sqcup\mathbf{F}_{\omega^\gamma}\sqcup\mathbf{G}_n\sqcup\mathbf{G}_n$ and $\mathcal{X}_\alpha=\mathcal{Y}_0\mathcal{U}_0\cup\mathcal{Y}_1\mathcal{U}_1\cup\mathcal{X}_{\omega^\gamma}^*\bar{\mathcal{U}}_0\bar{\mathcal{U}}_1$.

By 4.3 and 4.5, any $\mathbf{F}_\alpha^n$ is perfect. From 3.2 and $\mathcal{Y}_0\mathcal{U}_0\mathcal{U}_1=\mathcal{Y}_1\mathcal{U}_0\mathcal{U}_1$ it follows that $\mathcal{X}_\alpha^n\in\mathcal{S}_\alpha^n$. It remains to check that $\mathcal{X}_\alpha^n\notin\check{\mathcal{S}}_\alpha^n$. By 4.2, it suffices to find a μ_α^n-alternating tree for $\mathcal{X}_\alpha^n$. For the first three cases above this is trivial. Let $\alpha=\beta+1$ and β be a successor. By induction hypothesis, there is a μ_β^n-alternating tree $\{R_\sigma\}$ for $\mathcal{X}_\beta$. Let $T_\emptyset=T_0=R_\emptyset$, $T_{0\sigma}=R_{\bar\sigma}\cup\{v_0^n\}$ for $0<|\sigma|\le|\mu_\beta^n|$ and $T_{1\sigma}=R_{\bar\sigma}\cup\{v_0^n\}$ for $|\sigma|\le|\mu_\beta^n|$. Then $\{T_\rho\}$ is a μ_α^n-tree for $\mathcal{X}_\alpha^n$.

The remaining cases are similar to one another, so consider only the case $\alpha=\beta+\omega^\gamma$. By induction hypothesis, there are a μ_β-alternating tree $\{R_\sigma\}$ for $\mathcal{X}_\beta$ and a μ_{ω^γ}-alternating tree $\{S_\tau\}$ for $\mathcal{X}_{\omega^\gamma}$. Let $T_\tau=S_\tau$ for $|\tau|\le|\mu_{\omega^\gamma}|$ and $T_{\tau0\sigma}=S_\tau\cup R_\sigma\cup\{u_0\}$, $T_{\tau1\sigma}=S_\tau\cup R_{\bar\sigma}\cup\{u_0\}$ for $|\tau|=|\mu_{\omega^\gamma}|,|\sigma|\le|\mu_\beta^n|$. Then $\{T_\rho\}$ is a μ_α^n-alternating tree for $\mathcal{X}_\alpha^n$ completing the proof of (i).

(ii) Let $\mathbf{F}=\mathbf{F}_\alpha^0\sqcup\mathbf{H}_0$ and $\mathcal{X}=\mathcal{X}_\alpha^*\mathcal{U}_0\cup\bar{\mathcal{X}}_\alpha^*\mathcal{U}_1$, where $\mathcal{U}_i=\{\{v_i^0\}\}$. Then $\mathbf{F}$ is interpolable, $\mathcal{U}_0\mathcal{U}_1=\emptyset$, $\mathcal{U}_0\cup\mathcal{U}_1=\mathcal{F}$ and $\mathcal{U}_0,\mathcal{U}_1\in\mathcal{L}_0$, so $\mathcal{X}\in\tilde{\mathcal{S}}_{\alpha+1}$. As above, both $\mathcal{X}$ and $\bar{\mathcal{X}}$ have μ_α^0-alternating trees, so $\mathcal{X}\notin\mathcal{S}_\alpha\cup\check{\mathcal{S}}_\alpha$.

(iii) By induction on α we define the pairs $\mathbf{E}_\alpha^n$ and the classes $\mathcal{Z}_\alpha^n$, considering the following alternatives for α: ω, ω^λ for a limit λ, $\omega^{\gamma+1}$ for a nonzero γ, $\beta+\omega^\gamma$ for $\beta=\omega^\gamma\cdot\beta_1>0$ and $\gamma>0$.

Let $\mathbf{E}_\omega^n=\mathbf{H}_{n+1}$ and $\mathcal{Z}_\omega^n=\{\{v_0^{n+1}\}\}$. For $\alpha=\omega^\lambda$ let $\mathbf{E}_\alpha^n=\mathbf{E}_\lambda^{n+1}$ and $\mathcal{Z}_\alpha^n=\mathcal{Z}_\lambda^{n+1}$. For $\alpha=\omega^{\gamma+1}$ let $\mathbf{E}_\alpha^n=\mathbf{F}_{\omega^\gamma}^n\sqcup\mathbf{H}_{n+1}$ and $\mathcal{Z}_\alpha^n=\mathcal{Y}_0\mathcal{U}_0\cup\mathcal{Y}_1\mathcal{U}_1$, where $\mathcal{Y}_0=\mathcal{X}_{\omega^\gamma}^{n*}$, $\mathcal{Y}_1=\bar{\mathcal{Y}}_0$ and $\mathcal{U}_i=\{\{v_i^{n+1}\}\}$. For $\alpha=\beta+\omega^\gamma$ let $\mathbf{E}_\alpha^n=\mathbf{F}_\beta^n\sqcup\mathbf{E}_{\omega^\gamma}^n\sqcup\mathbf{H}_{n+1}$ and $\mathcal{Z}_\alpha^n=\mathcal{Y}_0\mathcal{U}_0\cup\mathcal{Y}_1\mathcal{U}_1\cup\mathcal{Y}_2\bar{\mathcal{U}}_0\bar{\mathcal{U}}_1$, where $\mathcal{Y}_0=\mathcal{X}_\beta^{n*}$, $\mathcal{Y}_1=\bar{\mathcal{Y}}_0$, $\mathcal{Y}_2=\mathcal{Z}_{\omega^\gamma}^n$ and $\mathcal{U}_i$ be as in the previous case.

Similarly to (i) and (ii), $\mathbf{E}_\alpha^n$ are discrete and $\mathcal{Z}_\alpha^n\in\tilde{\mathcal{S}}_\alpha^n$. So it remains to find a special μ_α^n-alternating tree for $\mathcal{Z}_\alpha^n$. For the first two cases this is trivial. For $\alpha=\omega^{\gamma+1}$ choose a $\mu_{\omega^\gamma}^n$-alternating tree $\{R_\sigma\}$ for $\mathcal{X}_{\omega^\gamma}$ and let $T_{0\sigma}=R_\sigma\cup\{v_0^{n+1}\}$ and $T_{1\sigma}=R_{\bar\sigma}\cup\{v_1^{n+1}\}$. Then $\{T_{0\sigma},T_{1\sigma}\}$ is a special μ_α^n-

alternating tree for $\mathcal{Z}^n_\alpha$, because $T_0 \equiv_n T_1$. For the case $\alpha = \beta + \omega^\gamma$ a special μ^n_α-alternating tree for $\mathcal{Z}^n_\alpha$ is constructed from a special $\mu^n_{\omega^\gamma}$-alternating tree $\{S_\tau\}$ for $\mathcal{Z}^n_{\omega^\gamma}$ and a μ^n_β-alternating tree $\{R_\sigma\}$ for $\mathcal{X}^n_\beta$ similarly to the last case in (i). This completes the proof of the proposition.

4.7. Remark. Let $\omega_0 = 1$ and $\omega_{k+1} = \omega^{\omega_k}$. From the proofs of 4.2 and 4.6 it follows that if we consider only sets F containing variables of types $< k$ for a given $k < \omega$, then we get the refinements of these propositions with ω_k in place of ε_0.

