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MODAL FOUNDATIONS FOR PREDICATE LOGIC

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Abstract

The complexity of any logical modeling reflects both the intrinsic structure of a topic described and the weight of the formal tools. Some of this weight seems inherent in even the most basic logical systems. Notably, standard predicate logic is undecidable. In this paper, we investigate 'lighter' versions of this general purpose tool, by modally 'deconstructing' the usual semantics, and locating implicit choice points in its set up. The first Part sets out the interest of this program and the modal techniques employed, while the second Part provides technical elaborations demonstrating its viability.

1 The Modal Core of Predicate Logic

The well-known standard semantics for predicate logic has the following key clause:

$$\mathbf{M}, \alpha \models \exists x\, \phi \qquad \text{iff} \qquad \textit{for some } d \in |\mathbf{M}|: \; \mathbf{M}, \alpha^x{}_d \models \phi\, .$$

Tarski's main innovation here was the use of assignments, which are essential in decomposing quantified statements, which leave free variables in their matrix. But much less than this is needed to give a compositional semantics for first-order quantification. The abstract core pattern which would make the latter work is this:

$$\mathbf{M}, \alpha \models \exists x\, \phi \qquad \text{iff} \qquad \textit{for some } \beta : R_x\alpha\beta \textit{ and } \mathbf{M}, \beta \models \phi\, .$$

Here, 'assignments' α, β become abstract states, and the concrete relation $\alpha =_x \beta$ (which holds between α and $\alpha^x{}_d$) has become just any binary update relation R_x . Evidently, this abstract pattern involves standard poly-modal models, of the form

$$\mathbf{M} \quad = \quad (S, \{R_x\}_{x \in VAR}, I)$$

where S is a set of 'states', R_x a binary relation for each variable x , and I a 'valuation' or 'interpretation function' giving a truth value to each atomic formula Px, Rxy, ... in each state α . In particular, existential quantifiers $\exists x$ become unary existential modalities $\langle x \rangle$. This modal state semantics for predicate logic has an independent dynamic appeal: first-order evaluation is an informational process which changes computational states. The first-order language then becomes a *dynamic logic*, with a special choice of atoms and without explicit compound programs.

From this modal point of view, conversely, 'standard semantics' arises by insisting on three additional mathematical choices, not enforced by the core semantics. (**1**) States are identified with variable assignments, (**2**) 'update' must be the specific relation $=_x$, and (**3**) all assignments in the function space D^{VAR} must be available to evaluation. The former are issues of implementation, the latter is a strong existence assumption. (Actually, standard predicate logic can get by with only locally finite assignments – but even that is a strong existence requirement.) Henceforth, we shall regard these further 'set-theoretic' choices as negotiable. This view lends further support to the abstract modal approach. E.g., it is often felt that the usual set-theoretic tricks making predicates sets of tuples should be orthogonal to the nature of logical validity. Finally, as an alternative to even assumptions (1), (2), Hollenberg & Vermeulen 1994 present a dynamic semantics for predicate logic manipulating states involving variable stacks whose update relations R_x differ considerably from the standard one.

The universal validities produced by a general modal semantics are well-known. One obtains a *minimal poly-modal logic*, whose principles consist of

- all classical Boolean propositional laws
- Modal Distribution: $\forall x\, (\phi \to \psi) \to (\forall x\, \phi \to \forall x\, \psi)$
- Modal Necessitation: if $\vdash \phi$, then $\vdash \forall x\, \phi$
- a definition of $\exists x\, \phi$ as $\neg\forall x \neg\phi$.

A *completeness theorem* with respect to the above abstract models may be proved via the standard modal Henkin construction with maximally consistent sets for the states in S , and the relations R_x defined via: $\Delta_1\ R_x\ \Delta_2$ iff *for all* $\phi \in \Delta_2 : \exists x\, \phi \in \Delta_1$. This logic can be analyzed in a standard modal fashion (cf. Andréka, van Benthem & Németi 1995 for a modern treatment), yielding usual meta-properties such as Craig Interpolation or Los-Tarski Preservation. Moreover, it is *decidable* by standard modal techniques (filtration, semantic tableaus). One can now usefully pursue standard first-order model theory in tandem with its modal counterpart. For instance, consider modal *bisimulations* for these models, relating states having the same atomic behaviour, with zigzag conditions for the relations R_x . Specialize these to standard Tarski models. The result is a notion of *partial isomorphism* between models, related but not equal to the standard one. (Essentially this analogy was observed in Fernando 1992.) Further analogies are elaborated in van Benthem 1991, 1995, De Rijke 1993.

The modal perspective suggests a whole landscape below standard predicate logic, with a 'minimal modal logic' at the base, and ascending up to 'standard semantics' via successive frame constraints. This seems the proper habitat of 'dynamic semantics' as currently explored in the logico-linguistic literature. In particular, this landscape contains *decidable* sublogics of predicate logic, sharing its desirable meta-properties. (The minimal modal base itself is an example.) Thus, the 'undecidability of predicate logic' largely reflects mathematical accidents of its Tarskian modeling, in particular, encoding set-theoretic facts about function spaces D^{VAR} – rather than the core logic of quantification and variable assignment. We shall explore the resulting view of first-order semantics, drawing upon the work of many authors. In particular, we find that, as with other 'fine-structure landscapes' underneath standard logic (e.g., the categorial or substructural hierarchy: van Benthem 1991, Dosen & Schroeder-Heister 1993), there is a rich family of natural calculi in our original language, but also one of richer languages reflecting the broader more sensitive semantics. In particular, abstract core models support distinctions between various forms of quantification ('monadic' and 'polyadic') that get collapsed in standard predicate logic.

2 Dependency Models

In addition to our two choices so far, there are further natural inhabitants of the landsape between standard logic and its minimal modal core. For instance, one may retain the general implementation of Tarski semantics (the above (1), (2)), while giving up its existence assumption (3). The result is a 'half-way house' where S is some family of assignments in the usual sense (not necessarily the full function space D^{VAR}), and the R_x are the standard relations $=_x$. For instance, with two variables {x, y}, a domain with objects {1, 2} supports 2^4 assignment sets. One is the standard model with all four maps from variables to objects. Another has just the two assignments $\{\alpha, \beta\}$ with $\alpha(x) = 1, \alpha(y) = 2$ and $\beta(x) = 2, \beta(y) = 1$. First-order evaluation will then be over generalized Tarskian models (**M**, V) having a range V of 'available assignments' as an extra parameter. An existential quantifier $\exists x\phi$ says that some x–modification of the current state exists inside V satisfying ϕ.

'Assignment gaps' turn out reflect an interesting phenomenon. Intuitively, one often wants to model 'dependencies' between variables: i.e., a situation where changes in value for one variable x may induce, or at least be correlated with, changes in denotation for another variable y. Examples include natural reasoning (Fine 1985), probabilistic logic (van Lambalgen 1991) and plural anaphoric discourse (van den Berg 1995). This phenomenon cannot be modeled in standard Tarskian semantics, where we can change values for variables completely independently: Starting from any state α, one can move to any $\alpha^x{}_d$. But in a model with assignment gaps, the only way to change values for x, starting from some assignment α, may be by incurring a change in y too. An example is the above two-assignment model, where any shift in value for x produces a corresponding one for y. Thus, standard models rather become those 'degenerate cases' where all dependencies between variables have been suppressed. This shows clearly in the standard validity of the quantifier exchange principle $\exists x \exists y \phi \leftrightarrow \exists y \exists x \phi$, which will become typically invalid on our generalized models. There are many possible modellings for dependence phenomena. For instance, Alechina 1995 proposes a semantics where (stated in our current framework) the key evaluation clause becomes

$$\mathbf{M}, \alpha \models \exists x \phi \quad \text{iff} \quad \textit{for some } \beta : R_{x,\mathbf{y}}\alpha\beta \textit{ and } \mathbf{M}, \beta \models \phi$$

where **y** is some sequence of 'relevant context variables' – which might consist, e.g., of the free variables in $\exists x \phi$. In this case, even Modal Distribution will fail. Along a different path, van den Berg 1995 makes assignment sets themselves into new states encoding dependencies, which can be modified in the dynamic process of evaluation.

3 What Do First-Order Axioms Say?

The above three semantic levels have further fine-structure. This may be brought out in two ways. First, one can study natural mathematical constraints on modal frames or generalized assignment models, reflecting various aspects of 'dependence'. But also, one can analyze possible validities expressible in our first-order language. The latter strategy involves modal *frame correspondences.* A modal formula ϕ expresses a relational constraint C on abstract state frames if:

C holds of $(S, \{R_x\}_{x\in VAR})$ iff
$(S, \{R_x\}_{x\in VAR}, I), \alpha \models \phi$ for all states α and interpretation functions I.

Let us see, over the minimal modal logic, what is expressed by the laws of predicate logic. Usually, all first-order validities are together in one big bag. But in our modal semantics, they come to express different requirements on states and accessibility, with a computational slant. For a concrete illustration, we use modal correspondence to 'deconstruct' the axioms in the well-known text book Enderton 1972. (But any text book axiomatization would do, with a different cut of the cake.) Enderton's list has all universal closures of Boolean propositional laws, plus the three quantifier axioms

(1) $\forall x(\phi\to\psi) \to (\forall x\phi \to \forall x\, \psi)$
(2) $\phi \to \forall x\, \phi$ *provided that* x *do not occur free in* ϕ
(3) $\forall x\, \phi \to [t/x]\, \phi$ *provided that* t *be free for* x *in* ϕ

The system has one inference rule, Modus Ponens. From a modal perspective, the propositional part is base valid (both axioms and rule). The first quantifer axiom is the base valid Modal Distribution. Moreover, universal closure of axioms is a technique which amounts to postulating a rule of Necessitation for universal quantifiers. Indeed, the first part of the Enderton axiomatization by itself is a complete calculus for the minimal modal logic! It is tempting to see the Hand of Providence at work here. Now, let us analyze the other quantifier axioms. We start with the least conspicuous one. From our present perspective, it is immensely powerful.

