Hajnal Andréka, Johan van Benthem, István Németi

Back and Forth between Modal Logic and Classical Logic

ML-95-04, *received*: February 1995

ILLC Research Report and Technical Notes Series
Series editor: Dick de Jongh

Mathematical Logic and Foundations (ML) Series, ISSN: 0928-3315

Institute for Logic, Language and Computation (ILLC)
University of Amsterdam
Plantage Muidergracht 24
NL-1018 TV Amsterdam
The Netherlands
e-mail: illc@fwi.uva.nl

BACK AND FORTH BETWEEN

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AND

CLASSICAL LOGIC

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Hajnal Andréka, Johan van Benthem & István Németi

Institute of Mathematics, Hungarian Academy of Sciences, Budapest

&

Institute for Logic, Language and Computation, University of Amsterdam

to appear in

Bulletin of the Interest Group in Pure and Applied Logics

European Association FoLLI for Logic, Language and Information

London & Saarbrücken

1 Modal Logic and Classical Logic

Modal Logic is traditionally concerned with the intensional operators "possibly" and "necessary", whose intuitive correspondence with the standard quantifiers "there exists" and "for all" comes out clearly in the usual Kripke semantics. This observation underlies the well-known translation from modal logic into the first-order language over possible worlds models (van Benthem 1976, 1984). In this way, modal formalisms correspond to *fragments* of a full first-order (or sometimes higher-order) language over these models, which are both expressively perspicuous and deductively tractable. In this paper, we shall enquire which features of 'modal fragments' are responsible for these attractions. Throughout, we shall concentrate on the basic language of modal propositional logic, which still serves as the 'pure paradigm' in a rapidly expanding field of more expressive modal formalisms (Venema 1991, De Rijke 1993). What precisely are 'modal fragments' of classical first-order logic? Perhaps the most influential answer is that of Gabbay 1981, which identifies them with so-called 'finite-variable fragments', using only some fixed finite number of variables (free or bound). This view-point has been endorsed by many authors (cf. van Benthem 1991). Our paper presents a critical review of its supporting evidence, adding some new results about finite-variable fragments, including failures of the Los-Tarski preservation theorem. But there is also a second answer to our question, implicit in much of the literature, which emphasizes so-called 'bounded quantification'. As our positive contribution, we shall develop the latter perspective here, showing its utility as a guide towards generalization of modal notions and techniques to larger fragments of classical logics. In particular, we prove decidability for a large 'bounded fragment' of predicate logic, and point out several applications. One can also combine the two views on modal logic, as will be illustrated. Finally, we shall make another move. The above analogy works both ways. Modal operators are like quantifiers, but quantifiers behave like modal operators. This observation inspires a generalized modal semantics for first-order predicate logic using accessibility constraints on assignments (cf. Németi 1986, 1992) which moves the earlier quantifier restrictions into the semantics. This provides a fresh look at the landscape of possible predicate logics, including candidates sharing various desirable features with basic modal logic – in particular, its decidability.

This paper is the first public version of a longer projected document – whose current working version is Andréka, van Benthem & Németi 1994A. Further off-spring of this Amsterdam–Budapest collaboration in the field of modal logic and universal algebra are Andréka, van Benthem & Németi 1993, and Andréka, van Benthem & Németi 1994B.

2 Basic Modal Logic

2.1 First-Order Translation

Consider the basic propositional modal logic, in the language with Booleans $\neg$ & $\vee$ and modalities $\Box$ $\Diamond$. The following effective translation takes modal formulas ϕ to first-order formulas $\underline{\phi}$ with one free variable (standing for the 'current world' of evaluation) recording their truth conditions on possible worlds models:

$\underline{p}$	Px	$\underline{\neg\phi}$	$\neg\,\underline{\phi}$
$\underline{\phi \vee \psi}$	$\underline{\phi} \vee \underline{\psi}$	$\underline{\phi \,\&\, \psi}$	$\underline{\phi} \,\&\, \underline{\psi}$
$\underline{\Diamond\phi}$	$\exists y\, (Rxy \,\&\, \underline{\phi}(y))$	$\underline{\Box\phi}$	$\forall y\, (Rxy \rightarrow \underline{\phi}(y))$

where y is some fresh variable over worlds in the last two clauses.

Other semantics for the modal language may inspire other forms of translation, but we stick with the standard case here. This embedding into predicate logic gives a number of facts about modal logic for free, namely those universal properties of first-order logic which are inherited by all its fragments, such as the Löwenheim-Skolem Theorem, or by all its effective fragments, such as recursive axiomatizability for universal validity. What the embedding does not give is specifics in the latter case: for that, more detailed analysis is needed (see below). Moreover, typically, we do not get more complex meta-properties for free, that make existential claims. E.g., consider Interpolation. If a modal formula ϕ implies another modal formula ψ , then, through the translation, some interpolant must exist in the first-order language – but there is no guarantee that this interpolant will itself be (equivalent to) a modal formula: we shall have to work for this (see again below).

2.2 Invariance for Bisimulation

The expressive power of the basic modal language with respect to classical logic is measured precisely by the following Invariance Theorem (van Benthem 1976, 1985):

Theorem A first-order formula $\phi(x)$ is equivalent to the translation of a modal formula iff it is invariant for bisimulation.

Here, a *bisimulation* is a binary relation between the universes of two models linking only points with the same unary predicates P , and satisfying back and forth conditions with respect to relational R-successors in both directions. (Thus, if x bisimulates y , and Rxz , then z bisimulates some u with Ryu , and vice versa. This is a kind of unbounded Ehrenfeucht Game with restricted choices of objects in each move, which has a natural generalization to the case with whole families of accessibility relations.)

'Invariance' means that a formula gets the same truth formula in any two models at states connected by a bisimulation. This result subsumes the usual facts in the modal literature about preservation under 'generated submodels', 'disjoint unions' and 'p-morphic images'.

Proof of the Theorem For later reference, we sketch a proof of the Invariance Theorem. First, modal formulas must all be invariant, by a simple induction on their construction. Here, the existential modality is taken care of, precisely, by the two zigzag clauses. Conversely, suppose that $\phi = \phi(x)$ is an invariant first-order formula. Let $\mathbf{mod}(\phi)$ be the set of all modal consequences of ϕ. We prove the following implication:

Claim $\mathbf{mod}(\phi) \models \phi$.

From this, by Compactness, ϕ is easily seen to be equivalent to some finite conjunction of its modal consequences. The proof of the Claim is as follows. Let $\mathbf{M}, x$ be any model for $\mathbf{mod}(\phi)$. Now consider the complete modal theory of x in $\mathbf{M}$ together with $\{\phi\}$. This set of formulas is finitely satisfiable, by a simple argument (using the fact that $\mathbf{mod}(\phi)$ holds at $\mathbf{M}, x$). By Compactness, it therefore has some model $\mathbf{N}, y$. Now, take any two ω-*saturated* elementary extensions $\mathbf{M}^+, x$ and $\mathbf{N}^+, y$ of $\mathbf{M}, x$ and $\mathbf{N}, y$, respectively. (These exist, by a simple argument in the style of Chang & Keisler 1973.)

Claim The relation of modal equivalence between worlds is a bisimulation between the two models $\mathbf{M}^+$ and $\mathbf{N}^+$, which connects x with y.

Here, of course, the key observation lies in the back-and-forth clauses. If some world u in $\mathbf{M}^+$ is modally equivalent with v in $\mathbf{N}^+$, and Rus holds, then the following set of formulas is finitely satisfiable in $\mathbf{N}^+$: $\{Rvt\}$ plus the full modal theory of s in $\mathbf{M}^+$. But then, by ω-saturation, some world t must exist satisfying all of this in $\mathbf{N}^+$: which is the required match. The converse argument is symmetric. Having thus proved the second claim, we return to the first, and clinch the argument by 'diagram chasing'. For a start, $\mathbf{N}, y \models \phi$, and hence $\mathbf{N}^+, y \models \phi$ (by elementary extension), whence $\mathbf{M}^+, x \models \phi$ (by bisimulation invariance), and so $\mathbf{M}, x \models \phi$ (an elementary submodel). ■

This style of argument can be extended in many directions, by modulating the key connection between zigzag clauses and restricted quantifier patterns. A much more elaborate discussion of this result and its generalizations to richer modal languages may be found in van Benthem and Bergstra 1993, De Rijke 1993. (These also discuss connections with the work by Hennessy & Milner 1985 on modal process equivalences.)

2.3 Decidability via Semantic Tableaus

A next pleasant feature of the modal formalism is the existence of a simple semantic tableau method for checking universal validity. Its rules include the usual decomposition principles for Boolean operators on sequents, of which the following are samples. (Here, we take semantic consequence between sets of formulas in the usual appropriate sense, as going from the truth of all premises to that of at least one conclusion.)

$\Sigma, \neg A \Rightarrow \Delta$ iff $\Sigma \Rightarrow A, \Delta$

$\Sigma \Rightarrow A\&B, \Delta$ iff $\Sigma \Rightarrow A, \Delta$ *and* $\Sigma \Rightarrow B, \Delta$

In modal tableaus, the key rule is that for existential modalities, which are best treated in a bunch, when no further propositional reductions are possible:

true: $\Diamond\phi_1, ..., \Diamond\phi_n$ $\bullet_w$ $\Diamond\psi_1, ..., \Diamond\psi_m$:false

create new worlds $v_1, ..., v_n$ *with* Rwv_i $(1 \le i \le n)$

and start these with sequents $\phi_i \bullet_{v_i} \psi_1, ..., \psi_m$.

Theorem Modal semantic tableaus are adequate for valid consequence in the minimal modal logic (over the universe of all possible worlds models).

Corollary Modal universal validity is decidable.

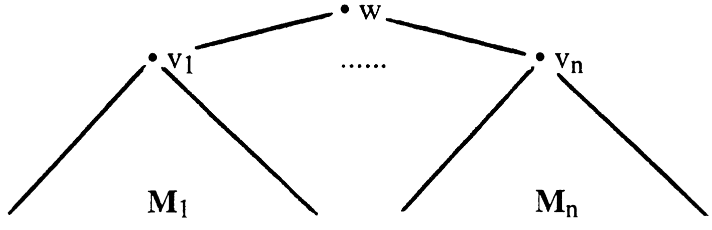
The corollary follows since all tableau rules decrease formula complexity of sequents (even though they may temporarily increase the number of parallel tasks). That tableaus are adequate for validity hinges on the semantic validity of the above $\Diamond$-Rule. Let P, Q be disjoint sequences of proposition letters. Then we have the following equivalence:

$$P, \Diamond\phi_1, ..., \Diamond\phi_n \models Q, \Diamond\psi_1, ..., \Diamond\psi_m \quad \text{iff} \quad \textit{for some } i\,(1 \le i \le n),\ \phi_i \models \psi_1, ..., \psi_m$$

This is immediate from right to left. The opposite part of the proof depends essentially on bisimulation invariance, through a well-known semantic construction of 'joint rooting':

any family of models $\mathbf{M}_i, v_i \models \phi_i \& \neg\psi_1 \& ... \& \neg\psi_m$ $(1 \le i \le n)$

can be 'glued disjointly' under one new common root:



Here, the models M_i lie embedded as 'generated submodels' (the identity relation is a bisimulation), whence no truth values change for modal formulas in their roots – and so the new top node w verifies $\Diamond\phi_1 \,\&\, ... \,\&\, \Diamond\phi_n \,\&\, \neg\Diamond\psi_1 \,\&\, ... \,\&\, \neg\Diamond\psi_m$, thereby refuting the top sequent. We shall return to this 'quantifier decomposition' in Section 4, extending these ideas to larger 'loose' decidable fragments of predicate logic.

2.4 Proof Theory via Sequent Calculus

Another way of describing validity is proof-theoretic. Read bottom-up, tableau rules become introduction rules in the 'Minimal Modal Logic' consisting of a Gentzen-style calculus of sequents (cf. Fitting 1993), with axioms

$\Sigma \Rightarrow \Delta$ *with* $\Sigma \cap \Delta$ *non-empty*

The following logical introduction rules are involved:

$$\frac{\Sigma, A \Rightarrow \Delta}{\Sigma \Rightarrow \neg A, \Delta} \qquad \frac{\Sigma \Rightarrow A, \Delta}{\Sigma, \neg A \Rightarrow \Delta}$$

$$\frac{\Sigma, A, B \Rightarrow \Delta}{\Sigma, A \,\&\, B \Rightarrow \Delta} \qquad \frac{\Sigma \Rightarrow A, \Delta \qquad \Sigma \Rightarrow B, \Delta}{\Sigma \Rightarrow A \,\&\, B, \Delta}$$

$$\frac{A \Rightarrow B_1, ..., B_m}{\Diamond A \Rightarrow \Diamond B_1, ..., \Diamond B_m}$$ the part "$B_1, ..., B_m$" may be empty

the rules for $\vee$ and $\Box$ are analogous.

Moreover, this calculus has two structural rules of

Permutation	inside the premises and the conclusions
Monotonicity	from $\Sigma \Rightarrow \Delta$ to $\Sigma', \Sigma \Rightarrow \Delta, \Delta'$

These are needed to get the exact correspondence with closed semantic tableaus right. Note that the other main classical structural rule of *Contraction* is redundant for the completeness proof. (It deduces $\Sigma, A \Rightarrow \Delta$ from $\Sigma, A, A \Rightarrow \Delta$.) In classical tableaus or sequent proofs for predicate logic, this rule is needed to ensure that false existential (and true universal) formulas can produce as many substitution instances as are required for the argument. With modal formulas, however, no such unbounded iteration is needed: we did all that is needed in one fell swoop. Thus, our calculus involves no shortening rules, and the proof search space is finite. (In a sense, then, at least as far as quantification is concerned, 'linear logic' is already complete for modal fragments of predicate logic.)