5 Fine Hierarchy and Boolean Terms

Here we prove some of our main results by relating the syntactic FH $\{\mathcal{S}_\alpha\}$ over $\mathcal{L} = \mathcal{L}^{\mathbf{F}}$ to the FH $\{S_\alpha\}$ over an arbitrary base L (the most important particular case is of course $L = \{\Sigma^0_{n+1}\}$). Relate to any $\mathcal{X} \subseteq P(F)$ the "disjunctive normal form" $d_\mathcal{X} = \cup\{e_R | R \in \mathcal{X}\}$, where $e_R = e^F_R$ are the "elementary conjunctions" from Section 2. It is well–known that $\mathcal{X} \mapsto d_\mathcal{X}$ is a bijection (even a BA-isomorphism) between $P(P(F))$ and the terms $t \in T_F$ modulo equivalence in the theory of BA's. As in Section 2, let $t[L, \mathbf{F}]$ be the set of all values $t[a]$ of $t \in T_F$ on the $\mathbf{F}$-assignments $a : F \to \cup_n L_n$.

The results of this section show that the map $\mathcal{X} \mapsto d_\mathcal{X}[L, \mathbf{F}]$ provides a close connection between our hierarchies.

5.1. Lemma. *If $\mathcal{X} \in \mathcal{S}_\alpha$ then $d_\mathcal{X}[L, \mathbf{F}] \subseteq S_\alpha$, and similarly for the classes $\check{S}_\alpha, \tilde{S}_\alpha$.*

Proof. It clearly suffices to prove the assertion for S_α. We have to show that $\mathcal{X} \in \mathcal{S}_\alpha$ implies $d_\mathcal{X}[a] \in S_\alpha$ for any $\mathbf{F}$-assignment a. By 3.11.(ii) it suffices to show that the map $\mathcal{X} \mapsto d_\mathcal{X}[a]$ is a morphism from $\mathcal{L}$ to L. It is clearly a BA-homomorphism from $P(\mathcal{F})$ to $\cup_n L_n$ (because $d_{\bar{\mathcal{F}}}[a] = 0$ and *a fortiori* $d_\mathcal{F}[a] = 1$). So it remains to check that $d_\mathcal{X}[a] \in L_n$ for $\mathcal{X} \in \mathcal{L}_n$. We can assume that $\mathcal{X}$ is a "cone" $\{S \in \mathcal{F} | R \leq_n S\}$ for some $R \in \mathcal{F}$, because $\mathcal{X}$ is a union of such "cones".

For $n = 0$ we have $\mathcal{X} = \{S \in \mathcal{F} | R^0 \subseteq S\}$, so $d_\mathcal{X}[a] = \cap\{a^0_k | v^0_k \in R\} \in L_0$. For $n > 0$ we have $\mathcal{X} = \mathcal{Y} \setminus \mathcal{Z}$, where $\mathcal{Y} = \{S \in \mathcal{F} | R^{\leq n} \subseteq S\}$ and $\mathcal{Z} = \{S \in \mathcal{F} | R^{<n} \subset S\}$. As for the case $n = 0$, $d_\mathcal{Y}[a] \in L_n$ and $d_\mathcal{Z}[a] \in L_{n-1}$, so $d_\mathcal{X}[a] = d_\mathcal{Y}[a] \setminus d_\mathcal{Z}[a] \in L_n$ completing the proof.

The next result is in a sense reverse of 5.1 for the reducible bases.

5.2. Lemma. *For any reducible base L and any $\mathcal{X} \subseteq \mathcal{F}$ we have:*

(i) *if* $\mathcal{X} \notin \check{\mathcal{S}}_\alpha$ *then* $S_\alpha \subseteq d_{\mathcal{X}}[L, \mathbf{F}]$;

(ii) *if* $\mathcal{X} \notin \cup_{\beta<\alpha}(\mathcal{S}_\beta \cup \check{\mathcal{S}}_\beta)$ *then* $\check{S}_\alpha \subseteq d_{\mathcal{X}}[L, \mathbf{F}]$.

Proof. (i) Let $\mathcal{X}$ and $b \in S_\alpha$ be given. We have to show that $b \in t[L, \mathbf{F}]$, where $t = d_{\mathcal{X}}$. By 4.2 and 3.10, there are a μ_α-alternating tree $\{R_\sigma\}$ for $\mathcal{X}$ and a reduced μ_α-tree $\{b_\sigma\}$ defining b. We have to find a $\mathbf{F}$-assignment $a = \{a_k^n | v_k^n \in F\}$ with $b = t[a]$. We claim that the assignment $a_k^n = \cup\{b_\sigma^* | v_k^n \in R_\sigma\}$ has the desired property.

First we check that $a_k^n \in L_n$. It suffices to show that if $v_k^n \in R_\sigma$, then $b_\sigma^* \subseteq c \subseteq a_k^n$ for some $c \in L_n$. Let σ_0 be the least (with respect to $\subseteq$) string such that $\sigma_0 \subseteq \sigma$ and $\mu(|\tau|) > n$ for all τ satisfying $\sigma_0 \subseteq \tau \subset \sigma$. Then $R_{\sigma_0} \leq_{n+1} R_\sigma$, so $v_k^n \in R_{\sigma_0}$. By minimality of σ_0, $b_{\sigma_0} \in L_n$ (for $\sigma_0 = \emptyset$ we have $b_{\sigma_0} = 1 \in L_n$, otherwise $l = \mu(|\sigma_0| - 1) \leq n$ and $b_{\sigma_0} \in L_l$).

Let m be the least number i satisfying $|\sigma| \leq i < |\mu|$ and $\mu(i) < n$ (if there is no such an i, let $m = |\mu|$). Finally, let $T = \{\tau \supseteq \sigma_0 : |\tau| \leq m\}$ and $c = \cup\{b_\tau^* | \tau \in T\}$. Then $\sigma \in T$ and $v_k^n \in R_\tau$ for all $\tau \in T$ (because $v_k^n \in R_{\sigma_0} \leq_n R_\tau$), so $b_\sigma^* \subseteq c \subseteq a_k^n$. But $c = b_{\sigma_0} \setminus \cup\{b_\tau : \tau \supseteq \sigma_0, |\tau| = m+1\}$, $b_{\sigma_0} \in L_n$ and $b_\tau \in L_{\mu(m)}$ for $|\tau| = m+1$, so $c \in L_n$ and a is a F-assignment.

Note that if $x \in b_\sigma^*$ then $v_k^n \in R_\sigma \leftrightarrow x \in a_k^n$ (the implication from left to right is by definition of a_k^n; in the opposite direction, if $x \in a_k^n$, then $x \in b_\tau^*$ for some τ with $v_k^n \in R_\tau$; but by 3.7.(iii) the components b_ρ^* are pairwise disjoint, so $\sigma = \tau$ and $v_k^n \in R_\sigma$). In other words, $x \in b_\sigma^*$ implies $x \in e_{R_\sigma}[a]$; so $b_\sigma^* \subseteq e_{R_\sigma}[a]$.

Now we can show that a is a $\mathbf{F}$-assignment, i.e. $e_R[a] = 0$ for $R \in P(F) \setminus \mathcal{F}$. It suffices to deduce $R \in \mathcal{F}$ from $e_R[a] \neq 0$, so let $x \in e_R[a]$ for some x. We have $\cup_\sigma b_\sigma^* = 1$, so $x \in b_\sigma^* \subseteq e_{R_\sigma}[a]$ for some σ. But the elements $e_S[a]$ $(S \subseteq F)$ are pairwise disjoint, so $R = R_\sigma \in \mathcal{F}$.

Finally, by 3.7.(i) we have $b = \cup_\sigma b_{\sigma 1}^* \subseteq \cup_\sigma e_{R_{\sigma 1}}[a] \subseteq t[a]$ and $\bar{b} = b_\emptyset^* \cup (\cup_\sigma b_{\sigma 0}^*) \subseteq e_{R_\emptyset}[a] \cup (\cup_\sigma e_{R_{\sigma 0}}[a]) \subseteq \overline{t[a]}$. So $b = t[a]$ completing the proof of (i).

