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(revised version of ML-93-14)

ML-95-06, *received*: March 1995

ILLC Research Report and Technical Notes Series
Series editor: Dick de Jongh

Mathematical Logic and Foundations (ML) Series, ISSN: 0928-3315

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Abstract

In this paper we study embeddings of Heyting algebras (*Ha*'s). It is pointed out that such embeddings are naturally connected with Derived Rules and with propositional theories. We consider the *Ha*'s embeddable in the *Ha* of the Intuitionistic Propositional Calculus (IPC), i.e. the free *Ha* on $\aleph_0$ generators, those embeddable in the *Ha* of Heyting's Arithmetic (HA) and those embeddable in the *Ha* of HA*, a 'natural' extension of HA. We prove the following theorems. The same *Ha*'s on finitely many generators are embeddable in the *Ha* of IPC and in the *Ha* of Boolean (or: Brouwerean) combinations of Σ-sentences of HA. The *Ha*'s on finitely many generators embeddable in the *Ha* of IPC are finitely presented. There is a non-recursive *Ha* on three generators that can be embedded in the *Ha* of HA. Every recursively enumerable prime *Ha* is embeddable in the *Ha* of HA*.

1 Preliminaries

1.1 Introduction

This paper sprung from an interest in the Heyting algebra's (*Ha*'s) of Constructive arithmetical theories. This interest was in its turn inspired by an interest in the propositional derived rules of constructive arithmetical theories.

An approach parallel to the study of Heyting algebras of constructive theories is the study of propositional theories faithfully interpretable in constructive theories. Consider a sequence of arithmetic sentences $A_1,\ldots,A_n$ (or similarly an ω-sequence of such sentences). In a theory like HA (Heyting Arithmetic) these sentences are said to (faithfully) interpret the propositional theory $\{B(p_1,\ldots,p_n) \mid HA \vdash B(A_1,\ldots,A_n)\}$. Such a theory of propositional formulas is closed under modus ponens, but not under substitution. Alternatively, one may consider the Lindenbaum algebra of HA (which is a Heyting algebra, for the formal definition, see section 1.4) and its subalgebra generated by $A_1,\ldots,A_n$. This subalgebra and the propositional theory determine each other, of course, so, the approaches of interpretability of propositional theories in an arithmetical theory and embeddability of Heyting algebras as a subalgebra of the Heyting algebra of an arithmetical theory are equivalent. Here we generally take the Heyting algebra point of view.

Closely connected are derived rules. If, for all $A_1,\ldots,A_n$, in the language of a theory T, $T\vdash B(A_1,\ldots,A_n)$ implies $T\vdash C(A_1,\ldots,A_n)$ for some propositional B,C, then the rule $B(p_1,\ldots,p_n)/C(p_1,\ldots,p_n)$ is called an admissible rule for T. It is well-known that standard theories like HA admit additional rules not derivable in intuitionistic logic, the best known being:

$\neg A\to(B\vee C)\,/\,(\neg A\to B)\vee(\neg A\to C)$ (Independence of Premise Rule)

This has as a consequence that propositional theories that are not closed under this rule are automatically disqualified for being interpretable in the standard theories. The same holds, of course, for Heyting algebras that do not validate the rule above in the obvious manner.

We study and compare four specific *Ha*'s in some detail:

- $\mathcal{H}_{IPC}$, the *Ha* of the Intuitionistic Propositional Calculus (IPC), in other words, the free *Ha* on $\aleph_0$ generators,.
- $\mathcal{H}_{HA}$, the *Ha* of Heyting's Arithmetic (HA).
- $\mathcal{S}_{HA}$, the *Ha* of $B\Sigma_1$-sentences in HA (here $B\Sigma_1$ is the set of Boolean, or perhaps more appropriately: Brouwerean, combinations of Σ_1-sentences).
- $\mathcal{H}_{HA^*}$, the *Ha* of HA^*, an arithmetical theory studied in Visser[82].

We ask ourselves which r.e. (recursively enumerable) *Ha*'s can be embedded in our target algebras. The restriction to recursively enumerable *Ha*'s is evident for finitely generated Heyting algebras, because only r.e. algebras of such a kind can be embeddable in r.e. theories T, and quite a reasonable one in general. The answer to our question also determines, as we will see, what the Propositional Derived Rules for the various theories are. A reasonably complete answer has only been obtained for $\mathcal{H}_{HA^*}$. All r.e. algebras of which one could reasonably expect it, i.e. those satisfying the property of primeness (corresponding to having the disjunction property), are embeddable in $\mathcal{H}_{HA^*}$, and, in consequence, the admissible rules of HA^* are precisely the trivial ones, i.e. A/B is an admissible rule of HA^* iff $IPC\vdash A\to B$ (corollary 5.9). This property of HA^* is a nice one—and in a surprising manner enables one to prove some properties of HA itself—but it does not seem to hold for more usual theories. Many of these algebras cannot be embedded in $\mathcal{H}_{HA}$, nor in $\mathcal{H}_{IPC}$, since, as mentioned both these theories validate the Independence of Premise Rule.

We will show that all *Ha*'s on finitely many generators embeddable in $\mathcal{H}_{IPC}$ are finitely presented (i.e. are the *Ha*'s of finitely axiomatized IPC-theories). In contrast there is a non-recursive (and hence not finitely presented) *Ha* on three generators, that can be embedded in $\mathcal{H}_{HA}$. It is an open question, whether there is a finitely presented *Ha* on finitely many generators that can be embedded in $\mathcal{H}_{HA}$ but not in $\mathcal{H}_{IPC}$. It is also open whether HA and IPC have the same admissible rules. On the other hand, we will show that the same *Ha*'s on finitely many generators are embeddable in $\mathcal{S}_{\mathrm{HA}}$ and in $\mathcal{H}_{\mathrm{IPC}}$. It follows that rules validated by HA when one restricts oneself to substitutions of propositional combinations of Σ_1-sentences, and rules validated by IPC are the same. It is open whether $\mathcal{S}_{\mathrm{HA}}$ and $\mathcal{H}_{\mathrm{IPC}}$ are isomorphic. We conjecture that they are not. We state some sample results with the places, where they can be found:

- Any r.e. prime *Ha* $\mathcal{H}$ can be embedded in $\mathcal{H}_{\mathrm{HA}*}$ (theorem 5.2).
- There are Σ_1-sentences A and B such that the subalgebra of $\mathcal{H}_{\mathrm{HA}*}$ generated by A and B is r.e., non-recursive (corollary 5.14).
- There are Σ_1-sentences A and B and a sentence C, such that the subalgebra of $\mathcal{H}_{\mathrm{HA}}$ generated by A, B and C is r.e., non-recursive. It follows that $\mathcal{H}_{\mathrm{HA}}$ is non-recursive (corollary 5.15).
- Let $\mathcal{H}$ be a *Ha* on finitely many generators which is embeddable in $\mathcal{H}_{\mathrm{IPC}}$. Then $\mathcal{H}$ is the *Ha* of a finitely axiomatizable IPC-theory (theorem 2.3).
- Let $\mathcal{H}$ be a *Ha* on finitely many generators. Then $\mathcal{H}$ is embeddable in $\mathcal{S}_{\mathrm{HA}}$ iff $\mathcal{H}$ is embeddable in $\mathcal{H}_{\mathrm{IPC}}$ (theorem 6.5).

The paper is organized as follows. In section 1, we define *Ha*'s in the presentation most useful to our purposes. In section 2, we introduce the notion of embedding and a connected notion of propositional formulas exactly provable for sentences of a theory. Propositional formulas with no iterations of implications on the left (NNIL formulas) turn out to play an important role. In sections 3 and 4 necessary facts about HA, and IPC and HA^* respectively, are given. In section 5 the above mentioned 'r.e. universality' of HA^* is proved. It is shown that, in consequence of the previous results, there is a non-recursive *Ha* on two generators that can be embedded in $\mathcal{H}_{\mathrm{HA}*}$, whereas such an algebra could never be embeddable in $\mathcal{S}_{\mathrm{HA}}$ or $\mathcal{H}_{\mathrm{IPC}}$. The theorem that there is a non-recursive *Ha* on three generators that can be embedded in $\mathcal{H}_{\mathrm{HA}}$ is an immediate consequence of this last result. It then follows that $\mathcal{H}_{\mathrm{HA}}$ is non-recursive. In section 6, $\mathcal{S}_{\mathrm{HA}}$ is treated. In an appendix the density of $\mathcal{H}_{\mathrm{T}}$ is proved for all reasonable arithmetic theories.

Technically, theorem 5.2 on the r.e. universality of HA^* can be considered as the main theorem of the paper, but, of course, the applications to the standard theory HA are of more general interest.

1.2 Acknowledgements

Most of the results of this paper were obtained during an exceptionally inspiring visit of both authors to the Katedra Logiky of Prague University and the Institute of Computer and Information Science of the Czech Academy of Sciences. We thank the Prague logicians for their wonderful hospitality. Volodya Shavrukov read a manuscript of the paper in a very thorough way. We are grateful for his many comments. We thank Lev Beklemishev for enlightening discussions and Troelstra for helping us out in our discussions about realizability. Finally, we thank the referee for valuable comments that greatly changed the appearance of the paper.

Some of the main methods employed in this paper were invented by Volodya Shavrukov (see Shavrukov[93]) and further developed and simplified by Domenico Zambella (see Zambella[94]). The work of Shavrukov and Zambella concerns embeddings of r.e. Magari algebras (up to recently usually called Diagonalizable algebras) into Magari algebras of Classical arithmetical theories.

A major tool of the present paper is also Pitts's Uniform Interpolation Theorem.

1.2.1 Uniform Interpolation Theorem (Pitts[92]) For each formula A of IPC and each list of propositional variables $\mathbf{q}$ there exists a *uniform pre-interpolant* $\forall\mathbf{q}A$ and a *uniform post-interpolant* $\exists\mathbf{q}A$, both containing only variables occurring in A that differ from the ones in $\mathbf{q}$, such that, for each B in IPC that does not contain $\mathbf{q}$,

(i) $\vdash B \to A$ if and only if $\vdash B \to \forall\mathbf{q}A$,

(ii) $\vdash A \to B$ if and only if $\vdash \exists\mathbf{q}A \to B$.

1.3 The classical case

Before going on, let us briefly look at the Boolean algebras of classical arithmetical theories. The Boolean algebras of all consistent r.e. arithmetical theories extending Q are isomorphic to the free Boolean algebra on $\aleph_0$ generators, i.e. to the Boolean algebra $\mathcal{B}_{CPC}$ of the Classical Propositional Calculus (CPC). As far as we can trace it this result is folklore. It follows from three observations. First, the Boolean algebras of all consistent r.e. arithmetical theories extending Q are countably infinite and (by Rosser's Theorem, see the appendix for some more details) atomless. Second, $\mathcal{B}_{CPC}$ is countably infinite and atomless. Third: all countably infinite atomless Boolean algebras are isomorphic.