2.5 Interpolation

The basic modal logic shares several important meta-properties with first-order predicate logic. One important example is Interpolation:

Theorem Let $\phi \models \psi$. Then there exists a modal formula α whose proposition letters are included in both ϕ and ψ such that $\phi \models \alpha \models \psi$.

Proof We present two proofs here, illustrating the above two perspectives at work.

Proof-theoretic Argument ('Tracing a Sequent Derivation')
By induction on derivations in the above Gentzen sequent calculus. . It is convenient to work only with formulas rewritten to the special format $(\neg)$ atom, &, $\vee$, $\Diamond$, $\Box$, $\bot$, T (Cf. Schütte 1962, Roorda 1991 for this technique.) The single axiom case is clear, and one constructs interpolants inductively following the successive rules in a derivation. ■

Model-theoretic Argument ('Amalgamation via a Bisimulation')
Let $L_{\phi\psi}$ be the joint language of ϕ and ψ . Consider the set $\mathbf{cons}_{\phi\psi}(\phi)$ of all modal consequences of ϕ in this language. We prove the following uniform assertion:

Claim $\mathbf{cons}_{\phi\psi}(\phi) \models \psi$.

By Compactness, then, some finite conjunction of formulas in $\mathbf{cons}_{\phi\psi}(\phi)$ implies ψ (and is implied by ϕ). It remains to prove the Claim. Let $\mathbf{M}, x$ be any L_ψ–model which verifies $\mathbf{cons}_{\phi\psi}(\phi)$. We must show that $\mathbf{M}, x \models \psi$. First, by a routine argument, the modal $L_{\phi\psi}$–theory of $\mathbf{M}, x$ is finitely satisfiable together with $\{\phi\}$. By Compactness again, there is an L_ϕ–model $\mathbf{N}, y \models \phi$ with the same modal $L_{\phi\psi}$–theory as $\mathbf{M}, x$. Next, as in the earlier proof of the Invariance Theorem, we can pass to ω-saturated models here, without loss of generality. By that earlier argument, there exists an $L_{\phi\psi}$ –bisimulation $\equiv$ between the two models which connects x to y . (Here, the language subscript reminds us that $\equiv$ only needs to respect proposition letters which are shared by ϕ and ψ .) Now, we construct a new product model **MN** out of these two bisimulating ones, which will be a kind of 'joint unraveling under bisimulation'. Its worlds are finite sequences of pairs $\langle(x_1, y_1), ..., (x_k, y_k)\rangle$, where always $x_i \equiv y_i$, and moreover, each world x_{i+1} must be an R–successor of x_i – and likewise for the sequence of worlds y_i – for $1 \leq i < k$. Now, consider the two natural projections from the final pairs of such sequences, one going to **M** and the other to **N** . Along these, we can lift the valuation for all proposition letters in L_ψ–$L_{\phi\psi}$ from **M** , and that for L_ϕ–$L_{\phi\psi}$ from **N** . The result is the desired model **MN** for the joint language $L_\phi \cup L_\psi$, whose two projections to **M** (**N**) have now become

L_ψ– (L_ϕ–) bisimulations. But then, we can argue 'clockwise'. First, **N**, y $\models \phi$, whence **MN** , <(x, y)> $\models \phi$ (by bisimulation). Now, since $\phi \models \psi$, also **MN** , <(x, y)> $\models \psi$. But then finally, **M** , x $\models \psi$ (again by bisimulation). ■

Remark This argument was first stated in correspondence between the authors in 1992. In the meantime, more elaborate versions have appeared independently in Visser et al. 1994, Marx 1994, clarifying its general categorial background.

Remark Modal Unraveling and Concrete Representation

Behind the preceding proof, as well as others to come, lies the well-known observation that each possible worlds model **M**, x bisimulates with an 'unraveled' model consisting of finite sequences of objects (namely, worlds that R-succeed one another), in which the accessibility relation is end extension by one additional element. This may be regarded as a semantic 'normal form' in which abstract accessibility has been replaced by a concrete uniform set-theoretic relation. (Marx 1994 has more elegant concrete representations of this kind for general modal logics, following the program of Henkin-Monk-Tarski 1985.)

2.6 Model Theory and Preservation

More generally, much of classical Model Theory holds within the modal fragment. One good example is the Los–Tarski Theorem, stated here in its 'upward version':

Theorem A modal formula is preserved under model extensions iff it can be defined using only propositional atoms and their negations, &, $\vee$, $\Diamond$.

Proof 1 *Original Model-Theoretic Version*

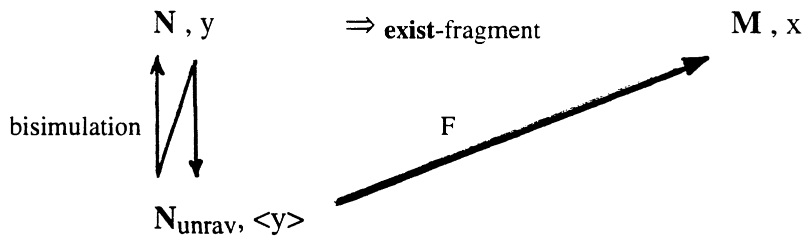
(The following argument was written up in correspondence with Albert Visser around 1985, but published only in van Benthem 1991.) A straightforward induction shows that existential forms are preserved under model extensions. Conversely, suppose that ϕ is so preserved. We shall prove the following semantic consequence:

$$\textbf{exist}(\phi) \models \phi , \qquad \text{where } \textbf{exist}(\phi) =_{def} \{ \psi \text{ existential} \mid \phi \models \psi \}.$$

Then the required existential modal form for ϕ will exist by Compactness for first-order logic, being of the form $\&\Delta$ for some finite set of formulas Δ contained in **exist** (ϕ) . Let **M**, x $\models$ **exist** (ϕ) . Without loss of generality, again, we can take the model **M**, x to be ω–saturated. Next, in the usual manner, we find a second model **N** , y such that

• **N** , y $\models \phi$ • **N** , y $\models \alpha \Rightarrow$ **M**, x $\models \alpha$ for all existential modal formulas α

Next, take the standard 'modal unraveling' of **N** , y via finite sequences of worlds, say, $\mathbf{N}_{unrav}$, <y> , which bisimulates **N** , y via end points of sequences. This will yield the following diagram:



Now, by induction on the length of sequences in $\mathbf{N}_{unrav}$, a map F may be defined from $\mathbf{N}_{unrav}$ to **M** sending <y> to x which is a homomorphism with respect to R , and which respects atomic facts in worlds. (The ω–saturation of **M**, x is used here to find suitable R-successors for points already mapped.) Finally, we perform a useful trick. Add a disjoint copy of **M**, x to $\mathbf{N}_{unrav}$, <y> , and extend the relation R as follows:

for all sequences Y *in* $\mathbf{N}_{unrav}$: Y R z *if* F(Y) R z .

<u>Claim</u> F united with the identity on **M**, x is a bisimulation between the two models $\mathbf{N}_{unrav}$, <y> + **M**, x and **M**, x .

<u>Proof</u> This follows by a simple inspection of cases. The point of using the unraveling $\mathbf{N}_{unrav}$ instead of **N** here is to get unambiguous relationships – while that of adding a copy of **M**, x to the left is to enforce the backward clause of bisimulation (on top of the already established 'forward' homomorphism). ■

To clinch the total argument, we again chase φ around the diagram:

N , y $\models \phi$	(by construction)
$\mathbf{N}_{unrav}$, <y> $\models \phi$	(bisimulation)
$\mathbf{N}_{unrav}$, <y> + **M**, x $\models \phi$	(model extension!)
M, x $\models \phi$	(bisimulation). ■

<u>Proof</u> 2 *Stream-Lined Modern Version*

In the meantime, simpler proofs of the above result have appeared. One version is essentially due to Dick de Jongh (cf. Visser et al. 1994). Here is a sketch of the idea, for the equivalent preservation theorem involving submodels and universal modal forms. Start again from some model **M**, x $\models$ **univ** (φ). Unravel this model to a bisimulation equivalent $\mathbf{M}^*$, <x> in the form of an intransitive acyclic tree.

Claim The atomic diagram of $\mathbf{M}^*$, <x> can be satisfied together with ϕ .

Once this is shown, the usual argument works. From the resulting model **N** , y , our ϕ can be transferred to its submodel (modulo isomorphism) **M**, x . To prove the claim, consider ϕ together with any finite set of (negated) atoms that are true in $\mathbf{M}^*$, <x> . The worlds mentioned in the latter can be described as some finite subtree, via branches going down all the way to the root <x> . Now, the resulting branching structure can be described completely via some (inductively constructed) existential modal formula. Moreover, ϕ cannot imply the (universal) negation of the latter, given the assumption that $\mathbf{M}^*$, <x> $\models$ **univ** (ϕ) . Hence ϕ can be satisfied together with this existential description in some model **N** . By unraveling **N** once more, this model can be taken to be an intransitive acyclic tree itself. But then, all atomic (negated) facts that were true in the above finite submodel of $\mathbf{M}^*$, <x> must also be true here. (No R-steps will be available except those explicitly demanded, which takes care of all negations.) ■

This second proof is close to the standard model-theoretic argument (cf. Chang & Keisler 1973), specialized to the modal fragment of the first-order language, more or less 'as is' . We shall return to this observation below in a more general setting. Further evidence for this analogy may be found with other model-theoretic preservation theorems, which may be obtained using similar methods. One example is the Lyndon homomorphism theorem for positive formulas (cf. van Benthem 1976). Here is a sketch for another classical case.

Example *Preservation Under Unions of Chains*
The first-order formulas that are preserved under unions of chains of models are precisely those that are definable with a universal-existential (Π_2) prenex form. In modal logic, the corresponding format must be extended (in the absence of prenex forms), just as above. We only allow formulas constructed from atoms and their negations, using & , $\vee$ as well as $\Diamond$, $\Box$, provided that the former never scopes over the latter. (In intuitionistic logic, this would become the natural class of formulas with 'implication rank' 2.) The classical argument again starts from a model **M** in which the **univ exist** consequences of ϕ hold. Then, two models **N** , **K** are found such that (1) **M** is a submodel of **N** and **N** of **K** , (2) **M** is an elementary submodel of **K** , (3) ϕ holds in **N** . Iterating this move, a chain of models arises, in whose union ϕ holds, which fact can then be transferred to the elementary submodel **M** . Inspecting the details of this standard argument, while using the above methods, similar triples of models may be constructed for the modal language. ■

2.7 Analyzing the General Situation: Transfer Results

The similarities between modal logic and standard first-order logic that have come to light so far call for more general explanation. There must be some general feature in the above arguments that can be isolated, and used to explore the full extent of the analogy. One obvious general point is the pervasive use of bisimulations, which are close to the fundamental notion of 'partial isomorphism' $\cong_p$ between first-order models ('cut off' at length 2). This observation may be found in van Benthem 1991, and it has inspired a systematic investigation of model theory for basic poly-modal logic in De Rijke 1993, whose results revolve around the 'heuristic equation'

Modal Logic : Bisimulation = Predicate Logic : Partial Isomorphism.

Another approach is to scrutinize the above arguments, and identify some key lemmas that allow for 'transfer' between modal and classical reasoning. One such result is easily extracted from the earlier proof of the Invariance Theorem. Two models **M**, x and **N**, y have the same modal theory iff they possess elementary extensions which bisimulate. (De Rijke 1993 observes that one can choose the latter to be countable ultrapowers.) Here is another result of this kind, which may be of independent interest. It shows how one can 'upgrade' modal equivalence to full elementary equivalence, up to bisimulation:

Lemma Two models **M**, x and **N**, y have the same modal theory if and only if they possess bisimulations with two models $\mathbf{M}^+$, x and $\mathbf{N}^+$, y (respectively) which are elementarily equivalent.

Proof From down to up, the assertion is immediate. Consider the downward direction. The required models are constructed using standard *Unraveling* by finite sequences of the form (x =) $x_1, x_2, ..., x_k$, where each x_{i+1} is an R-successor of x_i ($1 \leq i < k$), having 'immediate succession' for their accessibility relation, and bisimulating with the original model via their last elements. This unravels to the familiar intransitive acyclic trees. In addition, we perform *Multiplication*, making sure that each node (except the root x) gets copied infinitely many times. This can be done as follows, while maintaining a bisimulation at each stage. First, copy each successor of x at level 1 countably many times, and attach these (disjoint) copies to x . There is an obvious bisimulation here, identifying copies with originals. Next, consider successors at level 2 on all branches of the previous stage, and perform the same copying process at all level–1 worlds. Again, there is an obvious bisimulation with the original model. Iterating this process through all finite levels yields our intended models $\mathbf{M}^+$, x and $\mathbf{N}^+$, y .

Claim $\mathbf{M}^+$, x and $\mathbf{N}^+$, y are elementarily equivalent.

Proof The argument uses Ehrenfeucht Games. It suffices to show, for arbitrary finite n, how the Similarity Player can win in any game over n rounds between these structures. What we know at the outset is that the two roots x, y satisfy the same modal formulas. In fact, as we shall prove separately, they even satisfy the same *tense-logical* formulas. This observation will be used to describe the proper invariant for the Ehrenfeucht game. Assume that in round i of the game, a match $\equiv$ has been established already between certain finite groups of worlds in the two models which satisfies three conditions:

- if $a \equiv b$, then $\mathbf{M}^+$, a is equivalent with $\mathbf{N}^+$, b for all tense-logical formulas up to operator depth 2^{n-i}
- if $a \equiv b$ and $a' \equiv b'$, and the distance between a and a' is at most 2^{n-i}, then the distance between b and b' is the same on the other side, and it runs via an isomorphic path, all of whose members have been matched at this stage.