The axiom $\phi \to \forall x\, \phi$

We analyze this principle inductively, in a modified formulation with atoms and their negations, &, $\vee$, $\exists$, $\forall$. Our argument will be heuristic, determining the effect of various instances of this principle independently. The first instance is the atomic pair:

(2.1) $\mathbf{Py} \to \forall x\, \mathbf{Py}$ $\quad\quad \neg\, \mathbf{Py} \to \forall x\, \neg\, \mathbf{Py}$

These principles say that truth values for atoms without the variable x are unaffected by R_x–transitions. For assignments, with predicate interpretation as usual, this is equivalent to the condition that R_x imply $=_x$. In our abstract semantics, however, (2.1) does not naturally translate into a frame correspondence. It rather suggests a restriction on the range of our abstract interpretation functions I . These must satisfy a Heredity Principle stating that *if* I (α, Py), *then* I (β, Py) *for all states* β *with* $R_x\alpha\beta$. (Restrictions on valuations are known from Kripke semantics for intuitionistic logic.) Pure frame conditions do emerge with compound cases of axiom (2).

(2.2) *Boolean cases* $\phi_1 \& \phi_2$, $\phi_1 \vee \phi_2$

There is no new information to be extracted here. Suppose, inductively, that we already know that $\vdash \phi_1 \rightarrow \forall x\, \phi_1$ and $\vdash \phi_2 \rightarrow \forall x\, \phi_2$. Then, in the base logic, we have automatically (using Distribution) that $\vdash (\phi_1 \& \phi_2) \rightarrow \forall x\, (\phi_1 \& \phi_2)$. The case for disjunction is entirely analogous. The real impact is in the quantified cases.

(2.3) $\exists y\, \phi$, $\forall y\, \phi$

Here we must distinguish two subcases.

(2.3.1) *the quantified variable* y *is* x *itself.*

Then we have, without any assumptions, that

$$\exists x\, \phi \rightarrow \forall x\, \exists x\, \phi \qquad\qquad \forall x\, \phi \rightarrow \forall x\, \forall x\, \phi$$

Here we recognize two modal S5-axioms for <x> , namely

$$<x>\, \phi \rightarrow [x]\, <x>\, \phi \qquad\qquad [x]\, \phi \rightarrow [x]\, [x]\, \phi$$

Their frame content is well-known.

Fact
- $\forall x\, \phi \rightarrow \forall x\, \forall x\, \phi$ expresses that R_x is *transitive*
- $\exists x\, \phi \rightarrow \forall x\, \exists x\, \phi$ expresses that R_x is *euclidean*

If we add the simplest instance of Enderton's axiom (3), viz. $\forall x\, \phi \rightarrow \phi$ (expressing *reflexivity* of the R_x), we get full S5, where all R_x must be *equivalence relations*. (This modal character of first-order deduction is very clear in Chapter 1.2 of Henkin-Monk-Tarski 1985, which has much standard S5-deduction in an algebraic guise.) Henceforth, we assume the S5-principles, which hold in all generalized assignment models. (Without at least S4, the following analysis becomes somewhat messier.) Thus, consider the remaining case where genuine interactions take place between updates for different variables.

(2.3.2) *the variables* x, y *are distinct*

Inductively, we may assume that $\vdash \phi \to \forall x\, \phi$, and then we need

$$\exists y\, \phi \to \forall x\, \exists y\, \phi \qquad\qquad \forall y\, \phi \to \forall x\, \forall y\, \phi$$

Modulo S4, these two inference rules express two well-known quantifier shifts:

Claim
- The rule ' if $\vdash \phi \to \forall x\, \phi$, then $\vdash \exists y\, \phi \to \forall x\, \exists y\, \phi$ ' is equivalent with the axiom $\exists y\, \forall x\, \phi \to \forall x\, \exists y\, \phi$
- The rule ' if $\vdash \phi \to \forall x\, \phi$, then $\vdash \forall y\, \phi \to \forall x\, \forall y\, \phi$ ' is equivalent with the axiom $\forall y\, \forall x\, \phi \to \forall x\, \forall y\, \phi$

Proof (First case) 'Axiom to rule'. If $\vdash \phi \to \forall x\, \phi$, then in the minimal modal logic, $\vdash \exists y\, \phi \to \exists y \forall x\, \phi$ – whence by our axiom, $\vdash \exists y\, \phi \to \forall x \exists y\, \phi$. 'Rule to axiom'. (We use S4.) Start from the axiom $\vdash \forall x\, \phi \to \forall x\, \forall x\, \phi$, and apply the rule. This gives $\vdash \exists y\, \forall x\, \phi \to \forall x\, \exists y\, \forall x\, \phi$. Again, S4 has $\vdash \forall x\, \phi \to \phi$. In the minimal modal logic, this gives $\vdash \forall x\, \exists y\, \forall x\, \phi \to \forall x\, \exists y\, \phi$. Together: $\vdash \exists y\, \forall x\, \phi \to \forall x\, \exists y\, \phi$. ■

As for modal correspondence, both these quantifier shifts may be analyzed ad-hoc, or via a more sophisticated modal technique. Note that they are are *Sahlqvist Forms* to which a well-known algorithm applies computing their first-order frame equivalents:

Fact
- $\exists y\, \exists x\, \phi \to \exists x\, \exists y\, \phi$ expresses Path Reversal:
  $\forall \alpha\beta\gamma\, ((R_x\alpha\beta \,\&\, R_y\beta\gamma) \to \exists\delta\, (R_y\alpha\delta \,\&\, R_x\delta\gamma))$
- $\exists y\, \forall x\, \phi \to \forall x\, \exists y\, \phi$ expresses Confluence:
  $\forall \alpha\beta\gamma\, ((R_y\alpha\beta \,\&\, R_x\alpha\gamma) \to \exists\delta\, (R_x\beta\delta \,\&\, R_y\gamma\delta))$

In S5-models, Path Reversal and Confluence are semantically equivalent. And indeed, Henkin-Monk-Tarski 1985 has an algebraic proof of this fact. Our general conclusion so far is that first-order predicate-logical axioms express modal Sahlqvist forms, to which standard modal correspondence and completeness techniques may be applied. (Further illustrations may be found in Venema 1992, de Rijke 1993, Marx 1994.)

One may also use general facts about predicate logic to suggest natural constraints on dependency models. E.g., the Finiteness Lemma says that evaluation of formulas only depends on values for their free variables. This is no longer true in generalized assignment models, where free variables may carry implicit dependencies. But one can study Finiteness as an interesting condition per se. Such conditions may be on models rather than frames. Westerståhl 1995 redoes the above correspondence analysis on modal frames with heredity restrictions on admissible valuations.

Next, we must analyze Enderton's last quantifier axiom, stating that "$\forall x\ \phi \rightarrow [t/x]\ \phi$, *provided that* t *be free for* x *in* ϕ" . In the spirit of our analysis so far, it is natural to the substitution operator $[t/x]$ as a semantic update instruction in its own right. It will denote 'controlled value assignment' $x{:=}t$, which is the natural semantic companion to our 'random assignment' for the existential quantifier $\exists x$.

4 Quantifiers and Substitutions

There is a folklore idea in dynamic logic that syntactic substitutions $[t / x]$ work semantically as program instructions $x := t$. Goldblatt 1987 uses the latter notation to avoid syntactic complexities in Harel's quantified dynamic logic. Another instance of this duality shows up with the well-known Substitution Lemma for predicate logic:

$$\mathbf{M}, \alpha \models [t / x]\ \phi \quad \text{iff} \quad \mathbf{M}, \alpha^{x}{}_{\mathrm{value}(\mathbf{M}, \alpha, t)} \models \phi$$

This expresses a procedural equivalence between 'call by name' and 'call by value'. Finally, random assignment $[[\exists x]]$ naturally invites its specific counterpart $[[t / x]]$. The modal treatment of substitutions is quite like that for the earlier quantifiers.

Abstract Assignment Frames

We enrich the previous models by adding abstract relations $A_{x,y}$, whose concrete interpretation in standard models is as follows:

$$\alpha\ A_{x,y}\ \beta \quad \text{iff} \quad \beta(x) = \alpha(y) \ \text{and} \ \beta(z) = \alpha(z) \ \text{for all } z \text{ distinct from } x .$$

Henceforth, for convenience, we only consider substitutions of variables for variables. The truth definition treats the substitution operator $[\, y / x\,]$ literally as a modality (again, the Hand of Providence fixed this box notation long ago):

$$\mathbf{M}, \alpha \models [y/ x]\ \phi \quad \text{iff} \quad \textit{for all } \beta \textit{ with } A_{x,y}\, \alpha\beta : \mathbf{M}, \beta \models \phi .$$

The outcome is similar to that for existential quantifiers. There is a universally valid minimal logic, on top of which further principles express special constraints on the relations via frame correspondences. (All principles involved have Sahlqvist forms.) Interestingly, the usual syntactic 'definition' of substitution acquires semantic import:

Semantic Analysis of Substitution

Atomic Cases $\quad [y/x]\ Px \leftrightarrow Py$

$[y/x]\ Pz \leftrightarrow Pz \qquad$ (z *distinct from* x)

The outcome is as before. On abstract frames, these principles express heredity constraints on admissible valuations. On concrete assignment frames, however, they express that the relations $A_{x,y}$ are to behave as in the above concrete clauses.

Boolean Cases	$[y/x](\phi \& \psi)$	$\leftrightarrow$	$[y/x]\phi \& [y/x]\psi$
	$[y/x]\neg\phi$	$\leftrightarrow$	$\neg[y/x]\phi$

The first of these is universally valid in the minimal modal logic. The second is a well-known modal axiom, whose two halves together express that the relation $A_{x,y}$ is to be a *function*. For convenience, we make this assumption henceforth.

