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On forcing in bounded arithmetic

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Abstract. We present a simple and completely model-theoretic proof of a strengthening of a theorem of Ajtai's: the independence of the pigeonhole principle from $I\Delta_0(R)$. We illustrate a method for internalizing the notion of forcing that is both of methodological interest and technically convenient. Qua strength, the theorem proved here corresponds to the complexity/proof-theoretical results of [8] and [12] but a different combinatorics is used. Techniques inspired by Razborov [9] replace those derived from Håstad [6]. The switching lemma formalism is replaced with an approach that is in line with a model-theoretical framework.

1 Introduction

The (Δ_0-)pigeonhole principle is the assertion that there is no Δ_0-definable injective map from $[n+1)$ to $[n)$. Its provability in $I\Delta_0$ is one of the most famous open problems of bounded arithmetic. There is a second-order version of the same question. Can the pigeonhole principle be falsified in a second-order model of Σ_0^p-*comp*? (A theory that is, for $I\Delta_0$, what ACA_0 is for PA.) This corresponds to asking whether the pigeonhole principle is provable in $I\Delta_0(R)$, where $I\Delta_0(R)$ is the theory obtained from $I\Delta_0$ by adding a relation symbol R and allowing it in the induction schema (in this form the question has been posed by Paris and Wilkie in [10]). The second-order formulation of the question is interesting not only because of its similarity with the first-order problem but also because Σ_0^p-*comp* is the bottom level of a hierarchy whose union is BA (i.e., $I\Delta_0+\Omega_1$).

The second question appears to be more tractable then the original problem. A solution has been given by Ajtai [2]. Subsequently, Krajíček, Pudlák and Woods [8] and, independently, Pitassi, Beame and Impagliazzo [12] improved Ajtai's result. In Ajtai's proof the model constructed contained only standard powers of n, while, using the results in [8] and [12], one can construct a model where the numbers $\exp n^\epsilon$ exist for every infinitesimal rational ϵ while it still falsifies the pigeonhole principle between n and $n+1$ (notation: "$\exp n$" and "2^n" are interchangeable). This is an optimal result since the principle follows from the existence of $\exp n^{1/k}$ where k is

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any standard number [11]. This result implies also that assuming the totality of any Δ_0-definable function $g(x)$ such that all finite iterations of $g(x)$ are eventually majorized by 2^x does not help in proving the principle. In particular this holds for the the functions $\omega_0(x) := x^2$ and $\omega_{n+1}(x) := \exp \omega_n(|x|)$ where $|x|$ is $\lceil \log(x+1) \rceil$.

Our proof differs from those of [2], [8] and [12] in many respects. We prove the result directly, dealing exclusively with formulas and with models of arithmetic. The proof in [2] makes a detour via Boolean circuits while in [8] and [12] the problem is translated in a purely complexity/proof-theoretical formalism. (The connection between bounded arithmetic and propositional proof systems has been first pointed out in [10]. See, e.g., Krajíček's monograph [7] for a complete account.) Another difference involve the combinatorial methods used: [8] and [12] apply a variation of Håstad switching lemma that is proved with the probabilistic techniques developed in [6]. In place of these, we apply lighter methods inspired by some recent work of Razborov [9]. Finally, we base the argument of forcing on the notion of locally internal set introduced in Section 4. This allows a more direct formalization of forcing.

2 Preliminaries

The language L is that of second-order arithmetic. It consists of two constants: 0, 1, two binary functions: $+$, $\cdot$, and two binary relations: $<$, $\in$. Variables are of two sorts: first-order, $x, y, z,$... and second-order, $X, Y, Z,$... that are meant to range over numbers and, respectively, finite sets of numbers. The semantics of this language is the usual one but for the following interpretations: $X{<}y$ holds when all elements of X are less than y. Note that terms are just polynomial on first-order variables.