Here, *distance* is measured as follows: "go from node a to node b by descending the minimal distance needed to climb up to b again" . (This possible backward movement forces us to employ two-sided tense-logical formulas in the description of the invariant.)

- if the distance between a and a' is greater than 2^{n-i}, then on the other side, b and b' have distance greater than 2^{n-i}, too.

The upshot of all this is a number of 'matched islands' on both sides, all lying a distance of more than 2^{n-i} steps apart. Now, we have to show that this invariant can be maintained in the next step, whatever world the Difference Player chooses. Let the next choice be some point P in either tree.

Case 1 P has distance $\leq 2^{N-i-1}$ to some point Q that was already matched at the previous stage, say to some point Q' .

Consider the (unique) path of length k (say) between P and Q , and attach complete tense-logical descriptions δ to its nodes up to operator depth 2^{N-i-1} . This path may then be described, from the perspective of Q , by a tense-logical formula of the form

PAST (δ_1 & PAST (... & PAST (δ_i & FUT (δ_{i+1} & ... & FUT (δ_k)) ,
where δ_k is the full 2^{N-i-1}–description in tense logic of the point P .

The total operator depth of this formula is at most 2^{N-i-1} (being the length of the path) + 2^{N-i-1} (the quality of the descriptions at its nodes), which is at most 2^{N-i}. Now, at the previous stage i, Q and Q' agreed on tense-logical formulas up to the latter depth. Hence this path description is also true at Q', and we can find corresponding worlds on the other side, making the two paths isomorphic as required by our invariant, while also achieving the right degree of tense-logical equivalence.

Case 2 P lies at distance > 2^{N-i-1} from all previously matched points.

In this case, take the unique path from the root to P. Describe that path completely as before, with node descriptions up to level 2^{N-i-1}. The resulting tense-logical formula may be of high complexity (since there is no bound on the path length), but since the two roots agree on *all* tense-logical formulas, there must be a similar path on the other side, whose end-point is an appropriate match for P. Moreover, this path can be chosen so as to remain at a suitable distance from all nodes in already matched regions, because of the Multiplication of nodes (this is the only point where we use this feature). Thus, in this case too, the above invariant is maintained.

Finally, after n rounds, this invariant will lead to a partial isomorphism, which is a win for the Similarity Player. (To get a concrete feel for the strategy, compare two modally equivalent trees where one has an infinite branch and the other does not. This example also shows that we cannot improve our Lemma to the existence of a bisimulation between the unraveled multiplied models.) To wrap things up, it remains to prove the announced

Sublemma If the roots of two unraveled modal models have the same modal theory, then they also have the same tense-logical theory (in the language extended with a modal operator for "past").

Proof It suffices to observe a number of tense-logical validities on trees like this. First:

$$\text{FUT}(\text{PAST}\,\alpha \,\&\, \beta) \leftrightarrow \alpha \,\&\, \text{FUT}\,\beta \qquad \text{FUT}(\neg\,\text{PAST}\,\alpha \,\&\, \beta) \leftrightarrow \neg\,\alpha \,\&\, \text{FUT}\,\beta$$

As a result, using some standard modal manipulations, every formula is equivalent to one without future operators scoping over past ones. This just leaves compounds of 'pure future' (i.e., modal) formulas combined using ¬, & and PAST. The latter can still be simplified using two more valid equivalences:

$$\text{PAST}(\alpha \,\&\, \beta) \leftrightarrow \text{PAST}\,\alpha \,\&\, \text{PAST}\,\beta \qquad \text{PAST}\,\neg\,\alpha \leftrightarrow \neg\,\text{PAST}\,\alpha \,\&\, \text{PAST}\,\mathit{true}$$

As a result, every formula is equivalent to a Boolean combination of formulas PAST^{i} ϕ (with i repetitions) where ϕ is purely modal. But then, the roots must agree on all tense-logical formulas. They already agreed on all modal formulas, and they will both reject any PAST formula (lacking predecessors). ■

Using similar techniques of modal unraveling plus Ehrenfeucht Games, one can also show a related transfer result. Two models **M**, x and **N**, y *bisimulate* if and only if their multiplied unraveled versions are *partially isomorphic* .

2.8 Analyzing the General Situation: Predictions

Finally, as to the full extent of the analogies between modal logic and classical logic, let us risk a bold generalization. The set of all predicate-logical formulas may be viewed as the domain of a ('meta-')model which carries some natural structure. For instance, meta-theorems like Interpolation are themselves (Π_2) first-order statements about this model, in the following similarity type: one binary relation of "semantic consequence", and another binary relation of "vocabulary inclusion". (A closely related meta-model has been investigated in Mason 1985. The complete first-order meta-theory of propositional logic turned out to be effectively equivalent to True Arithmetic – thereby saving the logical profession from rapid extinction.) Similar observations can be made concerning other preservation theorems. For instance, the Los-Tarski Theorem can be restated as an equivalence between (1) $\phi \models (\phi)^A$ (i.e., ϕ implies its own relativization to some new unary predicate A) and (2) the existence of some universal formula equivalent to ϕ . Both assertions involve some slight expansion of the above meta-model to include further predicates encoding 'elementary syntax' into the similarity type. Thus formulated, the Los-Tarski theorem becomes a simple Π_2–sentence, too. Now, note that the modal fragment is a submodel of at least the first of these predicate-logical meta-models, in an obvious way. (With the second one, we have to be more careful, as the result needs to be restated due to the lack of modal prenex forms – though not of modal relativizations.) In this perspective, here is a guess which would explain why one always seems able to 'witness' existential quantifiers over formulas inside the modal fragment:

<u>Conjecture</u> The modal fragment is an *elementary submodel* of full predicate logic in the first similarity type given above.

With results like this, one could decide transfer of meta-theorems between first-order logic and modal logic by merely inspecting their form.

2.9 Poly-Modal Generalizations

One test for the naturalness of the above results for the basic modal language is how they survive generalization. At least, things work very smoothly for the practically important case of poly-modal languages with families of unary modalities $\Diamond_i$ $(i \in I)$, each with their corresponding accessibility relation R_i . One can virtually literally transcribe the above theory, putting in appropriate indices. Nevertheless, subtleties do arise occasionally. For instance, in the Interpolation Theorem, one can now also talk about the shared modalities of the two original formulas, and an interpolant should contain only these. But then, the above proof is incorrect as it stands. For, the amalgamation defined in Section 2.5 only yields bisimulating projections for (relations corresponding to) the shared modalities. In order to make the amalgamation bisimulate with the two separate models in their full language (as is required by the final argument), one has to add copies to the amalgam **MN** of those parts of **M** and **N** that branch off via non-shared successor relations, and extend the projection via the identity map on the new parts (cf. van Benthem 1994B). This is not a serious departure from the basic modal case, but it is not totally trivial either.

Next, we consider a more serious generalization, namely to *polyadic modalities*. Here, one needs (k+1)–ary accessibility relations for each k–ary existential modality:

$$\Diamond\, \phi_1 \dots \phi_k \quad \text{translates into} \quad \exists y_1 \dots y_k\, (\, R^{k+1}x,\, y_1 \dots y_k\, \&\, \&_{1\leq i\leq k}\, \phi_i\, (y_i))$$

We give a quick run-down of some basic results, showing what remains the same, and where some cosmetic changes are needed. Under translation, modal formalisms of this kind end up in what may be called the *restricted fragment* of first-order logic:

- start with all unary atoms Px , and allow
- closure under Boolean operations for compounds with the same variable
- closure under existential quantifiers of the form
  $\exists y_1 \dots y_n\, (\, R^{n+1}\, x,\, y_1 \dots y_n\ \&\ \phi_1\, (y_1)\, \&\, \dots\, \&\ \phi_n\, (y_n)\,)$.

Any set of restricting predicates R is allowed in the first-order language. The Restricted Fragment turns out to inherit all attractive properties of the original modal one, via obvious generalizations of earlier arguments (cf. van Benthem 1991, de Rijke 1993).

1 The 'restricted formulas' are precisely those first-order formulas $\phi\, (x)$ which are invariant for bisimulation with respect to the new extended set of relations. (One now has to find back-and-forth matches for triples $R^3\, x,\, y_1 y_2$, etcetera.) The earlier model-theoretic proof goes through with mere notational changes.

2 There is an adequate semantic tableau method which establishes decidability.

3 There is a complete sequent calculus axiomatization for universal validity, whose principles may be read off from closed tableaus.

Even though there is no change in principle here, practical complexity may increase in this proof system. For instance, the introduction rule for a binary existential modality that emerges from the tableau calculus will now read as follows:

$$\frac{\&_{\{1,\ldots,k\} \supseteq X} \; (\alpha \vdash \{ \gamma_i \mid i \in X \} \; \textit{or} \; \beta \vdash \{ \delta_1, \ldots, \delta_k \} - \{ \delta_i \mid i \in X \})}{\Diamond\alpha\beta \vdash \Diamond\gamma_1\delta_1, \ldots, \Diamond\gamma_k\delta_k}$$

4 Craig Interpolation holds, and it may still be proved constructively using the sequent proof calculus, as well as model-theoretically.

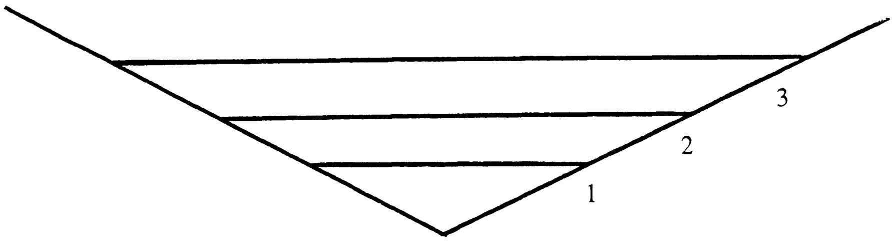
5 The Los-Tarski Preservation Theorem holds, by essentially the earlier argument with bisimulation invariance and copying. This requires a notion of 'unraveling' via suitable finite sequences for many relations at the same time.

The latter is again one of the cases where some care is needed in formulation of results. For instance, in unraveling a triple Ra,bc , one has to keep track of the whole ternary configuration, indicating that the step from a to b went via the triple (abc) , in order to distinguish this from a possible other situation Ra,bd . There are several notational solutions to problems like this, which we do not elaborate here.

3 Finite Variable Fragments

3.1 The Finite Variable Hierarchy

Finite-variable fragments of first-order logic were introduced for technical reasons in Henkin 1967. These consist of all formulas using only some fixed finite set of variables (free or bound): say {x}, {x, y}, {x, y, z}, etcetera – but otherwise allowing arbitrary combinations of quantifiers and connectives. An important connection with modal logic was pointed out in Gabbay 1981. Finite operator sets generate modal languages whose transcriptions involve only some fixed finite number of variables (free and bound). For instance, the basic modal language can make do with *two* world variables only (as may be seen by further analysis of the earlier translation, judiciously 'recycling' just x, y), while, e.g., the well-known temporal language with 'Since' and 'Until' uses essentially *three* variables. Thus, there is a *Finite Variable Hierarchy:*



inside whose levels one finds modal logics of ascending expressive strength. This is a natural division. Like the basic modal language itself, these successive levels can be characterized via a semantic invariance property (van Benthem 1991). To state the result, we follow the usual convention that $\phi\,(x_1, ..., x_k)$ denotes a formula whose free variables are all among $\{x_1, ..., x_k\}$. Moreover, by k-*variable formulas* we shall mean first-order formulas all of whose variables (whether free or bound) are among $\{x_1, ..., x_k\}$.

Theorem A first-order formula $\phi\,(x_1, ..., x_k)$ is equivalent to a k-variable formula if and only if it is invariant for k-partial isomorphism.

Here, k-*partial isomorphism* is a cut-off version of the well-known notion of 'partial isomorphism' from Abstract Model Theory. That is, we have a non-empty family **I** of partial isomorphisms between two models **M** and **N** , which is closed under taking restrictions to smaller domains, and where the standard Back-and-Forth properties are now restricted to apply only to partial isomorphisms of size at most k .

Proof (A complete argument is in van Benthem 1991.) An outline is reproduced here, for convenience. First, k-variable formulas are preserved under partial isomorphism, by a simple induction. More precisely, one proves, for any assignment A and any partial isomorphism $I \in \mathbf{I}$ which is defined on the A-values for all variables $x_1, ..., x_k$, that

$$\mathbf{M}, A \models \phi \quad \text{iff} \quad \mathbf{N}, I \circ A \models \phi .$$

The crucial step in the induction is the quantifier case. Quantified variables are irrelevant to the assignment, so that the relevant partial isomorphism can be restricted to size at most k–1, whence a matching choice for the witness can be made on the opposite side. This proves "only if". Next, "if" has a proof analogous to that of the Invariance Theorem (cf. Section 2.2). One shows that an invariant formula $\phi\,(x_1, ..., x_k)$ must be implied by the set of all its k-variable consequences. The key step in this argument goes as before. We find two models which are elementarily equivalent for all k-variable formulas. These then possess ω–saturated elementary extensions – for which the relation of k-elementary equivalence itself defines a family of partial isomorphisms between tuples of objects up to length k , which satisfies all the above requirements for k-partial isomorphism. ■

3.2 The Case For

In addition to the preceding natural semantic characterization, several arguments plead in favour of the preceding fragmentation of first-order logic. We list a few positive features:

- Gabbay's Functional Completeness Theorem (Gabbay 1981) also shows how, conversely, for each finite-variable level, a finite set of modal operators can be constructed effectively whose modal language yields precisely that fragment.
- Variables are 'semantic registers', whence the Finite Variable Hierarchy provides a natural fine-structure of expressive complexity. This leads to hierarchies of time complexity for verification of first-order statements (Immerman 1981, 1982).
- The controlled use of more variables, free and bound, in these fragments suggests natural 'many-dimensional completions' of modal languages (Venema 1991).
- Finite-variable fragments have also emerged naturally in other areas of mathematical logic, such as Relational, Polyadic and Cylindric Algebra (Németi 1991), and they are crucially involved in the formalization of set theory of Tarski and Givant 1987.
- Finite-variable fragments provide natural query languages supporting fixed-point operators in Finite Model Theory (cf. Kolaitis and Väänänen 1992).