Quantified Cases	$[y/x]\ \exists x\ \phi$	$\leftrightarrow$	$\exists x\ \phi$	
	$[y/x]\ \forall x\ \phi$	$\leftrightarrow$	$\forall x\ \phi$	
	$[y/x]\ \exists z\ \phi$	$\leftrightarrow$	$\exists z\ [y/x]\ \phi$	(z *distinct from* x, y)
	$[y/x]\ \forall z\ \phi$	$\leftrightarrow$	$\forall z\ [y/x]\ \phi$	(z *distinct from* x, y)

These express simple interactions between compositions of the abstract relations $A_{x,y}$ and R_x, which can be spelt out mechanically. The remaining quantified cases are $[z/x]\ \exists z\ \phi$ and $[z/x]\ \forall z\ \phi$. Here we allow only substitutions which are 'valid' in our intended semantics. Nothing holds in general when there are free occurrences of x in the matrix ϕ. But otherwise, we want to have the two equivalences

$$[z/x]\ \exists z\ \phi \leftrightarrow \exists z\ \phi \qquad\qquad [z/x]\ \forall z\ \phi \leftrightarrow \forall z\ \phi$$

More generally, here, we want a principle not unlike the earlier quantifier axiom (2):

$[z/x]\ \phi \leftrightarrow \phi$ *whenever* x *does not occur freely in* ϕ.

The direction from right to left here uses the quantifier axiom (2) in combination with axiom (3) (whose remaining force is gauged below). For, if x does not occur freely in ϕ, then $\forall x\ \phi$ follows, which again implies $[y/x]\ \phi$ (since y is free for x in ϕ). From left to right, we argue as follows. If x does not occur freely in ϕ, then we already have $\vdash \phi \rightarrow \forall x\ \phi$. In the minimal modal logic then $\vdash [z/x]\ \phi \rightarrow [z/x]\ \forall x\ \phi$. Now, by an earlier principle, we have $[z/x]\ \forall x\ \phi \rightarrow \forall x\ \phi$. Then, with one S4-axiom, we have $[z/x]\ \forall x\ \phi \rightarrow \phi$. Together, this yields what we need. Complete calculi for first-order substitutions occur in Németi 1985, Thompson 1981, and Venema 1993.

Finally, we return to the analysis of the initial quantifier axiom (3), which read:

$\forall x\ \phi \rightarrow [y/x]\ \phi$ *provided that* y *be free for* x *in* ϕ

Its business now becomes merely to relate the two modalities [x] and [y/x] :

$A_{x,y}$ *is contained in* R_x

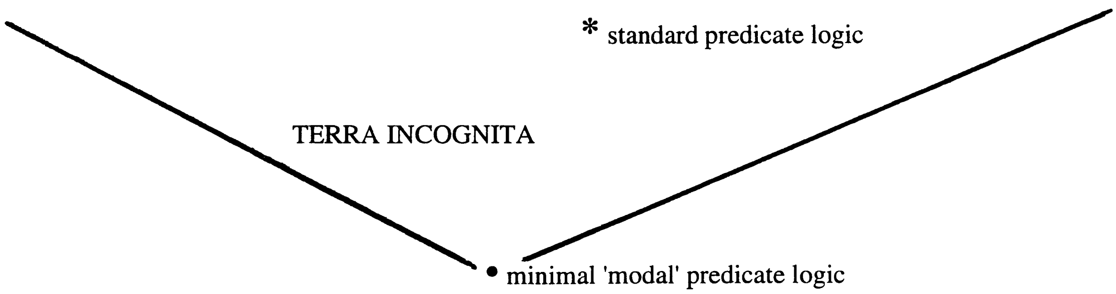
The proviso is taken care of by the earlier principles for 'cautious substitution'. This completes our semantic analysis of a complete axiomatic system for predicate logic. Its consists of all accumulated principles on expanded abstract assignment frames $(S, \{R_x\}_{x \in VAR}, \{A_{x,y}\}_{x,y \in VAR})$, plus some constraints on admissible valuations. Of course, this is just one pass through predicate logic, via Enderton's particular axiomatic presentation. One could also analyze different presentations (compare the natural deduction analysis in Meyer Viol 1995). E.g., Henkin-Monk-Tarski 1985 has:

if $\vdash \phi \leftrightarrow \psi$, then $\vdash \exists x\, \phi \leftrightarrow \exists x\, \psi$	(from minimal modal logic)
$\exists x \perp \leftrightarrow \perp$	(ditto)
$\exists x\, (\phi\, \&\, \exists x\, \psi) \leftrightarrow \exists x\, \phi\, \&\, \exists x\, \psi$	(well-known S5-principle)
$\phi \rightarrow \exists x\, \phi$	(T-axiom)

What comes out in general is the idea that the usual 'predicate-logical validities' form a very diverse bunch, which can be layered in many ways, for different purposes.

5 Landscape of Deductive Strength

Let us summarize the main line so far. First-order predicate logic may be viewed as a dynamic logic for variable assignment, whose atomic processes shift values in registers x, y, z, ... This view-point opens up a hierarchy of fine-structure underneath standard predicate logic. The latter system becomes the mere (undecidable) theory of a particular mathematical class of 'rich assignment models' in this perspective. What we get in this way is a broad semantic landscape (as also found in Modal Logic or Arrow Logic, cf. van Benthem 1991), with a minimal modal system at the bottom, where various intermediate systems arise by imposing some, though not all of the usual requirements on assignments and their R_x (and $A_{x,y}$) structure:



What are natural landmarks in this area? We would like to find logics (1) that are reasonably expressive, (2) that share the important meta-properties of predicate logic (such as Interpolation, Effective Axiomatizability, perhaps even 'Gentzenizability') and (3) that might even *improve* on this, by being decidable. The minimal predicate logic satisfies these three demands – but can we ascend in the above landscape and get more powerful logics with the same behaviour? Fortunately, this area is not totally unexplored. The existing body of research in Cylindric Algebra has already identified some very interesting intermediate systems (cf., e.g., Henkin-Monk-Tarski 1985, Andréka 1991, Németi 1985, Venema 1994, Marx 1994). Németi 1993 contains a number of interesting calculi, including the so-called 'non-commutative' version of cylindric algebra (first proposed in Thompson 1981), which becomes decidable by giving up the quantifier interchange axioms for $\exists x \exists y$ and $\forall x \forall y$. All this is much like the well-known lattice of modal logics (Bull & Segerberg 1984, Blok 1979).

One attractive candidate in this landscape is the earlier system CRS (Németi 1993). It may be described as the set of all predicate-logical validities that hold in those abstract state frames for quantification and substitution which satisfy all *universal frame conditions* true in standard assignment models. These obey all general logical properties of assignments, but they do not make any *existential* assumptions about the supply of available assignments. The former conditions seem more truly 'logical', whereas the latter would be more 'mathematical' or 'set-theoretic' in character. (This distinction between universal and existential principles in logical modelling has been defended more generally: e.g., for modelling temporal logic in van Benthem 1983.) For instance, in the above correspondences, universal S5-type conditions emerged, but also existential ones for quantifier interchange principles. Later on, we shall analyze the purely universal kind in more detail, by a representation method turning abstract state frames into assignment frames. Two important known facts about CRS are that it is decidable (Németi) and non-finitely axiomatizable (Andréka). Moreover, our representation method will show that it has a first-order definition by means of universal *Horn* clauses, from which one can derive Craig Interpolation (Marx 1994). But on top of CRS, one may continue, and add axioms up to the cliffs of complexity.

This landscape of dynamic predicate logic can be investigated model-theoretically using standard modal techniques (van Benthem 1985, Goldblatt 1987, Venema 1992 and De Rijke 1993). In particular, as observed before, basic modal notions such as 'bisimulation' between abstract models generalize model-theoretic counterparts over standard assignment models. There can still be some interesting discrepancies here. E.g., bisimulation relates complete variable assignments, whereas its model-theoretic counterpart of 'partial isomorphism' relates finite sequences of objects. This reflects another meta-property of standard first-order logic: no variable has a special identity. In the present abstract semantics, this is no longer the preferred option. With possible dependencies present, variables do gain 'individuality' (cf. van Lambalgen 1991, Meyer Viol 1994). Other relevant modal themes include axiomatization techniques and decision methods across this whole landscape (cf. Marx 1994, Mikulas 1995).

Remark Two First-Order Languages

Do not confuse two uses of 'first-order languages' here! One lives at an object level, as the 'dynamic modal language' of assignment change. Another is used at a meta-level, as our 'working language' for stating frame conditions. In particular, one can be a modal minimalist at the object level, and a standard logician at the meta-level.

The general picture here is like in Arrow Logic (van Benthem 1991, 1995, Venema 1994, Marx & Pólos 1994), with the same semantic thrust. Eventually, we do not just want to re-analyze predicate logic, but rather explore the expressive capacity of this new semantics (and the logics supported by it) because of its intrinsic intuitive appeal.

6 Rethinking the Language

The modal analogy suggests, in particular, that first-order predicate logic reflects only part of the expressive resources of abstract state models. In fact, there is an obvious first-order (meta-)language over these models, whose variables run over states (once again: please do not confuse this with our central modal first-order object language!). This is the language into which one 'translates' poly-modal logic in the usual sense (van Benthem 1983), which contains many assertions without a modal counterpart. One example is an *unrestricted* existential quantifier $\exists\alpha$ over states. By contrast, modal object quantifiers $\exists x$ induce *restricted* state quantifiers, which are of the form $\exists\beta$ ($R_x\alpha\beta$ & . (The 'modal fragment' of the full first-order state language is precisely determined by such quantifier patterns, which induce invariance for bisimulations. Up-to-date technical details are found in Andréka, Németi & van Benthem 1995.) 'Random assignment jumps' seem a natural meaning for isolated quantifier symbols $\exists$ not tagged by any variable. Likewsie, one might consider *global predicates* of states, not reducible to assertions about their object values at some finite set of variables. All this is just one instance of a broad theme mentioned at the beginning. A more general semantics below standard predicate logic usually suggests new notions, that were invisible in the 'classical system'. We list a few directions for such extensions.

Stronger Modalities Add modal operators, such as the "universal modality", or more complex ones about internal structure of state transitions ("since", "until").

Dynamic Operators Add program constructions, starting from individual variables as atomic programs. E.g., the Path Principles suggest addition of both sequential composition and conjunctive intersection. Propositional dynamic logic with these two operations is still decidable: and hence so is our minimal base logic.