It is not difficult to show that every countable Boolean algebra can be embedded into $\mathcal{B}_{CPC}$.

1.4 Heyting algebras

A *Heyting algebra* (*Ha*) $\mathcal{H}$ is a structure $\langle H, \leq, \wedge, \vee, \bot, \top, \to\rangle$, where $\langle H, \leq, \wedge, \vee, \bot, \top\rangle$ is a lattice with bottom $\bot$ and top $\top$, so that $x \leq y$ holds if and only if $x \vee y = y$, and $\to$ is a binary operation satisfying $x \wedge y \leq z$ iff $x \leq y \to z$. We demand that $\mathcal{H}$ is non-trivial, i.e. that H contains at least two elements. It is easily seen that, if a partial order can be extended to a *Ha*, such an extension is unique. *Ha*'s can be shown to be distributive lattices. Conversely, every *finite* distributive lattice determines a Heyting algebra. There are many good sources for *Ha*'s. We just mention Troelstra & van Dalen[88b].

The point of the present subsection is to clear up the connection between *Ha*'s and propositional theories, to make precise what it means for a *Ha* to be r.e. or recursive and to give a simple condition on *Ha*'s to be used later to decide that certain *Ha*'s are not recursive.

1.4.1 Definition We will write:

- $\neg x \quad := \quad x \to \bot$,
- $x \leftrightarrow y \quad := \quad (x \to y) \wedge (y \to x)$,
- $\mathcal{H} \models A(\mathbf{x}) \quad :\Leftrightarrow \quad A(\mathbf{x}) = \top$,

  where A is a polynomial in $\wedge, \vee, \bot, \top, \to$ and $\mathbf{x}$ is a sequence of elements of $\mathcal{H}$.
- $f{:}\,\mathcal{H} \leqq \mathcal{K} \quad :\Leftrightarrow \quad f$ is an embedding of $\mathcal{H}$ into $\mathcal{K}$
- $\mathcal{H} \leqq \mathcal{K} \quad :\Leftrightarrow \quad f{:}\,\mathcal{H} \leqq \mathcal{K}$ for some f
- $\mathcal{H} \equiv \mathcal{K} \quad :\Leftrightarrow \quad \mathcal{H} \leqq \mathcal{K}$ and $\mathcal{K} \leqq \mathcal{H}$

Clearly, $\leqq$ is a preorder on *Ha*'s with induced equivalence relation $\equiv$.

1.4.2 Example Equivalent *Ha*'s need not be isomorphic.
It is easily seen that any linear order with endpoints determines a *Ha*; one defines $x \to y = \top$ if $x \leq y$, and $x \to y = y$ if $y < x$. Moreover, an embedding of linear orderings determines an embedding of *Ha*'s. Consider the algebras given by the real interval $[0, 1]$ and by $[0, {}^1/_2] \cup \{1\}$. These algebras are equivalent but not isomorphic.

Let T be any consistent theory in constructive or predicate logic. We take $\mathcal{H}_T$ to be the obvious *Ha* given by the T-provable equivalence classes. Sometimes we will consider only equivalence classes of a subset X of the language of T which is closed under the propositional connectives. In this case we write $\mathcal{H}_T(X)$. In this manner, we can go from theory to algebra. Obviously, it is sometimes natural to go back and recover theories from algebras. We introduce some notions relevant to this motion, which is executed by equipping the algebra with a set of generators.

1.4.3 Definition

- A *numbered Ha* $\overline{\mathcal{H}}$ is a pair $\langle f, \mathcal{H} \rangle$, where
  (i) f is a function (not necessarily injective) from, either $n=\{0,...,n-1\}$, or ω, to $H_{\mathcal{H}}$;
  (ii) $\mathcal{H}$ is generated by the range of f.
- A numbered *Ha* is *finitely gripped* if dom(f) is finite.
- $\mathcal{L}_\nu$ is the language of IPC if $\nu=\omega$, and the language of IPC restricted to $p_0,...,p_{n-1}$ if $\nu=n$. We often write $\mathcal{L}$ for $\mathcal{L}_\omega$.
- For $A \in \mathcal{L}_{dom(f)}$, $\overline{\mathcal{H}} \vDash A :\Leftrightarrow \mathcal{H} \vDash A[f]$, where $A[f]$ is the result of substituting f(i) for p_i in A (for each relevant i). It is pleasant to use $\vDash$ also when A contains p_j for $j \notin dom(f)$; in this case we substitute $\top$ for p_j.
- $Th(\overline{\mathcal{H}}) := \{A \in \mathcal{L}_{dom(f)} \mid \overline{\mathcal{H}} \vDash A\}$.

1.4.4 Fact Let $\overline{\mathcal{H}}=\langle f, \mathcal{H} \rangle$ be a numbered *Ha*. Then $\mathcal{H}$ is isomorphic to $\mathcal{H}_{Th(\overline{\mathcal{H}})}$.

1.4.5 Definition

- A numbered *Ha* $\overline{\mathcal{H}}$ is r.e. (recursive) if Th($\overline{\mathcal{H}}$) is r.e. (recursive).
- A *Ha* is r.e. (recursive) if it can be extended to an r.e. (recursive) numbered *Ha*.
- A numbered *Ha* $\overline{\mathcal{H}}$ is finitely presented if Th($\overline{\mathcal{H}}$) is finitely axiomatizable

In other words, the recursiveness or recursive enumerability of numbered *Ha*'s is reduced the theories associated with them, and *Ha*'s themselves are considered to be recursive or r.e. if they are so with some designated generators. Note that the *Ha* of an r.e. (recursive) theory is r.e. (recursive).

1.4.6 Example Let $\overline{\mathcal{H}}_{PA} := \langle f, \mathcal{H}_{PA} \rangle$, where f(i) is the equivalence class of an arithmetical sentence A if i is the Gödelnumber of A, and f(i) is $\top$ if i is not the Gödelnumber of an arithmetical sentence. Let $\overline{\mathcal{H}}_{CPC} := \langle g, \mathcal{H}_{CPC} \rangle$, where g(i) is the equivalence class of p_i. Then $\overline{\mathcal{H}}_{PA}$ is r.e., non-recursive and $\overline{\mathcal{H}}_{CPC}$ is recursive. This is to be contrasted with the fact that, by section 1.3, $\mathcal{H}_{PA}$ and $\mathcal{H}_{CPC}$ are isomorphic and hence $\mathcal{H}_{PA}$ is recursive.

1.4.7 Lemma Suppose $\mathcal{H}$ is r.e. (recursive) and suppose $\mathcal{K} \leqq \mathcal{H}$, where $\mathcal{K}$ is an *Ha* on finitely many generators. Then any numbered finitely gripped *Ha* $\langle f, \mathcal{K} \rangle$ is r.e. (recursive).

Proof: If $\mathcal{H}$ is recursive or r.e., then, for some choice of generators, the theory is recursive (r.e.). The finite set of generators designated for $\mathcal{K}$ by f has a finite number of descriptions in terms of those generators and, hence their theory is also recursive (r.e.). ❑

Lemma 1.4.7 will be used to conclude that certain *Ha*'s cannot be recursive from the existence of some non-recursive finitely gripped numbered *Ha* that happens to be embeddable in them. The next corollary clarifies the issue somewhat further.

1.4.8 Corollary If $\langle f, \mathcal{K}\rangle$ and $\langle g, \mathcal{K}\rangle$ are both finitely gripped, then $\langle f, \mathcal{K}\rangle$ is recursive (r.e.) if and only if $\langle g, \mathcal{K}\rangle$ is recursive (r.e.).
Proof: Apply lemma 1.4.7 with $\mathcal{K}$ for $\mathcal{H}$. ❑

1.4.9 Definition Let T be any theory and let f be a function from the propositional variables to the language of T. We write A[f] for the result of substituting the $f(p_i)$ for p_i in A. Define:

- $A \vDash_T B :\Leftrightarrow \forall f(\text{if } T \vdash A[f], \text{ then } T \vdash B[f])$.

We say that the inference from A to B is an *IPC-admissible rule for* T.

Since all admissible rules considered in this paper are IPC-admissible we will suppress the 'IPC'. (Note that we could easily adapt the definition of an admissible rule to Heyting algebras instead of theories.) The IPC-admissible rules are studied in detail by V.V. Rybakov. A good reference is Rybakov[92], where it is shown that the IPC-admissible rules for IPC are decidable. The following fact lays down the exact relationship of numbered *Ha*'s with admissible rules.

1.4.10 Fact $A \vDash_T B$ if and only if for each $\langle f, \mathcal{K}\rangle$ with $\mathcal{K}$ embeddable in $\mathcal{H}_T$, if $\langle f, \mathcal{K}\rangle \vDash A$, then $\langle f, \mathcal{K}\rangle \vDash B$.

Example 1.4.2 is an indication of the fact that many properties of *Ha*'s are not captured by embeddability results. Properties of this kind are not the main subject of our paper, but they merit some attention in passing. In particular, many such properties of the Boolean algebra of classical arithmetical theories can be generalized to the constructive case. An example is the property of density. A proof that Heyting algebras of standard theories are dense is given in an appendix.

2 Embeddings into free Heyting algebras

Every *Ha* on countably many generators is the homomorphic image of $\mathcal{H}_{IPC}$. In other words, it is the *Ha* of some theory in IPC. On the other hand, not every *Ha* on countably many generators can be embedded into $\mathcal{H}_{IPC}$. First of all, $\mathcal{H}_{IPC}$ is *prime*, i.e., $\mathcal{H}_{IPC} \models x \vee y$ implies $\mathcal{H}_{IPC} \models x$ or $\mathcal{H}_{IPC} \models y$, or, in other words, IPC has the *disjunction property.* Clearly, subalgebras inherit primeness. In this section we illustrate that many countable prime *Ha*'s are not embeddable in $\mathcal{H}_{IPC}$, even some that are embeddable in $\mathcal{H}_{HA}$, thus showing up some distinctions between these algebras. We provide some information about the *Ha*'s on finitely many generators that are embeddable in $\mathcal{H}_{IPC}$. However, the problem of giving a neat characterization of the algebras embeddable in $\mathcal{H}_{IPC}$ is still open.

Whenever '$\vdash$' is used without exhibiting a theory we intend it to be read as provability in IPC.

2.1 Example There are many non-trivial admissible rules for IPC. For example:

- $(\neg\neg A \to A) \to A \vee \neg A \,/\, \neg\neg A \vee \neg A$ (De Jongh[82])
- $\neg A \to B \vee C \,/\, (\neg A \to B) \vee (\neg A \to C)$ (Independence of Premise Rule)

This means that every embeddable algebra $\mathcal{H}$ will satisfy:

- $\mathcal{H} \models (\neg\neg x \to x) \to x \vee \neg x$ implies $\mathcal{H} \models \neg\neg x \vee \neg x$
- $\mathcal{H} \models \neg x \to y \vee z$ implies $\mathcal{H} \models (\neg x \to y) \vee (\neg x \to z)$.