3.3 The Case Against

But there are also some negative counterpoints to the preceding observations. Notably,

- k-variable fragments have a poor proof theory. No finitely axiomatized Hilbert-style system exists (Monk 1969), and the complexity of the necessary axiom schemes is inevitably high (Andréka 1991).
- Craig Interpolation and Beth Definability fail (Sain 1989, Sain & Simon 1993, Andréka, van Benthem & Németi 1993).

Our contribution to this discussion will be both critical and constructive. First, we notice one more failure. The Los-Tarski Theorem fails for most finite-variable fragments, who therefore lose much of the meta-theory of full first-order logic. On the other hand, we do prove a natural modified version of the theorem, replacing submodels by a more general (and absolute) notion of 'partial embeddings'. Finally, we note that the theorem may also be reinstated by withdrawing to 'restricted quantifier fragments' (cf. Section 4.3 below), or by providing finite-variable fragments with a suitable kind of 'generalized semantics' (cf. Section 5 below). The latter move also provides a good proof theory, as well as interpolation and definability theorems (cf. Németi 1985, Németi 1992, Andréka, van Benthem & Németi 1994B, Marx 1994).

3.4 Failure of the Submodel Preservation Theorem

The standard Los–Tarski Theorem trivially fails for finite-variable fragments. E.g., the 1-variable formula $\forall xAx \lor \forall xBx$ is preserved under submodels, while lacking a universal prenex form with one variable (two are needed). But the more natural conjecture is this: a k-variable formula is preserved under submodels iff it can be defined as a k-universal formula. Here we define k-*universal formulas* as all those that can be constructed in the k-variable fragment using atoms and their negations, &, $\lor$ and $\forall$. (The above formula is 1-universal as it stands.) But this result, too, fails in finite-variable fragments.

<u>Theorem</u> For each $k \geq 3$, the k-variable fragment contains formulas that are preserved under submodels, while lacking any k-universal equivalent.

<u>Proof</u> We do the case k=3 for an illustration. The general case is completely analogous. Let R be some 3-place relation. Define $\delta(x) := \exists yz\, Rxyz$ (' x is in the head of R ') . Consider the following first-order formula:

$$|\delta| \leq 2 \quad (\text{"there are at most two objects satisfying } \delta \text{"}) \ \& \\ (|\delta| \leq 1 \ \lor \forall xx'yz\, ((Rxyz \ \& \ Rx'yz) \rightarrow x=x'))) \qquad \phi$$

This formula can be written as a purely universal prenex form (and even as a 4-universal formula, by proper variable management.) So, ϕ is preserved under submodels. Now, consider the following variant Φ of ϕ (involving existential quantifiers):

$$|\delta| \leq 2 \ \ \& \ (|\delta| \leq 1 \ \lor \forall xyz\, (Rxyz \rightarrow \exists x\, (\delta(x) \ \& \ \neg\, Rxyz))) \qquad \Phi$$

<u>Claim 1</u> ϕ and Φ are logically equivalent.

<u>Proof</u> This is a simple computation. In both directions, it suffices to consider the case where there are exactly two objects satisfying δ . ' From ϕ to Φ ' . Assume that Rxyz. Then there must be some $x' \neq x$ in δ which cannot have Rx'yz, by ϕ . This is the required witness for the existential quantifier. ' From Φ to ϕ ' . Assume that Rxyz, Rx'yz. Since $\delta(x)$, there must be some x'' in δ with $\neg$ Rx''yz. But then, x, x'' are different, and hence, x' must be equal to one of them. The only option here is x'=x , since Rx'yz, $\neg$ Rx''yz rules out x'=x''. ■

<u>Claim 2</u> Φ is in the 3-variable fragment.

<u>Proof</u> By some straightforward syntactic manipulation. ■

Now, we introduce two special models $\mathbf{M} = (D, Z)$, $\mathbf{N} = (D, U)$ for this language, where

$D = \{0, 1, 2, 3, 4, 5\}$
$U = \{0, 1\} \times \{2, 3\} \times \{4, 5\}$
$Z = \{ (i, j, k) \in U \mid i+j+k \text{ is even}\}$.

Claim 3 $\mathbf{M} \models \Phi$, but not $\mathbf{N} \models \Phi$.

Proof By direct inspection. In both cases, there are exactly two objects in δ. Note how the parity in the definition of Z ensures that there will be another object in δ which does not have the same 'tail'. The problem with U is that too many tails are allowed. ■

Now, the counter-example is complete once we prove our final

Claim 4 Every 3-universal formula which holds in $\mathbf{M}$ also holds in $\mathbf{N}$.

Proof There are several methods here. One way is to use a one-sided Ehrenfeucht game with three pebbles (cf. Immermann & Kozen 1987). Here we use another road, which will be suggestive for later developments. Consider the set **3PI** of all partial isomorphisms f with size at most 3 between $\mathbf{N}$ and $\mathbf{M}$, which satisfy the additional restriction that, whenever $f(a) = b$, then a and b lie in the same component $\{0, 1\}$, $\{2, 3\}$ or $\{4, 5\}$. We show that this family has the "Forth" property, from $\mathbf{N}$ to $\mathbf{M}$. More precisely, let $f(a) = b$ and let c be an arbitrary object in D. We find an object d such that the map $(f - \{(a, b)\}) \cup \{(c, d)\}$ is again in **3PI**. It suffices to consider the case where $c \neq a$. We distinguish some cases for the remaining f-arguments (after removal of the object a). *Case 1* "c equals some existing f-argument different from a". Then let its mate d be the corresponding f-value. (This must yield a partial isomorphism of the right kind.) *Case 2* "c is different from all existing f-arguments". *Case 2. 1* Suppose that c is in the same component as some existing f-argument. Then let d be the remaining possibility in this component. (In this case, no Z– or U–relation can hold on either side of the partial isomorphism.) *Case 2.2* Suppose that c is in a different component from the existing f-arguments. This case, too, will yield a partial isomorphism. *Case 2.2.1* The other f-arguments lie in the same component, and hence no Z– or U–relations can hold: let c be its own image. *Case 2.2.2* These arguments lie in different components. Then, the choice of an image for c in its component may be made according to the parity of the sum of the values assigned to the other arguments. (One of the two available options will always do.) Finally, an easy induction on 3-formulas shows that

Whenever $f \in$ **3PI** , A is some assignment whose values are in the domain of f ,
and α is some 3-existential statement such that **N** , A $\models \alpha$,
then **M** , Aof $\models \alpha$.

This shows that all true 3-existential statements in **N** are also true in **M** , from which the required assertion about 3-universal statements follows by duality. ■

Some Remaining Questions

(1) Does the Los Theorem hold for the 2-variable fragment?

(2) Our counter-examples involve k–ary relations for k-variable fragments. Can they also be given more uniformly employing just one binary predicate? (We have such an example for k=3, but not for k>3.)

(3) k-variable formulas that are preserved under submodels must have universal equivalents somewhere in the finite-variable hierarchy, by the ordinary Los-Tarski Theorem. Is there a *recursive* function f of k such that every k-variable formula preserved under submodels has an f(k)-variable universal equivalent? In particular, does the choice f(k) = k+1 work? And what about a similar function defined over formulas?

For further information, we refer to Andréka, van Benthem & Németi 1994B, which presents a more elaborate version of the above argument, that extends to first-order languages without equality. Moreover, related ideas are used in Andréka, van Benthem & Németi 1993 to construct uniform failures of Interpolation in finite-variable fragments.

3.5 Modified Preservation Theorems

The above negative argument contains the core of a positive result. (For convenience, we shift to a dual existential formulation here.) What can still be proved is a Los-Tarski Theorem for k-variable fragments characterizing their appropriate syntactic notion of 'k–existential definability' (in terms of atoms and their negations, &, $\vee$ and $\exists$) via preservation under so-called 'k-partial embeddings', being restriction-closed non-empty families of k-partial isomorphisms which satisfy the Forth-condition only. (The latter notion was introduced in van Benthem 1991.) Here is the relevant result, characterizing the class of existential formulas in the k–variable fragment model-theoretically:

Theorem A k-variable formula is k–existentially definable iff it is preserved under k–partial embeddings.

Proof That existential formulas are so preserved follows by a straightforward induction. Conversely, assume that ϕ is preserved under k–partial embeddings. Earlier kinds of argument apply. It suffices to show that ϕ is a consequence of the set k–**exist** (ϕ) of all k–existential logical consequences of ϕ . So, let **M**, A $\models$ k–**exist** (ϕ) . Then, by familiar reasoning, we can find a model **N**, B for ϕ , each of whose k–existential formulas is true in **M**, A . Without loss of generality, we may assume that **M**, A is ω–saturated. But then, the following stipulation defines a k–partial embedding from **N** into **M** :

> all partial isomorphisms f from **N** to **M** of size at most k ,
> such that, for all k–existential formulas α ,
> if **N** , A $\models \alpha$, then **M** , Aof $\models$ α .

In proving the "Forth" clause here, one has finite approximations of the new element by means of k–existential formulas, and then finds a simultaneous witness via Saturation. In particular, our embedding sends the sequence of B–values on our k variables to the corresponding A-values, whence ϕ must hold in **M** , A , too. ■

'Partial embedding' stands to 'submodel' as 'partial isomorphism' stands to 'isomorphism'. That is, it is the closest 'absolute' notion, in the sense of Abstract Model Theory. Similar finite-variable modifications exist for other classical model-theoretic results.

3.6 Failure of the Interpolation Theorem

Other classical key properties may fail, too. Here is a simple argument to this effect.

Theorem Craig Interpolation fails in all k–variable fragments ($k \geq 2$) .

Proof Consider any k–variable fragment L_k , and take k unary predicates $A_1, ..., A_k$. Let the first-order formula ϕ^k say that (i) each A_i holds for one object (this needs two variables), (ii) all A_i are disjoint (this needs one variable) and (iii) every object satisfies at least one A_i (again, one variable). Note that ϕ^k holds only in domains of size k . In a similar way, one constructs a formula ψ^{k+1} , using new unary predicates $B_1, ..., B_{k+1}$, true only in domains of size k+1 . Clearly $\phi^k \models \neg \psi^{k+1}$, with both formulas from L_k . By Interpolation, there should be a k–variable formula α containing only identity with $\phi^k \models \alpha \models \neg \psi^{k+1}$. But this is a contradiction. For, pure identity formulas using only k variables cannot make a diference between domains with k and with k+1 objects. ■

The counter-example generalizes to first-order languages without identity, by replacing = with a suitable equivalence relation.

3.7 Conclusion

Finite-variable fragments, though attractive, do not explain all there is to Modal Logic. Hence, we now turn to an alternative analysis, whose focus is restriction of quantifiers. This alternative is not exclusive. The finite variable hierarchy can be superimposed on it – and then, it will regain the positive properties that were shown to fail above.

4 Bounded Quantifier Fragments

4.1 The Basic Restriction Schema

Evidently, the basic modal fragment is only a subset of the full two-variable fragment, since its syntax satisfies additional constraints. In particular, all quantifiers in translations of modal formulas occur 'restricted', in the forms

$$\exists y\,(Rxy \;\&\; \phi\,(y))\,,\ \forall y\,(Rxy \rightarrow \phi\,(y))\,.$$

Semantically, the latter form correlates with the earlier definition of bisimulation, explaining its particular zigzag clauses. This observation suggests another classification. What we are dealing with are *quantifier restrictions*, which may be varied along various dimensions. The general schema here is as follows:

$$\exists \mathbf{y}\,(\,R\mathbf{y},\mathbf{x} \;\&\; \phi\,(\mathbf{x},\mathbf{y},\mathbf{z})\,)$$ where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are finite sequences of variables.

And the question is how much can be allowed as to variable occurrences without losing the attractive features of the basic modal logic, in particular, its decidability. We shall take this perspective as our point of departure in a hierarchy of 'restricted' or 'bounded' fragments of predicate logic. Some initial stages already occurred in Section 2.9 above. For instance, polymodal logic shows that families of different restricting predicates R_i are admissible. And polyadic modal logics showed that one can allow restrictions of the special form $\exists y\,(Rx{,}y \;\&\; \&_i\, \phi_i\,(y_i)\,)$ without major changes in theory and practice. We shall be concerned mainly with the following schemata in what follows:

Fragment 1	$\exists \mathbf{y}\,(\,R\mathbf{y},\mathbf{x} \;\&\; \phi\,(\mathbf{y})\,)$
Fragment 2	$\exists \mathbf{y}\,(\,R\mathbf{y},\mathbf{x} \;\&\; \phi\,(\mathbf{x},\mathbf{y})\,)$
Fragment 3	$\exists \mathbf{y}\,(\,R\mathbf{y},\mathbf{x} \;\&\; \phi\,(\mathbf{x},\mathbf{y},\mathbf{z})\,)\,.$

As for the attraction of this schema, restricted formulas play a crucial role in absoluteness in Set Theory (being called 'Δ_0–formulas'). The precise point of the latter analogy for modal logic remains to be understood (cf. van Benthem 1994A for some connections).

