An elementary construction of an ultrafilter on Aleph-One using the Axiom of Determinateness

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Abstract

In this article we construct a free and σ-complete ultrafilter on the set ω_1, using AD.
First we define for each V c ω_1 a game G(V). From the axiom AD we have that for each V c ω_1, either the first or the second player has a winning strategy in G(V). We then show, in several lemma's, how to obtain winning strategies in G(V) for several different constructions of V from other sets. Finally, we show that the collection { V c ω_1 | the first player has a winning strategy in G(V) } has several closure properties corresponding to the lemma's just proved, and that this set is in fact a free and σ-complete ultrafilter.

Introduction

An ultrafilter on a set X is a collection U c $\mathcal{P}$(X) of subsets of X satisfying:

- For all V ∈ U, if V c W, then also W ∈ U.
- For all V, W ∈ U, V∩W ∈ U.
- For all V c X, exactly one of V, X\V ∈ U.

An ultrafilter can be thought of as a partitioning of the subsets of X into 'big' subsets (those in U) and 'little' subsets (those not in U).
For any x ∈ X, the collection U_x = { V c X | x ∈ V } is an ultrafilter, of a rather trivial type.
An ultrafilter is called free if it is not trivial, i.e. it does not contain any singletons of X.
An ultrafilter U is called σ-complete if it also satisfies:

- For all V_1, V_2, V_3, ... ∈ U, $\cap_{i\in\omega}V_i$ ∈ U.

Ultrafilters are used in the study of certain classes of big ordinals, such as the measurable ordinals.

Aleph-One is the smallest cardinal strictly greater than Aleph-Zero, the cardinality of the set **N**.
Here we use the set ω_1 = { α∈ORD | α is finite or countably infinite }, which is a set of ordinals of cardinality Aleph-One.
Alternatively, we could use the set $\mathcal{P}$(**Q**)/~, where **Q** is the set of rational numbers and ~ is defined by:

For V, W c **Q**, V~W iff V and W are non-empty, well-ordered and order-isomorphic,
or both V and W are non-well-ordered or empty.

It is well-known that in any two-player finite game G without any ties, hidden information or random factors, one of the two players always has a winning strategy. The Axiom of Determinateness (AD) holds that this is also true for any countably infinite game G, i.e. any game G of countably infinite maximum duration, with a countably infinite selection of moves each turn.

The game G(V) used in this article can be visualized thus:

Two players independently construct countably many countable ordinals.
They construct these ordinals as subsets of the set of rational numbers **Q**.
Each player has his own countably infinite collection of (initially empty) subsets of **Q**.
Each turn each player adds a finite number of points to finitely many of his own subsets.
'After' playing countably infinitely many turns, each player has generated countably many subsets of **Q**, each one representing a countable ordinal (or 0, if the subset is not well-ordered).
The supremum of all the generated ordinals is obviously itself a countable ordinal.
Player A wins if this supremum is an ordinal in V; player B has won if it is not in V.

Hence player A tries to 'force' the supremum into V, and player B tries to force it outside V.

The property 'A can force the supremum to be in V' is immediately suggestive of a 'bigness' property. And indeed the collection of all sets V, such that player A can 'force' the supremum of the game to be an ordinal in V, is shown in this article to have all the properties of the required ultrafilter.

An elementary construction of an ultrafilter on Aleph-One, using AD

Notes

Since each move can be described as a finite sequence of pairs of an integer and a rational, there are only countably many possible moves at each turn, and AD applies. Hence, for any V, the outcome of the game can either be forced to be an ordinal in V, or it can be forced to be an ordinal outside V.

A recurring theme in the proof of the lemma's below is that the result of any finite sequence of moves can also be achieved by a single move, as the union of finitely many finite sets is itself a finite set. In a sense, this move is equivalent to the original finite sequence, and any response to this move will also suffice as a response to the original sequence.

For technical reasons, it is necessary at some places in the proof to be able to react to one's own moves as if they had been made by the opponent. Since the opponent cannot move in one's own subsets, it is necessary to 'see' some of one's own subsets as if they belonged to the opponent. To facilitate this an 'index-structure' ($\mathcal{A}$, $\mathcal{B}$) to the subsets-under-construction is explicitly defined and used. Note that this does not imply that any player can really add points to his opponents subsets.

In the proof of lemma 7 we will use AC-**N**, a weak form of AC which is derivable from AD.