(ii) is proved in the same way, one should only exclude $R_\emptyset$ from the consideration and remember that in this case $R_0 \equiv_l R_1$ for $l = \mu_\alpha(0) - 1$, and $b_\emptyset^* = 0$). This completes the proof of the lemma.

Now we prove the main result of this section.

5.3. Theorem. *Over any reducible base L we have:*

(i) $\{t[L, \mathbf{F}] : t \in T_F, \mathbf{F}$ *perfect*$\} = \{S_\alpha, \check{S}_\alpha : \alpha < \varepsilon_0\}$;

(ii) $\{t[L, \mathbf{F}] : t \in T_F, \mathbf{F}$ *interpolable*$\} = \{S_\alpha, \check{S}_\alpha, \tilde{S}_{\alpha+1} : \alpha < \varepsilon_0\}$;

(iii) $\{t[L, \mathbf{F}] : t \in T_F, \mathbf{F}$ *discrete*$\} = \{S_\alpha, \check{S}_\alpha, \tilde{S}_\lambda : \alpha, \lambda < \varepsilon_0, \lambda$ *limit*$\}$.

Proof. (i) Let $\mathbf{F}$ be perfect and $t \in T_F$. Take $\mathcal{X} \subseteq P(F)$ with $d_{\mathcal{X}}$ equivalent to t. We may assume that $\mathcal{X} \subseteq \mathcal{F}$, because $d_{\mathcal{X}}[L, \mathbf{F}] = d_{\mathcal{X}\mathcal{F}}[L, \mathbf{F}]$. By 4.1 we have $\mathcal{X} \in \cup_n \mathcal{L}_n$, so $\mathcal{X}$ belongs to a constituent of the FH over $\mathbf{F}$. By 4.4, this is either a $\mathcal{S}_\alpha$-, or a $\check{\mathcal{S}}_\alpha$-constituent. By 5.1 and 5.2, $t[L, \mathbf{F}]$ is one of $S_\alpha, \check{S}_\alpha$. It remains to find for a given α a perfect pair $\mathbf{F}$ and a term $t \in T_F$ with $t[L, \mathbf{F}] = S_\alpha$ (the assertion for $\check{S}_\alpha$ would then follow). By 4.6, there are a perfect $\mathbf{F}$ and a class $\mathcal{X} \in \mathcal{S}_\alpha \setminus \check{\mathcal{S}}_\alpha$. By 5.1 and 5.2, the term $t = d_{\mathcal{X}}$ has the desired property.

(ii) and (iii) are considered in the same way completing the proof.

This result immediately implies the assertion of Theorem 2.2 for any reducible base. But it is well–known that the base $\{\Sigma^0_{n+1}\}$ is reducible, so Theorem 2.2 is true. Theorem 2.1 follows from 5.3.(i) because $t[L] = t[L, \mathbf{F}]$ for $\mathbf{F} = (F; P(F))$ and $\mathbf{F}$ is perfect by 4.3 and 4.5 ($\mathbf{F}$ is isomorphic to a pair of the form $\mathbf{G}_{n_0} \sqcup \cdots \sqcup \mathbf{G}_{n_k}$).

6 Completeness Condition

Here we prove Theorem 2.3 and of some its refinements. These results significantly simplify and strengthen the completeness condition from Se91 (in particular, the next result shows that one can replace the perfectness in the condition from Se91 by a much weaker property of discreteness). Please recall the notion of a complete F-assignment A and notation like $\mathbf{F}_A = (F; \mathcal{F}_A)$ from Section 2. We call A *discrete* (*interpolable, perfect*), if so is the pair $\mathbf{F}_A$.

6.1. Proposition. (i) *Any Boolean combination of members of a complete F-assignment A is m-complete in one of* $\Sigma_\alpha, \Pi_\alpha, \Delta_{\alpha+1}$ $(\alpha < \varepsilon_0)$.

(ii) *Any Boolean combination of members of a complete discrete assignment A is m-complete in one of* $\Sigma_\alpha, \Pi_\alpha$ $(\alpha < \varepsilon_0)$.

Proof. First check that for any $t \in T_F$ the Boolean combination $t[A]$ is m-complete in $t[L, \mathbf{F}_A]$. We clearly have $t[A] \in t[L, \mathbf{F}_A]$, so it remains to reduce any $X \in t[L, \mathbf{F}_A]$ to $t[A]$. Let B be a $\mathbf{F}_A$-assignment with $X = t[B]$. By completeness of A, $B \leq_m A$. Let f be a recursive function reducing B to A. By induction on t, $t[B] = f^{-1}(t[A])$, so $X \leq_m t[A]$.

From 2.2 it now follows that $t[A]$ is m-complete in one of $\Sigma_\alpha, \Pi_\alpha, \Delta_{1+\alpha}$. It can not be complete in Δ_λ for a limit λ, because Δ_λ by a result in Se83 has no m-complete set. This states (i).

To prove (ii) choose $\mathcal{X} \subseteq \mathcal{F}_A$ satisfying $d_{\mathcal{X}}[L, \mathbf{F}_A] = t[L, \mathbf{F}_A]$ and note that by 4.4 $\mathcal{X}$ can not belong to a $\tilde{S}_{\alpha+1}$-constituent of the FH over $\mathbf{F}_A$. So $t[A]$ can not be m-complete in $\Delta_{\alpha+1}$. This completes the proof of the proposition.

6.2. Remarks. (i) We see that for a complete A the pair $\mathbf{F}_A$ can not be arbitrary (e.g. the $\tilde{S}_\alpha$-constituent of the FH over $\mathbf{F}_A$ is empty for any limit α). It seems interesting to characterize such pairs $\mathbf{F}_A$. In particular, is any such $\mathbf{F}_A$ interpolable?

(ii) From the proofs of 4.2 and 6.1 it follows that from (a canonical index of) $\mathbf{F}_A$ and t one can effectively compute the level of FH in which the set $t[A]$ is m-complete.

Our next goal is to prove that all the possibilities in 6.1 are realized but it requires some preliminary work. For $X, Y \subseteq \omega$, let $X \otimes Y = \{\langle x, y\rangle : x \in X, y \in Y\}$. For a F-assignment A and a G-assignment B with $FG = \emptyset$ define the $(F \cup G)$-assignment $C = A \sqcup B$ as follows: $C^n_k = A^n_k \otimes \omega$ for $v^n_k \in F$ and $C^n_k = \omega \otimes B^n_k$ for $v^n_k \in G$ (as in Section 4, the condition $FG = \emptyset$ is not essential).

6.3. Lemma. *If the assignments A and B are complete (discrete, interpolable, perfect) then so is also the assignment $A \sqcup B$.*

Proof. First we check that $\mathbf{H} = \mathbf{F} \sqcup \mathbf{G}$, where $\mathbf{F} = (F; \mathcal{F}_A)$, $\mathbf{G} = (G; \mathcal{F}_B)$, $H = F \cup G$ and $\mathbf{H} = (H; \mathcal{F}_C)$. For any $R \subseteq H$ we have $e^H_R = e^F_{RF} \cap e^G_{RG}$, so

$$e^H_R[C] = (e^F_{RF}[A] \otimes \omega) \cap (\omega \otimes e^G_{RG}[B]) = e^F_{RF}[A] \otimes e^G_{RG}[B].$$

This implies that $e^H_R[C]$ is nonempty iff both $e^F_{RF}[A]$ and $e^G_{RG}[B]$ are nonempty. So $\mathcal{F}_C = \{R \subseteq H | RF \in \mathcal{F}_A, RG \in \mathcal{F}_B\}$, i.e. $\mathbf{H} = \mathbf{F} \sqcup \mathbf{G}$.