Let us now be a bit more precise. Our fragments are defined inside a first-order predicate language. We start with arbitrary atoms, and allow further constructions with Boolean operators as well as the above restricted quantifier schemata (where the R can be any relation symbol). Many variations are possible here without affecting our main results. Here are some. (i) The **y**–arguments can occur at any place in R**y**, **x** , (ii) Boolean compounds of restrictions, such as 'Rxy_1 & Rxy_2', may be allowed in special cases (but we shall see below that this is not always admissible), (iii) some restricting relations might be forbidden to occur elsewhere as atoms, and (iv) [moving slightly, though not essentially, beyond standard first-order logic:] restricting predicates R may be allowed variable finite arity for their **x**– and **y**–arguments. More problematic seems the addition of an *identity* predicate. Nevertheless, in the full version Andréka, van Benthem & Németi 1994A, we extend our decidability results to the latter case as well. Note that the original basic modal fragment coincides with our Fragment 1 where we have only one special binary restricting relation symbol (that can occur in restrictions only).

These fragments may be understood in various ways. Model-theoretically, it is easy to extend the earlier notion of modal *bisimulation* to describe them (cf. Section 4.4 below), which can be used to see that we have a genuine upward hierarchy of expressive strength. But for the moment, we choose another, more combinatorial approach, which focuses on their 'looseness' and decidability (cf. Section 2.3 above). Fragment 1 is decidable, being still close to modal logic. Also, it is not too hard to see that Fragment 3 is undecidable. Our main result in this Section is that the rather powerful Fragment 2 is decidable and has a (uniform) finite model property. As we shall show, this theorem generalizes several existing results from the modal and algebraic literature.

4.2 Decomposition of Universal Validity

One way of understanding the nature of restricted fragments is by extension of the earlier semantic tableau analysis. The crucial point there was to find some decomposition rule for validity of a sequent involving only existential quantifers (plus perhaps atoms) on both sides. As a warming-up, here are two useful decomposition properties for the original modal fragment. The first is an earlier observation:

<u>Fact 1</u>

$$\exists y_1 (Rxy_1 \,\&\, \phi_1 (y_1)), \ldots, \exists y_k (Rxy_k \,\&\, \phi_k(y_k)) \models \exists y_1 (Rxy_1 \,\&\, \psi_1 (y_1)), \ldots, \exists y_m (Rxy_m \,\&\, \psi_m (y_m))$$

iff *for some* $i\ (1 \leq i \leq k)$ $\phi_i \models \psi_1, \ldots, \psi_m$

This fact depended on bisimulation invariance for this language. In fact, we only need invariance of these formulas for the generated submodels in the rooting construction. (The latter notion admits a larger class of predicate-logical formulas as invariants.) There is another convenient reduction for modal formulas involving different 'current worlds' (cf. Kracht 1993 for this generalization inside modal logic itself):

Fact 2 $\phi_1(x_1), \ldots, \phi_k(x_k) \models \psi_1(x_1), \ldots, \psi_k(x_k)$
iff *for some* $i\ (1 \leq i \leq k)\ \phi_i(x_i) \models \psi_i(x_i)$

This may be proved like Fact 1, but now, by mere disjoint union of counter-examples to the lower sequents. Thus, Fact 2 holds for all first-order formulas that are invariant for disjoint unions (van Benthem 1985 has a preservation theorem defining them). These points will return below, as our modal arguments can be generalized to first-order logic.

Fragment 1 is Decidable

We start with a simple warm-up case, which is still very close to basic modal logic.

Theorem Validity of formulas in Fragment 1 is decidable.

Proof (Outline) First, we perform all possible propositional reductions in a sequent, so that only atoms and existential quantifiers remain on both sides. Then we prove a reduction to matrix formulas like above. The point is to glue together counter-examples for sequents below where the quantifiers have been stripped off, so as to refute the original sequent with quantifiers. The presence of longer sequent arguments **x**, **y** does not make an essential difference to this construction. And neither does the presence of arbitrary arguments **y** in the matrix formula (rather than the 'separated conjunctions' employed in polyadic modal logic), provided that the former have the required invariance property. Indeed, even when starting with 'mixed' initial formulas $\exists y\,(Rx_1x_2,y\ \&\ \phi_1(y))$, $\exists y\,(Rx_3x_1,y\ \&\ \phi_2(y))$ and $\exists y\,(Rx_2,y\ \&\ \phi_3(y))$ followed by similarly heterogeneous conclusions, one can just match up premise/conclusion pairs with identical sequences of x–parameters, as there are no semantic dependencies between the behaviour of R–successors for sequences and their subsequences. (In more specialized model classes, however, with extra 'frame conditions' on R, this would all have to be re-checked.) ■

The preceding argument actually establishes a bit more than was stated. It is easy to see that all counter-examples constructed for non-valid formulas may be taken to be finite:

Corollary Fragment 1 has the finite model property.

Fragment 2 is Decidable

Things become more difficult at the next level, where the parameters **x** can also occur in the matrix formula (Fragment 2). For instance, here, we do have the valid consequence $\exists y\,(Rxy \,\&\, \neg Rxy) \models \exists y\,(Rxy \,\&\, \bot)$, but we do not have $\neg Rxy \models \bot$ or any obvious variant thereof. Nevertheless, several strategies are available for establishing decidability. First, we shall extend the above direct modal reduction to deal with the quantifier restriction schema $\exists \mathbf{y}\,(R\mathbf{x},\mathbf{y} \;\&\; \phi(\mathbf{x},\mathbf{y}))$ dividing up full occurrence sets of free variables. This reduction has some limitations, however, in that we must forbid occurrences of the relation R outside of restrictions. (On the other hand, R may be taken to allow arbitrary finite tuples of **x**-arguments, while it has some fixed finite arity for its **y**-arguments.) After this first argument, we shall present a second modified reduction strategy which deals with the whole of Fragment 2, including arbitrary occurrences of R–atoms.

First Reduction Strategy

For a start, we note a useful normal form.

<u>Fact</u> Every formula is equivalent to one whose immediate quantifier scope jumps, being of the form $[\alpha =]\ \exists \mathbf{y}\,(R\mathbf{x}, \mathbf{y} \,\&\, \ldots\ [\beta =]\ \exists \mathbf{z}\,(R\mathbf{u}, \mathbf{z} \,\&\, \ldots) \ldots)$, always have at least one variable y from **y** among the parameters **u** .

<u>Proof</u> If this normal form fails somewhere, then repair this, working inside out, by removing inner formulas outside of the scope of the outer ones – using the valid logical equivalence $\alpha \leftrightarrow (\beta \,\&\, [T/\beta]\alpha) \vee (\neg \beta \,\&\, [\bot/\beta]\alpha)$. ■

Next, we need a simple semantic fact of possible independent interest, inspired by the earlier 'unraveling' technique for modal logic. The result may be proved via bisimulations for Fragment 2 – but we give a direct argument here. Consider any first-order model **M** . The *unraveled model* $\mathbf{M}_{\text{unrav}}$ consists of all finite sequences of objects from **M** , with predicates defined as follows:

$R\,X_1 \ldots X_k\,Y_1 \ldots Y_m$ iff $\exists d_1 \ldots d_m : R_{\mathbf{M}}\ \text{last}\,(X_1) \ldots \text{last}\,(X_k)\ d_1 \ldots d_m$
$\qquad \& \;\; Y_i = X_1 {}^\frown \ldots {}^\frown X_k {}^\frown \langle d_i \rangle \quad (1 \le i \le m)$

and for all other Q ,

$Q\,X_1 \ldots X_n$ iff $Q_{\mathbf{M}}\ \text{last}\,(X_1) \ldots\ \text{last}\,(X_n)$

Note how unraveling forces R–successors to grow in size, while encoding the route which produced them. Here is the key semantic fact concerning this construction.

Lemma For all formulas $\phi = \phi\,(x_1, ..., x_n)$ in the current fragment,

$$\mathbf{M}_{unrav} \models \phi\ [X_1 , ..., X_n] \quad \text{iff} \quad \mathbf{M} \models \phi\ [\text{last}\,(X_1) , ..., \text{last}\,(X_n)]$$

Proof Use induction on the complexity of ϕ . The key step $\exists \mathbf{y}\,(R\mathbf{x},\mathbf{y}\ \&\ \phi(\mathbf{x}, \mathbf{y}))$ runs as follows. From $\mathbf{M}_{unrav}$ to $\mathbf{M}$, use the truth definition and the definition of R_{unrav} . Conversely, suppose that $\mathbf{M} \models \exists \mathbf{y}\,(R\mathbf{x},\mathbf{y}\ \&\ \phi(\mathbf{x}, \mathbf{y}))\ [\text{last}\,(X_1) , ..., \text{last}\,(X_n)]$. Then there exist objects $d_1 , ..., d_m$ with R_M $\text{last}\,(X_1) , ..., \text{last}\,(X_n), d_1 , ..., d_m$ such that $\mathbf{M} \models \phi(\mathbf{x}, \mathbf{y})\ [\text{last}\,(X_1) , ..., \text{last}\,(X_n), d_1 , ..., d_m]$. Now, the following unraveled successors of $X_1 , ..., X_n$ will ensure that $\mathbf{M}_{unrav} \models \phi(\mathbf{x}, \mathbf{y})\ [X_1 , ..., X_n, Y_1 , ..., Y_m]$: set $Y_i = X_1 {}^\frown ... {}^\frown X_n {}^\frown \langle d_i \rangle$ $(1 \leq i \leq m)$. ■

Remark Moving Upward in Evaluation

Combining the Unraveling Lemma with the earlier normal forms, we see that, for normal forms $\exists \mathbf{y}\,(R\mathbf{x},\mathbf{y}\ \&\ \phi(\mathbf{x}, \mathbf{y}))$, evaluation of the part $\phi(\mathbf{x}, \mathbf{y})$ 'moves upward'. That is, its truth value depends only on inspection of objects which are reachable through some finite chain of R–steps, starting from a tuple containing some y in $\mathbf{y}$. In particular, no immediate R–successors of $\mathbf{x}$ are ever encountered during the process of evaluation.

Now, we are ready to describe the desired general reduction. Consider any first-order consequence schema in Fragment 2 (in its current restricted form) which is of the form #

$$\underset{\exists \mathbf{y}\,(R\mathbf{x},\mathbf{y}\ \&\ \phi(\mathbf{x}, \mathbf{y}))}{\overset{\text{non R–atoms}}{\&}} \quad \models \quad \underset{\exists \mathbf{u}\,(R\mathbf{z},\mathbf{u}\ \&\ \psi(\mathbf{z}, \mathbf{u}))}{\overset{\text{non R–atoms}}{\bigvee}}$$

Without loss of generality, we may assume that no atom occurs on both sides, and that each 'parameter group' $\mathbf{x}$ occurs on both sides. (The latter can always be achieved by inserting 'inert' formulas with a constant 'true' or 'false' matrix for these parameters). Note that there can be more than one formula to the left (or right) for each parameter group.

Proposition A consequence # holds if and only if, for some totally disjoint choice of variables $\mathbf{y}$, we have a valid schema of the following form £ :

$$\underset{\phi(\mathbf{x}, \mathbf{y})}{\overset{\text{non R–atoms}}{\&_{\text{all } \mathbf{x}}}} \quad \models \quad \bigvee_{\text{all } \mathbf{x}} \bigvee_{\text{all } \phi(\mathbf{x}, \mathbf{y})} \underset{\psi(\mathbf{x}, \mathbf{y})}{\overset{\text{non R–atoms}}{}}$$

Example Let the schema # to be reduced have the form displayed below:

$$\exists y_1 y_2 (Rx_1x_2y_1y_2 \ \& \ \phi_1 (x_1, x_2, y_1, y_2))$$
$$\& \ \exists y_3 y_4 (Rx_1x_2y_3y_4 \ \& \ \phi_2 (x_1, x_2, y_3, y_4))$$
$$\& \ \exists y_5 y_6 (Rx_1y_5y_6 \ \& \ \phi_3 (x_1, y_5, y_6))$$
$$\models$$
$$\exists y_7 y_8 (Rx_1x_2y_7y_8 \ \& \ \psi_1 (x_1, x_2, y_7, y_8))$$
$$\vee \ \exists y_9 y_{10} (Rx_1y_9y_{10} \ \& \ \psi_2 (x_1, y_9, y_{10})) .$$

Then, its reducing schema £ described in the above Proposition will look as follows:

$$\begin{array}{lcl} \phi_1 (x_1, x_2, y_1, y_2) & & \psi_1 (x_1, x_2, y_1, y_2) \\ \& \ \phi_2 (x_1, x_2, y_3, y_4) & \models & \vee \ \psi_1 (x_1, x_2, y_3, y_4) \\ \& \ \phi_3 (x_1, y_5, y_6) & & \vee \ \psi_2 (x_1, y_5, y_6) \end{array}$$

Outline of a Proof for the Proposition First, from £ to #, a simple inspection suffices. Next, from # to £, suppose that the reducing sequent is not valid. Then it has a counterexample **M** with some assignment verifying its antecedent, while falsifying every disjunct in its consequent. Unravel **M**, and choose sequence-objects for the various y in the parameter groups **y** on the left which make all of them incomparable. In particular, then, their hereditary R–successors (recall the above remark about upward evaluation) will all be different. This gives us freedom for the following stipulation:

> For each of the parameters **x**, let its only R–successors be the vectors of objects for its associated **y** in the list of formulas to the left of #.