Definitions

Let $\mathcal{A}$ and $\mathcal{B}$ be two countably infinite, disjoint sets.
For any subset V of ω_1, we can define a game $G_{\mathcal{A},\mathcal{B}}(V)$ for two players, A and B:

A starts by naming a finite set $a_1 \in (\mathcal{A} \times \mathbf{Q})^{<\omega}$ of pairs (a, q), where $a \in \mathcal{A}$ and $q \in \mathbf{Q}$.
B names another finite set $b_1 \in (\mathcal{B} \times \mathbf{Q})^{<\omega}$ of pairs (b, r), where $b \in \mathcal{B}$ and $r \in \mathbf{Q}$.
A then names another finite set $a_2 \in (\mathcal{A} \times \mathbf{Q})^{<\omega}$ of pairs (a, q).
B then names another finite set $b_2 \in (\mathcal{B} \times \mathbf{Q})^{<\omega}$ of pairs (b, r).
. . . . . .
'until' a_k, b_k have been named for all natural numbers $k \geq 1$.
Define $I := \mathcal{A} \cup \mathcal{B}$, and the result $z := \cup\{a_1, b_1, a_2, b_2, a_3, b_3, ...\} \subset I \times \mathbf{Q}$
Define $\pi: \mathcal{P}(\mathbf{Q}) \to \mathrm{ORD}$ by

$$\pi(R) := \begin{cases} \text{the ordering type of } (R, \leq) & \text{if } (R,\leq) \text{ is well-ordered.} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\Pi_{\mathcal{A},\mathcal{B}} : P(I \times \mathbf{Q}) \to \mathrm{ORD}$ by $\Pi(z) := \text{supremum}(\{ \pi(\{ q \mid (i, q) \in z \}) \mid i \in I \})$.
Obviously, $\pi(R)$ and $\Pi_{\mathcal{A},\mathcal{B}}(z)$ are countable ordinals for any $R \subset \mathbf{Q}$ or $z \in I \times \mathbf{Q}$.
If $\Pi_{\mathcal{A},\mathcal{B}}(z) \in V$, then A has won the game, otherwise B has won the game.
Define $\underline{V}_{\mathcal{A},\mathcal{B}} := \Pi^{-1}[V] = \{ z \subset I \times \mathbf{Q} \mid \Pi(z) \in V \}$
Then the above becomes: if $z \in \underline{V}_{\mathcal{A},\mathcal{B}}$, then A has won the game, otherwise B has won.

When no confusion is possible, we will write $G(V)$, $\Pi(z)$ and $\underline{V}$ for $G_{\mathcal{A},\mathcal{B}}(V)$, $\Pi_{\mathcal{A},\mathcal{B}}(z)$ and $\underline{V}_{\mathcal{A},\mathcal{B}}$.
Note that $\Pi_{\mathcal{A},\mathcal{B}}$ and $\underline{V}_{\mathcal{A},\mathcal{B}}$ depend on $\mathcal{A} \cup \mathcal{B}$ only.

A strategy for A is a function f which takes as an argument a finite sequence of moves $a_1, b_1, ..., a_{k-1}, b_{k-1}$ (the moves 'so far') and gives as result a move a_k (the 'next' move).
For instance, a strategy for A in the game $G_{\mathcal{A},\mathcal{B}}(V)$ is a function from the set of even-length sequences of alternatingly finite sets of pairs (a, q), and of pairs (b, q), to the set of finite sets of pairs (a, q).
Player A plays according to a strategy f if $a_k = f(\langle a_1, b_1, ..., a_{k-1}, b_{k-1}\rangle)$ for all k.
A winning strategy for A in a game G is a strategy f such that, if A plays according to f, then A wins the game no matter what sequence of moves $b_1, b_2, ...$ player B plays.
In particular, if f is a winning strategy for A in $G_{\mathcal{A},\mathcal{B}}(V)$, then for any sequence $b_1, b_2, ...$, if $a_k = f(\langle a_1, b_1, ..., a_{k-1}, b_{k-1}\rangle)$ for all k, then $\cup\{a_1, b_1, a_2, b_2, a_3, b_3, ...\} \in \underline{V}$.
Strategies and winning strategies for B are defined in a like manner.

Proposition

$\{ V \in P(\omega_1) \mid$ A has a winning strategy in $G_{\mathcal{A},\mathcal{B}}(V) \}$ is a free and σ-complete ultrafilter on ω_1.

First we need an auxiliary lemma:

<u>Lemma 1</u>

If player A or B has a winning strategy in a game $G_{\mathcal{A},\mathcal{B}}(V)$, then A resp. B has a winning strategy in $G_{\mathcal{C},\mathcal{D}}(V)$ for any two disjoint countably infinite sets $\mathcal{C}$ and $\mathcal{D}$.

Proof Concept

We extend the bijective mappings $\mathcal{A}$<->$\mathcal{C}$ and $\mathcal{B}$<->$\mathcal{D}$ to a bijective mapping of all moves, games and (winning) strategies from $G_{\mathcal{A},\mathcal{B}}(V)$ onto those of $G_{\mathcal{C},\mathcal{D}}(V)$.

Proof

Since $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are countably infinite sets, there exist bijective funtions between them and ω.
Therefore there exist bijective functions between $\mathcal{A}$ and $\mathcal{C}$, and between $\mathcal{B}$ and $\mathcal{D}$.
Since $\mathcal{A}$ and $\mathcal{B}$ are disjoint, as well as $\mathcal{C}$ and $\mathcal{D}$, the union of the above function is a bijective function φ: $\mathcal{A}\cup\mathcal{B}$ <-> $\mathcal{C}\cup\mathcal{D}$.
We extend φ's domain to include moves by $\varphi(\{(i_j, q_j) \mid j<k\}) := \{\varphi(i_j), q_j) \mid j<k\}$,
and to include results by $\varphi(\{(i_j, q_j) \mid j\in\omega\}) := \{\varphi(i_j), q_j) \mid j\in\omega\}$
It is clear that φ is well-defined and remains bijective.
It is also obvious that for any z c $(A\cup B) \times \mathbf{Q}$, $\Pi_{\mathcal{A},\mathcal{B}}(z) = \Pi_{\mathcal{C},\mathcal{D}}(\varphi(z))$.
and for any result $z = \cup\{a_1, b_1, a_2, ...\}$, $\varphi(z) = \cup\{\varphi(a_1), \varphi(b_1), \varphi(a_2), ...\}$.