Polyadic Quantifiers A most interesting extension in expressive power is that to polyadic quantifiers. In standard predicate logic, a tuple notation $\exists xy \bullet \phi$ is just shorthand for either $\exists x \bullet \exists y \bullet \phi$ or $\exists y \bullet \exists x \bullet \phi$. But here, it becomes a notion sui generis. On generalized assignment models, $\exists xy \bullet \phi$ says that there exists some assignment agreeing with the current one up to $\{x, y\}$ values where ϕ holds. The corresponding transitions encode a form of *concurrency* vis-a-vis the single transition relations R_x and R_y. This is not reducible to either iterated version, which require the existence of 'intermediate states'. More generally, abstract state models admit natural definitions of quantifiers $\exists x_1 \ldots x_k \phi$ stating the existence of some $R_{<x1, \ldots, xk>}$–accessible state where ϕ holds. In standard logic, this assertion is equivalent to any of its linearized versions $\exists x_1 \ldots \exists x_k \phi$. But with possible 'gaps' in our models, it is not so reducible. Polyadic quantification has linguistic interest (cf. Keenan & Westerståhl 1994), and it comes into its own here. Thus, in formalizing natural reasoning, one may now treat sequences of variables as either 'dependent' or 'independent'. Moreover, adding the latter expressive resource leaves the basic predicate logic CRS *decidable* (cf. Mikulas 1995). There is a more general issue, of course, as to how adding vocabulary affects meta-properties of a logic in our landscape. Adding too much expressive power might reinstate standard first-order logic. (Marx 1995, Mikulas 1995 provide some case studies manipulating expressive power of vocabulary in 'Arrow Logic'.) Polyadic language extensions also make sense in the presence of explicit *substitutions*. For instance, the latter needs both sequential composition and 'concurrent conjunction' (to deal with irreducibly polyadic *multiple substitutions* of the form $[t_1/x_1, \ldots, t_k/x_k]$).

Our style of analysis extends to other semantic parameters. E.g., not just assignments can change in Tarski semantics, but also *interpretation functions*. (van Benthem & Cepparello 1994, Cepparello 1995). A modest, but natural extension arises as follows.

Partial-State Frames One new modeling proposed in dynamic semantics (Beaver 1994, van den Berg 1991, Vermeulen 1994) employs *partial assignments*. These account for the intuitive difference between 're-assignment' R_x, changing an old value for x, and 'new assignment' R_x^+, giving x a value for the first time. These actions have corresponding first-order quantifiers $\exists x$ and $\exists^+ x$, respectively.

In partial-state frames, R_x will remain transitive and Euclidean, but not reflexive (x-values are not always defined). We only have the weaker principle $\exists x\, T \,\&\, \phi \rightarrow \exists x\, \phi$. By contrast, R_x^+ is asymmetric, and it satisfies, e.g., $\forall\alpha\beta\, (R_x^+\alpha\beta \rightarrow \neg \exists\gamma\, R_x^+\beta\gamma)$. The connection between the two variable update relations is the valid quantifier principle $\exists x\, T \leftrightarrow \neg\, \exists^+ x\, T$. A central new notion in these generalized frames is *extension* of partial states. It will have a natural corresponding existential modality:

$$\mathbf{M}, \alpha \models \Diamond\phi \qquad \text{iff} \qquad \exists\, \beta \supseteq \alpha : \mathbf{M}, \beta \models \phi$$

Using it, one can also define substitutes for $\exists^+ x\phi$ such as $\Diamond\exists x\phi$. It would be of interest to axiomatize the complete modal logic of standard partial assignment frames.

7 Applications and Repercussions

The present perspective suggests a number of applications. In particular, *how much* of standard predicate logic is involved in natural language, common sense reasoning, or mathematical proof? E.g., can the present decidable subcalculi of predicate logic supply a 'natural logic' here? (Cf. Sanchez Valencia 1991, whose key principles of monotonicity and conservativity are derivable in weak calculi in our landscape.) Also, are there useful decidable systems of arithmetic or other parts of mathematics using these ideas? E.g., what is the theory of the natural numbers with all possible families of variable assignments? Perhaps, the usual predicate-logical base for applied theories is too strong for its purpose (cf. van Benthem 1993A). Other practical aspects concern the 'distance' between standard and generalized models for the first-order language. It is known that CRS has the Finite Model Property (cf. Andréka, van Benthem & Németi 1994). Thus, well-known formulas whose standard satisfaction enforces infinity must have finite generalized models. Do the latter have any practical uses?

The thrust of our modal program can also be extended. It does not just apply to the dynamics of changing variable assignments, but also to *updating information states*. Abstract models can carry further structure, such as 'composition' of states, which supports new dynamic connectives (van Benthem 1991, Kurtonina 1994). Similar issues to those discussed here will arise then, now affecting also the *propositional* base of first-order predicate logic – which remained inviolate in our analysis so far.

By way of conclusion, here is what we take to be the philosophical importance of this work. If our abstract models are indeed the natural semantics for first-order predicate logic, rather than a technical device, many received views of the field are challenged. In standard text books, 'predicate-logical validity' is one unique notion, specified definitively by Tarski, and justified by Gödel's Completeness Theorem. Moreover, it is complex, being undecidable by Church's Theorem: Leibniz's ideal of a 'Calculus Ratiocinator' just will not work. On the present view, however, 'standard predicate logic' has arisen historically from several semantic decisions that could have gone differently. The genuine logical core of first-order predicate reasoning may well be decidable – and the real interest lies not in one unique 'completeness theorem', but in the combined model-theoretic and proof-theoretic analysis of a rich family of options.

The remainder of this paper is a more technical exploration of the above framework (especially, CRS) with techniques from modal logic. These give a feel for how it really 'works'. Issues covered include (**i**) *representation* of abstract modal state frames in terms of concrete (generalized) Tarskian assignment models, (**ii**) *decidability* of logics over such generalized semantics via filtration and unraveling, (**iii**) weak and strong *interpolation* properties for weak predicate logics, (**iv**) *extended languages* for substitutions, (**v**) *effective translations* between varieties of dependency semantics, and (**vi**) 'generalized generalized semantics' employing *updates on assignment sets.*

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8 Representation

A systematic semantic view analyzes what it takes to represent any abstract modal frame as a family of assignments with the standard variable update relation $=_x$. The following proposal is very simple, and probably equivalent to some algebraic method. How can abstract states become assignments? The obvious idea is to create 'objects' (α, x) for each state α and variable x, and then set

$$\alpha^* (x) \quad := \quad (\alpha, x) .$$

This stipulation will indeed turn states into assignments, and represent abstract state frames as assignments frames with arbitrary abstract update relations R_x. (Thus, the latter option is not really different from the most general one.) But if the latter relations are to become the standard updates $=_x$, then some refinement is necessary.

Representing State Frames

For a start, we assume all universal properties of standard assignment frames. What is needed on the way will eventually be collected in the statement of our results. Let Z denote some sequence of variables. Extend the notion of accessibility as follows:

$$\alpha \, R_{\varnothing} \, \beta \quad := \quad \alpha = \beta$$

$$\alpha \, R_{Z \bullet y} \, \beta \quad := \quad \exists \gamma : \alpha \, R_Z \, \gamma \;\; \& \;\; \gamma \, R_y \, \beta$$

We use sequences here rather than sets, because we do not assume the existential quantifier interchange principles suppressing the ordering. Now define

$$(\alpha, x) \sim (\beta, y) \qquad \text{if} \qquad x=y \ \ \& \ \ \exists Z\!: \ x \notin Z \ \ \& \ \ \alpha \, R_Z \, \beta$$

It is easy to check that $\sim$ is an equivalence relation, using the symmetry of the update relations R_Z . This observation allows us to use equivalence classes for values:

$$\alpha^* (x) \qquad := \qquad (\alpha, x)^{\sim} .$$

Now, let us analyze what it takes to prove the following key equivalence:

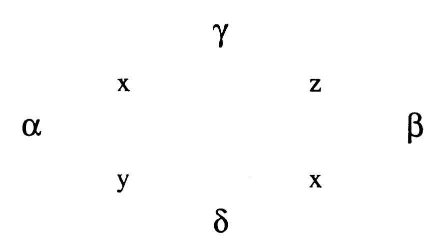
Adequacy of Representation

$$\alpha \ R_x \ \beta \qquad \textit{iff} \qquad \alpha^* =_x \beta^*$$

The direction from left to right is immediate. Let y be any variable distinct from x . Set $Z = \langle x \rangle$. Then, since $\alpha \ R_x \ \beta$, by the above definition, $(\alpha, y) \sim (\beta, y)$, and hence $\alpha^* (y) = \beta^* (y)$. From right to left, suppose that $\alpha^* =_x \beta^*$. By definition, this means that $\forall y \neq x \ \exists Z\!: \ y \notin Z \ \& \ \alpha \ R_Z \, \beta$. What we want from this bunch of facts is $\alpha \, R_x \, \beta$. Here is a special case. With only *two* variables, the latter information applied to the variable y says that α, β are related via some finite sequence (possibly empty) of R_x-steps . Using only reflexivity and transitivity, then, we get the desired conclusion. Thus, we have found (as more often in the algebraic literature) that the two-variable fragment of predicate logic is particularly simple:

<u>Proposition</u> With only two variables, an abstract state frame is representable as an assignment frame iff its relations R_x are equivalence relations.

The general situation is more complex. E.g., with *three* variables, we may have:



$Z_y = \langle x, z \rangle$

$Z_z = \langle y, x \rangle$

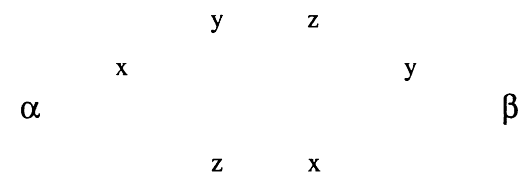
In the standard assignment model, this implies that α, β agree on both y and z , whence the arrows for y and z must be identity transitions, and we have $\alpha \ R_x \ \beta$. More generally, all *Path Principles* of the following form are valid under the standard Tarskian interpretation (notice that there are infinitely many of these):

- *If* $\alpha\, R_{Z1}\, \beta$, ..., $\alpha\, R_{Zk}\, \beta$, *and the only variable occurring in all of* $Z_1, ..., Z_k$ *is* x , *then* $\alpha\ R_x\ \beta$.
- *If no variable occurs in all connecting sequents, then* $\alpha = \beta$.