The preceding paragraph and Lemma 4.5 imply the assertion for the properties in parenthesis. It remains to m-reduce any $\mathbf{H}$-assignment D to C. Note that the restrictions D_F and D_G of D to F and G are respectively $\mathbf{F}$- and $\mathbf{G}$-assignments (e.g. for F we have: if $S \in P(F) \setminus \mathcal{F}_A$, then $S \cup T \notin \mathcal{F}_C$ for all $T \subseteq G$, so $e^F_S[D_F] = \cup\{e^H_{S \cup T}[D] : T \subseteq G\} = \emptyset$). Let f and g be recursive functions m-reducing respectively D_F to A and D_G to B. Then the function $h(x) = \langle f(x), g(x)\rangle$ reduces D to C completing the proof.

Now we are able to prove the "optimality" of 6.1.

6.4. Proposition. (i) *For any $\alpha < \varepsilon_0$ there exists a complete perfect F-assignment A such that some Boolean combination of its members is Σ_α-complete.*

(ii) *The same is true for the interpolable assignments and the levels $\Delta_{\alpha+1}$.*

Proof. Let $G_n = \{v^n_0\}$ and A^n be the G_n-assignment sending v^n_0 to a Σ^0_{n+1}-complete set. This assignment is clearly complete and $\mathbf{G}_n = (G_n; \mathcal{F}_{A^n})$, where $\mathbf{G}_n$ is from 4.3. Repeating the proof of 4.6.(i) with A^n in place of $\mathbf{G}_n$ and applying 6.3, we get the assertion (i). Now let $H_0 = \{v_0^0, v_1^0\}$ and B be the H_0-assignment sending v_0^0 to the set of all even and v_1^0 to the set of all odd numbers. Then B is universal and $\mathbf{H}_0 = (H_0; \mathcal{F}_B)$, where $\mathbf{H}_0$ is from 4.3. Taking B in place of $\mathbf{H}_0$ in the proof of 4.6.(ii), we get the desired assignment for (ii). This completes the proof.

Propositions 6.1 and 6.4 imply Theorem 2.3. They imply also Corollary 2.4. To see this, consider "index sets" $A^n = \{\langle x_1, \ldots, x_k\rangle | P_n(x_1, \ldots, x_k)\}$ of the predicates P_n in the trivial identity numeration of ω. The assignment $\{A^n\}$ is universal, so suitable Boolean combinations of these sets are m-complete in all levels Σ_α. But these combinations are "index sets" of Boolean combinations of the predicates P_n, so 2.4 is true. Note also that the proof of 6.4 and Remark 4.7 imply the refinement of 2.4 whith "$n < k$" in place of "$n < \omega$" and ω_k in place of ε_0, for every $k < \omega$.

We conclude this section by relating the F-assignments to a bit more general finite sequences $\{A_i\}_{i \in I}$ naturally arising in applications. The notions of "elementary conjunction" $e_R(R \subseteq I)$, of its value $e_R[A_i]$ and of m-reducibility for such sequences $\{A_i\}$ are similar to the corresponding notions for the assignments. We call a sequence $\{B_i\}_{i \in I}$ *compatible* with $\{A_i\}$, if $e_R[A_i] = \emptyset$ implies $e_R[B_i] = \emptyset$ for all $R \subseteq I$. For a given map $M \to \{\Sigma_n^0, \Pi_n^0 | n > 0\}$, call $\{A_i\}_{i \in I}$ a *M-sequence* if $A_i \in M_i$ for all $i \in I$. Such a sequence $\{A_i\}$ is *universal*, if any M-sequence compatible with $\{A_i\}$ is m-reducible to $\{A_i\}$. Call sequences $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ *isomorphic* if $A_i = B_{g(i)}$ for a bijection g between I and J. Of course all "nontrivial" notions on sequences are invariant under isomorphism.

Now let A be a F-assignment and $T \subseteq F$. Define the maps M^T and A^T as follows: $M_v^T = \Sigma_{n+1}^0$ and $A_v^T = A_k^n$ for $v = v_k^n \in T$, and $M_v^T = \Pi_{n+1}^0$ and $A_v^T = \bar{A}_k^n$ for $v = v_k^n \in F \setminus T$. Then $\{A_v^T\}_{v \in F}$ is clearly a M^T-sequence, and any M-sequence is isomorphic to a sequence of this form.

Let us state some relations between introduced notions for assignments and sequences. Let $\mathcal{G}$ be the set of all $R \subseteq F$ with $e_R[A_v^T] \neq \emptyset$ (note that $\mathcal{G}$ is in a natural bijective correspondence with atoms of the BA generated by the sets A_k^n). Relate to any $R \in \mathcal{G}$ the unique $R^* \in \mathcal{F}_A$ with $e_R[A_v^T] = e_{R^*}[A]$; this is a bijection between $\mathcal{G}$ and $\mathcal{F}_A$. Let $\preceq_n$ be the preordering on $\mathcal{G}$ induced by the preordering $\leq_n$ in this bijection.

6.5. Proposition. (i) *A is a universal F-assignment iff $\{A_v^T\}$ is a universal M^T-sequence.*

(ii) $R \preceq_n S$ *iff* $R^{<n} = S^{<n}$, $R^nT \subseteq S^nT$ *and* $S^n\bar{T} \subseteq R^n\bar{T}$.

Proof. (i) is clear, so consider (ii). It is easy to see that $R^* = F\backslash(R\Delta T)$, where Δ is the symmetric difference. From definition of $\leq_n$ in Section 4 one by cases easily gets the desired equivalence. This completes the proof.

The reduction of M-sequences to assignments simplifies technical details because definition of $\leq_n$ is simpler than the description (ii). This is a reason why the proof of the completeness condition here is simpler and clearer than the proof of a weaker condition in Se91.

7 Definable Index Sets

The main goal of this section is to prove Theorem 2.5. This needs some preliminary work and a citation from Se91.

By a *r.e. BA* we mean a pair $\mathbf{B} = (B, \beta)$ consisting of a BA $B = (B; \cup, \cap, ^-, 0, 1)$ and a map β from ω onto B (called a numeration of B) in which the Boolean operations are representable by recursive functions and the equality relation on B is r.e. Two such objects $\mathbf{A}$ and $\mathbf{B}$ are *recursively isomorphic* (in symbols $\mathbf{A} \simeq_r \mathbf{B}$) if there exists a BA-isomorphism between A and B representable by a recursive function in the corresponding numerations. Call a r.e. BA $\mathbf{B}$ *universal*, if any r.e. BA $\mathbf{A}$ is recursively isomorphic to an initial segment $\widehat{b} = \{x \in B | x \leq b\}, b \in B$ (with the induced numeration). In Pe91 and Se91 it was shown that any two universal r.e. BA's are recursively isomorphic. In Han75 and Pe82 it was shown that the Lindenbaum algebra from 2.5 is universal. For technical reasons it is more convenient to work with another universal r.e. BA constructed as follows.

Call a sequence $\{\mathbf{A}_n\}$ of r.e. BA's *uniform*, if the Boolean operations in these algebras are recursive, and the equality relations are r.e. in the corresponding numerations uniformly in $n < \omega$. By an *acceptable numeration* of r.e. BA's we mean a uniform sequence $\{\mathbf{B}_n\}$ such that for any uniform sequence $\{\mathbf{A}_n\}$ there exists a recursive function f satisfying $\mathbf{A}_n \simeq_r \mathbf{B}_{f(n)}$ for all n. In Se91 we constructed such an acceptable numeration by taking $\mathbf{B}_n$ essentially as the quotient structure of the free countable BA under the congruence relation induced by the r.e. set with the standard number n. From $\{\mathbf{B}_n\}$ we constructed a universal r.e. BA $\mathbf{U} = (U, \nu)$ as a "direct sum" $\mathbf{B}_0 \sqcup \mathbf{B}_1 \sqcup \cdots$. Namely, let $U_0(U_1)$ be the set of all sequences $\{b_n\}$ such that $b_n \in \mathbf{B}_n$ for all n and $b_n = 0_{B_n}$ (resp. $b_n = 1_{B_n}$) for almost all n. Define the

Boolean operations on $U = U_0 \cup U_1$ componentwise and let ν be the natural numeration for which $\mathbf{U}$ is a r.e. BA and both sets $\nu^{-1}(U_i)$ are recursive. This $\mathbf{U}$ is universal, so it is recursively isomorphic to the algebra from 2.5 and it suffices to prove 2.5 for the algebra $\mathbf{U}$.