Now suppose that f is a winning strategy for player A in $G_{\mathcal{A},\mathcal{B}}(V)$.
Then for any sequence $b_1, b_2, ...$, if $a_k=f(<a_1, b_1, ..., a_{k-1}, b_{k-1}>)$, then:
$\Pi_{\mathcal{A},\mathcal{B}}(\cup\{a_1, b_1, a_2, b_2, ...\}) \in V$.
Define the strategy g for C in $G_{\mathcal{C},\mathcal{D}}(V)$ by
$g(<c_1, d_1, ..., d_k>) := \varphi(f(<\varphi^{-1}(c_1), \varphi^{-1}(d_1), ..., \varphi^{-1}(d_k)>))$.
Let $d_1, d_2, ...$, be any sequence of moves in $G_{\mathcal{C},\mathcal{D}}(V)$, and let $c_k=g(<c_1, d_1, ..., d_{k-1}>)$ for all k.
By defining $a_k:=\varphi^{-1}(c_k)$, $b_k:=\varphi^{-1}(d_k)$, we have:
$a_k = \varphi^{-1}(c_k) = \varphi^{-1}(g(<c_1, d_1, ..., d_{k-1}>)) = \varphi^{-1}(\varphi(f(<\varphi^{-1}(c_1), \varphi^{-1}(d_1), ..., \varphi^{-1}(d_{k-1})>)))$
$= f(<a_1, b_1, ..., b_{k-1}>)$ for all k
and hence:
$\Pi(z) = \Pi(\cup\{c_1, d_1, c_2, d_2, ...\}) = \Pi(\varphi(\cup\{a_1, b_1, a_2, b_2, ...\}) = \Pi(\cup\{a_1, b_1, a_2, b_2, ...\}) \in V$.
So g is a winning strategy for A in $G_{\mathcal{C},\mathcal{D}}(V)$.

Now suppose that f is a winning strategy for player B in $G_{\mathcal{A},\mathcal{B}}(V)$.
Then for any sequence $a_1, a_2, ...$, if $b_k=f(<a_1, b_1, ..., b_{k-1}, a_k>)$, then:
$\Pi_{\mathcal{A},\mathcal{B}}(\cup\{a_1, b_1, a_2, b_2, ...\}) \in \omega_1\backslash V$.
Define the strategy g for D in $G_{\mathcal{C},\mathcal{D}}(V)$ by
$g(<c_1, d_1, ..., c_k>) := \varphi(f(<\varphi^{-1}(c_1), \varphi^{-1}(d_1), ..., \varphi^{-1}(c_k)>))$.
Let $c_1, c_2, ...$, be any sequence of moves in $G_{\mathcal{C},\mathcal{D}}(V)$, and let $d_k=g(<c_1, d_1, ..., c_k>)$ for all k.
By defining $a_k:=\varphi^{-1}(c_k)$, $b_k:=\varphi^{-1}(d_k)$, we have:
$b_k = \varphi^{-1}(d_k) = \varphi^{-1}(g(<c_1, d_1, ..., c_k>)) = \varphi^{-1}(\varphi(f(<\varphi^{-1}(c_1), \varphi^{-1}(d_1), ..., \varphi^{-1}(c_k)>)))$
$= f(<a_1, b_1, ..., a_k>)$ for all k
and hence:
$\Pi(z) = \Pi(\cup\{ c_1, d_1, c_2, d_2, ...\}) = \Pi(\varphi(\cup\{a_1, b_1, a_2, b_2, ...\}) = \Pi(\cup\{a_1, b_1, a_2, b_2, ...\}) \in \omega_1\backslash V$.
So g is a winning strategy for B in $G_{\mathcal{C},\mathcal{D}}(V)$.

Note

This lemma justifies our use of the notation G(V) for $G_{\mathcal{A},\mathcal{B}}(V)$ when no confusion is possible.

The lemma's 2-7 correspond to properties of an ultrafilter.

Lemma 2

If A has a winning strategy in the game G(V), and VcW, then A has a winning strategy in G(W).

Proof Concept

Any winning strategy for A in G(V) is also a winning strategy for A in G(W).

Proof

Let f be a winning strategy for A in G(V).
Then for any sequence $b_1, b_2, ...$, if $a_k=f(<a_1, b_1, ..., a_{k-1}, b_{k-1}>)$, then:
$\cup\{a_1, b_1, a_2, b_2, ...\} \in \underline{V}$.
Now consider f as strategy for A in G(W).
Then for any sequence $b_1, b_2, ...$, if $a_k=f(<a_1, b_1, ..., a_{k-1}, b_{k-1}>)$, then:
$z = \cup\{a_1, b_1, a_2, b_2, ...\} \in \underline{V} \subset \underline{W}$, hence $z \in \underline{W}$.
So f is a winning strategy for A in G(W).