2.2 Example We give an *infinitary admissible rule.* Let $F_n(p)$ be an enumeration of the countably many formulas presenting the non-top elements of the Rieger-Nishimura Lattice. (For information about this lattice, see e.g., Troelstra & van Dalen[88a], p49.) We have for arbitrary A and B:

- If, for all n, $\vdash F_n(A) \to B$, then $\vdash B$.

It follows that in an embedded *Ha*, for no x, can there be an element between all of the $F_n(x)$ and the top. (We will illustrate in 5.12 that this rule is not admissible for HA.)

Proof: Suppose, for all n, $\vdash F_n(A) \to B$. Let p be a propositional variable not in A or B. It follows that, for all n, $\vdash F_n(p) \to ((p \leftrightarrow A) \to B)$. By Pitts' theorem (1.2.1) a uniform pre-interpolant $\forall \mathbf{q}((p \leftrightarrow A) \to B)$ of $((p \leftrightarrow A) \to B)$ w.r.t. to the variables in this formula other than p exists. This means that, for any formula D containing no variables of A or B, we have $\vdash D \to ((p \leftrightarrow A) \to B)$ if and only if $\vdash D \to \forall \mathbf{q}((p \leftrightarrow A) \to B)$. It follows that, for every n, $\vdash F_n(p) \to \forall \mathbf{q}((p \leftrightarrow A) \to B)$. By the properties of the Rieger-Nishimura lattice ($\forall \mathbf{q}((p \leftrightarrow A) \to B)$ only contains p !), $\vdash \forall \mathbf{q}((p \leftrightarrow A) \to B)$ and hence $\vdash (p \leftrightarrow A) \to B$. Substituting A for p we find $\vdash B$. ❑

2.3 Theorem Every *Ha* on finitely many generators that is embeddable in $\mathcal{H}_{IPC}$ is finitely presented, i.e. the *Ha* of a finitely axiomatizable IPC theory.

Proof: Suppose the generators of the algebra go to $A_1,...,A_n$. We have:

$\vdash B(A_1,...,A_n)$ if and only if $\vdash (p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n) \rightarrow B(p_1,...,p_n)$.

We suppose that $\{p_1,...,p_n\} \cap VAR(A_i) = \varnothing$ and $VAR(B) \subseteq \{p_1,...,p_n\}$. Now take the Pittsean post-interpolant $\exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n))$ of $(p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n)$ w.r.t. the variables in the A_i. The only variables of $\exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n))$ are the p_i and we have $\vdash B(A_1,...,A_n)$ if and only if $\vdash \exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n)) \rightarrow B$, i.e., $\exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n))$ axiomatizes the propositional theory of $A_1,...,A_n$. ❑

2.4 Corollary Every *Ha* on finitely many generators that is embeddable in $\mathcal{H}_{IPC}$ is recursive.

Proof: This is immediate from the fact shown in the proof of theorem 2.3 that the theory generated by $A_1,...,A_n$ is axiomatized by $\exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n))$ and, as this formula can be found effectively, even decidable in the parameters $A_1,...,A_n$. ❑

As we will see in 5.15, three elements can be found in $\mathcal{H}_{HA}$ that generate a non-recursive subalgebra. A fortiori, this subalgebra is not a finitely presented algebra. Thus, no analogue of 2.3 holds for $\mathcal{H}_{HA}$.

Which formulas C are axioms of *Ha*'s on finitely many generators that are embeddable in $\mathcal{H}_{IPC}$? Such C are called *IPC-exactly provable*. In this paper we will abbreviate *IPC-exactly provable* by *exact*. So, $C(p_1,...,p_n)$ is exact if there are $A_1,...,A_n$ such that, for all $B(p_1,...,p_n)$, $\vdash B(A_1,...,A_n)$ if and only if $\vdash C \rightarrow B$. Clearly, by the above, the exactly provable formulas are precisely those which are provably equivalent to Pitts formulas of the form $\exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n))$ where $\mathbf{q}$ contains all and only the variables occurring in the A_i and where none of the p_j is in $\mathbf{q}$. The notion of exactly provable formula was introduced in De Jongh[82]. More discussion relating the concept of exact provability and the notions presented here can be found in De Jongh-Chagrova[95]. In this paper a specific instance of corollary 2.4 was proved and discussed, namely, the case that $\exists \mathbf{q}((p_1 \leftrightarrow A_1) \wedge ... \wedge (p_n \leftrightarrow A_n))$ is provable in IPC. Sentences $A_1,...,A_n$ with this property are called *independent*. By the remark in the proof of corollary 2.4, it follows that the property of dependency is decidable.

We write IPC+A for the propositional *theory* axiomatized by A over IPC, rather than the *logic* given by A. Thus IPC+A need not be closed under substitution.

2.5 Definition We say that A is *prime* if $\mathcal{H}_{IPC+A}$ is prime, i.e., IPC+A is consistent and IPC+A has the *disjunction property*:

- for all B, C$\in\mathcal{L}$, if $\vdash A\rightarrow B\vee C$, then $\vdash A\rightarrow B$ or $\vdash A\rightarrow C$.

In an alternative formulation, adhering to the convention that the empty disjunction is $\bot$, A is prime if, for every finite set of formulas X, $\vdash A\rightarrow\bigvee X$ implies $\exists B\in X(\vdash A\rightarrow B)$. The notion of primeness extends to sets of formulas; a set Δ of formulas is also called prime if it has the disjunction property.

We give some properties of exact formulas and provide some special classes of such formulas. We have not as yet succeeded in providing a complete characterization of the exact formulas in simple semantical or syntactical terms. For our purposes in section 6.2 we need to show that all prime NNIL-formulas are exact and we will restrict ourselves to that here although our methods go a little further. The *NNIL-formulas* are the formulas with No Nestings of Implications to the Left, but let us define them more precisely. We work in a language where $\neg$ and $\leftrightarrow$ are defined symbols. Let Sub(A) be the set of sub-formulas of A. We have:

- A is in NNIL iff, for all $B\rightarrow C\in$ Sub(A), B does not contain $\rightarrow$.

The class of NNIL-formulas is studied in Visser[85], Renardel[86], Visser[94] and Visser et al[95].

2.6 Lemma If A is exact, then A is prime.
Proof. Just apply the disjunction property of IPC itself. ❑

Remember that the converse of lemma 2.6 doesn't hold. Counterexamples are $(\neg\neg p\rightarrow p)\rightarrow p\vee\neg p$ and $\neg p\rightarrow q\vee r$ (see example 2.1). If *f* witnesses the exactness of A, we will say that A is exact *via f*. Written out, this means the following:

2.7 Fact The formula A is exact via *f* iff for all B with only the propositional variables of A, $\vdash B[f]$ if and only if $\vdash A\rightarrow B$.

2.8 Lemma Suppose that A is exact via *f*, and that B[*f*] is exact via *g*. Then $A\wedge B$ is exact via $f\circ g$.
Proof: We have $\vdash A\wedge B\rightarrow C$ if and only if $\vdash A\rightarrow(B\rightarrow C)$ if and only if (by fact 2.7) $\vdash(B\rightarrow C)[f]$ if and only if $\vdash B[f]\rightarrow C[f]$ if and only if (by fact 2.7) $\vdash C[f][g]$. ❑

2.9 Lemma (i) p is exact via $[p{:=}\top]$, (ii) $p\to A$ is exact via $[p{:=}p\wedge A]$.

Proof: (i) is trivial. We prove (ii). Without loss of generality we may assume that p does not occur in A, since $\vdash(p\to A)\leftrightarrow(p\to A[p{:=}\top])$ and $\vdash p\wedge A\leftrightarrow(p\wedge A[p{:=}\top])$. We have $\vdash(p\to A)\to C$ iff $\vdash(p\leftrightarrow p\wedge A)\to C$ iff $\vdash C[p{:=}p\wedge A]$. ❑

2.10 Definition We say that a formula is *confined* if it is a conjunction of formulas of the form $p\to B$. A formula is *strictly confined* if it is confined and if for any two distinct conjuncts the antecedent variables are different. (We consider $\top$ as the empty conjunction, so $\top$ is strictly confined.)

2.11 Corollary Any confined formula is exact.

Proof: Suppose A is confined. First rewrite A to a strictly confined formula A' by merging different conjuncts $p\to B$ and $p\to C$ to $p\to B\wedge C$. Suppose A' is of the form $(p\to D)\wedge E$. This formula is equivalent to $A'':=(p\to D[p{:=}\top])\wedge E$. According to lemmas 2.8, 2.9, A'' is exact if $A^*{:=}E[p{:=}(p\wedge D[p{:=}\top])]$ is. Clearly A^* is again a strictly confined formula with less conjuncts than A'. Repeat the procedure till all conjuncts are eliminated and we end up with $\top$ which is exact by the identity substitution. ❑

Note that it follows that confined formulas are prime.

2.12 Lemma Suppose p does not occur in A. Then A is prime if $p\wedge A$ is.

Proof: Suppose $p\wedge A$ is prime. Let X be a finite set of formulas and suppose $\vdash A\to\bigvee X$. Without loss of generality we may assume that p does not occur in X. It follows that $\vdash p\wedge A\to\bigvee X$ and, hence, $\vdash p\wedge A\to B$ for some $B\in X$. By substituting $\top$ for p we find $\vdash A\to B$. ❑

2.13 Theorem Every prime NNIL-formula is exact.

Proof: Let A be a NNIL-formula. We will reduce A to a NNIL-formula A_0 (not necessarily equivalent to A in IPC) satisfying:

(a) If A_0 is exact, then A is exact.

(b) A_0 is confined *or* a prime NNIL-formula with strictly less propositional variables than A.

If A_0 is confined, we are done; otherwise we keep repeating the procedure until we do get a confined formula.

Step 1: An occurrence of $\bot$ in a given formula A is *trivial* if it is not equal to A and if it does not occur as conclusion of an implication. An occurrence of $\top$ in a given formula A is *trivial* if it is not equal to A. We eliminate all trivial occurences of $\top$ and of $\bot$. Note that the procedure may end up in $\top$, but, by primeness, not in $\bot$.

Step 2: Write A in disjunctive normal form (treating the implications as atoms). Since A is prime, the disjunction is non-empty and A is equivalent with one of its disjuncts, say A'. A' is a conjunction of atoms and implications. Primeness precludes the occurrence of $\bot$ as a conjunct. So, by step 1, all atomic conjuncts of A' are propositional variables. If the number of atoms is zero, go on to step 3. Otherwise, write A' in the form $p\wedge C$. Clearly $p\wedge C$ is equivalent to $p\wedge(C[p:=\top])$. Put $A_0:=C[p:=\top]$. Note that A_0 is again prime by lemma 2.12, and that A is exact if A_0 is, by lemmas 2.8, 2.9.