We start with the following easy fact:

7.1. Lemma. (i) *$U_0 U_1 = \emptyset$, $U_0 \cup U_1 = U$ and $U_1 = \check{U}_0$.*

(ii) *U_0 is an ideal, and U_1 is a filter of U.*

(iii) *For any uniform sequence $\{\mathbf{A}_n\}$ of r.e. BA's there exists a ν-recursive sequence $\{a_n\}$ of pairwise disjoint elements of U_0 with $\mathbf{A}_n \simeq_r \widehat{a}_n$.*

(iv) *For any $a \in U_1$, $\widehat{a} \simeq_r \mathbf{U}$.*

Proof. (i) and (ii) are clear. Let $\{\mathbf{A}_n\}$ be the sequence from (iii) and f be a recursive function satisfying $\mathbf{A}_n \simeq_r \mathbf{B}_{f(n)}$. From definition of $\{\mathbf{B}_n\}$ it follows that we may assume f to be injective. Now let a_n be the sequence $\{b_k\}$ defined by $b_n = 1_{\mathbf{B}_{f(n)}}$ and $b_k = 0_{\mathbf{B}_{f(n)}}$ for $k \neq n$. The sequence $\{a_n\}$ has the desired properties.

(iv) It suffices to show that $\widehat{a}$ is a universal r.e. BA, and this is immediate by definition of $\{\mathbf{B}_n\}$ and U. This completes the proof.

Recall definitions of some objects relevant to the elementary classification of BA's due to A.Tarski, see e.g. CK73 or Er80. For any BA B, let $T(B)$ be the ideal of B generated by the atomic and by the atomless elements of B. Let $B^{(0)} = B$ and $B^{(n+1)}$ be the quotient of $B^{(n)}$ under $T(B^{(n)})$. Let $\mathcal{A}^n_k(\mathcal{B}^n_k, \mathcal{A}^n)$ be the class of BA's B such that $B^{(n)}$ has $\leq 2^k$ elements (resp. has $\leq k$ atoms, is an atomic BA). Let $\mathcal{U}$ be the class of all BA's, and for any sentence φ let $\mathcal{M}_\varphi = \{B \in \mathcal{U} | B \models \varphi\}$. From the elementary classification of BA's it follows (see Se90) that the classes $\mathcal{M}_\varphi$ (i.e. the finitely axiomatizable classes of BA's) are exactly the Boolean combinations of the classes $\mathcal{A}^n_k, \mathcal{B}^n_k, \mathcal{A}^n (n, k < \omega)$ inside $\mathcal{U}$, and there is an algorithm computing from a given sentence φ a Boolean combination coinciding with $\mathcal{M}_\varphi$.

Let us return to the main problem and describe the predicates definable in U (as well as in any other BA); this realizes the stage (i) of the program in Section 2. For $m < \omega, I \subseteq m = \{i | i < m\}$ and a class $\mathcal{C}$ of BA's, let $\mathcal{C}_I = \mathcal{C}^m_I = \{a \in U^m : \widehat{e_R[a]} \in \mathcal{C}\}$, where $a = (a_0, \ldots, a_{m-1})$ and $e_I = e^m_I$ are again the "elementary conjunctions" with m variables.

7.2. Proposition. *A predicate on U is definable iff it is a Boolean combination of the predicates $\mathcal{C}^m_I (m < \omega, I \subseteq m, \mathcal{C} \in \{\mathcal{A}^n_k, \mathcal{B}^n_k, \mathcal{A}^n\})$. For any formula $\varphi(v_0, \ldots, v_{m-1})$ in the language of BA's one can effectively find a*

Boolean combination of the specified predicates coinciding with P_φ.

Proof. One direction is immediate because the specified predicates are definable. In the other direction, let a formula $\varphi(v_0,\ldots,v_{m-1})$ be given. By Se91 and Se92, one can effectively find a number l and sentences $\theta_i^I(I \subseteq m, i \leq l)$ in the language of BA's such that

$$U \models \varphi(a_0,\ldots,a_{m-1}) \ iff \ \bigvee_{i \leq l}(\bigwedge_{I \subseteq m} \widehat{e_I[a]} \models \theta_i^I)$$

for all $a \in U^m$. So it suffices to show that for any sentence θ the predicate $\widehat{e_I[a]} \models \theta$ is a Boolean combination of the specified predicates. But this is immediate by the remarks above completing the proof.

For any fixed $m, q > 0$, let $\mathcal{G} = \mathcal{G}_p = \{\mathcal{A}_k^n, \mathcal{B}_k^n, \mathcal{A}^n | n, k \leq q\}$ and $\mathcal{H} = \mathcal{H}_q^m = \{\mathcal{C}_I | \mathcal{C} \in \mathcal{G}, I \subseteq m\}$. By 7.2, the U-definable predicates are exactly the Boolean combinations of the predicates from $\mathcal{H}_q^m$ for all m, q. So for the realization of the stage (ii) of the program in Section 2 we have to find a complete sequence containing the members of $\{\nu^{-1}(X) | X \in \mathcal{H}_q^m\}$ (the last sequence itself is unfortunately not complete). For this aim consider the predicates $R_J = R_J^m = \{a \in U^m : e_J[a] \in U_1\}$ for $J \subseteq m$.

7.3. Lemma. (i) *The sets $\nu^{-1}(U_J)$ are recursive, pairwise disjoint and exhaust ω.*

(ii) *The map $a \mapsto \{e_J[a]\}_{J \subseteq m}$ is a bijection between U^m and the set of sequences $\{b_J\}_{J \subseteq m}$ such that $b_J \in U$, b_J are pairwise disjoint and $\cup_J b_J = 1$.*

(iii) *For all $\mathcal{C} \in \mathcal{G}$ and $I \subseteq m$, $R_I \mathcal{C}_I = \emptyset$.*

Proof. (i) and (ii) are clear, so consider (iii). Let $a \in R_I$, then by 7.1.(iv) $\widehat{e_I[a]} \simeq_U$. It is well-known that for any $\mathcal{C} \in \mathcal{G}$ there is a recursive BA not belonging to $\mathcal{C}$. So $U \notin \mathcal{C}$ and $a \notin \mathcal{C}_I$ completing the proof.

Now we realize the stage (ii) of our programm as follows:

7.4. Proposition. *The sequence $\{\nu^{-1}(\mathcal{C}_I), \nu^{-1}(R_J) | \mathcal{C} \in \mathcal{G}; I, J \subseteq m\}$ is complete.*

Proof. First we show that the sequence $A_I = \{\nu^{-1}(\mathcal{C}_I) | \mathcal{C} \in \mathcal{G}\}$ is complete for any $I \subseteq m$. In Se91 we proved that $B = \{\theta\mathcal{C} | \mathcal{C} \in \mathcal{G}\}$, where $\theta\mathcal{C} = \{x | \mathbf{B}_x \in \mathcal{C}\}$, is a complete sequence satisfying

$$\theta\mathcal{A}_0^n \in \Sigma_{4n+1}^0,\ \theta\mathcal{A}_{k+1}^n \in \Pi_{4n+2}^0,\ \theta\mathcal{B}_k^n \in \Pi_{4n+3}^0,\ \theta\mathcal{A}^n \in \Pi_{4n+4}^0 \qquad (3)$$

(see the notion of a complete M-sequence in Section 6). So it suffices to state that $A_I \equiv_m B$. For $x = \langle x_0, \ldots, x_{m-1} \rangle$ let $a^x = (\nu x_0, \ldots, \nu x_{m-1})$, then

$\{\widehat{e_I[a^x]}\}$ is a uniform sequence of r.e. BA's, so $\widehat{e_I[a^x]} \simeq_r \mathbf{B}_{f(x)}$ for a recursive function f. This function m-reduces A_I to B. To state $B \leq_m A_I$ it suffices to find a recursive function g with $\mathbf{B}_x \simeq_r \widehat{e_I[b^x]}, b^x = a^{g(x)}$. By 7.1.(iii), there is a recursive function h with $\mathbf{B}_x \simeq \widehat{\nu_{h(x)}}$. Define $b^x \in U^m$ by: $b_i^x = \nu_{h(x)}$ for $i \in I$ and $b_i^x = \bar{\nu}_{h(x)}$ for $i \notin I$; this b^x has the desired property.