Lemma 3

If V is a singleton, then player B has a winning strategy in G(V).

Proof Concept

B constructs an ordinal greater than the ordinal in V.
Note: A cannot prevent this, since he cannot make moves $(\mathcal{b}, q)$ with $\mathcal{b} \in \mathcal{B}$.

Proof

Suppose $V=\{v\}$, $v \in \omega_1$.
Now v+1 is countable, therefore there exists a subset Sc**Q** such that $(S, \leq)$ has type v+1.
S is countable, so there exists a surjective function h: $\mathbf{N}^+ \to S$.
Fix a $\mathcal{b} \in \mathcal{B}$, and let f be defined by $f(<a_1, b_1, ..., b_{k-1}, a_k>) := \{(\mathcal{b}, h(k))\}$.
Then for any sequence $a_1, a_2, ...$, if $b_k=f(<a_1, b_1, ..., b_{k-1}, a_k>)$, then:
$z = \cup\{a_1, b_1, a_2, b_2, ...\} = \{\mathcal{b}\}xS \cup \cup\{a_1, a_2, a_3, ...\}$.
Hence $\Pi(z) \geq \pi(S)=v+1$, and then $\Pi(z) \neq v$, $\Pi(z) \notin V$.
So f is a winning strategy for B in G(V).

<u>Lemma 4</u>

If B has a winning strategy in the game G(V), then A has a winning strategy in $G(\omega_1 \setminus V)$.

Proof Concept

Player A first plays Ø, and then plays according to B's strategy for G(V).

Proof

Suppose player B has a winning strategy in the game G(V).
By lemma 1 there exists a winning strategy f for B in the game $G_{\mathcal{B},\mathcal{A}}(V)$.
Then for any sequence $a_1, a_2, \ldots$, in $(\mathcal{B} \times \mathbf{Q})^{<\omega}$, if $b_k = f(\langle a_1, b_1, \ldots, b_{k-1}, a_k \rangle)$ for all k, then:

$$\cup\{ a_1, b_1, a_2, b_2, a_3, \ldots\} \in (\underline{\omega_1 \setminus V})_{\mathcal{B},\mathcal{A}} = \underline{\omega_1 \setminus V} \ (= P(I \times \mathbf{Q}) \setminus \underline{V}).$$

Define the strategy g for A by:

$$g(\langle\rangle) := \varnothing$$
$$g(\langle c_1, d_1, \ldots, c_k, d_k \rangle) := f(\langle d_1, c_2, \ldots, c_k, d_k \rangle)$$

Let $d_1, d_2, \ldots$, be any sequence of moves in $(\mathcal{B} \times \mathbf{Q})^{<\omega}$, and let $c_k = g(\langle c_1, d_1, \ldots, c_{k-1}, d_{k-1} \rangle)$ for all k.
By defining $a_k := d_k$, $b_k := c_{k+1}$, we have:

$$a_k \in (\mathcal{B} \times \mathbf{Q})^{<\omega} \text{ for } k \geq 1$$
$$b_k = c_{k+1} = g(\langle c_1, d_1, \ldots, c_k, d_k \rangle) = f(\langle d_1, c_2, \ldots, c_k, d_k \rangle) = f(\langle a_1, b_1, \ldots, a_k \rangle) \text{ for all k}$$

and hence:

$$z = \cup\{ c_1, d_1, c_2, d_2, c_3, \ldots\} = \cup\{\varnothing, a_1, b_1, a_2, b_2, \ldots\} = \cup\{a_1, b_1, a_2, b_2, \ldots\} \in \underline{\omega_1 \setminus V}.$$

So g is a winning strategy for A in $G(\omega_1 \setminus V)$.

Lemma 5

If A has a winning strategy in the game G(V), then B has a winning strategy in $G(\omega_1 \setminus V)$.

Proof Concept

Player B plays according to A's strategy for G(V), except that B's first move is a combination of the opening move he should have made and the response to A's move.
A key notion in this and the next lemma's is that any finite sequence of moves is equivalent to the single move corresponding to the finite union of the finite sets of the moves in the sequence.

Proof

Suppose player A has a winning strategy in the game G(V).
By lemma 1 there exists a winning strategy f for A in the game $G_{\mathcal{B},\mathcal{A}}(V)$.
Then for any sequence $b_1, b_2, \ldots$, in $(\mathcal{A} \times \mathbf{Q})^{<\omega}$, if $a_k = f(\langle a_1, b_1, \ldots, a_{k-1}, b_{k-1} \rangle)$ for all k, then:

$$\cup\{ a_1, b_1, a_2, b_2, a_3, b_3, \ldots\} \in \underline{V}_{\mathcal{B},\mathcal{A}} = \underline{V}.$$