Proposition   An abstract frame is representable as an assignment frame iff its relations R_x are equivalence relations satisfying all Path Principles.

Proof Continue the above argument. Suppose that $\forall y \neq x\ \exists Z\!: \ y \notin Z\ \ \&\ \ \alpha\, R_Z\, \beta$. Let y be any specific variable distinct from x , and select a connecting path Z_y . For any of the (finitely many!) variables u occurring on this path distinct from x , select some connecting path Z_u on which u fails to occur. Then x is the only variable in the intersection, and the path principle for Z_y and the Z_u's will say that $\alpha\ R_x\ \beta$. ■

Three points may clarify this. (1) Transitivity for relations R_x follows from the Path Principles. (2) Reflexivity is needed when the intersection of all occurrence sets of variables on the paths is empty. (3) One should take care with the Path Principle. For instance, it does *not* say that the two R_y-transitions displayed must be identical ones:



Next, the second Path Principle also implies that our representation is one-to-one.

Fact   In the above case, the map from states α to assignments α^* is injective.

Proof (We need at least two variables x, y .) Suppose that $\alpha^* = \beta^*$. Then we have, in particular, that $\alpha^* =_x \beta^*$ and $\alpha^* =_y \beta^*$. By the above observation, this implies that $\alpha\ R_x\ \beta$ and $\alpha\ R_y\ \beta$. But then, by the second Path Principle, $\alpha = \beta$. ■

Analyzing this simple representation from a logical point of view – especially, the crucial family of Path Principles – we see the following:

- The class of representable abstract frames is definable by a set of first-order sentences which are all *universal Horn.*
- This definition employs infinitely many frame conditions.

The former property has pleasant consequences, including Interpolation for predicate-logical validity over this frame class (by general results in modal logic; Section 10).

The second property hints at a certain complexity (cf. the non-finite axiomatizability result by Andréka). Finally, it is easy to see that few Path Principles correspond to a modal formula in the predicate-logical language. This completes our analysis of CRS.

Remark   Sets Instead of Sequences
In the full standard case, with the two quantifier exchange axioms, it suffices to define a relation $\alpha\, R_X\, \beta$, where X is a *set* of variables, postulating some connecting sequence of transitions indexed by variables in X. The Path Principles then reduce to

$$\textit{if } \alpha\, R_X\, \beta \textit{ and } \alpha\, R_Y\, \beta, \textit{ then } \alpha\, R_{X\cap Y}\, \beta \;.$$

Representing State Models
Our representation extends to models $\mathbf{M}$ that interpret structured atomic formulas. These are abstract frames $(S, \{R_x\}_{x\in VAR})$ plus an interpretation function I interpreting atoms over states (a 'modal valuation'), needed to interpret the predicate-logical object language. Define the following standard interpretation function over the represented frame (with one binary predicate letter Q, for convenience):

$$I^*(Q) \quad = \quad \{ \, ((\alpha, x)^\sim, (\alpha, y)^\sim) \mid I(\alpha, Qxy) \, \}$$

Whe need to show the following assertion of adequacy:

Claim $\mathbf{M}, \alpha \models \phi$ iff $\mathbf{M}^*, \alpha^* \models \phi$,   for all predicate-logical ϕ.

Unfortunately, we do not quite succeed. The following is as far as we get.

Proof Attempt   The assertion is automatic for Booleans, and it holds for quantifiers by the above proof. The atomic case presents a difficulty, though. From left to right, its assertion is trivial. If $\mathbf{M}, \alpha \models Qxy$, then $I(\alpha, Qxy)$ holds, and hence $I^*(Q)$ holds of $(\alpha, x)^\sim, (\alpha, y)^\sim$ (i.e., $\alpha^*(x), \alpha^*(y)$) by definition. From right to left, however, we encounter an obstacle. Let $I^*(Q)$ hold in $\mathbf{M}^*$ of $\alpha^*(x), \alpha^*(y)$, that is, of $(\alpha, x)^\sim, (\alpha, y)^\sim$. Thus, there exists γ with $(\alpha, x) \sim (\gamma, x)$, $(\alpha, y) \sim (\gamma, y)$ such that $I(\gamma, Qxy)$. By the definition of $\sim$, then, there exist two finite sequences Z_x (not containing x) and Z_y (not containing y) with $\alpha\, R_{Z_x}\, \gamma$, $\alpha\, R_{Z_y}\, \gamma$. Now, what we need to show is that $I(\alpha, Qxy)$. Here, evidently, the earlier atomic invariance principles $Py \rightarrow \forall x\, Py$ and $\neg Py \rightarrow \forall x \neg Py$ should help. But these are not strong enough. We need a more complex path principle stating that $Qxy \rightarrow [Z_x \cap Z_y]Qxy$. This is beyond our modal predicate-logical language, however – as it involves what is essentially a further operation of 'program intersection'. ■

One way of overcoming this difficulty uses an extension of our representation to a richer predicate-logical language. Two options are presented in digressions below. Westerståhl 1995 presents the most elegant solution so far, combining our previous representation with ideas from Section 9 below. One can extend these representation arguments to abstract frames with transition relations $A_{x,y}$ reflecting the earlier *substitution*. We forego this extension here (Section 11 has some relevant details).

Option 1 Pointwise Equality of States

The following useful relation turns up implicitly in the above arguments:

$$\alpha R^x \beta \qquad \text{iff} \qquad \alpha(x) = \beta(x)$$

This suggests the use of enriched state models $(S , \{R_x\}_{x\in VAR} , \{R^x\}_{x\in VAR} , I)$. The new relations R^x are easier to handle than the old R_x , being equivalence relations. They can be used to define the latter, via the following equivalence:

$$\alpha R_x \beta \qquad \leftrightarrow \qquad \&_{y\neq x} \ \alpha R^y \beta$$

With this definition, all Path Principles are simply derivable. Representation becomes much easier, with objects as equivalence classes for the new relations R^x . But there are draw-backs. The above equivalence is *infinitary* for general first-order languages. Also, the existential modality $\langle\langle x \rangle\rangle$ for the new relations, though interesting, is less natural from a predicate-logical point of view. For $\phi = \phi (x, \mathbf{y})$, $\langle\langle x \rangle\rangle \phi$ becomes like an existential closure $\exists \mathbf{y}\, \phi (x, \mathbf{y})$ ("for some values of all other parameters").

Option 2 Polyadic Quantifiers

Another natural extension of our framework (cf. Section 6) uses indifference relations between states involving finite sets of variables X :

$$\alpha R_X \beta \qquad \text{iff} \qquad \alpha(y) = \beta(y) \text{ for all variables } y \text{ outside of } X$$

The corresponding 'polyadic' first-order quantifiers $\exists X\, \phi$ are no longer equivalent to sequential forms $\exists x_1 \ldots \exists x_k\, \phi$. In this case, we can use the old representation, setting $(\alpha, x) \sim (\beta, x)$ iff there exists some finite set Y not containing x with $\alpha R_Y \beta$. Moreover, the earlier Path Principles may be replaced by the following ones:

- $\alpha R_X \beta$ and $\alpha R_Y \beta$ imply $\alpha R_{X\cap Y} \beta$
- $\alpha R_X \beta$ and $\beta R_Y \gamma$ imply $\alpha R_{X\cup Y} \gamma$

A typical law of the corresponding polyadic predicate logic is $\phi \rightarrow \forall X\, \phi$ *(provided that no free variable in* ϕ *occur in* Z).

9 Decidability

This Section presents a new proof for decidability and finite model property of CRS (without substitutions) – using modal filtration and unwinding, plus representation.

Filtration of Generalized Assignment Models

Consider generalized models $(\mathbf{M}, \mathbb{V})$ for predicate logic, where $\mathbb{V}$ is the range of 'available assignments'. Here are two relevant extra constraints. 'Atomic Locality' says that assignments $\alpha, \beta \in \mathbb{V}$ which agree on all free variables $FV(\phi)$ of an atomic formula ϕ must give ϕ the same truth value. 'Locality' says the same for all formulas. In what follows, we fix some formula ϕ with variables VAR_ϕ (free or bound) and subformulas SUB . Everything will be restricted to such *finite* syntax sets. First, we define a multi-S5 finite filtration over generalized assignment models.

<u>Definition</u> For $\alpha, \beta \in \mathbb{V}$, set $\alpha \sim \beta$ if α, β give the same truth values to all formulas in SUB. The *ϕ–filtration* of $(\mathbf{M}, \mathbb{V})$ is the Kripke model $FILT_\phi(\mathbf{M}, \mathbb{V}) = (S, \{R_x\}_{x \in VAR}, V)$ obtained as follows. State Universe: S consists of all ~–equivalence classes $\alpha^\sim$. Accessibility: $\alpha^\sim R_x \beta^\sim$ holds if α, β give the same truth value to all relevant formulas $\exists x\psi$ and to all relevant atomic formulas not containing x. Valuation: $V(\alpha^\sim, \psi) = 1$ for relevant atoms ψ iff ψ is true at α in $(\mathbf{M}, \mathbb{V})$.