A formula is in Σ_0^p if all of its quantifiers are first-order and bounded, that is, they appear in the context: $(Qx{\in}X)\varphi$ or $(Qx{<}t)\varphi$ where Q is either $\forall$ or $\exists$ and t is a term in which x does not occur. In this paper we concentrate on the class Σ_0^p. This class is the ground level of a hierarchy of formulas, Σ_i^p, Π_i^p that is obtained by counting the alternations of second-order bounded quantifiers: $(QX{<}t)\varphi$. For $i{>}0$ this hierarchy coincides with the polynomial time hierarchy. In complexity theory Σ_0^p corresponds to (a uniform version of) AC_0. The theory Σ_0^p-*comp* is axiomatized by

- the axioms of Robinson arithmetic,
- extensionality: $A{=}B \leftrightarrow \forall x\ (x{\in}A \leftrightarrow x{\in}B)$
- the definition of "<": $A{<}a \leftrightarrow (\forall x{\in}A)(x{<}a)$,
- the least number principle: $A{\neq}\emptyset \rightarrow (\exists x{\in}A)(\forall y{<}x)(y{\notin}A)$,
- the axiom of finiteness: $\exists x\ (A{<}x)$,
- comprehension for all Σ^p_0-formulas: $(\exists X{<}a)(\forall x{<}a)\ [x{\in}X \ \leftrightarrow\ \varphi(x)]$

This theory is the ground level of a hierarchy of theories that is obtained extending the schema of comprehension to Σ^p_i formulas. We will not consider higher levels of this hierarchy here but we refer the reader to [15]. Observe that Σ^p_0-*comp* is a conservative extension of $I\Delta_0$. In fact every model of $I\Delta_0$ or of $I\Delta_0(R)$ can be expanded to a model of Σ^p_0-*comp* (the expansion is canonical: add all finite Δ_0, respectively $\Delta_0(R)$, definable sets). So, there is no essential difference if we present results in terms of Σ^p_0-*comp* or of $I\Delta_0(R)$.

The set of natural numbers together with the set of its finite subsets is the standard model. Our construction is based on a countable elementary extension $\mathcal{M}$ of the standard model. This model $\mathcal{M}$ will be our ground model and it will stay fixed throughout this notes, the truth value of a formula is always evaluated in $\mathcal{M}$ (unless we specify differently). We shall expand an initial segment of $\mathcal{M}$ to a model $\mathcal{N}$ that contains the graph of a bijection

$$G\ :\ N\ \rightarrow\ M$$

between $N, M \in \mathcal{M}$ that have cardinality n and respectively $n{+}n^{\epsilon}$ (ϵ is an arbitrary infinitesimal rational number of $\mathcal{M}$).

Unless we explicitly assert differently, the language L is expanded to include a name for every element of $\mathcal{M}$. We call these new constants parameters. The class of Σ^p_0-formulas is naturally extended to include formulas with parameters. A set of numbers is called *internal* when it belongs to $\mathcal{M}$. Otherwise, we will call it *external.* The (graph of the) bijection G that we intend to add to $\mathcal{M}$ is, clearly, an external object.

3 Antipasto

This section is added for expository reasons, the self-confident reader can skip directly to the next section. Here we aim to present the combinatorial structure of the method in its simplest form. For this reason we will not construct any model and we use only a trivial notion of forcing (i.e., X, as anything else, ranges over the internal sets). Still, the theorem proved here is of major importance because it implies a famous result on the undefinability of parity.

Fix for the rest of this section a set N of non standard cardinality n. In this section X will range over the (graphs of the) total maps from N into $\{0,1\}$ (i.e., over the characteristic functions of the subsets of N). A *condition* is a partial map P from the set N into $\{0,1\}$. We call $\|P\|$ (the cardinality of P), the *length* of the condition. We denote by $\mathcal{R}_p$ the set of conditions of length p. If Q (or X) extends P as partial function we write $Q{\prec}P$ (respectively $X{\prec}P$). If P and Q are compatible, i.e., they have a common extension, we denote by $Q^\frown P$ the set-theoretical union of Q and P. We write $P \Vdash \varphi(X)$, when $\varphi(X)$ holds for all $X{\prec}P$. In words we say that P *forces* $\varphi(X)$. Given a condition P, a set $A{\subseteq}N$ is called a *P-support* of $\varphi(X)$ if

$$P \Vdash \Big[\varphi(X) \equiv \varphi(P \cup X{\restriction_A})\Big]$$

i.e., if $X{\prec}P$, the truth of $\varphi(X)$ depends only on the restriction of X to A. Alternatively we say that P forces φ to have support A. We often say simply "support" in place of "$\emptyset$-support" or when the context clearly suggests a particular condition P. Note also that if Q *extends* P *to* the P-support of φ (i.e., if $Q{\prec}P$ and $\mathrm{dom}(Q)$ contains the P-support of φ) then Q *decides* $\varphi(X)$, i.e., either $Q \Vdash \varphi(X)$ or $Q \Vdash \neg\varphi(X)$.