Define the strategy g for B by

$$g(\langle c_1 \rangle) := f(\langle\rangle) \cup f(\langle f(\langle\rangle), c_1 \rangle)$$
$$g(\langle c_1, d_1, c_2, \ldots, d_{k-1}, c_k \rangle) := f(\langle f(\langle\rangle), c_1, f(\langle f(\langle\rangle), c_1 \rangle), c_2, \ldots, d_{k-1}, c_k \rangle) \quad \text{for } k \geq 2.$$

Let $c_1, c_2, \ldots$, be any sequence of moves in $(\mathcal{A} \times \mathbf{Q})^{<\omega}$, and let $d_k = g(\langle c_1, d_1, \ldots, c_k \rangle)$ for all k.
By defining $a_1 := f(\langle\rangle)$, $a_2 := f(\langle f(\langle\rangle), c_1 \rangle)$, $a_{k+1} := d_k$ for $k \geq 2$, $b_k = c_k$, we have:

$$b_k \in (\mathcal{A} \times \mathbf{Q})^{<\omega} \text{ for } k \geq 1.$$
$$a_1 = f(\langle\rangle)$$
$$a_2 = f(\langle f(\langle\rangle), c_1 \rangle) = f(\langle a_1, b_1 \rangle)$$
$$a_k = g(\langle c_1, d_1, \ldots, c_{k-1} \rangle) = f(\langle f(\langle\rangle), c_1, f(\langle f(\langle\rangle), c_1 \rangle), c_2, d_2, \ldots, c_{k-1} \rangle$$
$$= f(\langle a_1, b_1, a_2, b_2, a_3, \ldots, b_{k-1} \rangle) \text{ for } k \geq 2.$$

and hence:

$$z = \cup\{c_1, d_1, c_2, d_2, \ldots\} = \cup\{b_1, a_1 \cup a_2, b_2, a_3, \ldots\} = \cup\{a_1, b_1, a_2, b_2, a_3, \ldots\} \in \underline{V}.$$

So g is a winning strategy for B in $G(\omega_1 \setminus V)$.

Lemma 6

If B has a winning strategy in G(V) and in G(W), then B has a winning strategy in G(V∪W).

Proof Concept

Player B plays G(V) and G(W) simultaneously by alternating between using her strategies for G(V) and G(W), and interpreting the moves she makes for G(V) as part of her opponents moves when playing in G(W) (w.r.t. the 'input' the strategy gets), and vica versa.
In order to do this, we split the 'index'-set $\mathcal{B}$ into two index-sets $\mathcal{B}_1$ and $\mathcal{B}_2$,.
B plays all moves for G(V) in $\mathcal{B}_1$, and all moves for G(W) in $\mathcal{B}_2$, interpreting the index-set not used as part of the opponents index-set.

Proof

Suppose B has winning strategies in G(V) and in G(W).
Let $(\mathcal{B}_1, \mathcal{B}_2)$ be a partitioning of $\mathcal{B}$ into two disjoint countably infinite sets.
Define $\mathcal{A}_i := (\mathcal{A}\cup\mathcal{B})\backslash\mathcal{B}_i$.
Then for i=1,2, $\mathcal{A}_i$ and $\mathcal{B}_i$ are disjoint countably infinite sets, and $\mathcal{A}_i\cup\mathcal{B}_i=\mathcal{A}\cup\mathcal{B}$.
By lemma 1, there exist winning strategies f_1, f_2 for B in $G_{\mathcal{A}_1, \mathcal{B}_1}(V)$ and $G_{\mathcal{A}_2, \mathcal{B}_2}(W)$.
For i=1,2, for any sequence $a_1, a_2, ...,$ in $(\mathcal{A}_i \times \mathbf{Q})^{<\omega}$, if $b_k=f_i(<a_1, b_1, ..., a_k>)$ for all k, then:

$$\cup\{ a_1, b_1, a_2, b_2, a_3, b_3, ...\} \in (\underline{\omega_1\backslash V_i})_{\mathcal{A}_i, \mathcal{B}_i} = \underline{\omega_1\backslash V_i}$$

Define the strategy g for B by:

$$g(<c_1, d_1, ..., d_{2k-2}, c_{2k-1}>) := \begin{cases} f_1(<c_1, d_1, c_2\cup d_2\cup c_3, d_3, ..., d_{2k-3}, c_{2k-2}\cup d_{2k-2}\cup c_{2k-1}>) \\ \quad \text{if } <...> \text{ is a proper sequence of moves w.r.t. } G_{\mathcal{A}_1, \mathcal{B}_1}(V) \\ \emptyset \quad \text{otherwise} \end{cases}$$

$$g(<c_1, d_1, ..., d_{2k-1}, c_{2k}>) := \begin{cases} f_2(<c_1\cup d_1\cup c_2, d_2, c_3\cup d_3\cup c_4, ..., d_{2k-2}, c_{2k-1}\cup d_{2k-1}\cup c_{2k}>) \\ \quad \text{if } <...> \text{ is a proper sequence of moves w.r.t. } G_{\mathcal{A}_2, \mathcal{B}_2}(W) \\ \emptyset \quad \text{otherwise} \end{cases}$$

(Here, a sequence of moves is called 'proper' with respect to a game $G_{\mathcal{A}_i, \mathcal{B}_i}(V_i)$, if it consists of, alternatingly, finite subsets of $\mathcal{A}_i \times \mathbf{Q}$ and of $\mathcal{B}_i \times \mathbf{Q}$. The strategies f_i need not be defined on improper sequences, hence the extra clause in the definition of g.)