Step 3: A' is a conjunction of implications. Reduce, IPC-equivalently, subformulas of the form $B\wedge C\rightarrow D$ to $B\rightarrow(C\rightarrow D)$ and subformulas of the form $B\vee C\rightarrow D$ to $(B\rightarrow D)\wedge(C\rightarrow D)$. Repeat the procedure till no such subformulas are left. Let A_0 be the result. Since A' was in NNIL and since by step 1 we cannot end up with conjuncts of the form $\bot\rightarrow(.)$ or $\top\rightarrow(.)$, clearly A_0 is confined. ❑

3 Some useful facts about IPC and HA

In this section we provide some technical preliminaries to the result of section 5. We suppose the reader is familiar with Kripke models for IPC (see Troelstra & van Dalen[88a], or Smoryński[73]). To fix notations, a *Kripke model* is a structure $\mathbb{K}=\langle K,\leq,\models\rangle$, where K is a non-empty set of nodes, $\leq$ is a partial ordering, $\models$ is the atomic forcing relation, a relation between nodes and propositional atoms satisfying *persistence*, i.e., if $k\leq k'$ and $k\models p$, then $k'\models p$. The relation $\models$ can be extended to the full language of IPC in the standard way. We write $\mathbb{K}\models A$ for $\forall k\in K.\, k\models A$. A *rooted* Kripke model $\mathbb{K}$ is a structure $\langle K,k_0,\leq,\models\rangle$, where $\langle K,\leq,\models\rangle$ is a Kripke model and where $k_0\in K$ is the bottom element w.r.t. $\leq$. For any $k\in K$, $\mathbb{K}[k]$ is the model $\langle K',k,\leq',\models'\rangle$, where $K':=\{k'\mid k\leq k'\}$ and where $\leq'$ and $\models'$ are the restrictions of $\leq$ respectively $\models$ to K'.

The object of the present section is the following. In section 5, we will show that any r.e. prime $\mathcal{H}$ can be embedded in $\mathcal{H}_{HA^*}$, or in other words, that any r.e. prime Δ can be faithfully interpreted in HA^*. The proof needs as input a Δ that is provably prime in HA. To get the result we want it is therefore sufficient to show that any r.e. prime Δ is an HA-provably r.e. prime Δ, or put in a more precise way, that we can get a presentation Δ' of Δ in HA that proves the same IPC-formulas and is provably prime. This is executed in the present section. For the purpose we develop some Kripke model theory and formalize it in HA.

3.1 Definition (The Henkin construction) A set X is *adequate* if it is finite, closed under subformulas, and contains $\bot$. A set Γ is *X-saturated* if:

(i) $\Gamma \subseteq X$

(ii) $\Gamma \nvdash \bot$

(iii) if $\Gamma \vdash A$ and $A \in X$, then $A \in \Gamma$,

(iv) if $\Gamma \vdash B \vee C$ and $B \vee C \in X$, then $B \in \Gamma$ or $C \in \Gamma$.

The *Henkin model* $\mathbb{H}_X$ has the X-saturated sets as its nodes and $\subseteq$ as its accessibily relation. The atomic forcing in the nodes is given by $\Gamma \vDash p$ iff $p \in \Gamma$. We have by a standard argument, for $A \in X$, $\Gamma \vDash A$ iff $A \in \Gamma$.

3.2 Definition

(i) Let $\mathscr{X}$ be a set of Kripke models. Then $M(\mathscr{X})$, the *disjoint union of* $\mathscr{X}$, is the model with nodes $\langle k, \mathbb{K} \rangle$ for $k \in \mathbb{K} \in \mathscr{X}$ and ordering $\langle k, \mathbb{K} \rangle \leq \langle m, \mathbb{M} \rangle :\Leftrightarrow \mathbb{K} = \mathbb{M}$ and $k \leq_{\mathbb{K}} m$. As atomic forcing we take $\langle k, \mathbb{K} \rangle \vDash p :\Leftrightarrow k \vDash_{\mathbb{K}} p$. (In practice we will forget the second components of the new nodes, pretending the domains to be disjoint already.)

(ii) Let $\mathbb{K}$ be a Kripke model. Then $B(\mathbb{K})$ is the rooted model obtained by adding a new bottom b to $\mathbb{K}$ and by taking $b \vDash p :\Leftrightarrow \mathbb{K} \vDash p$. We write $\text{Glue}(\mathscr{X}) := BM(\mathscr{X})$ and e.g. $\text{Glue}(\mathscr{X}, \mathbb{K})$ for $\text{Glue}(\mathscr{X} \cup \{\mathbb{K}\})$.

3.3 Push Down Lemma Let X be adequate. Suppose Δ is X-saturated and $\mathbb{K} \vDash \Delta$. Then $\text{Glue}(\mathbb{H}_X[\Delta], \mathbb{K}) \vDash \Delta$.

Proof: We show, by induction on $A \in X$, that $b \vDash A$ if and only if $A \in \Delta$. The cases of atoms, conjunction and disjunction are trivial. If $B \to C \in X$ and $b \vDash B \to C$, then $\Delta \vDash B \to C$ and hence $B \to C \in \Delta$. Conversely, suppose $B \to C \in \Delta$. If $b \nvDash B$, we are easily done. If $b \vDash B$, then $B \in \Delta$, hence $C \in \Delta$, and, by the Induction Hypothesis, $b \vDash C$. ❑

3.4 Theorem If X is adequate and Δ is X-saturated, then Δ is prime.

Proof: Δ is consistent by definition. Suppose $\Delta \vdash C \vee D$ and $\Delta \nvdash C$, $\Delta \nvdash D$. Suppose, moreover, $\mathbb{K} \vDash \Delta$, $\mathbb{K} \nvDash C$, $\mathbb{M} \vDash \Delta$ and $\mathbb{M} \nvDash D$. Consider $\text{Glue}(\mathbb{H}_X[\Delta], \mathbb{K}, \mathbb{M})$. By the Push Down Lemma we have $b \vDash \Delta$. On the other hand, by persistence, $b \nvDash C$ and $b \nvDash D$, a contradiction. ❑

3.5 Lemma Take the Kripke model with the set of nodes $\langle \Gamma, X \rangle$, where X is adequate and Γ is X-saturated, and $\langle \Gamma, X \rangle \leq \langle \Delta, Y \rangle$ iff $\Gamma \subseteq \Delta$ and $X \subseteq Y$, and $\langle \Gamma, X \rangle \vDash p$ iff $p \in \Gamma$. Then, for all $A \in \mathscr{L}$, $\langle \Gamma, X \rangle \vDash A$ iff $\Gamma \vdash A$. Thus, a big Henkin model is obtained.

Proof: Use theorem 3.4. ❑

We need to formalize these lemmas in HA.

3.6 Lemma HA$\vdash$"IPC-provability is decidable".

Proof: We first formalize Kripke completeness of IPC for finite models in Peano Arithmetic (PA). Noting that the model existence theorem yields a multi-exponential bound E on the size of the Henkin model we formulate the result as follows:
$PA \vdash \forall A((\forall \mathbb{K} \leq E(A).\, \mathbb{K} \models A) \rightarrow IPC \vdash A)$.

The formula proved is Π_2, so we see, by Kreisel's theorem that PA is Π_2-conservative over HA, that $HA \vdash \forall A((\forall \mathbb{K} \leq E(A).\, \mathbb{K} \models A) \rightarrow IPC \vdash A)$. Since the converse is readily verifiable in HA, we find $HA \vdash \forall A((\forall \mathbb{K} \leq E(A).\, \mathbb{K} \models A) \leftrightarrow IPC \vdash A)$. ❑

In intuitionistic theories even subsets of the singleton set are not decidable. We take however the finite sets that we are using, e.g. in the construction of the Henkin model, to be *coded as numbers* and hence provably finite and decidable.

3.7 Lemma HA proves "it is decidable whether a finite set is X-saturated", it proves statements 3.3-3.5 (and, hence, it proves e.g. the disjunction property for IPC).

Proof: Straightforward using the convention about finite sets and lemma 3.6, the latter e.g. to show the reductio reasoning in theorem 3.4 to be harmless. ❑

We assume infinite r.e. sets of formulas Δ to be given by a recursive increasing sequence Δ_i of finite approximations Δ_i with $\Delta_0 = \varnothing$. Such a sequence is presented by a (provably functional) Δ_1-formula $\delta(i,x)$ in such a way that $A \in \Delta_i$ iff $a \in b$ (where a codes A) for the (unique) b such that $\delta(i,b)$. We also assume the sequence to be provably increasing: $HA \vdash \forall i,j,x,y,z(i<j \wedge x \in y \wedge \delta(i,y) \wedge \delta(j,z) \rightarrow x \in z)$. We will reason informally and will have no need to refer to the formulas $\delta(i,x)$ explicitly. We fix an increasing sequence U_i of adequate sets such that for every A we can effectively find an i such that $Sub(A) \subseteq U_i$.

3.8 Theorem Let Δ be a prime, r.e. set of IPC-formulas, closed under IPC-consequence, represented by a sequence Δ_i. Then we can find a presentation Δ'_i of Δ such that HA proves that $\bigcup \Delta'_i$ is prime.

Proof: We reason informally in HA. Assume Δ to be a prime, r.e. set of IPC-formulas, closed under IPC-consequence (for short: "Δ is prime, etc.") represented by Δ_i. We construct two sequences Δ'_i and X_i in parallel where:

(i) $\Delta = \bigcup \Delta'_i$,

(ii) Δ'_i and X_i are functional and increasing,

(iii) X_i is adequate,

(iv) Δ_i' is X_i-saturated.

Before proving the theorem, we prove two claims. Define:

- $\mathrm{Sat}(n,m,k) :\Leftrightarrow$ for all $B \vee C \in U_n$ (if $\Delta_m \vdash B \vee C$, then $\Delta_k \vdash B$ or $\Delta_k \vdash C$).

Note that, by our lemmas, HA proves that Sat is decidable.

Claim I: HA $\vdash$ "Δ is prime, etc." $\rightarrow \forall n,m \exists k\, \mathrm{Sat}(n,m,k)$.

Claim II: HA $\vdash$ "Δ is prime, etc." $\rightarrow \forall n,m \exists k (k \geq m \wedge \mathrm{Sat}(n,k,k))$.

Proof of claim I: Since U_n is finite, we can exhaustively enumerate the U_n-disjunctions $E \vee F$ proved by Δ_m. Since Δ is prime, we can find for any such $E \vee F$ an i such that $\Delta_i \vdash E$ or $\Delta_i \vdash F$. By the collection principle (which is provable in HA), we can find an upper bound k to these i's.

Proof of claim II: Define F(p) as the smallest $q \geq p$ such that $\mathrm{Sat}(n,p,q)$. By claim I, F is recursive. Let $N := |U_n|$. Consider the sequence $\langle m, F(m), \ldots, F^{N+2}(m) \rangle$. If this sequence were strictly increasing, there would be $N+1$ different disjunctions in U_n. This is impossible by the Pigeon Hole Principle for recursive injections and decidable finite sets, which is verifiable in HA. Since everything in sight is decidable, we may conclude that there is a k with $0 \leq k < N+2$ and $\mathrm{Sat}(n,k,k)$.