Now let $D = \{S_{\mathcal{C},I}, T_J | \mathcal{C} \in \mathcal{G}; I, J \subseteq m\}$ be any sequence compatible with the sequence C from the formulation of the proposition; we have to show $D \leq_m C$. For any $I \subseteq m$ the sequence $\{S_{\mathcal{C},I} | \mathcal{C} \in \mathcal{G}\}$ is compatible with A_I, so by the preceding paragraph it is m-reducible to A_I. Let $\{b_I^x\}$ be a ν-recursive sequence of elements of U^m such that $x \in S_{\mathcal{C},I}$ iff $b_I^x \in \mathcal{C}_I$. By 7.1.(iii) we may assume that the elements $d_I^x = e_I[b_I^x]$ are pairwise disjoint and belong to U_0.

By compatibility of D with C and by 7.3.(i), the sets T_J are recursive, pairwise disjoint and exhaust ω. So for a given x we can compute the unique $J = J_x$ with $x \in T_J$. By 7.3.(ii), there is a unique $c^x \in U^m$ such that $e_I[c^x] = d_I^x$ for all $I \subseteq m, I \neq J$ (then automatically $c^x \in R_J$). The sequence $\{c^x\}$ is ν-recursive, so $c^x = a^{g(x)}$ for a recursive function g. From the construction it follows that g m-reduces D to C completing the proof of the proposition.

So the structure $\mathbf{U}$ is easy. By the effectiveness in 7.2 and by 6.2.(iii), it is easy in the effective sense described in Section 2.

The realization of the stages (iii) and (iv) in Section 2 requires some additional considerations. Recall from Section 6 that instead of the complete sequence B above we can consider an isomorphic complete assignment A, and $\mathcal{F}_A$ is in a natural bijective correspondence with the set of atoms of the BA generated by $\theta\mathcal{C}(\mathcal{C} \in \mathcal{G})$ inside $P(\omega)$, and *a fortiory* with the set $at(\mathcal{G})$ of atoms of the BA generated by $\mathcal{G}$ inside $\mathcal{U}$. We will work with the last set. Let $\leq_i$ be the preorderings on $at(\mathcal{G})$ induced by the corresponding preorderings on $\mathcal{F}_A$. We need an explicit description of these preorderings.

We have the following evident inclusions of the introduced classes of BA's:

$$\mathcal{A}_k^n \subseteq \mathcal{A}_{k+1}^n \subseteq \mathcal{A}^n \subseteq \mathcal{A}_0^{n+1},\ \mathcal{B}_k^n \mathcal{A}^n = \mathcal{A}_k^n,\ \mathcal{A}_k^n \subseteq \mathcal{B}_k^n \subseteq \mathcal{B}_{k+1}^n \subseteq \mathcal{A}_0^{n+1}.$$

Let $a_k^n(b_k^n, a^n)$ be the difference of $\mathcal{A}_k^n$ (resp. of $\mathcal{B}_k^n, \mathcal{A}^n$) and the "lesser" classes with the indices $\leq q$ (e.g. $b_1^n = \mathcal{B}_1^n \bar{\mathcal{B}}_0^n \bar{\mathcal{A}}_1^n$). Let $c = \bar{\mathcal{A}}^q \bar{\mathcal{B}}_q^q$, then $at(\mathcal{G}) = \{a_k^n, b_k^n, a^n, c | n, k \leq q\}$. By definition, *the types* of the atoms $a_0^n, a_{k+1}^n, b_k^n, a^n, c$ are respectively $4n, 4n+1, 4n+2, 4n+3$ and $4n+3$ (so the types correspond to the levels in (3)). From 6.5 we immediately get the following description of the preorderings $(at(\mathcal{G}); \leq_i)$:

7.5. Lemma. *In $(at(\mathcal{G}); \leq_i)$ the atoms of type $< i$ are incomparable with all other atoms, and the atoms ot type $> i$ are pairwise equivalent. In addition, we have:*

(i) *for $i = 4j, j \leq q$, the atoms of type $> i$ are less than a_0^j;*

(ii) *for $i = 4j+1, j \leq q$, the atoms of type $> i$ are greater than a_q^j, and $a_q^j > \cdots > a_1^j$;*

(iii) *for $i = 4j+2, j \leq q$, the atoms of type $> i$ are greater than b_q^j, and $b_q^j > \cdots > b_0^j$;*

(iv) *for $i = 4j+3, j < q$, the atoms of type $> i$ are greater than a^j;*

(v) *for $i = 4q+3$, $a^q < c$.*

The operation $A \times B$ of Cartesian product of BA's induces the operation $\mathbf{x} \times \mathbf{y}$ (denoted also by $\mathbf{xy}$) on $at(\mathcal{G})$ as follows: if $A \in \mathbf{x}$ and $B \in \mathbf{y}$ then $\mathbf{x} \times \mathbf{y}$ is the atom containing $A \times B$. The definition is clearly correct. Let us state some properties of the introduced operation.

7.6. Lemma. (i) *For all* $\mathbf{x}$, $a_0^0\mathbf{x} = \mathbf{x}$.

(ii) *If* $\mathbf{x} \leq_i \mathbf{x}_1$ *and* $\mathbf{y} \leq_i \mathbf{y}_1$ *then* $\mathbf{xy} \leq_i \mathbf{x}_1\mathbf{y}_1$.

Proof. (i) immediately follows from the well–known property of BA's $(A \times B)^{(n)} \simeq A^{(n)} \times B^{(n)}$. This property implies also the equalities on the atoms like $\mathbf{x}c = c$, $a_1^n a_2^n = a_3^n$, $a_1^n b_2^n = b_3^n$ (if $3 \leq q$), $a^n a^n = a^n$, $a_1^n a_q^n = a^n$.

The assertion (ii) follows from these equalities, Lemma 7.5 and the evident facts that the introduced operation is associative and commutative, and that the type of $\mathbf{xy}$ is not less than that of $\mathbf{x}$. This completes the proof.

We need also a description of atoms of the BA's generated by $\mathcal{H}$ and by $\mathcal{K} = \mathcal{K}_q^m = \{\mathcal{C}_I, R_J | \mathcal{C} \in \mathcal{G}; I, J \subseteq m\}$ inside $P(U^m)$ and of the preorderings $\leq_i$ on $at(\mathcal{K})$ defined similarly to the preorderings on $at(\mathcal{G})$. In the next lemma (i) and (ii) follow from 7.3, and (iii) follows from 6.5. For $f : P(m) \to at(\mathcal{G})$, let $f^* = \cap_J f(J)_J$.

7.7. Lemma. (i) $at(\mathcal{H}) = \{f^* | f : P(m) \to at(\mathcal{G}), c \in rng(f)\}$.

(ii) $at(\mathcal{K}) = \{f^* R_I | f : P(m) \to at(\mathcal{G}), f(I) = c\}$.