By previous observations, this does not affect truth values in **M** for matrix formulas ϕ. Thus, this slightly modified model verifies all restricted formulas $\exists \mathbf{y} (R\mathbf{x},\mathbf{y} \ \& \ \phi(\mathbf{x}, \mathbf{y}))$ to the left of schema #, and it falsifies all formulas $\exists \mathbf{u} (R\mathbf{z},\mathbf{u} \ \& \ \psi(\mathbf{z}, \mathbf{u}))$ on its right. ■

Comments

(1) This construction depends crucially on the independence of the successor sets R[**x**], even for parameter groups **x** that may be structurally related (say, as subsequences). Imposing *structural constraints* on the relation R may affect our results. For instance, one can easily allow permutation or contraction of **x**-parameters (cf. Alechina 1994), but it is less clear what happens when some form of (generalized) transitivity is imposed. (Here, the modal analogy with the logic K4 would still suggest decidability, though.)

(2) These reductions may also be used to measure *complexity* of decidability.

Second Reduction Strategy
Next, we introduce another semantic method, which yields the desired result for our full fragment – without the earlier syntactic proviso on restricting atoms R .

Theorem Universal validity for formulas in Fragment 2 is decidable.

As before, it will follow from the proof that Fragment 2 has the finite model property.

Proof Again, use standard propositional reductions to arrive effectively at a consequence problem # of the following kind. As usual, the notation ' $\phi(\mathbf{u})$ ' indicates that all free variables of ϕ are among those listed in $\mathbf{u}$:

$$\mathop{\&}_{\exists \mathbf{y}\,(R\mathbf{x},\mathbf{y}\ \&\ \phi(\mathbf{x},\mathbf{y}))}^{\text{atoms}} \quad \models \quad \bigvee_{\exists \mathbf{v}\,(S\mathbf{z},\mathbf{v}\ \&\ \psi(\mathbf{z},\mathbf{v}))}^{\text{atoms}}$$

More explicitly, this schema says that

$$\alpha, \exists \mathbf{y}_1\,(R_1\,\mathbf{y}_1, \mathbf{x}_1\ \&\ \phi_1\,(\mathbf{y}_1, \mathbf{x}_1))\,, \ldots, \exists \mathbf{y}_n\,(R_n\,\mathbf{y}_n, \mathbf{x}_n\ \&\ \phi_n\,(\mathbf{y}_n, \mathbf{x}_n))$$
$$\models$$
$$\beta, \exists \mathbf{z}_1\,(S_1\,\mathbf{z}_1, \mathbf{v}_1\ \&\ \psi_1\,(\mathbf{z}_1, \mathbf{v}_1))\,, \ldots, \exists \mathbf{z}_k\,(S_k\,\mathbf{z}_k, \mathbf{v}_k\ \&\ \psi_k\,(\mathbf{z}_k, \mathbf{v}_k))\,,$$

where α, β are sequences of atoms. Here, we may assume that all bound variables are distinct from each other, and from all free variables $\mathbf{x}_i, \mathbf{v}_j$. Also, by earlier observations, we can make sure that the free variables of ϕ_i properly overlap $\mathbf{y}_i$ – and also for ψ_j , $\mathbf{z}_j$. Next, we collect all free variables in the preceding schema into a single sequence, say $\mathbf{x}$. For convenience, at this stage, we introduce an *identity predicate* = into our argument, but only in a limited form. We fix any formula stating that the values of the variables in $\mathbf{y}$ are all different from the values of the variables in the parameter sequence $\mathbf{x}$:

$$\&\ \{ \neg\, y{=}x \mid y \in \mathbf{y}, x \in \mathbf{x} \} \qquad\qquad \mathbf{y}{\neq}\mathbf{x}$$

Then, by substituting variables and using standard reasoning with identity, we have the following general equivalence, using new quantifiers only over objects 'disjoint from $\mathbf{x}$ ' :

$$\exists \mathbf{y}\, \phi \ \leftrightarrow\ \{ \exists \mathbf{y_i}\ (\mathbf{y_i}{\neq}\mathbf{x}\ \&\ \phi')\mid \mathbf{y}{\supseteq}\mathbf{y_i},\ \phi' \text{ some substituted form of } \phi \}\,.$$

For instance, with $\mathbf{x} = x_1x_2$, the formula $\exists y_1y_2\, \phi$ is equivalent with the disjunction of the following variants involving substitutions: $\phi(x_1, x_1)$, $\phi(x_1, x_2)$, $\phi(x_2, x_1)$, $\phi(x_2, x_2)$,

$\exists y\,(y{\neq}x_1\ \&\ y{\neq}x_2\ \&\ \phi(y, x_2))$, $\exists y\,(y{\neq}x_1\ \&\ y{\neq}x_2\ \&\ \phi(y, x_1))$, $\exists y\,(y{\neq}x_1\ \&\ y{\neq}x_2\ \&\ \phi(x_1, y))$, $\exists y\,(y{\neq}x_1\ \&\ y{\neq}x_2\ \&\ \phi(x_2, y))$, $\exists y_1 y_2\,(y_1{\neq}x_1\ \&\ y_1{\neq}x_2\ \&\ y_2{\neq}x_1\ \&\ y_2{\neq}x_2\ \&\ \phi(y_1, y_2))$. Note that these variants remain inside Fragment 2. Next, we define an auxiliary notion:

Definition Let $\mathbf{x}$ be any sequence of variables. We say (inductively) that a formula ϕ is $\mathbf{x}$–*normal* if ϕ is either an atom or it is of the form

$$\exists \mathbf{y}\,(\mathbf{y}{\neq}\mathbf{x}\ \&\ R\mathbf{y}, \mathbf{z}\ \&\ \mathop{\&}_{i\in I}\ (\neg)\, \phi_i\,(\mathbf{y}, \mathbf{z}))\,,$$

where $\mathbf{y}$ is distinct from $\mathbf{x}, \mathbf{z}$, R is a relation symbol, and the ϕ_i in the matrix are all $\mathbf{x}$–normal formulas whose free variables overlap with $\mathbf{y}$.

By previous observations, any formula in Fragment 2 can be transformed algorithmically into a Boolean combination of $\mathbf{x}$–normal formulas. Let us assume that this has been done to the above schema # . Now, we can state our key quantifier reduction – this time, focusing on the left hand side (recall that α, β were sequences of atoms):

Proposition The following two statements are equivalent:

(i) $\alpha,\ \mathop{\&}_{i\in I}\ \exists \mathbf{y_i}\,(\mathbf{y_i}{\neq}\mathbf{x}\ \&\ R_i\,\mathbf{y_i}, \mathbf{x_i}\ \&\ \phi_i\,(\mathbf{y_i}, \mathbf{x_i}))\ \models$
$\beta,\ \bigvee_{j\in J}\ \exists \mathbf{z_j}\,(\mathbf{z_j}{\neq}\mathbf{x}\ \&\ S_j\,\mathbf{z_j}, \mathbf{x_j}\ \&\ \psi_j\,(\mathbf{z_j}, \mathbf{x_j}))$

(ii) $\alpha,\ \mathbf{y_i}{\neq}\mathbf{x}\ \&\ R_i\,\mathbf{y_i}, \mathbf{x_i}\ \&\ \phi_i\,(\mathbf{y_i}, \mathbf{x_i})\ \models$
$\beta,\ \bigvee_{j\in J}\ \exists \mathbf{z_j}\,(\mathbf{z_j}{\neq}\mathbf{x}\ \&\ S_j\,\mathbf{z_j}, \mathbf{x_j}\ \&\ \psi_j\,(\mathbf{z_j}, \mathbf{x_j}))$, for some $i\in I$.

Proof From (ii) to (i), the implication is immediate. From (i) to (ii), suppose that no conjunct $\alpha,\ \mathbf{y_i}{\neq}\mathbf{x}\ \&\ R_i\,\mathbf{y_i}, \mathbf{x_i}\ \&\ \phi_i\,(\mathbf{y_i}, \mathbf{x_i}))$ implies the consequent. Hence, there exist models $\mathbf{M}_i$ with assignments s_i verifying that antecedent and falsifying the consequent. Without loss of generality, these models may be taken to be disjoint except for the objects $\mathbf{d}$ assigned to the variables $\mathbf{x}$, where all of them coincide. Now, let $\mathbf{M}$ be the union of all models $\mathbf{M}_i$ ('glued together' at the $\mathbf{x}$–objects). We claim that this provides a counter-example to the implication (i), with the obvious joint assignment to all variables involved. The crucial observation here is this: $\mathbf{M}$–evaluation of $\mathbf{x}$–normal formulas can only involve one specific 'lobe' $\mathbf{M}_i$ of the glued disjoint union. More precisely, we have this

Claim Let ψ be an $\mathbf{x}$–normal formula whose free variables are not all in $\mathbf{x}$.
Let the assignment s send $\mathbf{x}$ to the common objects $\mathbf{d}$, and the other free variables of ψ to the rest of $\mathbf{M}_i$. Then we have $\mathbf{M}, s \models \psi$ iff $\mathbf{M}_i, s \models \psi$.

The Claim is proved by induction. The crucial quantifier case has the following shape: $\exists \mathbf{y}\,(\mathbf{y}\neq\mathbf{x}\ \&\ \mathrm{R}\mathbf{y},\mathbf{z}\ \&\ \&_{i\in I}\ (\neg)\ \phi_i(\mathbf{y},\mathbf{z}))$. If such a formula is true in $\mathbf{M}_i$, then it holds for objects (assigned to) $\mathbf{y}$ that are distinct from all $\mathbf{d}$. Moreover, not all free variables of the $\mathbf{x}$–normal formulas ϕ_i are included in $\mathbf{x}$ (at least one variable $y\in\mathbf{y}$ must have a free occurrence). Therefore, the inductive hypothesis applies to the ϕ_i : these hold in $\mathbf{M}$, and therefore, so does the whole formula. Conversely, if the above formula holds in $\mathbf{M}$, then the objects (assigned to) $\mathbf{y}$ must lie inside the domain of $\mathbf{M}_i$. (This is because of the true restriction $\mathrm{R}\mathbf{y},\mathbf{z}$: if one of its objects occurs in some lobe, then all of them do.) But then again, the inductive hypothesis applies to the ϕ_i : and we are done.

From the Claim, we see that the truth of the above separate $\mathbf{x}$–normal formulas $\mathbf{y_i}\neq\mathbf{x}\ \&\ \mathrm{R}_i\,\mathbf{y_i},\mathbf{x_i}\ \&\ \phi_i(\mathbf{y_i},\mathbf{x_i})$ (whose free variables are not all among $\mathbf{x}$) is preserved in going from $\mathbf{M}_i$ to $\mathbf{M}$, whence $\exists\mathbf{y_i}\,(\mathbf{y_i}\neq\mathbf{x}\ \&\ \mathrm{R}_i\,\mathbf{y_i},\mathbf{x_i}\ \&\ \phi_i(\mathbf{y_i},\mathbf{x_i}))$ is true in $\mathbf{M}$ as well. But also, if any formula $\exists\mathbf{z_j}\,(\mathbf{z_j}\neq\mathbf{x}\ \&\ \mathrm{S}_j\,\mathbf{z_j},\mathbf{x_j}\ \&\ \psi_j(\mathbf{z_j},\mathbf{x_j}))$ were to become true in $\mathbf{M}$, this can only be the case for objects $\mathbf{e}$ (assigned to $\mathbf{z_j}$) which lie in one component $\mathbf{M}_j$ (in its part disjoint from the common $\mathbf{d}$) : by the definition of relations S_j in a disjoint union. Again, this will imply that the $\mathbf{x}$–normal formula $\mathbf{z_j}\neq\mathbf{x}\ \&\ \mathrm{S}_j\,\mathbf{z_j},\mathbf{x_j}\ \&\ \psi_j(\mathbf{z_j},\mathbf{x_j})$ does not change its truth value in going to $\mathbf{M}_j$: which contradicts its falsity there. ■

In conjunction with general predicate-logical validities, the preceding Proposition allows us to reduce any consequence problem to one with only atoms α in the antecedent (involving free variables $\mathbf{x}$), and with atoms β plus existential forms in the consequent (quantifying about objects disjoint from all $\mathbf{x}$). It remains to give the 'coup de grâce'. Indeed, it is easy to see (by considering any 'minimal model' for the antecedent atoms) that this final consequence holds iff $\alpha \models \beta$: which is trivial to decide. ■

Applications will come below. We end with a negative result about our third fragment.

Fragment 3 is Undecidable

The general parametrized quantifier restriction schema $\exists y\,(\mathrm{R}\mathbf{x},\mathbf{y}\ \&\ \phi(\mathbf{x},\mathbf{y},\mathbf{z}))$ becomes as powerful as predicate logic itself.

<u>Fact</u> Predicate-logical satisfiability is effectively reducible to satisfiability in the parametrized restriction language.

<u>Proof</u> The relevant reduction takes any predicate-logical sentence ϕ to its relativization $\rho(\phi)$ to some unary predicate U not occurring in ϕ . $\rho(\phi)$ lies inside the parametrized restriction language. It is easy to see that ϕ is satisfiable if and only if $\rho(\phi)$ is. ■

4.3 Meta-Properties of Bounded Fragments

Other modal techniques from earlier sections may also be generalized to these fragments. We will merely formulate a short list of results obtained along the above lines. (The two question marks indicate open problems.) Detailed statements and proofs may be found in Andréka, van Benthem & Németi 1994A.