We define weakly monotonic functions $f, g : \omega \rightarrow \omega$ and take $X_i := U_{f(i)}$ and $\Delta_i' := \{B \in X_i \mid \Delta_{g(i)} \vdash B\}$.

- $f(0) := 0$, $g(0) := 0$
- Consider $U_{f(n)+1}$. In case $\{B \in U_{f(n)+1} \mid \Delta_{n+1} \vdash B\}$ is $U_{f(n)+1}$-saturated (i.o.w. if $\mathrm{Sat}(f(n)+1, n+1, n+1)$), put $f(n+1) := f(n)+1$, $g(n+1) := n+1$.
  Otherwise $f(n+1) := f(n)$, $g(n+1) := g(n)$.

The functions f and g are recursive, since, by lemma 3.6, IPC-provability is decidable. Moreover, it is clear that (ii)-(iv) are satisfied. Therefore, by the formalization of theorem 3.4, every Δ_i' is prime (in the case $i=0$, this uses the fact that IPC is prime), and hence $\bigcup \Delta_i'$ is prime.

We will show that both f and g tend to infinity, and hence (i), $\Delta = \bigcup \Delta_i'$. Consider any n. Let k be the smallest number such that $\mathrm{Sat}(f(n)+1,k,k)$ and $k \geq n+1$. Then, evidently, the first clause of the recursion step of the definitions of f and g will be activated at k. So, for every n, there is a $k > n$ such that f and g increase at k. By a simple induction it follows that f and g tend to infinity.

Surveying the proof, one sees that, actually, "Δ is prime,etc." $\rightarrow \Delta = \bigcup \Delta_i'$ has been proved in HA. ❑

4 What is HA*?

In this section we describe the theory HA*. This theory was introduced in Visser[82]. The natural way to define HA* is by a fixed point construction as HA plus the *Completeness Principle for HA**. (Here it is essential that the construction is verifiable in HA, see below.) The Completeness Principle can be viewed as an arithmetically interpreted modal principle. The Completeness Principle viewed modally is:

C $\quad \vdash A \to \Box A$

The Completeness Principle for a specific theory T is:

C[T] $\vdash A \to \Box_T A$.

Here $\Box_T$ stands for the formalization of provability in T. In the statement of the principle the syntactical variable 'A' ranges over formulas. Free occurrences of variables inside the box are interpreted according to the convention that $\Box_T A$ means $Prov_T(t(\mathbf{x}))$, where $t(\mathbf{x})$ is the term 'the Gödelnumber of the result of substituting the Gödelnumbers of the numerals of the **x**'s for the variables in A'.

We have $HA^* = HA + C[HA^*]$. Many results on HA* and using HA* have been obtained by Visser[82]. We just state the one fact that we will use.

4.1 Fact Let $\mathcal{A}$ be the smallest class closed under atoms and all connectives,and quantifiers except implication, and satisfying that, if $A \in \Sigma_1$ and $B \in \mathcal{A}$, then $A \to B \in \mathcal{A}$. Note that, modulo provable equivalence in HA, all formulas of the classical arithmetical hierarchy (in standard form) are in $\mathcal{A}$. Then HA* is conservative w.r.t. $\mathcal{A}$ over HA.

Consider the *Löb conditions.*

L1 If $\vdash A$, then $\vdash \Box A$

L2 $\vdash \Box(A \to B) \to (\Box A \to \Box B)$

L3 $\vdash \Box A \to \Box\Box A$

L4 $\vdash \Box(\Box A \to A) \to \Box A$

The logic i-K is given by IPC+L1, L2; i-L is i-K+L3, L4. We write i-K{P} for the extension of i-K with some principle P. Note that i-L{C} is valid for provability interpretations in HA*.

A principle closely connected to C is the *Strong Löb Principle*:

SL $\vdash (\Box A \to A) \to A$

We allow free variables in SL. Note that, as a special case, we have $\vdash \neg\neg \Box \bot$.

4.2 Lemma The logics i-L{C}and i-K{SL} are the same.

Proof: L4 is immediate from SL. To prove C from SL, first note that in i-K, $\vdash A \to (\Box(A \wedge \Box A) \to A \wedge \Box A)$. Applying SL to the succedent one obtains that $\vdash A \to A \wedge \Box A$ and, hence, $\vdash A \to \Box A$.

To prove SL from C, apply L4 to $\vdash (\Box A \to A) \to \Box(\Box A \to A)$ to obtain $\vdash (\Box A \to A) \to \Box A$. Since also $\vdash (\Box A \to A) \to (\Box A \to A)$, $\vdash (\Box A \to A) \to A$ follows. ❑

5 A Shavrukov Style Embedding result for HA^*

Shavrukov proved that every r.e. Magari algebra satisfying an appropriate Disjunction Property is embeddable in the Magari algebra of Peano Arithmetic. Is an analogous result possible for HA and prime Heyting algebras? The answer must clearly be negative, since HA is closed under non-trivial admissible rules. This closure is inherited by $\mathcal{H}_{HA}$ and its Heyting subalgebras. However, in this section we show that an analogue of Shavrukov's theorem can be obtained for the theory HA^*, and even in such a way that the interpretation of all propositional formulas is Σ_1. We give a number of applications, first to $\mathcal{H}_{HA^*}$ of course, but in a surprising manner applications to $\mathcal{H}_{HA}$ can be given too. Both $\mathcal{H}_{HA^*}$ and $\mathcal{H}_{HA}$ are proved to be non-recursive. Before proving the theorem we briefly look at an illustrative example to give the reader some feeling of how it is possible that an embedded algebra can exclusively consist of equivalence classes of Σ_1-sentences.

5.1 Example Consider the algebra $\mathcal{H}$, IPC-axiomatized by $\neg\neg p \to p$. To be precise, $\mathcal{H} = \mathcal{H}_{IPC+(\neg\neg p \to p)}(\mathcal{L}_1)$. We have that $\mathcal{H}$ can be embedded into $\mathcal{H}_{IPC}$, by e.g. $[p := \neg p]$ and that $\mathcal{H}$ can be embedded into $\mathcal{H}_{HA}$, by e.g. $[p := \neg\Box_{HA}\bot]$. On the other hand, $\mathcal{H}$ cannot be embedded into $\mathcal{H}_{HA}$ by sending p to a Σ_1-sentence, since for any Σ_1-sentence B, we have that $HA \vdash \neg\neg B \to B$ implies that $HA \vdash B$ or $HA \vdash \neg B$.

(For a proof of this: Let $HA \vdash \neg\neg B \to B$ with $B := \exists x Cx$, C in Δ_0. By the independence of premise rule for $\exists$ in HA, $HA \vdash \exists x (\neg\neg B \to Cx)$, and by the existence property for HA, $HA \vdash \neg\neg B \to Cn$ for some n. Depending on the truth of Cn, either $HA \vdash Cn$, or $HA \vdash \neg Cn$. In the first case, $HA \vdash \exists x Cx$ and hence, $HA \vdash B$. In the second case, $HA \vdash \neg B$.)

We turn to HA^*. Let R be the ordinary Σ_1 Rosser sentence for HA^*, and S its dual, i.e.,

$$HA^* \vdash R \leftrightarrow \Box_{HA^*}\neg R \leq \Box_{HA^*} R$$

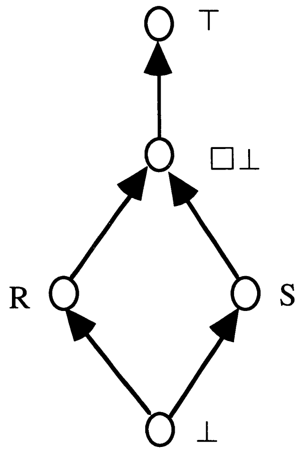
$$HA^* \vdash S \leftrightarrow \Box_{HA^*} R < \Box_{HA^*}\neg R.$$

Here $\leq$ and $<$ are the *witness comparison relations*, which are defined between formulas having an outer existential quantifier in the following manner:

- $\exists x\, Dx \leq \exists y\, Ey) := \exists x (Dx \wedge \forall y < x \neg Ey)$,
- $\exists x\, Dx < \exists y\, Ey) := \exists x (Dx \wedge \forall y \leq x \neg Ey)$.

We have by the ordinary Rosser property that $HA^* \nvdash R$ and $HA^* \nvdash S$. On the other hand, we have that $HA^* \vdash \neg R \leftrightarrow S$ and $HA^* \vdash \neg S \leftrightarrow R$.

We prove the first equivalence. "$\leftarrow$" Trivially $HA^* \vdash S \rightarrow \neg R$. "$\rightarrow$" Reason in HA^*. Suppose $\neg R$. To prove S, by SL we may assume $\Box_{HA^*} S$. From the latter it follows that $\Box_{HA^*} \neg R$ and hence that $R \vee S$. Combining $R \vee S$ with $\neg R$ we get S.



Using the above facts it is easy to see that the subalgebra of $\mathcal{H}_{HA^*}$ generated by R is given by the non-equivalent Σ_1-sentences: $\bot$, R, S, $\Box_{HA^*}\bot$, $\top$. This algebra is clearly isomorphic to $\mathcal{H}$.

The surprising fact that the particular Σ_1-sentences occurring in the interpretations in this example are closed under Boolean operations, i.e. under negation, will reappear in the next theorem. We will show however in lemma 5.9 that it does not generalize to all Σ_1-sentences. The proof of Theorem 5.2 is a Solovay like proof (stemming from Solovay[76]). It combines the proof strategy from Zambella[94] with an idea from Visser[85] (on how to handle implication using the SL). In fact, it follows Zambella's quite closely modulo some inessential stylistic differences (like our use of a kind of Henkin model).

5.2 Theorem Every r.e. prime *Ha* $\mathcal{H}$ can be embedded in $\mathcal{H}_{HA^*}$. Moreover, each of the equivalence classes in the range of the embedding contains a Σ_1-sentence.

Proof: Let $\Box$ stand for $\Box_{HA^*}$ and Proof for Proof_{HA^*}. Consider the following Kripke model $\mathbb{H}$, which is a variant of the big model of lemma 3.5. Its nodes are of the form $\langle i, U, V\rangle$, where $i \in \{0,1\}$, V is an adequate set of formulas, $U \subseteq V$ and U is V-saturated.

Define $\preccurlyeq$ and $\vDash$ as follows:

$\langle i,U,V\rangle \preccurlyeq \langle j,W,T\rangle :\Leftrightarrow i \leq j,\ U \subseteq W,\ V \subseteq T$ and, if $i=1$ then $V=T$,

$\langle i,U,V\rangle \vDash p :\Leftrightarrow p \in U$.