(iii) *For all $f^*R_I, g^*R_J \in at(\mathcal{K})$ and $i < \omega$, $f^*R_I \leq_i g^*R_J$ iff $I = J$ and $f(K) \leq_i g(K)$ for all $K \subseteq m$.*

Our next (and the last) lemma relates the introduced notions for different m. Define the projection $p : U^{m+1} \to U^m$ by $p(a_0, \ldots, a_m) = (a_0, \ldots, a_{m-1})$. Let $\{\mathcal{S}_\alpha\}$ be the FH over $\mathcal{L} = \{\mathcal{L}_n\}$, where $\mathcal{L}_n$ consists of unions of subsets of $at(\mathcal{K}_q^m)$ closed upwards under $\leq_n$ (by the end of Section 6 and by 3.11.(ii), this hierarchy is isomorphic to the corresponding syntactic FH). For $I \subseteq m$, let $I_m = I \cup \{m\}$.

7.8. Lemma. (i) *For any* $a \in U^{m+1}$, $e_I[p(a)] = e_I[a] \cup e_{I_m}[a]$.

(ii) *For any* $I \subseteq m$, $p^{-1}(R_I^m) = R_I^{m+1} \cup R_{I_m}^{m+1}$.

(iii) *For any* $f : P(m) \to at(\mathcal{G})$, $p^{-1}(f^*)$ *is the union of* g^* *for all* $g : P(m+1) \to at(\mathcal{G})$ *satisfying* $f(I) = g(I) \times g(I_m)$ *for any* $I \subseteq m$.

(iv) *For any* $\mathcal{X} \subseteq U^m$, *if* $\mathcal{X} \in \mathcal{S}_\alpha$ *then* $p^{-1}(\mathcal{X}) \in \mathcal{S}_\alpha$.

Proof. (i) and (ii) are clear, so consider (iii). Let $a \in p^{-1}(f^*)$, i.e. $e_I\widehat{[p(a)]} \in f(I)$ for all $I \subseteq m$. Define $g : P(m+1) \to at(\mathcal{G})$ as follows: for any $I \subseteq m$, let $g(I)$ and $g(I_m)$ be respectively the elements of $at(\mathcal{G})$ containing the BA's $\widehat{e_I[a]}$ and $\widehat{e_{I_m}[a]}$. By (i) and by the well–known fact that $\widehat{x \cup y} \simeq \widehat{x} \times \widehat{y}$ for all disjoint $x, y \in U$, we have $a \in g^*$ and $f(I) = g(I) \times g(I_m)$. This proves a half of our assertion; the other half is proved in the same way.

(iv) By 3.11.(ii), it suffices to show that the map $\mathcal{X} \mapsto p^{-1}(\mathcal{X})$ is a morphism from the base $\mathcal{L}$ over $\mathcal{K}_q^m$ into that over $\mathcal{K}_q^{m+1}$. This is of course a BA-monomorphism, so it remains to check that $p^{-1}(\mathcal{X}) \in \mathcal{L}_n$ for $\mathcal{X} \in \mathcal{L}_n$. It suffices to show that for all $\mathbf{x}, \mathbf{y} \in at(\mathcal{K}_q^{m+1})$, if $\mathbf{x} \subseteq p^{-1}(\mathcal{X})$ and $\mathbf{x} \leq_n \mathbf{y}$ then $\mathbf{y} \subseteq p^{-1}(\mathcal{X})$. By 7.7, $\mathbf{x} = g^* R_I$ and $\mathbf{y} = h^* R_I$ for some $I \subseteq m+1$ and g, h satisfying $g(I) = h(I) = c$ and $g(K) \leq_n h(K)$ for $K \subseteq m+1$. Assume that $I \subseteq m$ (the other alternative is considered similarly). Define the functions $g_1, h_1 : P(m) \to at(\mathcal{G})$ by $g_1(J) = g(J) \times g(J_m)$ and similarly for h. Let $\mathbf{x}_1 = g_1^* R_I^m$ and $\mathbf{y}_1 = h_1^* R_I^m$. By (ii) and (iii), $\mathbf{x}_1 \subseteq \mathcal{X}$. By 7.7, $\mathbf{x}_1 \leq_n \mathbf{y}_1$, so $\mathbf{y}_1 \subseteq \mathcal{X}$. Again by (ii) and (iii), $\mathbf{y} \subseteq p^{-1}(\mathcal{X})$. This completes the proof.

Now we are able to prove the main statements. The next result classifies definable index sets in the structure $(\mathbf{U}; U_0)$ otained by enrichment of the BA U by the unary predicate "$a \in U_0$".

7.9. Theorem. *Any* $(\mathbf{U}; U_0)$*-definable index set is m-complete in one of levels* $\Sigma_\alpha, \Pi_\alpha, \Delta_{\alpha+1} (\alpha < \varepsilon_0)$, *and all the posibilities are realized.*

Proof. A quantifier elimination similar to that in 7.2 shows that the $(U; U_0)$-definable predicates are exactly the Boolean combinations of the predicates from $\mathcal{K}_q^m (m, q > 0)$. So one half of the theorem follows from 7.4. It remains to show that any of the specified levels is realized. In Se90 and Se91 we proved that the sets in (3) are m-complete in the corresponding levels. This fact and the proof of 7.4 imply that for any $n < \omega$ there is a U-definable unary predicate $\mathcal{X} \subseteq U$ with $\nu^{-1}(\mathcal{X})$ m-complete in Σ_{n+1}^0. By 2.4, for any $\alpha < \varepsilon_0$ there is a Boolean combination $\mathcal{X}_\alpha$ of these predicates with $\nu^{-1}(\mathcal{X}_\alpha)$ m-complete in Σ_α. Note that the predicates $\mathcal{X}_\alpha$ are U-definable.

It remains to realize the level $\Delta_{\alpha+1}$. Choose $m, q > 0$ for which $\mathcal{X}_\alpha$ is in the BA generated by $\mathcal{K}_q^m$ inside $P(U^m)$. By the isomorphism of $\{\mathcal{S}_\alpha\}$ with the corresponding syntactic FH and by the proof of 6.1, $\mathcal{X}_\alpha \in \mathcal{S}_\alpha \setminus \check{\mathcal{S}}_\alpha$. By 4.2, there is a μ_α-alternating tree $\{\mathbf{x}_\sigma\}$ for $\mathcal{X}$ (we use the isomorphic copies of the trees in Section 4, so in particular $\mathbf{x}_\sigma \in at(\mathcal{K}_q^m)$ and $\mathbf{x}_{\sigma 1} \subseteq \mathcal{X}$). By 7.7, $\mathbf{x}_\sigma = f_\sigma^* R_I^m$ for some fixed $I \subseteq m$ and functions f_σ satisfying $f_\sigma(I) = c$. Let $\mathcal{Y} = p^{-1}(\mathcal{X}_\alpha)R_I^{m+1} \cup \overline{p^{-1}(\mathcal{X}_\alpha)R_{I_m}^{m+1}}$, then $\mathcal{Y} \in \tilde{\mathcal{S}}_{\alpha+1}$ by 7.8 and 3.2. Define $g_\sigma, h_\sigma : P(m+1) \to at(\mathcal{G}_q)$ as follows: for any $K \subseteq m$, let $g_\sigma(K) = h_\sigma(K_m) = f_\sigma(K)$ and $g_\sigma(K_m) = h_\sigma(K) = a_0^0$. From 7.8.(iii) and 7.7.(i) it follows that $\{g_\sigma^* R_I\}$ and $\{h_\sigma^* R_{I_m}\}$ are the μ_α-alternating trees respectively for $\mathcal{Y}$ and $\bar{\mathcal{Y}}$. So $\mathcal{Y}$ is in the $\tilde{\mathcal{S}}_\alpha$-constituent. By the proof of 6.1, $\nu^{-1}(\mathcal{Y})$ is $\Delta_{\alpha+1}$-complete. This concludes the proof of the theorem.