	Gentzenizable Validity	Decidability	Los-Tarski	Interpolation
Fragment 1	+	+	+	+
Fragment 2	+	+	+	+
Fragment 3	–	–	?	?

The first negative outcome for Fragment 3 is true in a very strong sense, as the non-finite-axiomatizability arguments of Andréka 1991 generalize to this case. We conjecture that the questions left open for Fragment 3 will receive positive solutions. But, the Table also suggests further issues. In Section 3 above, we saw that the finite variable hierarchy does not behave as well with respect to first-order logic as its modal counterpart. But this behaviour may improve when a finite-variable hierarchy is super-imposed on restricted quantifier fragments. In particular, we have proofs for complete finite axiomatizability, Craig Interpolation and Los-Tarski for all k–variable levels of Fragment 2. By contrast, one can obtain negative solutions to the above open questions for all finite–variable levels of Fragment 3. This may be proved by suitably relativized versions of earlier results.

<u>Example</u> Failure of Submodel Preservation in F3.3

Consider the counter-example ϕ to the Submodel Preservation Theorem from Section 3. Its relativization ϕ^U to some new unary predicate U is in Fragment 3 . It is easy to see that ϕ^U , too, is preserved under submodels. Now, suppose that it had a universal equivalent α in the three-variable fragment of Fragment 3 . This cannot happen, because we can now compare the two models constructed in the proof of Claim 4 in Section 3.4, both having U as the universal predicate. There is a 3–simulation from one to the other, which refutes the adequacy of α just as before.

4.4 Bounded Fragments and Bisimulation

The above fragments may be analyzed in terms of modal bisimulations. For this purpose, one can fix the earlier Invariance Theorem as a target result, and use its model-theoretic proof as a heuristic device for generating the appropriate notions of semantic simulation. We omit technical details here. Here are the outcomes for this style of analysis:

- in all cases, bisimulations will be binary relations between *tuples* of objects in the two models being compared (the modal fragment made do with single objects)
- all matches between tuples correspond to *partial isomorphisms*
- in general, bisimulations must be closed under *restrictions*, in the obvious way
- the *back and forth* conditions reflect the existential quantifier patterns of the fragments in a straightforward fashion (we display one side only):

Assume that two tuples **a** and **b** are already matched, then we must require

fragment 1
given any tuple **c** R–dependent on **a** , one chooses some *new* tuple **d** on the other side, matching **c** in the bisimulation, which must be R-dependent on **b**

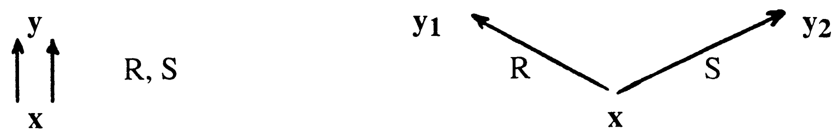
fragment 2
given any tuple **c** R–dependent on **a** , one chooses a tuple **d** on the other side, R-dependent on **b** such that the $\mathbf{a}^*\mathbf{c}$ and $\mathbf{b}^*\mathbf{d}$ match up in the bisimulation

fragment 3
here, R–restricted choices may depend on subsequences of **a, b** .

The crucial test, in each case, is that, in the earlier proof of the Invariance Theorem, elementary equivalence in the relevant fragment between tuples of objects defines a bisimulation of the appropriate kind, at least between ω-saturated models. The outcome of this analysis is a sequence of Preservation Theorems, characterizing Fragments 1, 2, 3 as consisting of precisely those first-order formulas (up to logical equivalence) which are invariant for these three types of bisimulation. Here is an application of this analysis.

<u>Theorem</u> Our three fragments form a properly ascending hierarchy.

<u>Proof</u> (1) The formula $\exists y\,(Rxy \,\&\, Sxy)$ is in Fragment 2, but not in Fragment 1. For, the following two models have different truth values for this formula – even though a Fragment–1 bisimulation exists between them, consisting of the following matches between single objects: (x, x), (y, y_1), (y, y_2) .



(2) The formula $\exists y\,(Ay \,\&\, \neg\, Rxy)$ is in Fragment 3, but not in Fragment 2 . For, it can distinguish between the following two models (both without any R–links), even though they admit a Fragment–2 bisimulation, consisting of only the match (x, x) :

y A

x x

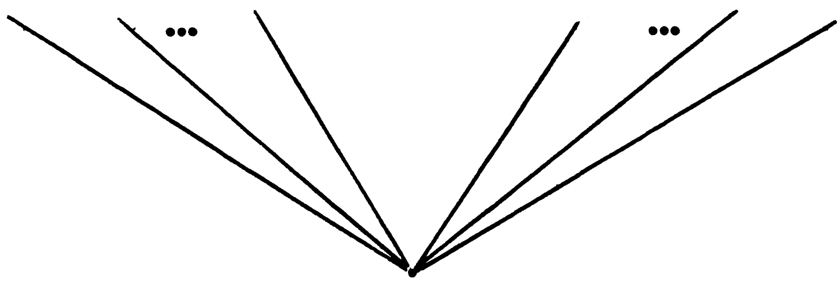
(3) Finally, Fragment 3 is still poorer than predicate logic as a whole. For instance, the formula $\forall x Ax$ is beyond it. This may be shown by the Fragment–3 bisimulation (x, x) between the following two models:

y

x A x A

■

More generally, the bounded fragments serve as a point of departure for a new hierarchy in predicate logic, orthogonal to the finite-variable levels:



What is the natural layering here? In addition to the degrees of freedom in the above restriction schema, one can vary the format of the restricting predicates themselves. E.g., Temporal Logic would typically involves "Betweenness": $\exists z\ (Rxz\ \&\ Rzy\ \&\ \phi(z))$. Or, Dynamic Logic has predicate operations, with the pattern: $\exists z\ (\mathbf{O}\ (R, ...)\ xz\ \&\ \phi\ (z))$. Some operations **O** produce formulas inside our restricted fragments (e.g., sequential composition and choice), while others lead outside of them (e.g., predicate intersection and complement, which violate 'bisimulation invariance': cf. van Benthem 1993). Such more expressive fragments, too, can be analyzed via our previous techniques.

The preceding results can all be specialized to Finite-Variable fragments of Bounded fragments. For instance, it may be of interest to consider 'joint simulations' combining the restrictions of Sections 4 and 3 . Even so, the two hierarchies seem largely orthogonal.

5 Generalized First-Order Semantics

5.1 A Modal View of Tarski Semantics

In our discussion so far, the main impact of modal techniques on standard logic has been the identification of first-order *fragments* that behave well over standard Tarski models. But one can also turn the tables, and interpret the full language of first-order predicate logic over generalized first-order models – where assignments (or objects) may only be 'available' subject to certain constraints (again, regulated by certain accessibility relations R), retaining the core of Tarski's truth definition. This proposal originates in algebraic logic (cf. Németi 1981, 1990, 1992 on relativizations of representable algebras of logics, with applications to the field of logic itself). This approach has been pursued since with modal techniques as well (cf. Venema 1991, Marx 1994, van Benthem 1994B). This second research program is not the subject of the present paper, but it is closely related in motivation and content. Here is a brief sketch of its main features and outcomes.

Modal first-order models are triples of the form $\mathbf{M} = (S, \{R_x\}_{x \in VAR}, I)$ where S is a set of 'states', R_x a binary relation between states for each variable x , and I is an 'interpretation function' giving a truth value to all atomic formulas Px, Rxy, Ryx, ... in each state α . This abstract modal format turns out to be all that is needed to set up the standard inductive truth definition for first-order logic:

$\mathbf{M}, \alpha \models Px$	iff	$I(\alpha, Px)$
Boolean connectives		as usual
$\mathbf{M}, \alpha \models \exists x\, \phi$	iff	*for some* $\beta : R_x\alpha\beta$ *and* $\mathbf{M}, \beta \models \phi$.

Thus, predicate logic becomes a poly-modal logic with $\exists x$ as an existential modality. In this full generality, models may deviate considerably from the standard paradigm. Notably, the interpretation of intuitively related atoms like Rxy and Ryx may become completely independent. And the same holds for such related formulas as Px and $\exists y Px$. Nevertheless, one can easily enforce such desired behaviour by additional stipulations (cf. Andréka, Gergely & Németi 1977, Németi 1986: section on "NA", Németi 1992: sections 7, 12, or Németi 1994: section 8). In particular, also, one might insist that the binary relations R_x be equivalence relations, as they are in standard Tarski models. This happens in a natural 'half-way house', in between modal first-order models and standard Tarski semantics, which we shall call *generalized assigment models.* Here S is some family of assignments in the usual sense (not necessarily the full function space D^{VAR}) , while the accessibilities R_x are the standard relations $=_x$ of identity up to x-values.

The resulting assignment gaps have positive virtues. They model the natural phenomenon of 'dependencies' between variables: when changes in value for one variable x may induce, or be correlated with, changes in value for another variable y. (Examples in natural deduction and probabilistic reasoning are in Fine 1985, van Lambalgen 1991). Dependence cannot be modeled in standard Tarskian semantics, wich modifies values for variables completely arbitrarily. Finally, to get an even closer approximation to the standard first-order language, one must introduce substitutions in the models (see below).

There is a growing literature on this generalized semantics. In particular, modal first-order models validate a 'minimal predicate logic', which is really just the minimal poly-modal logic, with all the positive properties studied in this paper (including decidability). On top of that lies a landscape of further calculi, all the way up to full predicate logic: which now becomes the particular (undecidable) mathematical theory of full function-space assignment models. The modal logic of the above generalized assignment models is an interesting intermediate possibility (called 'cylindric-relativized-set algebras' in the algebraic literature), which is decidable and has positive meta-properties (Los-Tarski, Craig Interpolation). Natural extensions arise by imposing constraints on the admissible assignments, such as 'local squareness' or the 'patchwork property' (cf. Németi 1992). Many results on completeness, correspondence and interpolation for modal first-order logics in this sub-classical landscape, as well as representation theorems for its abstract models, may be found in Németi 1992 and other papers cited above. (Cf. also van Benthem 1994B, Marx 1994.) One novel feature of this approach is that the generalized semantics invites the introduction of *new vocabulary*, reflecting distinctions not usually found in first-order logic. Examples are irreducibly polyadic quantifiers $\exists \mathbf{y}$ binding tuples of variables $\mathbf{y}$, or modal calculi of substitutions (cf. van Lambalgen & Simon 1994, Andréka & Németi 1994, Németi 1994, Venema 1993).

Illustration *Modal Analysis of Substitutions*

For each variable x, we already have an accessibility relation R_x in modal first-order models, corresponding to 'random assignment'. Next, we can also introduce "determinate assignment' mirroring the syntactic operation of substitution. Here is a way of doing this. For each pair of variables x, y, introduce a new unary modality S_{xy} with the intended meaning that, for each formula ϕ of predicate logic, $S_{xy}\phi$ is equivalent to the formula $[y/x]\phi$ in which all free occurrences of x have been replaced by y in the usual way. That is, $S_{xy}\phi$ is equivalent with $\exists x\,(x=y\ \&\ \phi)$. Now, modal first-order models will carry extra accessibility relations $A_{x,y}$, that can be subjected to the following constraints 'corresponding with' substitution principles in the modal sense (cf. van Benthem 1984). (1) $A_{x,y}$ is a function (this reflects commutation of S_{xy} with the Booleans). (2) $A_{x,y}$ is contained in R_x (this reflects the axiom $S_{xy}\phi \rightarrow \exists x\phi$) . (3) As a function, $A_{x,y} A_{x,y} = A_{x,y}$ (this reflects the axiom $S_{xy}S_{xy}\phi \leftrightarrow S_{xy}\phi$) . (4) Likewise, the substitution axiom $S_{xy}S_{yx}\phi \leftrightarrow S_{xy}\phi$ corresponds to $A_{x,y} A_{y,x} = A_{y,x}$. (5) Finally, the interpretation function I can be restricted to satisfy all atomic substitution laws, such as $S_{xy}Rxz \leftrightarrow Ryz$, etcetera. The resulting modal logic displays all the positive properties of our basic modal logic. For generalized assignment models, these definitions become even easier.

5.2 Back-and-Forth Between Modal Logic and Predicate Logic

Comparing the main thrust of this paper and the program outlined in Section 5.1, two main approaches emerge towards 'taming' classical first-order logic: i.e., localizing what may be called a well-behaved decidable 'core part'. One can either use standard semantics over non-standard language fragments, or use non-standard generalized semantics over the full standard first-order language. The former approach is more 'syntactical' in nature, the latter more 'semantical'. (Eventually, as so often in logic, this distinction is relative. For instance, one can also translate 'semantic' modal discourse about the above modal first-order models into a restricted syntactic fragment of a *two-sorted* first-order logic, with direct reference to both 'individuals' and 'states'. But also conversely, ... etcetera.) More specifically, evident technical analogies exist between existing proof methods for generalized semantics in the sense of Section 5.1 and those of the present paper. We feel that there is a mathematical duality lurking in the background here, largely unexplored – which we illustrate by some simple observations. In particular, our earlier analysis of bounded first-order fragments may be used to derive results about generalized assignment semantics, or equivalently, about relativized cylindric algebras (i.e., Crs-models).