Using lemmas 3.3 and 3.5, it is easy to see (in HA) that, for any formula A, $\langle 0,U,V\rangle \vDash A$ iff $U \vdash A$, and for $A \in V$, $\langle 1,U,V\rangle \vDash A$ iff $U \vdash A$. It follows that the relation $k \vDash A$ is decidable.

Let Δ_i with X_i enumerate a propositional theory presenting $\mathcal{H}$, satisfying the properties promised in theorem 3.8 for $\Delta_i^!$, X_i. We define a Solovay function h from ω to the nodes of $\mathbb{H}$. A function s with $s(x)$ called the *state* of h at x, is defined as $(hx)_0$ with $s(0)$ set at 0. Until a certain Catastrophic Event happens, the state will remain 0 and h will run upwards through nodes $\langle 0,\Delta_i,X_i\rangle$. As soon as (and *if*) the Event happens, the state will definitively move to 1 and our function runs upwards through nodes of the form $\langle 1,U,V\rangle$. Define by the Recursion Theorem h as follows:

- $[A] :\leftrightarrow \exists x\, h(x) \vDash A$
- $h(0) := \langle 0,\Delta_0,X_0\rangle$
- $h(n+1) := k$ if

  Proof(n, [A]), $h(n) \nvDash A$, k is a 1-node, $h(n) \preccurlyeq k$, k maximal such that $k \nvDash A$ $\quad (*)$
- $h(n+1) := \langle 0,\Delta_{n+1},X_{n+1}\rangle$ if case $(*)$ does not obtain and $s(n)=0$
- $h(n+1) := h(n)$ if case $(*)$ does not obtain and $s(n)=1$.

Since $\vDash$ is (provably in HA) decidable, it follows that h is a well defined recursive function.

Note that the Catastrophic Event is the first time that $(*)$ obtains. Before the Event the function enumerates nodes representing better and better approximations of $\mathcal{H}$. After the event it behaves like an ordinary Solovay function traveling upwards through a converse well-founded (w.r.t. $<$) part of the model. We can now prove the usual lemmas of a Solovay like proof.

5.3 Lemma

$HA \vdash x \leq y \rightarrow h(x) \preccurlyeq h(y)$

$HA \vdash x \leq y \wedge h(x) \vDash A \rightarrow h(y) \vDash A$

Proof: Obvious. ❑

5.4 Lemma $HA \vdash s(x)=1 \rightarrow \Box \exists y\, h(x) < h(y)$.

Proof: Reason in HA. Suppose $s(x)=1$. The function h must have arrived at $h(x)$ by case $(*)$. So, for some A and $p<x$, $Proof_{HA^*}(p,[A])$, $h(p+1)=h(x) \nvDash A$. By Σ-complete-

ness we have $\Box h(x) \nvDash A$. Combining this with $\Box\exists y(h(y) \vDash A)$, we obtain, using lemma 5.3, the desired result. ❑

5.5 Lemma $s(n)=0$ for any n.

Proof: Suppose $s(n)\neq 0$. By lemma 5.4, $\Box\exists y(h(n)<h(y))$. Remember that HA^* is Π_2-conservative over HA. Thus, HA^* will certainly satisfy Σ-reflection. It follows that, for some m, $h(n)<h(m)$. Repeating the argument, we can construct an infinite strictly ascending chain above $h(n)$. This contradicts $s(n)\neq 0$. ❑

5.6 Lemma [.] commutes, modulo HA^*-provability, with the propositional connectives.

Proof: Reason in HA^*. Clearly $[\bot]\leftrightarrow\bot$ and $[\top]\leftrightarrow\top$.
Suppose $[A\wedge B]$. Then for some x, $h(x)\vDash A\wedge B$. It follows that $h(x)\vDash A$ and $h(x)\vDash B$, and hence, $[A]\wedge[B]$. Conversely, suppose $[A]\wedge[B]$. Say $h(y)\vDash A$ and $h(z)\vDash B$. Let u be $\max(y,z)$. Then, by lemma 5.3, $h(u)\vDash A$ and $h(u)\vDash B$ and thus $h(u)\vDash A\wedge B$. We may conclude that $[A\wedge B]$.
Suppose $[A\vee B]$. Then for some x, $h(x)\vDash A\vee B$. It follows that $h(x)\vDash A$ or $h(x)\vDash B$, and hence, $[A]\vee[B]$. Conversely, suppose $[A]\vee[B]$. Suppose e.g., $h(y)\vDash A$. It is immediate that also $h(y)\vDash A\vee B$, and so, $[A\vee B]$. Similarly, in case $h(z)\vDash B$.
Suppose $[A\rightarrow B]$ and $[A]$. Then, for some x and y, $h(x)\vDash A\rightarrow B$ and $h(y)\vDash A$. Take u to bemax(x,y). Clearly, $h(u)\vDash A\rightarrow B$ and $h(u)\vDash A$. Ergo, $h(u)\vDash B$, and thus [B].
Conversely, suppose $[A]\rightarrow[B]$. We show $[A\rightarrow B]$ using SL. So, we may also assume $\Box[A\rightarrow B]$. Suppose Proof(p,$[A\rightarrow B]$). In case $h(p)\vDash A\rightarrow B$, we have $[A\rightarrow B]$. Suppose $h(p)\nvDash A\rightarrow B$. In this case $h(p+1)$ is a maximal $k\geqslant h(p)$ such that $k\nvDash A\rightarrow B$. It follows that $k\vDash A$ and $k\nvDash B$. From $h(p+1)=k\vDash A$, we have [A], and hence, by assumption, [B]. But [B] immediately implies $[A\rightarrow B]$. So, in both cases we find $[A\rightarrow B]$. ❑

We finish the proof of theorem 5.2 by showing that $A\in\Delta$ if and only if $HA^*\vdash[A]$.

Suppose $A\in\Delta$. Then, for some n, $A\in\Delta_n$. By lemma 5.5, $s(n)=0$, and hence, $h(n)=\langle 0,\Delta_n,X_n\rangle$. Ergo, $h(n)\vDash A$, and so, $HA^*\vdash(h(n)\vDash A)$, and thus, $HA^*\vdash[A]$.

Conversely, suppose $HA^*\vdash[A]$. Say, m codes a proof of [A]. Suppose $\Delta_m\nvdash A$. Since $h(m)=\langle 0,\Delta_m,X_m\rangle$, it follows that $h(m)\nvDash A$. So, clause $(*)$ would become active and the Catastrophic Event would take place. But lemma 5.5 tells us this cannot happen at a standard stage. ❑

An attractive alternative formulation of the proof of theorem 5.2 is to take, on the one hand, as nodes of the Henkin model the more traditional pairs $\langle U,V\rangle$, but to work, on the other hand, with two accessibility relations:

- $\langle U,V\rangle \preccurlyeq_0 \langle W,T\rangle \quad :\Leftrightarrow \quad U\subseteq W$ and $V\subseteq T$
- $\langle U,V\rangle \preccurlyeq_1 \langle W,T\rangle \quad :\Leftrightarrow \quad U\subseteq W$ and $V=T$

Corresponding to these different accessibility relations we have forcing relations $\models_0$ and $\models_1$. We define a suitably adapted Solovay function simultaneously with an auxiliary state function. Which accessibility relation and which forcing relation is relevant, will depend on the state. We leave it to the reader to work out more details.

Using the notation of 1.4.8, we have that the admissible rules for HA* are precisely the trivial ones.

5.7 Corollary $A\models_{HA^*}B$ if and only if $IPC\vdash A\to B$.

Proof: '$\Leftarrow$' is trivial. '$\Rightarrow$' Suppose $IPC\nvdash A\to B$. Then there is a finite rooted Kripke model $\mathbb{K}$ such that $\mathbb{K}\models A$ and $\mathbb{K}\not\models B$. Let $\mathcal{H}$ be the *Ha* of the upwards closed sets of $\mathbb{K}$. Obviously, $\mathcal{H}$ is finite and hence recursively enumerable. Embedding $\mathcal{H}$ into $\mathcal{H}_{HA^*}$ gives us an interpretation *f* such that $HA^*\vdash A[f]$ and $HA^*\nvdash B[f]$. ❑

Before giving more applications we first have two comments on the proof. The first comment is that the proof cannot be extended in any obvious way to give a completeness theorem for the provability logic of HA^*, since nodes of our Henkin model where we have $\Box\bot$ (i.e. the end nodes) also satisfy the Excluded Third. But HA^* does not prove the Excluded Third from $\Box_{HA^*}\bot$. We sketch a proof of this fact. First we prove a lemma that we will need again in our second comment. It is a simple adaptation of Kripke's result on flexible sentences to the constructive case.

5.8 Lemma Let T be any consistent extension of HA. Then there is a Σ_1-sentence Ω such that, for no Σ_1-sentence S, $T\vdash\neg(\Omega\leftrightarrow S)$.

Proof: Let T be a consistent extension of HA. Take Ω such that:

$$HA\vdash\Omega\leftrightarrow True_{\Sigma}(\epsilon S.\Box_T\neg(\Omega\leftrightarrow S)).$$

Here $True_{\Sigma}$ is the usual truth predicate for Σ_1-sentences and $\epsilon S.\Box_T\neg(\Omega\leftrightarrow S)$ is the first S such that $\Box_T\neg(\Omega\leftrightarrow S)$ that we find if we run through the T-proofs. Clearly, $\Omega\in\Sigma_1$. Suppose for some $S'\in\Sigma_1$, $T\vdash\neg(\Omega\leftrightarrow S')$. Let S" be the first such S' that we encounter, when running through the T-proofs. We have $HA\vdash S''=\epsilon S'.\Box_T\neg(\Omega\leftrightarrow S')$, and hence, $HA\vdash\Omega\leftrightarrow True_{\Sigma}(S'')$. We may conclude that $HA\vdash\Omega\leftrightarrow S''$. But, since $HA\subseteq T$, this implies that T is inconsistent. Quod non. ❑