<u>Filtration Lemma</u> For all relevant formulas ψ and all assignments α ,

$$(\mathbf{M}, \mathbb{V}), \alpha \models \psi \quad \text{iff} \quad FILT_\phi(\mathbf{M}, \mathbb{V}), \alpha^\sim \models \psi .$$

<u>Proof</u> Induction on ψ . Atoms: by definition. Booleans: use routine. Existentials. By the truth definition in generalized models, $(\mathbf{M}, \mathbb{V}), \alpha \models \exists x\psi$ implies that there exists some $\beta \in \mathbb{V}$ with $\alpha =_x \beta$ & $(\mathbf{M}, \mathbb{V}), \beta \models \psi$. Hence (by the inductive hypothesis) $FILT_\phi(\mathbf{M}, \mathbb{V}), \beta^\sim \models \psi$, and also $\alpha^\sim R_x \beta^\sim$ (mapping assignments to their equivalence classes is a homomorphism). Therefore, $FILT_\phi(\mathbf{M}, \mathbb{V}), \alpha^\sim \models \exists x\psi$. Conversely, $FILT_\phi(\mathbf{M}, \mathbb{V}), \alpha^\sim \models \exists x\psi$ implies the existence of $\beta^\sim$ such that $\alpha^\sim R_x \beta^\sim$ and $FILT_\phi(\mathbf{M}, \mathbb{V}), \beta^\sim \models \psi$. Then, by the inductive hypothesis, $(\mathbf{M}, \mathbb{V}), \beta \models \psi$ – and so $(\mathbf{M}, \mathbb{V}), \beta \models \exists x\psi$, whence $(\mathbf{M}, \mathbb{V}), \alpha \models \exists x\psi$ by definition of the relations R_x . ■

Filtration also works for generalized models with Locality (for all relevant formulas), to yield a finite model with that property. One makes two equivalence classes R_z–accessible when they agree on all formulas in which variable z does not occur free.

Unwinding Kripke Models

The above (filtrated) Kripke models are abstract. They may lack some key properties of generalized assignment models. Notably, the earlier 'Path Principles' may fail.

(E.g., there may be two different links R_x, R_y between two distinct states.) We can improve this behaviour by path unravelling, to get a basis for concrete representation.

Definition The *unwinding* UNW(**M**) of a rooted Kripke model (**M**, s) consists of all finite sequences $(s, x_1, ..., s_{k-1}, x_{k-1}, s_k)$ where all s_i are worlds in **M**, and always $s_i\ R_{xi}\ s_{i+1}$. The relations R_{xi} are the reflexive symmetric transitive closures of the relations consisting of all pairs $(X, X \frown \langle z, w\rangle)$ with $\text{last}(X)\ R_z\ w$ in **M** . Finally, the valuation V for sequences X is copied from that for last(X) in **M** .

Unwinding Lemma For all formulas ψ, and all sequences X ,

$$\text{UNW}(\mathbf{M}), X \models \psi \quad \text{iff} \quad \mathbf{M}, \text{last}(X) \models \psi .$$

Proof The function sending X to last(X) is a bisimulation. ■

The only non-routine fact here is that the map 'last' is a homomorphism with respect to the relations R_z in the unwinding. (This part of the argument will work as long as our frame conditions are universal Horn.) One further observation may be made.

Corollary Formulas satisfiable in finite Kripke models are also satisfiable in finite unwound Kripke models.

Proof This is the multi–S5 version of the well-known modal Finite Depth Lemma. Evaluating a formula from the root involves only finitely many alternations in depth across different relations R_z – as may be seen through normal forms for multi-S5. ■

This 'cut-off' at the modal depth of ϕ preserves Atomic Locality (in its abstract Kripke version, as a constraint on the valuation V) – though not necessarily full Locality. Finally, we note that unwound Kripke models do satisfy all Path Principles.

Representing Unwound Kripke Models

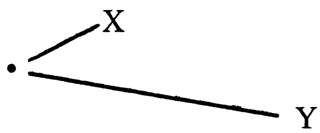
Unwound multi–S5 models can be represented as generalized assignment models. (This is a more concrete version of the representation in Section 8.) The idea is easily explained. Take an arbitrary assignment (x_i, d_i) $(1 \leq i \leq k)$ of different objects to the relevant variables at the root. Then, follow longer sequences X upward. If an assignment ass(X) has already been defined, then choose a supply of *new objects*, and change values at z only for steps from X to $X \frown \langle z, w\rangle$. This is well-defined.

Definition The *object representation* OBJ(**M**) of an unwound Kripke model **M** has just been described. Its admissible assignments are those produced in the process. Its interpretation function I(Q) for predicates Q collects all tuples **d** contributed by those ass(X) where some atom Q**z** was true in **M** at X .

Representation Lemma For all formulas ψ , and all sequences X,

$$\mathbf{M}, X \models \psi \quad \text{iff} \quad OBJ(\mathbf{M}), ass(X) \models \psi \ .$$

Proof The map from X to ass(X) is a bisimulation. Atomic clause. If $\mathbf{M}, X \models Q\mathbf{z}$, then $OBJ(\mathbf{M}), ass(X) \models Q\mathbf{z}$, by the definition of I(Q) . Next, if $OBJ(\mathbf{M}), ass(X) \models Q\mathbf{z}$, then by that same definition, $\mathbf{M}, Y \models Q\mathbf{z}$ for some sequence Y whose ass(Y) agrees with ass(X) on all variables $z \in \mathbf{z}$. Hence, by construction, X, Y are equal or connected by a chain of relations R_u with u outside of $\mathbf{z}$. Atomic Locality, i.e., the truth of all formulas $Q\mathbf{z} \to \forall u Q\mathbf{z}$ in $\mathbf{M}$, then yields $\mathbf{M}, X \models Q\mathbf{z}$. (Unicity of the relevant atoms is guaranteed by our 'free' assignment of objects.) Zigzag clauses. Inspection of the above construction shows that, if $X R_z Y$, then $ass(X) =_z ass(Y)$ – and also vice versa. From left to right, this is easy. From righ to left, suppose that *not* $X R_z Y$. Then, in the most general case, they must lie in some tree situation



where on the minimal connecting path shown, some value has changed for a variable u distinct from $\mathbf{z}$. As our representation chooses different objects all the time, upwards and sideways in branches, this difference will still show up in the pair ass(X), ass(Y): which therefore lacks the relationship $=_z$. ■

We can now apply these results to obtain

Theorem Validity in generalized assignment semantics is decidable.

Proof Combining the previous facts, a formula ϕ is satisfiable in a generalized assignment model iff it has a finite abstract Kripke model satisfying Atomic Locality whose size is bounded by $2^{|SUB\phi|}$. The latter property is decidable. ■

Theorem Generalized assignment semantics has the Finite Model Property.

Proof By the above Cut-Off Property, finite generalized models will suffice. ■

In Andréka, van Benthem & Németi 1995, this reasoning is also applied to obtain decidability for large 'bounded fragments' of predicate logic over standard models. That paper also investigates some connections in the following heuristic equation:

full predicate language : generalized dependency semantics
= bounded quantifier fragments: standard semantics.

10 Interpolation

We show how a typical meta-property of predicate logic fares in our modal landscape. Modal interpolation theorems come in different forms. The usual version of Craig Interpolation concerns just shared proposition letters (Weak Interpolation):

> If $\phi \models \psi$, then there exists some modal formula α over the shared propositional vocabulary of ϕ, ψ (including T) such that $\phi \models \alpha \models \psi$.

Strong Interpolation requires, in addition, that the interpolant α contain only modal operators that are shared between ϕ and ψ. Both versions hold for the basic poly-modal logic. Here is a proof sketch (cf. Andréka, van Benthem & Németi 1995).

Weak Interpolation
Consider shared proposition letters only. Let $cons_{\phi\psi}(\phi)$ be the set of valid modal consequences of ϕ in the language over the shared proposition letters. We show that

$$cons_{\phi\psi}(\phi) \models \psi$$

Then the interpolant comes from $cons_{\phi\psi}(\phi)$ by Compactness. So, let **M**, x be any model for the language L_ψ satisfying $cons_{\phi\psi}(\phi)$. By a standard model-theoretic argument, we find a model **N**, y for the language L_ϕ with the same modal theory as **M**, x in the shared language $L_{\phi\psi}$ where ϕ holds. Now, move from these models to ω-saturated elementary extensions $\mathbf{M}^+$, x and $\mathbf{N}^+$, y. The states x, y still share the same $L_{\phi\psi}$–theory in the latter models – and moreover, this sharing relation $\equiv$ is an $L_{\phi\psi}$–bisimulation between such states. Now, consider the following model **MN**. Its states are pairs (u, v) with u in the domain of $\mathbf{M}^+$ and v in the domain of $\mathbf{N}^+$ such that $u \equiv v$. Its relations R_i are the usual ones in a direct product: R_i (u, v) (u', v') iff R_i uu' and R_i vv'. Here, the two obvious projections are L_ψ– and L_ϕ–p-morphisms from **MN** onto the generated submodels of $\mathbf{M}^+$ and $\mathbf{N}^+$ by x and y, respectively. For proposition letters in the common language $L_{\phi\psi}$, we can define an unambiguous valuation on these pairs (since the bisimulation $\equiv$ left these invariant). For proposition letters in $L_\phi - L_\psi$ and $L_\psi - L_\phi$ we obtain suitable interpretations, too, by copying along the projections. The result is a model for the full language $L_\phi \cup L_\psi$, which has an L_ψ–bisimulation with $\mathbf{M}^+$, x and an L_ϕ–bisimulation with $\mathbf{N}^+$, y. Then we can argue as follows. ϕ holds in **N**, y (by construction) and therefore in $\mathbf{N}^+$, y (by L_ϕ –elementary extension), **MN**, <x, y> (L_ϕ–bisimulation). Since $\phi \models \psi$ (here is where we use our key assumption), ψ must be true in **MN**, <x, y>, and hence also in $\mathbf{M}^+$, x (L_ψ–bisimulation) and **N**, x (L_ψ–elementary submodel). ■

This proof goes through for any modal logic whose characteristic frame class is defined by *universal Horn conditions*, since these are preserved under submodels and direct products of frames. The essential model **MN** in the above argument is a submodel of a direct product. (It is a categorial 'pull-back': cf. Marx 1995 for further category-theoretic background.) Note that this situation cannot be too common, since Interpolation is known to be scarce for modal logics (Maksimova 1979).