A quantifier free formula $\varphi(\bar{x}, X)$ can be forced to have a finite support $A_{\bar{x}}$ (depending, of course, on $\bar{x}$). In fact, take a condition P of finite length that forces X to be different from every second-order constant occurring in $\varphi(X)$. Also, in order to decide all atomic formulas of the form $X{<}\,t(\bar{x})$, let P map the largest element of N to 1 (we assume the standard coding of pairs). Then, the truth of $\varphi(X)$ for $X{\prec}P$ depends only on the terms $t(\bar{x})$ appearing in the subformulas of the form $t(\bar{x}){\in}X$. In the theorem below we generalize this fact to more complex formulas. We isolate in the following lemma the most relevant step in the proof of the theorem. The lemma shows how the support increases with the complexity of the formula. The counting part of the lemma is essential to prove the theorem: we need to show that some P works simultaneously for all possible parameters. But in N there is no space for diagonalization, so some counting is necessary.

Lemma 1 Fix some numbers t, s, l such that $t<s<l<n$. Suppose that $\varphi(x, X)$ has support of cardinality $<t$ for all x. Choose a P in $\mathcal{R}_{n-l}$ at random with uniform distribution. The probability that $\forall x\varphi(x, X)$ has no P-support of cardinality $<s$ is at most

$$\left(\frac{2tl}{n}\right)^s$$

Proof. We construct a partial injection from $\mathcal{R}_{n-l}$ to pairs consisting of: an extension of P of length $n-l+s$ and a subset of cardinality s of $[s]\times[t]$ (that is $\{\langle s', t'\rangle : s'<s,\, t'<t\}$). All the conditions that do not force $\forall x\varphi(x, X)$ to have a support of cardinality $<s$ are in the domain of this injection. So, the (normalized) cardinality of the codomain (that is immediate to count) yields the upper bound that we are looking for. The map is defined below. To prove injectivity we check its (partial) invertibility. The whole procedure may be thought as the construction of a unambiguous coding of the conditions in $\mathcal{R}_{n-l}$ by means of pears like those above.

For every a let A_a be the minimal support of $\varphi(a, X)$. Define inductively a_i and R_i as follows. Let a_i be the minimal a such that $\neg\varphi(a, X)$ is forced by some condition Q that extends P to A_a and such that

$$(*) \qquad A_a \setminus \mathrm{dom}(P^\frown R_0^\frown..^\frown R_{i-1}) \neq \emptyset,$$

let R_i be the condition of Q to (*). If possible, continue until the first m such that $s \leq \|R_0^\frown..^\frown R_{m-1}\|$. If we cannot reach this m, we say that we *succeed* on P and leave the injection undefined. We claim that if the procedure succeeds at stage i, then $\mathrm{dom}(R_0^\frown..^\frown R_{i-1})$ is a P-support of $\forall x\varphi(x, X)$. Let D stand for $\mathrm{dom}(R_0^\frown..^\frown R_{i-1})$. Suppose first that $\neg\forall x\varphi(x, X)$ holds for some $X\prec P$. Then $\neg\varphi(a, X)$ for some a and, consequently, $\neg\varphi(a, P\cup X\restriction_{A_a})$. Since $A_a\subseteq D$, (otherwise we could take a Q that extends $X\restriction_{A_a}$), we have $\neg\varphi(a, P\cup X\restriction_{D,\restriction A_a})$. Finally, using the property of the support, we get $\neg\varphi(a, P\cup X\restriction_D)$ and hence $\neg\forall x\varphi(x, P\cup X\restriction_D)$. The converse implication is proved by reversing the implications just shown. So, D is a support of $\forall x\varphi(x, X)$.