5.9 Theorem For the Σ_1-sentence Ω of lemma 5.7, $HA^* \nvdash \Box_{HA^*}\bot \to \Omega \vee \neg\Omega$.

Proof: Take for T, in lemma 5.8, $HA + \Box_{HA}\bot$ and find the corresponding Ω. Note that, by fact 4.2, $HA \vdash \Box_{HA^*}\bot \leftrightarrow \Box_{HA}\bot$ and that $HA \vdash \Box_{HA}\bot \to \Omega \vee \neg\Omega$ if and only if $HA^* \vdash \Box_{HA}\bot \to \Omega \vee \neg\Omega$, since $\Box_{HA}\bot \to \Omega \vee \neg\Omega \in \mathcal{A}$. So, it is sufficient to show that $HA \nvdash \Box_{HA}\bot \to \Omega \vee \neg\Omega$. Suppose, to get a contradiction, that $HA \vdash \Box_{HA}\bot \to \Omega \vee \neg\Omega$. So HA proves, for some index e, that it q-realizes this formula, $HA \vdash e\,\mathbf{q}\,(\Box_{HA}\bot \to \Omega \vee \neg\Omega)$. Also HA proves, by the fact that $\Box_{HA}\bot$ is in Σ_1, that $\Box_{HA}\bot$ is is equivalent to its own r-realizability and q-realizability and that there is a specific (Kleene-bracket) term $\psi_{\Box_{HA}\bot}$ that realizes it, if it is realizable at all (see Troelstra & van Dalen[88a] or Troelstra[92]):

$$HA \vdash \Box_{HA}\bot \to \psi_{\Box_{HA}\bot}\,\mathbf{q}\,\Box_{HA}\bot.$$

This implies:

$$HA + \Box_{HA}\bot \vdash \{e\}(\psi_{\Box_{HA}\bot})\downarrow \wedge \{e\}(\psi_{\Box_{HA}\bot})\,\mathbf{q}\,(\Omega \vee \neg\Omega),$$

$$HA + \Box_{HA}\bot \vdash ((\{e\}(\psi_{\Box_{HA}\bot}))_0 = 0 \to \Omega) \wedge ((\{e\}(\psi_{\Box_{HA}\bot}))_0 \neq 0 \to \neg\Omega),$$

$$HA + \Box_{HA}\bot \vdash \neg\Omega \leftrightarrow (\{e\}(\psi_{\Box_{HA}\bot}))_0 \neq 0.$$

Ergo, $HA + \Box_{HA}\bot \vdash \neg(\Omega \leftrightarrow (\{e\}(\psi_{\Box_{HA}\bot}))_0 \neq 0)$. Hence, since $(\{e\}(\psi_{\Box_{HA}\bot}))_0 \neq 0$ is Σ_1, lemma 5.8 leads to $HA + \Box_{HA}\bot \vdash \bot$, but this contradicts the second incompleteness theorem. ❑

The [A]'s in the proof of theorem 5.2 are Σ_1. So, our embedding is into the Σ_1-formulas modulo HA^*-provable equivalence. The surprising property of the [A]'s is that they are closed under negation and implication (modulo HA^*-provable equivalence). Our second comment is that it is not true in general that the Σ-sentences of HA^* are closed under negation.

5.10 Corollary The Σ-sentences of HA^* are not closed under negation.

Proof: This is immediate from lemma 5.7. For, take $T := HA^*$. Then, for no S, we have $HA^* \vdash \neg\Omega \leftrightarrow S$, since $HA^* \vdash \neg\Omega \leftrightarrow S$ implies $HA^* \vdash \neg(\Omega \leftrightarrow S)$. ❑

For other applications it is necessary to find a finitely axiomatizable extension of HA to replace HA^*. Looking back one finds that the only place in the proof of theorem 5.2 where HA^* is used in an essential way is the application of SL to handle the case of implication in the proof of lemma 5.5. These applications all have the form $(\Box_{HA^*}S \to S) \to S$ for $S \in \Sigma_1$. Let Tr_Σ be the Σ_1-truth predicate. Clearly, all applications of SL that we need follow from the single sentence:

$SL_0 \quad \forall x \in \Sigma((\Box_{HA^*}Tr_\Sigma(x) \to Tr_\Sigma(x)) \to Tr_\Sigma(x)).$

In its turn, SL_0 follows from SL, since SL is a scheme in which we allow free variables. A pleasant lazy notation for SL_0 is $\forall S((\Box_{HA^*}S \to S) \to S)$ where the variable 'S' ranges over Σ_1-sentences. Since HA proves Π_2-conservativity of HA^* over HA (fact 4.2), SL_0 is HA-provably equivalent to $\forall S((\Box_{HA}S \to S) \to S)$. The complexity of SL_0 is $\forall((\Sigma_1 \to \Sigma_1) \to \Sigma_1)$, which is a subclass of both $\forall(\Pi_2 \to \Sigma_1)$ and $\forall B\Sigma_1$. By the preceding considerations we find:

5.11 Corollary Every r.e. and prime *Ha* can be embedded in the *Ha* of $HA+SL_0$.

Corollary 5.11 is the source of the applications of theorem 5.2 to HA.

5.12 Corollary The infinitary derived rule of 2.2 is not admissible for HA.
Proof: Consider the *Ha* $\mathcal{H}$ obtained by adding a new top to the Rieger-Nishimura Lattice (see also 2.2). This algebra $\mathcal{H}$ is finitely generated. The generators are, say, the generator p of the Rieger-Nishimura lattice and the old top q. It is not difficult to see that $\mathcal{H}$ is recursive. We can embed $\mathcal{H}$ in $\mathcal{H}_{HA+SL_0}$ by assigning, say, A to p and B to q. It is easy to see that the element $SL_0 \to B$ is situated between the $F_n(A)$ and $\top$ in $\mathcal{H}_{HA}$. This is enough for our result, but in fact it is not difficult to see that the subalgebra of $\mathcal{H}_{HA}$ generated by A and $SL_0 \to B$ is precisely $\mathcal{H}$. ❑

To get the other applications we have to get an old result into the right form.

5.13 Theorem There is an r.e., non-recursive, prime *Ha* on two generators.
Proof: It is sufficient to produce an infinite decidable set X of IPC-formulas in p, q such that:

- for every finite $X_0 \subseteq X$ and every $A \in X \setminus X_0$, $X_0 \nvdash A$,
- every finite $X_0 \subseteq X$ has the disjunction property.

The desired algebra is obtained by taking an r.e. and non-recursive subset Y of X as axiomatization.

In de Jongh[80] infinite sequences are produced of finite rooted Kripke models $\mathbb{L}_i$, of formulas A_i (ψ_i^* in de Jongh[80], p107) and of formulas B_i (ψ_i in de Jongh[80], p107) such that:

- $\mathbb{L}_i \models A_j \Leftrightarrow i \neq j$
- $\mathbb{L}_i \models B_j \Leftrightarrow i = j$
- A_i is of the form $B_i \to C_i$ for some C_i
- It is decidable whether a formula is of the form A_i

(This result is originally due to Jankov, see Jankov[68].)

We take X to be the set of A_i. Consider a finite $X_0 \subseteq X$ and $A \in X \backslash X_0$. Suppose $A = A_i$. Then clearly $\mathbb{L}_i \vDash X_0$ and $\mathbb{L}_i \nvDash A_i$. Hence $X_0 \nvdash A_i$.

To prove the disjunction property, consider any finite $X_0 \subseteq X$. Suppose $X_0 \vdash E \vee F$, but $X_0 \nvdash E$ and $X_0 \nvdash F$. Let $\mathbb{K} \vDash X_0$ and $\mathbb{K} \nvDash E$ and $\mathbb{M} \vDash X_0$ and $\mathbb{M} \nvDash F$. Let j be such that A_j is not in X_0. We have, $\mathbb{L}_j \vDash X_0$ and $\mathbb{L}_j \nvDash B_i$, for A_i in X_0. Consider Glue($\mathbb{K}, \mathbb{M}, \mathbb{L}_j$). Clearly, $b \nvDash E$ and $b \nvDash F$. Consider any $A_i \in X_0$. Note that $b \nvDash B_i$, since $\mathbb{L}_j \nvDash B_i$. Since A_i is of the form $B_i \rightarrow C$ and $\mathbb{K}$, $\mathbb{M}$ and $\mathbb{L}_j$ all force A_i, it follows that $b \vDash A_i$. We may conclude that $b \vDash X_0$, but $b \nvDash E$ and $b \nvDash F$, a contradiction. ❑

We draw some obvious conclusions from the existence of this *Ha*.

5.14 Corollary There exist Σ_1-sentences A and B such that the subalgebra of $\mathcal{H}_{HA*}$ generated by A and B is non-recursive.

In contrast, every finitely generated *Ha* embeddable in $\mathcal{H}_{IPC}$ is decidable by corollary 2.4, and, in consequence, as we will see in theorem 6.5, every finitely generated *Ha* embeddable in $\mathcal{S}_{HA}$ as well.

5.15 Theorem There are Σ_1-sentences A and B and a $\forall((\Sigma_1 \rightarrow \Sigma_1) \rightarrow \Sigma_1)$-sentence C such that the subalgebra of $\mathcal{H}_{HA}$ generated by A, B and C is non-recursive. By 1.4.7 it follows that $\mathcal{H}_{HA}$ is non-recursive.

Proof: Let A and B be as in corollary 5.14, and take $C := SL_0$. Let the algebra generated by A and B in $\mathcal{H}_{HA*}$ be $\mathcal{F}$, and let the algebra generated by A, B and C in $\mathcal{H}_{HA}$ be $\mathcal{G}$. For all propositional D(p, q) we have that $\mathcal{F} \vDash D(A,B)$ if and only if $\mathcal{G} \vDash C \rightarrow D(A,B)$. So, if $\mathcal{G}$ were recursive, $\mathcal{F}$ would be. Quod non. ❑

5.16 Open Questions

(i) Are there sentences A and B, such that the subalgebra of $\mathcal{H}_{HA}$ generated by A and B is non-recursive?

(ii) Are there Σ_1-sentences A and B and a sentence C of complexity less than $\forall((\Sigma_1 \rightarrow \Sigma_1) \rightarrow \Sigma_1)$ (e.g. Π_2), such that the subalgebra of $\mathcal{H}_{HA}$ generated by A, B and C is non-recursive?

(iii) Is there a *finitely presented Ha* on finitely many generators, which is a subalgebra of $\mathcal{H}_{HA}$ and not of $\mathcal{H}_{IPC}$? (In other words, is there a formula which is exact for HA, but not for IPC.)

In the next section we prove that the A and B in question (i) and the C in question (ii) cannot be Σ_1. The same holds for the generators in question (iii).

6 Results concerning Σ_1-sentences in HA

In section 5, we completely solved the question which algebras are embeddable in $\mathcal{H}_{HA^*}$. In final instance, we are, of course, more interested in embeddability in $\mathcal{H}_{HA}$. We have not solved the embeddability question for that algebra. A partial result is given in this section, where we restrict our attention to Brouwerian combinations of Σ_1-sentences and study $\mathcal{S}_{HA} := \mathcal{H}_{HA}(B\Sigma_1)$. We show that $\mathcal{H}_{IPC}$ can be embedded in $\mathcal{S}_{HA}$, and that the same *Ha*'s on finitely many generators are embeddable in $\mathcal{S}_{HA}$ and in $\mathcal{H}_{IPC}$.