Strong Interpolation

To prove this stronger property, the above bisimulation $\equiv$ can be merely assumed to satisfy zigzag clauses with respect to accessibility relations R_i whose modalities <i> occur in both ϕ and ψ . But since we need full L_ψ– and L_ϕ–bisimulations connecting **MN** and $\mathbf{M}^+$, x and $\mathbf{N}^+$, y , respectively, we have to modify the product model. Here is a sketch. In particular, the new model needs enough successors to verify the zigzag clauses for all accessibility relations of modalities in L_ϕ–L_ψ and L_ψ–L_ϕ . For this purpose, one adds disjoint copies of both $\mathbf{M}^+$ and $\mathbf{N}^+$, making the obvious links (u, v) R_i v' if v R_i v' towards the $\mathbf{M}^+$ copy, for all modalities <i> in L_ϕ–L_ψ – and similarly for L_ψ–L_ϕ towards the copy of $\mathbf{N}^+$. For this extended frame, the obvious projections to $\mathbf{M}^+$, $\mathbf{N}^+$ are again bisimulations of the right kind, whence the final argument goes through as before. This method can be made to prove Strong Interpolation for the *minimal poly-modal logic.*

With additional frame conditions, however, even universal Horn clause ones, matters may be much more complicated. For instance, the modal logic with axiom <1>ϕ $\rightarrow$ <2>ϕ has weak interpolation, but it evidently lacks the strong version. With simple Horn clause conditions, the above proof may work. In particular, for modal *multi–S5*, where all R_i are equivalence relations, the preceding construction works with two extra stipulations. In the original product part, one must add all links (u, v) R_i (u', v') with v R_i v' for all <i> in L_ϕ–L_ψ and L_ψ–L_ϕ – while all earlier links between that part and the two copies of $\mathbf{M}^+$, $\mathbf{N}^+$ are to be made symmetric. The resulting model is fit for multi–S5, and the above projections are suitable p-morphisms automatically. We sum up our results in the following

<u>Theorem</u> Minimal Predicate Logic and CRS have Strong Interpolation.

<u>Proof</u> Minimal predicate logic is just our minimal poly-modal logic. For CRS, one further, perhaps surprising, observation is needed (cf. Németi 1991, Thm 8).

<u>Fact</u> The complete modal logic of CRS is nothing but multi–S5.

In one direction, all S5–laws are clearly validated by the Path Principles. But also conversely, any model for multi-S5 can be *unravelled* to one which satisfies all Path Principles for free. This requires careful unraveling by sequences to make sure that worlds share loops for all relations R_i , while apart from that, all proper successor steps are to be unique for each such relation. More precisely, the new worlds become finite sequences of worlds $< ..., w, i, v , ... >$, whose immediate successor steps select some R_i-successor v of w , marking this transition uniquely. Over these sequences, the new relation R_i is defined as the reflexive, symmetric transitive closure of the set of all tuples $(X, <X, i, y>)$. It follows that two sequences X, Y can only be related via some finite sequence of transitions (using possibly different indices) iff Y can be reached from X by first dropping successive X-tails, and then adding new tails. (There is a unique shortest link of this kind. Longer connections may arise by making excursions en route.) This observation implies the Path Principles for CRS. If there exists a route between X and Y in which a relation R_i is missing, then the minimal connecting path does not involve R_i . Repeating this for any given finite family of linkings, a pure minimal connection must exist for the remaining modality. ■

By this Fact, Strong Interpolation for CRS follows from that for multi-S5. ■

This argument resembles the representation of Section 9. Over models, it axiomatizes CRS as multi-S5 plus Atomic Locality. It also shows that virtually no Path Principle has a modal definitions over state frames. The axiomatic description of CRS becomes more complicated with accessibility relations for substitutions, and other vocabulary for a full predicate-logical language (cf. Németi 1991, 1993, Mikulas 1995). Standard predicate logic lacks Strong Interpolation. The valid consequence $\exists xPx \models \exists yPy$ has no interpolant *in shared variables.* For details on interpolation properties for standard predicate logic, its finite variable fragments, and cylindric-algebraic approximations, cf. Sain 1990, Marx 1995. This negative outcome is compatible with the above positive result. Under translation of modal formulas into a first-order state language, the previous result is like standard Interpolation, but with respect to the *accessibility relations* for the shared modalities.

11 Substitutions Revisited

This Section lists some supplementary observations on substitutions and assignments.

Valid principles in a 'pure substitution calculus'

These include the following, assuming that the relations $A_{x,y}$ are total functions:

x:=y ; u:=v	↔	u:=v ; x:=y	x:=y ; x:=v	↔	x:=v
x:=y ; u:=x	↔	x:=y ; u:=y	x:=x	↔	*id*

Extending the earlier representation method for abstract state frames

For representing abstract frames $(S, \{A_{x,y}\}_{x, y \in VAR})$, one can use either fixed points α with $A_{x,y}\, \alpha\alpha$, or the earlier method of equivalence relations:

$$(\alpha, x) \sim (\beta, x) \quad : \quad \exists y \neq x\ \exists z : \alpha\, A_{y,z}\, \beta$$
$$(\alpha, x) \sim (\beta, y) \quad : \quad \alpha\, A_{y,x}\, \beta$$

and take the reflexive symmetric transitive closure. Again, Path Principles arise in the analysis of the key equivalence

$$\alpha\, A_{x,y}\, \beta \qquad \textit{iff} \qquad \beta^* = \alpha^{*\,x}{}_{\alpha^*(y)}$$

When this representation is combined with the earlier one, over modal state frames $(S, \{R_x\}_{x \in VAR}, \{A_{x,y}\}_{x, y \in VAR})$, adjustments are needed for the other equivalence:

$$\alpha\, R_x\, \beta \qquad \textit{iff} \qquad \alpha^* =_x \beta^*$$

These involve several earlier modal interaction principles between substitutions and quantifiers. The functions $A_{x,y}$ are not very complex. For instance, in CRS, an 'existential principle' like $[y/x]\, \exists z\, \phi \leftrightarrow \exists z\, [y/x]\, \phi$ (modulo distinctness) can be treated quasi-universally (cf. Marx 1994). On standard models, the principles of (1) contract finite sequences of substitutions to normal forms for standard simultaneous substitutions $\mathbf{x} := \mathbf{u}$ (with all x_i distinct, and no u_i occurring among the x_i).

Enriching the language

Again, these models suggest introducing richer languages. Notably, the relations $A_{x,y}$ are not symmetric, and hence it makes sense to also look *backward* along them. This requires a 'temporal logic' with two directions for substitution:

$\mathbf{M}, \alpha \models F_{y/x}\, \phi$	iff	$\exists\beta : \alpha\, A_{x,y}\, \beta\ \&\ \mathbf{M}, \beta \models \phi$	*Future*
$\mathbf{M}, \alpha \models P_{y/x}\, \phi$	iff	$\exists\beta : \beta\, A_{x,y}\, \alpha\ \&\ \mathbf{M}, \beta \models \phi$	*Past*

As an illustration, take the Hoare-style assignment axiom for program correctness:

$$\{\ [t/x]\, \phi\}\ \ x{:=}t\ \ \{\phi\}$$

In our temporal logic, this is the basic conversion axiom (H: "has always been"):

$$\phi \rightarrow H_{t/x}\, F_{t/x}\, \phi$$

One can also express standard identity statements using backward modalities:

$$x{=}y \quad \leftrightarrow \quad P_{y/x}\, T\,.$$

The backward substitution modality combines identity and ordinary quantification:

$$P_{y/x}\ \phi(x, y) \quad \leftrightarrow \quad x=y\ \&\ \exists z\ \phi(z, y)$$

With this additional expressive power, it would be of interest, even in standard predicate logic, to axiomatize a version of this back-and-forth substitution calculus. The substitution calculus may also be extended to deal with multiple substitutions, as was suggested for polyadic quantifiers in Sections 6, 8. The same points apply.

12 Translations

Modal languages may be translated into first-order ones over standard state models. This reflects a broad perspective on dependency semantics relating different models and languages. Indeed, there are two main approaches towards 'taming' classical first-order logic, localizing a decidable 'core'. One uses standard semantics over non-standard 'bounded' language fragments, the other non-standard generalized semantics over the standard first-order language. The former approach is more 'syntactical' in nature, the latter more 'semantical'. (Eventually, as so often in logic, this distinction is relative. For instance, one can also translate 'semantic' modal discourse about the above modal first-order models into a restricted syntactic fragment of a *two-sorted* first-order logic, with direct reference to both 'individuals' and 'states'. But also conversely, ... etcetera.) There is a mathematical duality lurking in the background here, largely unexplored – which we illustrate by some simple observations from Andréka, van Benthem & Németi 1995, which involve *one-sorted* translations.

From Bounded Fragments to Generalized Models

Consider any k–variable language $L\{x_1, ..., x_k\}$. Let R be a new k–ary predicate. We define a translation tr_g from k–variable formulas to bounded first-order ones:

Global Relativization

$tr_g(\phi)$ arises from ϕ by relativizating all its quantifiers to the atom $Rx_1...x_k$

Next, we define a corresponding operation on models. Let **M** be any generalized assignment model for $L\{x_1, ..., x_k\}$ (as yet without the new predicate symbol R).

Restricted Standard Models

The standard model $\mathbf{M}_{rest}$ is **M** , viewed as a standard model,
and expanded with the following interpretation for the new predicate:
$R(d_1, ..., d_k)$ iff the assignment $x_i := d_i$ $(1 \leq i \leq k)$ is available in **M**.

The purpose of this construction shows in the following fact.

Proposition For all available assignments α in $\mathbf{M}$, and all formulas ϕ,

$$\mathbf{M}, \alpha \models \phi \quad \text{iff} \quad \mathbf{M}_{\text{rest}}, \alpha \models tr_g(\phi)$$

As a consequence, one can effectively reduce universal validity over all generalized assignment models (i.e., in CRS) to standard validity for bounded formulas.

Corollary $\models_{CRS} \phi$ iff $\models_{standard} Rx_1 \ldots x_k \rightarrow tr_g(\phi)$

There is more to this analysis. Special classes of generalized assignment models arise by imposing constraints on admissible assignments. The first-order theory of such classes, too, will be decidable, as long as their additional conditions can be stated into suitably bounded first-order forms. In particular, this applies to so-called 'locally square' generalized assignment models, in which every permutation or identification of valuse in an admissible assignment yields another admissible assignment. (These are needed for the full substitution version of CRS.) So far, we know less about converse translations, running from bounded fragments to generalized semantics.