We stipulate that on unsuccessful conditions our function outputs the condition $P^\frown R_0^\frown..^\frown R_{m-1}$ and the set $B\subseteq[s]\times[t]$ defined below. With B we want to code $\mathrm{dom}(R_0^\frown..^\frown R_{m-1})$. We use the following injection of $\bigcup_{i<m} A_{a_i}$ into $[s]\times[t]$: the k-th element of A_{a_i} that does not belong to $\bigcup_{j<i} A_{a_j}$, maps it into the pair $\langle i, k\rangle$. So, we

let B be the image of $\bigcup_{j<i} A_{a_j} \cap \mathrm{dom}(R_0^\frown..^\frown R_{m-1})$ under this injection. Note that having $a_0,..,a_i$ and a pair $\langle i,k\rangle$ we can invert this map.

The map just defined does not output conditions of length exactly $n{-}l{+}s$. To enlighten the exposition of the following argument let us neglect this detail. (It is immediate how to adjust it: in fact, there is no loss of information if we truncate R_{m-1} to the required length and act consequently with the set B. The reader can work out the details.) Now, to prove the injectivity of the function, we show that it is possible to invert it. Namely, that given the condition $P^\frown R_0^\frown..^\frown R_{m-1}$ and the set B we can decode P uniquely. It suffices to reconstruct the sequence $a_0,..,a_m$ because from B we can obtain the $\mathrm{dom}(R_0^\frown..^\frown R_{m-1})$ and, finally, P by subtraction. So, suppose that we have found a_j for all $j{<}i$. Using B and $a_0,..,a_{i-1}$ we can compute the set $\mathrm{dom}(R_0^\frown..^\frown R_{i-1})$ and determine the condition $P^\frown R_i^\frown..^\frown R_{m-1}$. Look for the minimal a such that $\neg\varphi(a,X)$ is forced by some condition that extends $P^\frown R_i^\frown..^\frown R_{m-1}$ to A_a. Clearly, this a must coincide with a_i.

We conclude that conditions in $\mathcal{R}_{n-l}$ that do not force a support of cardinality $<s$ cannot exceed the number of unsuccessful conditions and, so, the number of conditions of length $n{-}l{+}s$ times the number of subsets of cardinality s of $[s]{\times}[t)$. That is

$$\binom{n}{l-s} 2^{n-l+s} \binom{s\cdot t}{s}$$

To obtain a probability we divide by the cardinality of $\mathcal{R}_{n-l}$,

$$\frac{\binom{n}{l-s} 2^{n-l+s} \binom{s\cdot t}{s}}{\binom{n}{l} 2^{n-l}} \quad \leq \quad \left(\frac{2tl}{n-l}\right)^{s}.$$

This yields the lemma. □

We abbreviate with "$<n^\epsilon$" the sentence "$<n^{1/k}$ for every standard k". Similarly, for all the other inequalities involving ϵ.

Theorem 2 Let N and n be as above and assume that $N<2^{n^\epsilon}$. Let $\varphi(\bar{x},X)$ be a Σ_0^p formula of with parameters $<\exp n^\epsilon$. There is a condition P of length $<n{-}n^{1-\epsilon}$ and a family $A_{\bar{x}}$ of sets of cardinality $<n^\epsilon$ such that for all $\bar{x}$ $<\exp n^\epsilon$ the formula $\varphi(\bar{x},X)$ has P-support $A_{\bar{x}}$.

Proof. The theorem is obvious when φ is open. The inductive step for negation is trivial, so, it suffices to prove inductive step for the universal quantifier. Note that all universal quantifiers may be assumed to be of the form $(\forall x{<}a)$ where a is some constant $< \exp n^\epsilon$. (Indeed, the hypothesis $N < \exp n^\epsilon$ is essential to deal with quantifiers like $\forall x{\in}X$.)