From Bounded Fragments to Cylindric Algebra

Consider any k–variable language $L\{x_1, ..., x_k\}$. Let R be some new k–ary predicate. We define a translation tr_g from k–variable formulas to restricted first-order formulas:

> Global Relativization
>
> $tr_g(\phi)$ arises from ϕ by relativization of all its quantifiers
>
> to the same atom $Rx_1...x_k$

Note that the relativized images $tr_g(\phi)$ will even lie inside our bounded Fragment 2. Next, we define a corresponding operation on models. Let **M** be any generalized assignment model for $L\{x_1, ..., x_k\}$ (as yet without the new predicate symbol R).

Restricted Standard Models

The standard model $\mathbf{M}_{rest}$ is $\mathbf{M}$, viewed as a standard model, and expanded with the following interpretation for the new predicate: $R(d_1, ..., d_k)$ iff the assignment $x_i := d_i$ $(1 \le i \le k)$ is available in $\mathbf{M}$.

The purpose of this construction shows in the following fact.

Proposition For all available assignments α in $\mathbf{M}$, and all formulas ϕ ,

$$\mathbf{M}, \alpha \models \phi \quad \text{iff} \quad \mathbf{M}_{rest}, \alpha \models tr_g(\phi)$$

Proof Induction on first-order formulas. The crucial case is that of existential quantifiers. In particular, suppose that $\mathbf{M}_{rest}, \alpha \models tr_g(\exists x_i \phi) = \exists x_i (Rx_1...x_k \ \& \ tr_g(\phi))$. Then, there exists a satisfying k–tuple of objects in R for $tr_g(\phi)$, which corresponds to an available assignment in $\mathbf{M}$ which is an i–variant of α . I.e., $\mathbf{M}, \alpha \models \exists x_i \phi$. ■

As a consequence, one can effectively reduce universal validity over all generalized assignment models (i.e., in Crs) to standard validity in Fragment 2.

Corollary $\models_{gen'd} \phi$ iff $\models_{standard} Rx_1...x_k \rightarrow tr_g(\phi)$

Proof 'Only if'. If ϕ has a generalized counter-example $\mathbf{M}, \alpha$, then the above model $\mathbf{M}_{rest}$ falsifies $Rx_1...x_k \rightarrow tr_g(\phi)$. 'If'. Suppose, conversely, that the latter formula has a standard counter-example $\mathbf{M}, \alpha$. Now define a corresponding generalized model $\mathbf{M}_g$ by retaining only those assignments whose values for $x_1, ..., x_k$ stand in the relation $R_\mathbf{M}$ (in particular, the falsifying assignment α itself remains available). Then ϕ is falsified in $\mathbf{M}_g$ by α as above. ■

This result provides a new 'modal' proof for the following theorem (cf. Nemeti 1992).

Theorem Validity in Crs is decidable, and Crs has the finite model property.

Proof This follows from the corresponding results for Fragment 2 . ■

Remark Uniform Translation

The above translation can be made to work for the whole first-order language at once, using a slightly more complex model construction. (The idea is to assign one additional 'dummy object' to all but finitely many variables in our 'available assignments'. Cf. Andréka, van Benthem & Németi 1994A.)

There is more to the above analysis, however. Special classes of generalized assignment models have arisen by imposing more specific constraints on admissible assignments. Now, the first-order theory of such classes, too, will be decidable, as long as their additional conditions can be stated in first-order forms translatable *into Fragment 2* . Checking some examples found in Németi 1992, we see that this analysis applies to 'D_α' or 'G_α' . In particular, we have a new proof for the following result.

Theorem Universal validity is decidable on the class of generalized assignment models which are locally square.

Proof The reason is that the requirements for being locally square are all expressible inside Fragment 2. Here is an example of the relevant kind of formula:

$$\forall xy\ (Rxy \to (Ryx \,\&\, Rxx \,\&\, Ryy))$$ ■

By contrast, we know that validity is undecidable in the class of generalized assignment models satisfying the Patchwork Property. Again, this checks out. In first-order form, the latter constraint involves statements like

$$\forall xyzuv\ ((Rxyz \,\&\, Ruyv) \to (Rxyv \,\&\, Ruyz))$$

Note that these are not in Fragment 2: variable inclusion holds from matrix to restriction, but the latter is not one single atom. (This shows that we cannot allow arbitrary Boolean combinations of restrictions in our earlier results.) This style of analysis is quite powerful, and it can be used to predict decidability of many other combinations of algebraic axioms on top of Crs, as long as their complete frame properties fall inside Fragment 2.

We conclude with a natural converse question. Can one also derive the behaviour of our modal bounded fragments from algebraic results about generalized assignment models? In particular, is there a *converse reduction* going from standard validity of formulas ψ in Fragment 2 to generalized validity of suitable formulas red(ψ) over generalized assignment models? At least, the identical translation does not work here. The following Fragment 2 formula is valid, but it is not in Crs:

$$\exists x\ (Ax \,\&\, \exists y\ (Rxy \,\&\, Ay)) \to \exists y\ (Ay \,\&\, \exists x\ (Rxy \,\&\, Ax))$$

We do have some partial converse results, that work for suitably 'uniformly relativized' formulas in Fragment 2 – but we shall leave this matter open here.

<u>Remark</u> Dependency Semantics

The preceding analysis also suggests a comparison between generalized assignment models and the 'dependency models' for generalized quantifiers $Qx \bullet \phi$ proposed in van Lambalgen 1991, Alechina & van Benthem 1993. These quantifiers are read there as stating the existence of some object 'depending' on the range of the assignment so far. The two semantics are evidently related in spirit, but still they are not quite isomorphic. For instance, generalized assignment semantics validates unrestricted Monotonicity for the existential quantifier (i.e., $\exists x\, \phi \rightarrow \exists x\, (\phi \vee \psi)$), whereas dependency semantics does not. (It only retains Monotonicity and Distribution with suitably 'balanced' variables.) On the other hand, dependency semantics validates the unrestricted axiom $\exists x\, \phi \rightarrow \phi$ (when x is not free in ϕ), which does not hold on all generalized assignment models. We briefly analyze the situation from the preceding point of view. Dependency semantics may be said to arise from first-order logic through a 'local translation' tr_l which is much like the above 'global translation' tr_g , but with the following delicate difference. At each subformula $\exists x_i\, \psi$, one only relativizes to an atom $R\mathbf{x}$ where $\mathbf{x}$ enumerates all free variables of the local context ψ . This difference explains all the deviant behaviour. E.g., consider the effect of the two translations on the above-mentioned Monotonicity. The global one makes this principle valid in Fragment 2, whereas the local one does not:

$\forall y\ (\forall x\, (Ax \rightarrow Bxy) \rightarrow (\exists xAx \rightarrow \exists xBxy))$

translation tr_g

$Rxy \rightarrow \forall y\, (Rxy \rightarrow$

$(\forall x\, (Rxy \rightarrow (Ax \rightarrow Bxy)) \rightarrow (\exists x\, (Rxy \;\&\; Ax) \rightarrow \exists x\, (Rxy \;\&\; Bxy))\,)$

translation tr_l

$\forall y\, (Ry \rightarrow$

$(\forall x\, (Rxy \rightarrow (Ax \rightarrow Bxy)) \rightarrow (\exists x\, (Rx \;\&\; Ax) \rightarrow \exists x\, (Rxy \;\&\; Bxy)))$

Nevertheless, all our general results apply – since tr_l , like tr_g , takes first-order formulas to formulas in Fragment 2. Therefore, we can derive the decidability result of Alechina 1994, and also, we can predict decidability for all stronger dependency logics having their characteristic frame conditions inside Fragment 2 (cf. Alechina & van Lambalgen 1994).

Bisimulation and Ehrenfeucht Games

Next, consider basic notions of equivalence between models in the respective domains. As was pointed out in van Benthem 1991, de Rijke 1993, bisimulation stands to modal logic as Ehrenfeucht games (or rather, 'partial isomorphism') to standard first-order logic. As before, comparisons in the present setting are promising but somewhat inconclusive.

For instance, a modal bisimulation $\equiv$ between generalized assignment models which relates assignments α, β only if they have the same domain, induces an obvious relation PI between (those tuples of objects that form) the ranges of α and β . PI is a family of partial isomorphisms. It will satisfy the usual back-and-forth extension conditions for 'partial isomorphism' iff our generalized first-order model satisfies the following Update Postulate: 'For any object, any assignment has an extension which assigns that object to some fresh variable'. Then, the bisimulation clause for the latter variable will do the job. Conversely, given a partial isomorphism PI between two models, we can define a modal bisimulation $\equiv$ between partial assignments over them by checking whether their ranges are a matching pair of object tuples in PI . But perhaps, the more interesting comparison lies in the differences. Generalized assignment models suggest a *change* in the standard model-theoretic notion of partial isomorphism, using a finer-grained connection between partial assignments (rather than flat sequences of objects) in the two models involved.

Thus, the mathematical analogies between generalized assignment models in Cylindric Algebra and possible worlds semantics for Modal Logic have proved of evident benefit.

6 Further Directions

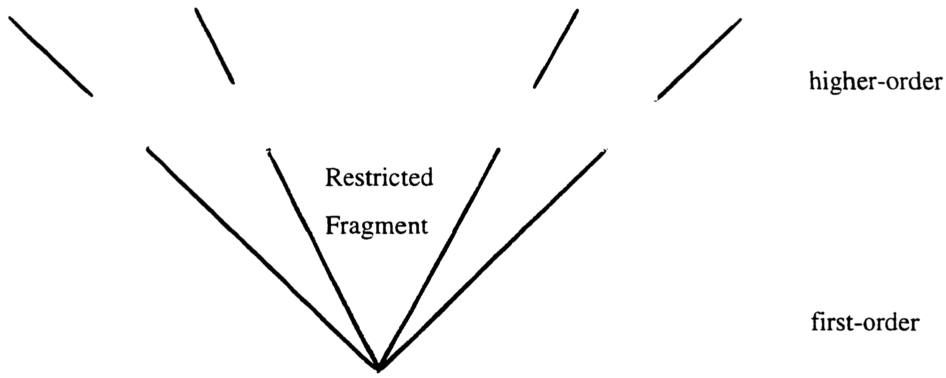
At various places so far, we indicated open research questions. These concerned both technical elaboration within our framework (details of Los-Tarski theorems in Section 3, meta-properties of further first-order fragments: cf. Section 4) and extensions to a broader environment (e.g., mathematical connections with generalized assignment or dependency semantics: cf. Section 5). In this last Section, we briefly mention some further directions.

6.1 Special Frame Constraints

Basic modal logic is usually enriched with special frame conditions , as with S4, S5 or more complex systems, imposing transitivity or other natural frame conditions. How do these additional conditions systematically affect the above picture? (Note, e.g., that the generalized Tarski Semantics of section 5 employs, at least, S5-like frame conditions.) What we also need to understand here is the possibility of a systematic 'trade-off' between 'packages' of first-order translations plus frame constraints. For instance, S5 can also be translated faithfully without any frame constraints by dropping the accessibility restriction on quantifiers. And more subtle examples of this phenomenon exist as well. (Cf. the modelling for the modal logic B of all symmetric frames in van Benthem 1983.) To complicate the picture even further, other first-order translations may be studied from this perspective, too, such as the 'path formulas' of Ohlbach 1991.

6.2 Infinitary Extensions

'Restriction' works just as well in fragments of *higher-order* languages, such as $L_{\omega_1\omega}$ or $L_{\infty\omega}$ or second-order logic. We can transfer our 'modal hierarchy' up to here:



Possible analogies to be explored lie in second-order logic (cf. Gallin 1975 on the good behaviour of restricted 'extensional fragments') and in admissible set theory (cf. Barwise 1975). Possibly significant here is the simple folklore characterization of bisimulation between arbitrary models via their elementary equivalence in the $L_{\infty\omega}$–version of modal logic with arbitrary set conjunctions and disjunctions (van Benthem and Bergstra 1993). (De Rijke 1993 present more sophisticated results, e.g., inside $L_{\omega_1\omega}$.) Can we also generalize other results from the above, such as the Los-Tarski Theorem?

6.3 Extended Modal Logics

What we have not considered here are enriched modal operator formalisms, in the style of Gargov, Passy & Tinchev 1987 or de Rijke 1993, which allow both modest additions (such as the 'difference modality') and strong extensions in expressive power (such as temporal logics of 'since' and 'until'). It would be of interest to extend our analysis in this direction. This would also fit in with the move towards introducing richer vocabularies in generalized assignment semantics, mentioned in Section 5. The important point to note here is that our strategy of 'fragments' by no means implies logical poverty.

6.4 Alternative Semantics

There are still further alternatives in modelling first-order predicate logic which may be relevant to our present concerns. For instance, starting from a more computational motivation in 'dynamic semantics', Hollenberg & Vermeulen 1994 propose a stack-based account of first-order logic which makes the latter's two-variable fragment as powerful as the whole language. (In essence, this says that two variables over finite sequences are as good as arbitrary finite numbers of variables over individual objects.) It remains to be seen how our considerations fare in such sequence semantics.

7 Acknowledgments

Various participants at the Amsterdam Modal Logic Colloquium (organized by Maarten de Rijke) provided useful comments on earlier versions of this work, including Peter van Emde Boas and Tim Fernando. We were also inspired by the lively participants at the FoLLI Summer School in Lisbon 1993. Finally, we would like to thank a group of inspiring colleagues, in particular, Natasha Alechina, Michiel van Lambalgen, Maarten Marx, Szabolcs Mikulas, Maarten de Rijke, Ildiko Sain, Andras Simon and Yde Venema.

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