Let $c_1, c_2, ...$ be any sequence of moves in $(\mathcal{A} \times \mathbf{Q})^{<\omega}$, and let $d_k=g(<c_1, d_1, ..., c_k>)$ for all k.
Then for all k, $d_{2k-1} \in (\mathcal{B}_1 \times \mathbf{Q})^{<\omega} \subset (\mathcal{A}_2 \times \mathbf{Q})^{<\omega}$, and $d_{2k} \in (\mathcal{B}_2 \times \mathbf{Q})^{<\omega} \subset (\mathcal{A}_1 \times \mathbf{Q})^{<\omega}$.
Also, for any m≥1, $c_m \in (\mathcal{A}_1 \times \mathbf{Q})^{<\omega}$ and $c_m \in (\mathcal{A}_2 \times \mathbf{Q})^{<\omega}$.
This, with the observation that unions of finitely many finite sets yield finite sets, implies:

- $c_1 \in (\mathcal{A}_1 \times \mathbf{Q})^{<\omega}$
- $c_{2k-2}\cup d_{2k-2}\cup c_{2k-1} \in (\mathcal{A}_1 \times \mathbf{Q})^{<\omega}$ for k≥2.
- $c_{2k-1}\cup d_{2k-1}\cup c_{2k} \in (\mathcal{A}_2 \times \mathbf{Q})^{<\omega}$ for all k≥1.

Therefore, for all k, the sequence <...> in the definition of $g(<c_1,..., c_{2k-1}>)$ resp. $g(<c_1, ..., c_{2k}>)$ is a proper sequence of moves w.r.t. $G_{\mathcal{A}_1, \mathcal{B}_1}(V_1)$ resp. $G_{\mathcal{A}_2, \mathcal{B}_2}(V_2)$, and the first clause in the definition of g always applies..

So by defining $a_1:=c_1$, $a_k:=c_{2k-2}\cup d_{2k-2}\cup c_{2k-1}$ for k≥2, and $b_k:=d_{2k-1}$ we have:

$$b_k = g(<c_1, d_1, ..., d_{2k-2}, c_{2k-1}>) = f_1(<c_1, d_1, c_2\cup d_2\cup c_3, ..., d_{2k-3}, c_{2k-2}\cup d_{2k-2}\cup c_{2k-1}>).$$
$$= f_1(<a_1, b_1, a_2, ..., b_{k-1}, a_k>) \text{ for all k}$$

and hence:

$$z = \cup\{c_1, d_1, c_2, d_2, c_3, ...\} = \cup\{c_1, d_1, c_2\cup d_2\cup c_3, d_3, ...\} = \cup\{a_1, b_1, a_2, b_2, ...\} \in \underline{\omega_1\backslash V}.$$

Also, by defining $a_k:=c_{2k-1}\cup d_{2k-1}\cup c_{2k}$, $b_k:=d_{2k}$ we have:

$$b_k = g(<c_1, d_1, ..., d_{2k-1}, c_{2k}>) := f_2(<c_1\cup d_1\cup c_2, d_2, c_3\cup d_3\cup c_4, ..., c_{2k-1}\cup d_{2k-1}\cup c_{2k}>)$$
$$= f_2(<a_1, b_1, ..., b_{k-1}, a_k>)$$

and hence:

$$z = \cup\{c_1, d_1, c_2, d_2, c_3, ...\} = \cup\{c_1\cup d_1\cup c_2, d_2, c_3\cup d_3\cup c_4, ...\} = \cup\{a_1, b_1, a_2, ...\} \in \underline{\omega_1\backslash W}.$$

Hence $z \in (\underline{\omega_1\backslash V})\cap(\underline{\omega_1\backslash W}) = \underline{\omega_1\backslash(V\cup W)}$.
So g is a winning strategy for B in G(V∪W).

<u>Lemma 7</u>

If $(V_i)_{i\in\omega}$ is a countable set of subsets of ω_1, and B has a winning strategy in $G(V_i)$ for all $i\in\omega$, then B has a winning strategy in $G(\cup_{i\in\omega}V_i)$.

Proof Concept

Player B plays the games $G(V_i)$ simultaneously by using her strategies for each game in turn, and in each game $G(V_i)$, interpreting the moves made for other games $G(V_j)$ as part of her opponents moves (with respect to the 'input' the strategy gets).
In order to do this, we split the 'index'-set $\mathcal{B}$ into countably many index-sets $\mathcal{B}_i$.
B plays all moves for $G(V_i)$ in $\mathcal{B}_i$, interpreting all the other $\mathcal{B}_j$ as part of the opponents index-set.