6.1 Theorem $\mathcal{H}_{IPC}$ is embeddable in $\mathcal{S}_{HA}$.
Proof: Let [.] denote the embedding in theorem 5.2 of $\mathcal{H}_{IPC}$ into $\mathcal{H}_{HA^*}$ that sends propositional letters to Σ_1-sentences. Let *f* be given by *f*(p) := [p]. We show that IPC ⊢ A if and only if HA ⊢ A[*f*]. From left to right is, of course, obvious. So, let us asume HA ⊢ A[*f*]. This implies HA* ⊢ A[*f*], which is nothing but HA* ⊢ [A]. By theorem 5.2, IPC ⊢ A. ❑

Note that theorem 6.1 is the uniform version of De Jongh's Completeness Theorem for IPC w.r.t. interpretations in HA, using theorem 5.2 to restrict the interpretation of the propositional variables to Σ_1-sentences. We are now able to prove that embeddability in $\mathcal{S}_{HA}$ and in $\mathcal{H}_{IPC}$ comes down to the same thing by borrowing the following three facts from Visser[85] (alternatively, see Visser[94], or, for 6.2 and 6.4, Visser et al[94]).

6.2 Fact For each IPC-formula A there is a formula A^* in NNIL such that:
(i) All propositional variables of A^* occur in A,
(ii) For all B ∈ NNIL, IPC ⊢ B → A if and only if IPC ⊢ B → A^*.

Note that 6.2(ii) tells us in terms of $\mathcal{H}_{IPC}$ that $\{B \in NNIL \mid B \leq A\}$ both has and contains a supremum A^*. Thus, A^* may be called the *greatest lower NNIL-approximant* of A.

6.3 Fact Let *f* assign Σ_1-sentences to the propositional variables. Then, for any propositional A, if HA ⊢ A[*f*], then HA ⊢ A^* [*f*].

6.4 Fact The number of NNIL-formulas in $p_1,\ldots,p_m$, modulo IPC-provable equivalence, is finite.

6.5 Theorem Let $\mathcal{H}$ be a *Ha* on finitely many generators. Then $\mathcal{H}$ is embeddable in $\mathcal{S}_{HA}$ iff $\mathcal{H}$ is embeddable in $\mathcal{H}_{IPC}$.

Proof: Let $\mathcal{H}$ be a *Ha* on finitely many generators. The direction from right to left follows immediately from theorem 6.1, so suppose $\mathcal{H}$ is embeddable in $\mathcal{S}_{HA}$ and let us prove that $\mathcal{H}$ is embeddable in $\mathcal{H}_{IPC}$. Let the generators of $\mathcal{H}$ be $A_1,\ldots,A_n$. These generators are in their turn Boolean combinations of Σ_1-sentences, say, $S_1,\ldots,S_m$. So, $A_i=B_i(S_1,\ldots,S_m)$ for some propositional B_i. Let $\mathcal{K}$ be the subalgebra of $\mathcal{S}_{HA}$ generated by $S_1,\ldots,S_m$. Since $\mathcal{H}$ is embedded in $\mathcal{K}$ by assigning B_i to p_i, it is sufficient to show that $\mathcal{K}$ is embeddable in $\mathcal{H}_{IPC}$. Let C^* be the greatest lower NNIL-approximant of an arbitrary formula C, as promised by fact 6.2. We find by fact 6.3, that, if $HA\vdash C(S_1,\ldots,S_m)$, then $HA\vdash C^*(S_1,\ldots,S_m)$. So, if $\mathcal{K}\models C$, then $\mathcal{K}\models C^*$. Since the set of NNIL-formulas in $p_1,\ldots,p_m$ is finite (modulo IPC-provable equivalence) by fact 6.4, there are only finitely many possible C^*. Let C^+ be the conjunction of the C^*. We find for D in $p_1,\ldots,p_m$, $\mathcal{K}\models D$ iff $IPC\vdash C^+\rightarrow D$. Clearly C^+ is a prime NNIL-formula. By corollary 2.11, C^+ is exact. Ergo $\mathcal{K}$ is embeddable in $\mathcal{H}_{IPC}$. ❑

6.6 Corollary The IPC-admissible rules for IPC are the same as the IPC-admissible rules for HA w.r.t. substitutions involving only $B\Sigma_1$-sentences.

6.7 Open question Is $\mathcal{S}_{HA}$ isomorphic to $\mathcal{H}_{IPC}$? We conjecture that it is not.

Appendix
The density of Heyting algebras of arithmetical theories

We first sketch a proof of the density theorem for the classical case.

Theorem A.1 The $\mathcal{H}_T$ of any consistent r.e. extension T of Q (Robinson's arithmetic) is dense, i.e. between every two points in the Boolean algebra there is a third one.

Sketch of proof: Assume $T\vdash A\rightarrow B$, $T\nvdash B\rightarrow A$. Take the Rosser sentence R of $T+B+\neg A$, i.e., something like $T\vdash R\leftrightarrow\Box(B\wedge\neg A\rightarrow\neg R)\leq\Box(B\wedge\neg A\rightarrow R)$ holds. The element between A and B will be $A\vee(B\wedge R)$. ❑

In constructive logic one cannot even conclude from the data that $T+B+\neg A$ is consistent. Nevertheless, the correct constructive proof is just a slight variation on the classical argument.

Theorem A.2 For any consistent r.e. extension T of i-Q (the constructive version of Robinson's arithmetic), $\mathscr{H}_T$ is dense.

Proof: Under the same circumstances as in the previous proof, define by the fixed point theorem a sentence R such that $T \vdash R \leftrightarrow \Box(B \wedge R \rightarrow A) \leq \Box(B \rightarrow A \vee R)$ and let $S := \Box(B \rightarrow A \vee R) < \Box(B \wedge R \rightarrow A)$ and $C := A \vee (B \wedge R)$. Clearly, $T \vdash A \rightarrow C$ and $T \vdash C \rightarrow B$. Reason in T. Assume $\Box(C \rightarrow A)$. Then $\Box(B \wedge R \rightarrow A)$. By the properties of witness comparisons this implies $R \vee S$, i.e. $(\Box(B \wedge R \rightarrow A) \wedge R) \vee (\Box(B \wedge R \rightarrow A) \wedge S)$. The first disjunct implies, by Σ-completeness, $\Box(B \wedge R \rightarrow A) \wedge \Box R$, and hence, $\Box(B \rightarrow A)$. The second disjunct implies $\Box(B \wedge R \rightarrow A) \wedge \Box(B \rightarrow A \vee R)$ and hence, $\Box(B \rightarrow A)$ as well. As, by assumption, $T \nvdash B \rightarrow A$, we find $T \nvdash C \rightarrow A$.

Next, assume $\Box(B \rightarrow C)$. Then, $\Box(B \rightarrow A \vee R)$. By the properties of witness comparisons this implies $R \vee S$, i.e. $(\Box(B \rightarrow A \vee R) \wedge R) \vee (\Box(B \rightarrow A \vee R) \wedge S)$. The second disjunct implies, by Σ-completeness, $\Box(B \rightarrow A \vee R) \wedge \Box S$, and hence, $\Box(B \rightarrow A \vee R) \wedge \Box \neg R$, from which $\Box(B \rightarrow A)$ follows. The first disjunct implies $\Box(B \rightarrow A \vee R) \wedge \Box(B \wedge R \rightarrow A)$, from which $\Box(B \rightarrow A)$ follows as well. As by assumption $T \nvdash B \rightarrow A$, we find $T \nvdash B \rightarrow C$. ❑

References

Jankov, V.A., 1968, *Constructing a sequence of strongly independent superintuitionistic propositional calculi,* Sov. Math. Dok., 9, 806-807.

de Jongh, D.H.J., 1970, *A characterization of the intuitionistic propositional calculus,* in Kino, Myhill, Vesley[70], 211-217.

de Jongh, D.H.J., 1982, *Formulas of one propositional variable in intuitionistic arithmetic,* in: Troelstra & van Dalen[82], 51-64.

de Jongh, D.H.J., Chagrova, L.A., 1995, The decidability of dependency in intuitionistic propositional logic, To be published in JSL.

Kino, A., Myhill, J., Vesley, R.E., (eds.), 1970, *Intuitionism and proof theory,* North Holland, Amsterdam.

Pitts, A., 1992, *On an interpretation of second order quantification in first order intuitionistic propositional logic,* JSL 57, 33-52.

Renardel de Lavalette, G.R., 1986, *Interpolation in a fragment of intuitionistic propositional logic,* Logic Group Preprint Series 5, Dept. of Philosophy, University of Utrecht, Heidelberglaan 8, 3584 CS Utrecht.

Rybakov, V.V., 1992, *Rules of Inference with Parameters for Intuitionistic Logic*, JSL 57, 912-923.
Shavrukov, V., 1993, *Subalgebras of diagonalizable algebras of theories containing arithmetic,* Dissertationes Mathematicae, Polska Akademia Nauk., Mathematical Institute.
Smoryński, C., 1973, *Applications of Kripke Models,* in Troelstra[73], 324-391.
Smoryński, C., 1985, *Self-Reference and Modal Logic,* Springer Verlag.
Solovay, R., 1976, *Provability interpretations of modal logic,* Israel Journal of Mathematics 25, 287-304.
Troelstra, A.S. (ed.), 1973, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis,* Springer Lecture Notes 344, Springer, Berlin.
Troelstra, A.S., and van Dalen, D., (eds.), 1982, *The L.E.J. Brouwer Centenary Symposium,* North Holland, Amsterdam.
Troelstra, A.S., and van Dalen, D., 1988a, *Constructivism in Mathematics, vol 1,* North Holland, Amsterdam.
Troelstra, A.S., and van Dalen, D., 1988b, *Constructivism in Mathematics, vol 2,* North Holland, Amsterdam.
Troelstra, A.S., 1992, *Realizability*, ILLC-prepublication ML-92-09, ILLC, University of Amsterdam, Plantage-Muidergracht 24, 1018TV Amsterdam, to be published as a chapter of the *Handbook of Proof Theory* (S. Buss, ed.), North-Holland, Amsterdam.
Visser, A., 1982, *On the Completeness Principle,* Annals of Mathematical Logic 22, 263-295.
Visser, A., 1985, *Evaluation, provably deductive equivalence in Heyting's Arithmetic of substitution instances of propositional formulas,* Logic Group Preprint Series 4, Dept. of Philosophy, University of Utrecht, Heidelberglaan 8, 3584 CS Utrecht.
Visser, A., 1994, *Propositional combinations of Σ-sentences,* Logic Group Preprint Series 117, Dept. of Philosophy, University of Utrecht, Heidelberglaan 8, 3584 CS Utrecht.
Visser, A., van Benthem, J.F.A.K., de Jongh, D.H.J., Renardel de Lavalette, G.R., 1994, NNIL, a study in intuitionistic propositional logic, Logic Group Preprint Series 111, Dept. of Philosophy, University of Utrecht, Heidelberglaan 8, 3584 CS Utrecht. To be published in *Modal Logic and Process Algebra, a bisimulation perspective* (A. Ponse, M. de Rijke, Y. Venema eds.), CSLI Lecture Notes, CSLI, Stanford, California.
Zambella, D., 1994, Shavrukov's Theorem on the subalgebras of diagonalizable algebras *for theories containing* $I\Delta_0$+*Exp,* Notre Dame Journal of Formal Logic 35, 147-157.

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