So, let φ have the form $(\forall y{<}a)\psi(y, \bar{x}, X)$. As induction hypothesis we assume the existence of a definable support $A_{y,\bar{x}}$ of ψ that has cardinality $< n^\epsilon$. There is no loss of generality if we assume that the condition Q forcing this support is $\emptyset$. Otherwise, we can take $N' := N \setminus \mathrm{dom}(Q)$ as new N and $\psi'(y, \bar{x}, X) := \psi(y, \bar{x}, Q \cup X)$ as new ψ. The support of ψ' will be $A'_{y,\bar{x}} := A_{y,\bar{x}} \cap N'$. Since N' has cardinality $n^{1/h}$ for some standard h, $A'_{y,\bar{x}}$ has cardinality $< n'^\epsilon$. Also, if we find a P' (with domain $\subseteq N'$) of length $n' - n'^{1/k}$ forcing $(\forall y{<}a)\psi'(y, \bar{x}, X)$ to have support $< n'^\epsilon$, the condition $P := Q \cup P'$ of length $n - n^{1/(hk)}$ will forces φ to have a support of cardinality $< n^\epsilon$.

Now, apply the lemma. For every fixed $\bar{x} < 2^{n^\epsilon}$ there are at least $1 - 2^{-n^\epsilon}$ conditions of length $n - n^{1/k}$ that forces $\varphi(\bar{x}, X)$ to have support of cardinality $< n^\epsilon$. By counting there is a condition that forces all formulas $\varphi(\bar{x}, X)$ (simultaneously for all $\bar{x} < \exp n^\epsilon$) to have a support of cardinality $< n^\epsilon$. □

This theorem has a famous corollary. There is no Σ_0^p-formula that defines the parity of X, that is a formula that holds iff the set X has an odd number of elements. In fact for obvious reason parity cannot have a P-support smaller than $N \setminus \mathrm{dom}(P)$. This theorem has been proved by Ajtai [1] and independently by Furst, Saxe and Sipser [5] for formulas with parameters $< n$. Subsequently, the result has been extended by Yao [14] to formulas as in the theorem above. The setting in these articles in not quite the same as ours: in [1], the formalism used is that of finite model theory, while [5] and [14] prove a theorem on Boolean circuits complexity. In Boolean circuits complexity jargon Yao's theorem is called "exponential lower bound for parity". The proofs that appear in the most recent expositions make use of a so-called "switching lemma" and use probabilistic techniques developed in [6]. The switching lemma corresponds (in spirit) to the induction lemma proved above.

One of the interesting facts about the proof of the theorem above is that it formally remains the same if we allow X to range over external objects that are locally internal. That is, $X{\restriction}_A$ is standard for all $A \in \mathcal{M}$ that have cardinality $< n^\epsilon$ or, less precise but more suggestively: for those A in $\mathcal{M}$ that are codable by first-order numbers $< \exp n^\epsilon$. This trick becomes essential in following section.

4 Pigeonhole principle

The construction of a model falsifying the pigeonhole principle is divided in two parts. The second part is a standard model theoretical argument of compactness. We shall consider an initial segment of $\mathcal{M}$, (those elements that are $< \exp n^\epsilon$ for some infinitesimal ϵ), add to it a new second-order object (the graph of a bijection falsifying the pigeonhole principle) and, finally, take the closure under Σ_0^p definability. A forcing-like argument is used to ensure that the least number principle holds in the new model.

The first part is of combinatorial nature. It is similar to the proof presented in the previous section. Here X will be a bijection, again, we want to prove that the truth of $\varphi(X)$ does not depend on the whole of X. We claim that it depends only on the image of some small set A and the inverse image of some small set B. There would not be much new if we considered X to be a bijection between sets of equal cardinality. But this is not the case we are most interested in. Since there is no internal bijection between sets of different cardinality, a definition of forcing similar to that of the previous section would quantify on the empty set. So, we have to allow X to be an external object. Fortunately, to have the proof working it is sufficient to require that X is locally internal.

4.1 The forcing lemma

First of all some notation. We abbreviate with "$< n^\epsilon$ " the sentence "$< n^{1/k}$ for all standard k". Similarly for "$< \exp n^\epsilon$ ". Fix a set N of non-standard cardinality n and a set M of cardinality $n+d$. *Conditions* are graphs of partial injections of N into M (but often we speak of them as functions). They are denoted by the letters P and Q. We call $\|P\|$ the *length* of the condition P and we denote by $\mathcal{R}_p$ the set of the condition of length p.