Translations help in comparing different models for dependency. Recall the analysis of generalized quantifiers in Section 2. The latter arises from first-order logic through a 'local translation' tr_l like the 'global translation' tr_g, but with a delicate difference. At subformulas $\exists x_i\, \psi$, one only relativizes to an atom $R\mathbf{x}$ where $\mathbf{x}$ enumerates all free variables of the local context ψ. This difference explains all deviant behaviour. E.g., Tr_g makes Modal Distribution a valid bounded principle, whereas tr_l does not:

$$\forall y\ (\forall x\ (Ax \rightarrow Bxy) \rightarrow (\exists x Ax \rightarrow \exists x Bxy))$$

tr_g $\quad Rxy \rightarrow \forall y\ (Rxy \rightarrow (\forall x\ (Rxy \rightarrow (Ax \rightarrow Bxy)) \rightarrow (\exists x\ (Rxy\ \&\ Ax) \rightarrow \exists x\ (Rxy\ \&\ Bxy))))$

tr_l $\quad \forall y\ (Ry \rightarrow (\forall x\ (Rxy \rightarrow (Ax \rightarrow Bxy)) \rightarrow (\exists x\ (Rx\ \&\ Ax) \rightarrow \exists x\ (Rxy\ \&\ Bxy))))$

13 Higher Dependency Models

Generalized assignment semantics can be taken further, for richer languages. We discuss a logical system inspired by the account of plurality in van den Berg 1995.

From Singular to Plural States

The semantic literature on collectives and plurals uses assignments mapping variables to sets of objects. Thus, states move up, from type $(v \rightarrow e)$ to type $(v \rightarrow (e \rightarrow t))$. But in a next step, one can identify states with sets of standard assignments, in the type $((v \rightarrow e) \rightarrow t)$. This allows for finer discrimination, with possible dependencies between values for individual variables (needed to account for linguistic anaphora). In set-theoretic terms, from a repeated power $(2^{DOM})^{VAR}$, we go to $2^{(DOM^{VAR})}$.

Generally speaking, the latter will be much larger in size – which reflects our greater freedom for encoding dependencies between objects assigned. Thus, the above generalized assignment models re-emerge from a quite different angle. There are some natural connections between the two state domains. The following map sends plural assignments A to sets S(A) of individual assignments:

$$S(A) \quad = \quad \{ f \mid \text{for all } x, f(x) \text{ is in } A(x) \}$$

Another map sends sets S of individual assignments to plural assignments A(S):

$$A(S) \quad = \quad \lambda x \bullet \{ f(x) \mid \text{all } f \text{ in } S \} .$$

The map S delivers special 'full' sets of assignments. It is 1–1, unlike the map A . The difference between the two levels depends on the formal language interpreted over them. With a standard first-order predicate logic for plurality, nothing changes. This is the import of the Equivalence Theorem in van den Berg 1995, for a language having new operators for 'individualization' $\delta_x \bullet \phi$ and 'participation' $\pi_x \bullet \phi$. E.g.,

$\mathbf{M}, S \models \delta_x \bullet \phi$ iff *for some set* S' *which consists of all functions in* S *set to one specific individual value* d *for* x, $\mathbf{M}, S' \models \phi$

Richer Logics of Dependency

'Generalized assignment models' interpret first-order languages in a traditional format:

$\mathbf{M}, S, \alpha \models \phi$ (*singular state* α *verifies* ϕ *in 'plural context'* S)

But we can also interpret in the following format:

$\mathbf{M}, S \models \phi$ (*plural state* S *itself verifies formula* ϕ)

where formulas ϕ may involve *new* logical operators, exploiting the richer structure of collective states. We have at least two kinds of existential quantification now, reflecting two natural transition relations over states:

$\exists_{coll}\, x \bullet \phi$ is true at S iff ϕ is true at some S' with $S =^x S'$, i.e., S, S' have the same assignments up to values for the variable x

$\exists_{ind}\, x \bullet \phi$ is true at S iff ϕ is true at some S' with $S =_x S'$, i.e., S, S' have the same assignments but all x-values in S' are set to one object.

The resulting modal logic encodes a theory of interaction between individual and collective quantification. It can be explored via modal frame correspondences, with axioms reflecting structural properties of and connections between the above two types of accessibility relation, say, $R_{coll, x}$ and $R_{ind, x}$ (for all variables x).

Example Plural Frame Correspondences

- All $R_{coll, x}$ are equivalence relations. Hence, each quantifier $\exists_{coll}\, x$ satisfies S5.
- The $R_{ind, x}$ are not equivalence relations. They are transitive, but not reflexive or symmetric. Thus, we have $\exists_{ind}\, x\bullet \exists_{ind}\, x\bullet \phi \rightarrow \exists_{ind}\, x\bullet \phi$, but not, e.g., $\phi \rightarrow \exists_{ind}\, x\bullet \phi$ (as ϕ might be true as a collective assertion, but not for any individual value of x) or $\exists_{ind}\, x\bullet \forall_{ind}\, x\bullet \phi \rightarrow \phi$. Even so, $R_{ind, x}$ satisfies further S5-like properties like:

$$\exists_{ind}\, x\bullet \phi \rightarrow \exists_{ind}\, x\bullet \exists_{ind}\, x\bullet \phi$$

$$\exists_{ind}\, x\bullet \forall_{ind}\, x\bullet \phi \leftrightarrow \forall_{ind}\, x\bullet \phi$$

- There are also interactions between $R_{coll, x}$ and $R_{ind, x}$. Each $R_{ind, x}$ is contained in $R_{coll, x}$, whence $\exists_{ind}\, x\bullet \phi \rightarrow \exists_{coll}\, x\bullet \phi$ (truth for an individual is a boundary case of collective truth). The converse fails, of course. Next, $R_{coll, x}$ followed by $R_{ind, x}$ reduces to an $R_{ind, x}$ step, and vice versa:

$$\exists_{coll}\, x\bullet \exists_{ind}\, x\bullet \phi \leftrightarrow \exists_{ind}\, x\bullet \phi$$

$$\exists_{ind}\, x\bullet \exists_{coll}\, x\bullet \phi \leftrightarrow \exists_{coll}\, x\bullet \phi$$

The same is true with universal quantifiers: the last occurrence always counts.

- Interactions between relations with different variable index occur, too. First, it is easy to see that we have the following Permutation Principles:

$$\exists_{coll}\, x\bullet \exists_{coll}\, y\bullet \phi \leftrightarrow \exists_{coll}\, y\bullet \exists_{coll}\, x\bullet \phi$$

$$\exists_{ind}\, x\bullet \exists_{ind}\, y\bullet \phi \leftrightarrow \exists_{ind}\, y\bullet \exists_{ind}\, x\bullet \phi$$

$$\exists_{coll}\, x\bullet \exists_{ind}\, y\bullet \phi \leftrightarrow \exists_{ind}\, y\bullet \exists_{coll}\, x\bullet \phi$$

We also get Church-Rosser independence properties, reflecting semantic *confluence* for all different $R_{ind, x}$, $R_{ind, y}$, $R_{coll, x}$, $R_{coll, y}$. This gives us principles like

$$\exists_{coll}\, x\bullet \forall_{ind}\, y\bullet \phi \rightarrow \forall_{ind}\, y\bullet \exists_{coll}\, x\bullet \phi$$

$$\exists_{ind}\, x\bullet \forall_{ind}\, y\bullet \phi \rightarrow \forall_{ind}\, y\bullet \exists_{ind}\, x\bullet \phi$$

The above high-lights a more general issue. Sets of assignments S encode several kinds of 'dependence' between variables. There may not be one single intuition. 'Dependence' may mean functional dependence (if two assignments in S agree on x, they also agree on y), but also other kinds of 'correlation' among value ranges. E.g., let $S\,[x \mid y{:=}d]$ be the set of x–values for all assignments in S whose y–value equals d. Then, one may require that not all values $S\,[x \mid y{:=}d]$ are the same, as d runs over the domain of individual objects. Different dependence relations may have different mathematical properties, and suggest different logical formalisms.

14 Appendix: Some Sources

The proposal made in this paper is not new. It brings out a common pattern behind a number of interesting relevant developments in the recent literature.

Dynamic Logic

Our modal semantics reflects standard dynamic logic (Pratt 1976, Harel 1984) – also in details like 'substitutions as assignments'. In particular, it reflects current dynamic accounts of anaphora in natural language, such as 'dynamic predicate logic' (Groenendijk & Stokhof 1991, Vermeulen 1994), which change variable assignments, viewing an existential quantifier $\exists x$ as a 'random assignment' to x . So far, dynamic predicate logic has employed standard set-theoretic semantics, and hence it inherits the latter's undecidability. But that is merely a conservative feature of its original presentation. A more abstract state view is already discernible in Janssen 1983, when dealing with Montague-style semantics for programming languages.

Probabilistic Logic

The core semantics is also close to that for probabilistic quantifiers proposed in van Lambalgen 1991, with an 'independence relation' restricting the choice of new individuals when evaluating first-order formulas. Alechina & van Benthem 1993 put this into a more general form with 'structured domains', without probabilistic concerns – but with some technical complications (due to their use of sequences of individuals, rather than assignments, as the states). Alechina 1995 is a sustained study of dependency models, comparing state-based and object-based views of dependence.

Algebraic Logic

Technically, our proposals involve the generalized semantics for predicate logic proposed by Andréka & Németi, on the basis of their earlier work in cylindric algebra (see their beautiful lectures at the conference "Logic at Work"; Marx & Pólos 1995). What we consider here are essentially 'atom structures' in cylindric algebra, viewed from a modal perspective. This perspective has been developed in depth and applied quite extensively in the dissertation Venema 1992 on multi-modal logic, whose 'cylindric modal algebra' is our technical paradigm here (cf. Marx & Venema 1995).

What we have to add to all this is a more radical intrinsic motivation, as well as the thesis that it is the landscape of new options itself that is of intrinsic value. It needs to be developed – rather than just serving to provide side-lights on an unchallenged orthodoxy. In the course of this story, we also found a number of new technical results, showing the viability and interest of this kind of semantic analysis.

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