Proof

Suppose B has winning strategies in $G(V_i)$ for $i\in\omega$.
Let $(\mathcal{B}_i)_{i\in\omega}$ be a partitioning of $\mathcal{B}$ into countably infinite many disjoint countably infinite sets.
Define $\mathcal{A}_i := (\mathcal{A}\cup\mathcal{B})\backslash\mathcal{B}_i$.
Then for $i\in\omega$, $\mathcal{A}_i$ and $\mathcal{B}_i$ are disjoint countably infinite sets, and $\mathcal{A}_i\cup\mathcal{B}_i=\mathcal{A}\cup\mathcal{B}$.
By lemma 1 and AC-**N**, there exists a winning strategy f_i for B in $G_{\mathcal{A}_i,\mathcal{B}_i}(V_i)$ for $i\in\omega$.
For $i\in\omega$, for any sequence $a_1, a_2, \ldots$ in $(\mathcal{A}_i \times \mathbf{Q})^{<\omega}$, if $b_k=f_i(<a_1, b_1, \ldots, a_k>)$ for all k, then:

$$\cup\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\} \in (\underline{\omega_1\backslash V_i})_{\mathcal{A}_i,\mathcal{B}_i} = \underline{\omega_1\backslash V_i}$$

Define the strategy g for B by $g(<c_1, d_1, \ldots, d_{(2k-1)*2^i-1}, c_{(2k-1)*2^i}>) :=$

$$:= \begin{cases} f_i(< c_1\cup d_1\cup\ldots\cup d_{2^i-1}\cup c_{2^i},\ d_{2^i},\ c_{2^i+1}\cup d_{2^i+1}\cup\ldots\cup d_{3*2^i-1}\cup c_{3*2^i},\ d_{3*2^i}, \\ \quad c_{3*2^i+1}\cup d_{3*2^i+1}\cup\ldots\cup c_{5*2^i},\ \ldots,\ d_{(2k-3)*2^i},\ c_{(2k-3)*2^i+1}\cup d_{(2k-3)*2^i+1}\cup\ldots\cup c_{(2k-1)*2^i}>) \\ \qquad \text{if <...> is a proper sequence of moves w.r.t. } G_{\mathcal{A}_i,\mathcal{B}_i}(V_i), \\ \qquad\quad \text{(i.e. all moves are finite subsets of, alternatingly, } \mathcal{A}_i \times \mathbf{Q} \text{ and } \mathcal{B}_i \times \mathbf{Q}) \\ \emptyset \qquad \text{otherwise} \end{cases}$$

Since for any $n\geq1$ there are unique $i\geq0$, $k\geq1$ such that $n=(2k-1)*2^i$, this is a proper definition.

Let $c_1, c_2, \ldots$ be any sequence of moves in $(\mathcal{A} \times \mathbf{Q})^{<\omega}$, and let $d_k=g(<c_1, d_1, \ldots, c_k>)$ for all k.
Now for any $i\in\omega$, $z = \cup\{c_1, d_1, c_2, d_2, \ldots\} \in \omega_1\backslash V_i$.
Proof:

Let $i\in\omega$.
For any $k\geq1$, and any $m\geq1$ with $(2k-3)*2^i+1 \leq m \leq (2k-1)*2^i-1$, there are unique $j\geq0$, $l\geq1$ such that $m = (2l-1)*2^j$ and $j\neq i$, and then $d_m = d_{(2l-1)*2^j} \in (\mathcal{B}_j \times \mathbf{Q})^{<\omega} \subset (\mathcal{A}_i \times \mathbf{Q})^{<\omega}$ (because $\mathcal{B}_j \subset \mathcal{A}_i$).
Also for any $m\geq1$, $c_m \in (\mathcal{A} \times \mathbf{Q})^{<\omega} \subset (\mathcal{A}_i \times \mathbf{Q})^{<\omega}$.
This and the observation that the union of finitely many finite sets is a finite sets yield:

- $c_1\cup d_1\cup\ldots\cup d_{2^i-1}\cup c_{2^i} \in (\mathcal{A}_i \times \mathbf{Q})^{<\omega}$.
- $c_{(2k-3)*2^i+1}\cup d_{(2k-3)*2^i+1}\cup\ldots\cup c_{(2k-1)*2^i} \in (\mathcal{A}_i \times \mathbf{Q})^{<\omega}$ for all $k\geq2$.

Therefore, for all k, the sequence <...> in the definition of $g(<c_1, \ldots, c_{(2k-1)*2^i}>)$ is a proper sequence of moves w.r.t. $G_{\mathcal{A}_i,\mathcal{B}_i}(V_i)$, and the first clause in the definition of g always applies.