Proof of 2.5. First we check that all the specified levels are realized. For the levels Σ_α the predicates $\mathcal{X}_\alpha$ above work. For a limit $\alpha < \varepsilon_0$, consider the U-definable predicate $\mathcal{Y} = p^{-1}(\mathcal{X}_\alpha)\mathcal{U}_0 \cup \overline{p^{-1}(\mathcal{X}_\alpha)}\mathcal{U}_1$, where $\mathcal{U}_0 = \bigcap_{J \subseteq m}(\mathcal{A}_0^0)_{J_m}$ and $\mathcal{U}_1 = \bigcap_{J \subseteq m}(\mathcal{A}_0^0)_J$. The trees $\{g_\sigma^* R_I\}$ and $\{h_\sigma^* R_{I_m}\}$ above are μ_α-alternating trees respectively for $\mathcal{Y}$ and $\bar{\mathcal{Y}}$, so $\mathcal{Y} \notin \mathcal{S}_\alpha \cup \check{\mathcal{S}}_\alpha$. We have $\mathcal{U}_i \in \mathcal{L}_0, \overline{R_J, R_{J_m}} \in \check{\mathcal{L}}_0$ and $R_{J_m}\mathcal{U}_0 = R_J\mathcal{U}_1 = \emptyset$ for all $J \subseteq m$. By 3.2, $\mathcal{Y}R_{J_m} = \overline{p^{-1}(\mathcal{X})\mathcal{U}_1 R_{J_m}} \in \check{\mathcal{S}}_\alpha$ and $\mathcal{Y}R_J = p^{-1}(\mathcal{X})\mathcal{U}_0 R_J \in \mathcal{S}_\alpha$. So the sets $R_K (K \subseteq m+1)$ are $\tilde{\mathcal{L}}_0$, pairwise disjoint, exhaust U^{m+1} and satisfy $\mathcal{Y}R_K \in \mathcal{S}_\alpha \cup \check{\mathcal{S}}_\alpha$. By 3.2.(v), $\mathcal{Y} \in \tilde{\mathcal{S}}_{\alpha+1}$, so $\nu^{-1}(\mathcal{Y})$ is $\Delta_{\alpha+1}$-complete.

It remains to show that the levels $\Delta_{\alpha+2}$ are not realized, i.e. the $\tilde{\mathcal{S}}_{\alpha+2}$-constituent is empty for any $\alpha < \varepsilon_0$. It suffices to show that if $\mathcal{X}$ is in the BA generated by $\mathcal{H}_q^m$ and $\mathcal{X} \notin \mathcal{S}_{\alpha+1} \cup \check{\mathcal{S}}_{\alpha+1}$ then $\mathcal{X} \notin \tilde{\mathcal{S}}_{\alpha+2}$. Let $f(I) = c$ for any $I \subseteq m$. By 7.7, $f^* \in at(\mathcal{H}_q^m)$, so either $f^* \subseteq \bar{\mathcal{X}}$ or $f^* \subseteq \mathcal{X}$. Consider e.g. the first alternative. We have $\mathcal{X} \notin \mathcal{S}_{\alpha+1}$, so there is a $\mu_{\alpha+1}$-alternating tree $\{\mathbf{x}_\sigma\}$ for $\bar{\mathcal{X}}$. By 7.7, $\mathbf{x}_\sigma$ are of the form $g_\sigma^* R_I$ for some fixed $I \subseteq m$ and functions g_σ with $g_\sigma(I) = c$. Let $\mathbf{y}_\emptyset = \mathbf{y}_0 = f^* R_I$, $\mathbf{y}_1 = \mathbf{x}_\emptyset$ and $\mathbf{y}_{i\sigma} = \mathbf{x}_{\bar{\sigma}}$ for $i < 2$ and $0 < |\sigma| < |\mu_{\alpha+1}|$. By 7.5 and 7.7, $\mathbf{y}_0 \leq_0 \mathbf{y}_\sigma$ for all σ. By definition of $\mu_{\alpha+2}$ in Section 3, $\mu_{\alpha+2} = 00\mu_\alpha$, so $\{\mathbf{y}_\sigma\}$ is a $\mu_{\alpha+2}$-alternating tree for $\mathcal{X}$ and *a fortiori* $\mathcal{X} \notin \check{\mathcal{S}}_{\alpha+2}$. This completes the proof of the theorem.

Let $(DI_{\mathrm{U}}^n; \leq_m)$ be the stucture of m-degrees of index sets of n-ary U-definable predicates. In Se91 and Se92 we have shown that the structure $(DI_{\mathrm{U}}^1; \leq_m)$ is almost well–ordered with the corresponding ordinal ω^2. An analysis of the proofs above shows that for any n the structure $(DI_{\mathrm{U}}^n; \leq_m)$ is almost well–ordered with some ordinal $< \varepsilon_0$. By 2.5, the structure $(DI_{\mathrm{U}}; \leq_m)$

is almost well–ordered with the ordinal ε_0. From these estimations we can deduce the following result on nondefinability in U.

By *a coding* on U we mean any injection of $U \times U$ into U. By remarks on the universal r.e. BA's at the beginning of this section, there is a coding on U which is even a recursive isomorphism (because $\mathbf{U} \times \mathbf{U}$ is universal). Is there a definable coding on U (i.e. a coding with the definable graph)?

7.10. Corollary. *There is no definable coding on U.*

Proof. Suppose the contrary: $(a_0, a_1) \mapsto [a_0, a_1]$ is such a coding. Let $\gamma(v_0, v_1, v)$ be a formula defining the graph $[a_0, a_1] = a$ in U, then γ is a Σ^0_n-formula for some nonzero $n < \omega$. Define $[a_0, \dots, a_k]$ by induction on k as follows: $[a_0] = a_0$, $[a_0, \dots, a_{k+1}] = [[a_0, \dots, a_k], a_{k+1}]$. For any k, the function $(a_0, \dots, a_k) \mapsto [a_0, \dots, a_k]$ is an injection of U^{k+1} into U, and it is definable by a Σ^0_n-formula $\gamma_k(v_0, \dots, v_k, v)$ (e.g. for $k = 2$ we have $[a_0, a_1, a_2] = a$ iff $\exists b([a_0, a_1] = b \wedge [b, a_2] = a)$, so we can take $\gamma_2 = \exists u(\gamma(v_0, v_1, u) \wedge \gamma(u, v_2, v)))$.

Relate to any formula $\theta(v_0, \dots, v_k)$ the unary formula

$$\theta^*(v) = \exists v_0 \cdots \exists v_k(\gamma_k(v_0, \dots, v_k, v) \wedge \theta(v_0, \dots, v_k)).$$

For all $a_0, \dots, a_k \in \mathbf{U}$ we have: $\mathbf{U} \models \theta(a_0, \dots, a_k)$ iff $\mathbf{U} \models \theta^*([a_0, \dots, a_k])$.

Now let $h = \emptyset^{n+1}$ and $\leq^h_m$ be the relativization of $\leq_m$ to h. It is easy to see that $\nu^{-1}(P_\theta) \equiv^h_m \nu^{-1}(P_{\theta^*})$ for any $\theta(v_0, \dots, v_k)$, so the quotients $\mathbf{S}$ and $\mathbf{S}_1$ of respectively $DI_{\mathbf{U}}$ and $DI^1_{\mathbf{U}}$ modulo $\equiv^h_m$ are the same. But from the cited result it follows that $\mathbf{S}_1$ is well–ordered with some ordinal $\leq \omega^2$, and from 2.5 and 3.11.(i) it follows that $\mathbf{S}$ is almost well–ordered with the ordinal ε_0. This contradiction completes the proof.

A similar argument shows that for any $n > 0$ there is no definable injection of U^{n+1} into U^n.

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