The variable X ranges over the bijections between N and M. We allow X to be an external object but we require that X is *locally internal.* That is, for every $A \subseteq N$ and $B \subseteq M$ both of cardinality $< n^\epsilon$, the two-sided restriction of X to A and B,

$$X_{\restriction A,B} \;:=\; X \cap \; (A \times M \,\cup\, N \times B)$$

is internal. When we quantify for all X we always mean that X is ranging over these set of locally internal bijections. No other object considered in the sequel is external. Since X is an external object, the truth value of $\varphi(X)$, has to be computed by expanding the language with a symbol for an unary predicate (on first-order objects) and interpreting this symbol with the set X. In our notation we confuse the new symbol and its interpretation.

We write $Q \prec P$ (or $X \prec P$) if Q (respectively, X) extends P. We say that P and Q are compatible if there is an injection extending both P and Q, i.e., if their set theoretical union is still a condition. Note that two conditions with disjoint domains and disjoint ranges are always compatible. If P and Q are compatible we denote their union by $P \frown Q$. We say that P *forces* $\varphi(X)$ if $\varphi(X)$ holds for every $X \prec P$; we use the notation $P \Vdash \varphi$. Let $A \subseteq N$ and $B \subseteq M$, we say that a condition P forces A, B to be a *support* of $\varphi(X)$ if

$$P \Vdash \left[\varphi(X) \equiv \varphi(P \cup X{\restriction}_{A,B})\right],$$

i.e., when the truth of $\varphi(X)$ for $X \prec P$ depends only on the image of A and the inverse image of B. Equivalently, we say that A, B is a P-support of $\varphi(X)$. We say simply "support" for $\emptyset$-support or when P is clear from the context. We say that a support is $< s$ if both of the two sets have cardinality $< s$. The following remark is of fundamental importance. It motivates the use of locally internal sets in the definition of forcing.

Remark. If the support of $\varphi(x, X)$ is definable (in terms of x) and is $< n^\epsilon$, then $Q \Vdash \varphi(x, X)$ is definable in terms of x and Q.

We say that Q *covers* A, B if $A \subseteq \mathrm{dom}(Q)$ and $B \subseteq \mathrm{range}(Q)$. Note that if Q covers the support of φ then Q decides φ. The restriction of Q to A, B is denoted by $Q{\restriction}_{A,B}$ and it is defined similarly as the restriction of X. We say that Q *extends P to A, B* if $Q \prec P$ and Q covers A, B.

Lemma 3 Fix l, t and s such that $t < s < n^\epsilon < l < n$. Suppose A_x, B_x is a definable sequence of supports of $\varphi(x, X)$ that are $< t$ for all x. There is a definable function and a definable subset $\mathcal{S}$ of $\mathcal{R}_{n-l}$ such that every P in $\mathcal{S}$ is mapped to some P-support of $\forall x \varphi(x, X)$ that is $< s$ and such that the cardinality of $\mathcal{R}_{n-l} \setminus \mathcal{S}$ is at most

$$\|\mathcal{R}_{n-l}\| \cdot \left(\frac{4l(l+d)t^2}{n-l}\right)^s$$

Moreover, if $\varphi(x, X)$ depends on some (hidden) parameters and A_x, B_x are given uniformly, then the function above is also uniform.

Proof. We will define a function from $\mathcal{R}_{n-l}$ to either a support $< s$ of $\forall x\varphi(x,X)$ (this will give the set $\mathcal{S}$) or to a triple consisting of: a condition of length $n-l+s$ and two subsets of cardinality s of $[2s)\times[t)$. To find an upper bound to the cardinality of the complement of $\mathcal{S}$ we show that the function is one-to-one there. So the conditions in $\mathcal{R}_{n-l}\setminus\mathcal{S}$ can not exceed the number of triples like those above (these are immediate to count). The function is defined below. To prove injectivity we shall show the (partial) invertibility. The whole procedure may be thought of as the attempt of finding a support that is $< s$. We code our failures unambiguously with triples like the above.