So by defining $a_1 := c_1\cup d_1\cup\ldots\cup d_{2^i-1}\cup c_{2^i}$,
$a_k := c_{(2k-3)*2^i+1}\cup d_{(2k-3)*2^i+1}\cup\ldots\cup c_{(2k-1)*2^i}$ for $k\geq2$,
and $b_k := d_{(2k-1)*2^i}$,
we have for all k:

$$\begin{aligned} b_k &= g(<c_1, d_1, \ldots, d_{(2k-1)*2^i-1}, c_{(2k-1)*2^i}>) \\ &= f_i(< c_1\cup d_1\cup\ldots\cup d_{2^i-1}\cup c_{2^i},\ d_{2^i},\ c_{2^i+1}\cup d_{2^i+1}\cup\ldots\cup d_{3*2^i-1}\cup c_{3*2^i}, \\ &\qquad d_{3*2^i}, \ldots, d_{(2k-3)*2^i},\ c_{(2k-3)*2^i+1}\cup d_{(2k-3)*2^i+1}\cup\ldots\cup c_{(2k-1)*2^i}>) \\ &= f_i(<a_1, b_1, a_2, \ldots, b_{k-1}, a_k>) \end{aligned}$$

and hence:

$$\begin{aligned} z &= \cup\{c_1, d_1, c_2, d_2, c_3, d_3, \ldots\} \\ &= \cup\{c_1\cup d_1\cup\ldots\cup d_{2^i-1}\cup c_{2^i},\ d_{2^i},\ c_{2^i+1}\cup d_{2^i+1}\cup\ldots\cup d_{3*2^i-1}\cup c_{3*2^i},\ d_{3*2^i},\ \ldots\} \\ &= \cup\{a_1, b_1, a_2, b_2, \ldots\} \in (\underline{\omega_1\backslash V_i})_{\mathcal{A}_i,\mathcal{B}_i} = \underline{\omega_1\backslash V_i}. \end{aligned}$$

So $z = \cup\{c_1, d_1, c_2, d_2, \ldots\} \in \cap_{i\in\omega}\underline{\omega_1\backslash V_i} = \underline{\omega_1\backslash\cup_{i\in\omega}V_i}$
So g is a winning strategy for B in $G(\cup_{i\in\omega}V_i)$.

An elementary construction of an ultrafilter on Aleph-One, using AD

Proposition

$U := \{ V \in P(\omega_1) \mid$ A has a winning strategy in $G(V) \}$ is a free and σ-complete ultrafilter on ω_1.

Proof

Every move is one of countably many choices, since $\mathcal{A}$, $\mathcal{B}$ and **Q** are countably infinite, and hence there are only countably many finite sets of pairs $(a, q) \in \mathcal{A} \times \mathbf{Q}$ resp. $(b, r) \in \mathcal{B} \times \mathbf{Q}$.
Therefore, the Axiom of Determinateness applies.
By AD, for any V, either player A has a winning strategy in G(V), or B has a winning strategy.
Therefore: $V \in U$ <=> player A has a winning strategy in G(V),
and $V \notin U$ <=> player B has a winning strategy in G(V).

Lemma's 2-7, therefore, can be translated to properties of U:

(2'): If $V \in U$, and V c W, then $W \in U$.
(3'): If V is a singleton, then $V \notin U$.
(4'): If $V \notin U$, then $\omega_1 \setminus V \in U$.
(5'): If $V \in U$, then $\omega_1 \setminus V \notin U$.
(6'): If $V, W \notin U$, then $V \cup W \notin U$.
(7'): If $V_i \notin U$ for $i \in \omega$, then $\cup_{i \in \omega} V_i \notin U$.

From 4', 5' and 6' we can derive: (6'') If $V, W \in U$, then $V \cap W \in U$.
From 4', 5' and 7' we can derive: (7'') If $V_i \in U$ for $i \in \omega$, then $\cap_{i \in \omega} V_i \in U$.
Proof:

$V, W \in U \Rightarrow_{5'} \omega_1 \setminus V, \omega_1 \setminus W \notin U \Rightarrow_{6'} (\omega_1 \setminus V) \cup (\omega_1 \setminus V) \notin U, \Rightarrow_{4'} V \cap W = \omega_1 \setminus ((\omega_1 \setminus V) \cup (\omega_1 \setminus V)) \in U$

$\forall i \in \omega: V_i \in U \Rightarrow_{5'} \forall i \in \omega: \omega_1 \setminus V_i \notin U \Rightarrow_{7'} \cup_{i \in \omega} \omega_1 \setminus V_i \notin U \Rightarrow_{4'} \cap_{i \in \omega} V_i = \omega_1 \setminus (\cup_{i \in \omega} \omega_1 \setminus V_i) \in U$

2', 4', 5' and 6'' are the three defining properties of an ultrafilter.
3' and 7'' imply that U is free and σ-complete, respectively.
So U is a free and σ-complete ultrafilter, Q.E.D.

Examples

Examples of $V \in U$ are:

$V = \omega_1$: trivial.

$V = \{ \alpha \in \omega_1 \mid \alpha > \omega \}$

Strategy for G(V): construct the ordinal $\omega+1$.

V is co-countable.

Strategy for G(V): construct the ordinal $\sup(\omega_1 \setminus V)+1$.

$V = \{ \omega \cdot \alpha \mid \alpha \in \omega_1 \}$

Strategy for G(V): There exists an order-isomorphic bijection h: $\mathcal{B} \times \mathbf{Q} \rightarrow \mathcal{A} \times \mathbf{Q}_{<0}$.
Each turn player A copies the moves player B makes using the bijection h, and then adds the points {0, 1, 2, ..., k} to each one of his non-empty sets (where k is the number of the turn being played). The end-result is that for each subset of **Q** that B has produced, A has an order-isomorphic subset followed by a 'tail' of ω points.

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