For every P in $\mathcal{R}_{n-l}$ define inductively a_i and R_i as follows. Let a_i be the minimal a such that $\neg\varphi(a,X)$ is forced by some condition Q that extends P to A_a, B_a (see the remark above) and such that A_a, B_a is not already covered by $P^\frown R_0^\frown..^\frown R_{i-1}$ (i.e., such that the sets in (*) below are not both empty). Let R_i be the restriction of Q to

$$(*)\qquad A_a\setminus\mathrm{dom}(P^\frown R_0^\frown..^\frown R_{i-1}),\quad B_a\setminus\mathrm{range}(P^\frown R_0^\frown..^\frown R_{i-1}).$$

If possible, continue until the first m is reached such that $s\le\|R_0^\frown..^\frown R_{m-1}\|$. If we are forced to stop before, we say that the procedure *succeed* and we let the function output

$$(**)\qquad \mathrm{dom}(R_0^\frown..^\frown R_{i-1}),\ \mathrm{range}(R_0^\frown..^\frown R_{i-1})$$

We claim that in case of success the output is a P-support of $\forall x\varphi(x,X)$. To prove the claim, suppose that the procedure succeed at stage i. That is, for all a if $\neg\varphi(a,X)$ is forced by some condition $Q\prec P$ then A_a, B_a is covered by $P^\frown R_0^\frown..^\frown R_{i-1}$. Let D, R stand for the pair displayed in (**). Suppose first that $\neg\forall x\varphi(x,X)$ for some $X\prec P$. Then $\neg\varphi(a,X)$ for some a and, consequently, $\neg\varphi(a,P\cup X\lceil_{A_a,B_a})$. Since $A_a\subseteq D$ and $B_a\subseteq R$, (otherwise, since $X\lceil_{A_a,B_a}$ is internal, we could take a Q that extends it), we have $\neg\varphi(a,P\cup X\lceil_{D,R}\lceil_{A_a,B_a})$. Finally, using the property of the support, we get $\neg\varphi(a,P\cup X\lceil_{D,R})$ and hence $\neg\forall x\varphi(x,P\cup X\lceil_{D,R})$. The converse implication is proved by reversing the implications above. So, D, R is a support of $\forall x\varphi(x,X)$. The set of successful conditions is denoted by $\mathcal{S}$.

On the unsuccessful condition the function outputs $P^\frown R_0^\frown..^\frown R_{m-1}$ together with two subsets C and D of $[2s)\times[t)$ that are meant to encode $\mathrm{dom}(R_0^\frown..^\frown R_{m-1})$ and/or $\mathrm{range}(R_0^\frown..^\frown R_{m-1})$. The sets D and C are constructed in the following way. Note that $m < 2s$, in fact, at each stage either $\mathrm{dom}(R_0^\frown..^\frown R_{i-1})$ or $\mathrm{range}(R_0^\frown..^\frown R_{i-1})$ gets a new elements. We inject both $\bigcup_{i<m} A_{a_i}$ and $\bigcup_{i<m} B_{a_i}$ into $[2s]\times[t)$. The k-th element of the A_{a_i} that does not belong to $\bigcup_{j<i} A_{a_j}$, is mapped to the pair $\langle i, k\rangle$. Similarly, for $\bigcup_{i<m} B_{a_i}$. Let C be the image of the intersection of $\bigcup_{j<i} A_{a_j}$ with $\mathrm{dom}(R_0^\frown..^\frown R_{m-1})$ and let D be the image of the intersection of $\bigcup_{i<m} B_{a_i}$ with $\mathrm{range}(R_0^\frown..^\frown R_{m-1})$. Note that having $a_0,..,a_i$ and a pair $\langle i,k\rangle$ we can invert this maps (whenever an inverse exists).

The function defined in the previous paragraph is not quite what we need. In fact, it does not output conditions of length exactly $n{-}l{+}s$. It is immediate how to adjust this. In fact, there is no loss of information if we truncate R_{m-1} to the required length and act consequently with the sets D and C. For legibility, we assume that $P^\frown R_0^\frown..^\frown R_{m-1}$ has always length exactly $n{-}l{+}s$. The necessary changes to obtain a general proof are left to the reader. To prove injectivity on the unsuccessful conditions, we show that from the condition $P^\frown R_0^\frown..^\frown R_{m-1}$ and the sets C and D we can decode P. It suffices to reconstruct the sequence $a_0,..,a_m$ because from C and D we can obtain the $\mathrm{dom}(R_0^\frown..^\frown R_{m-1})$ and, finally by subtraction, P. So, suppose that we have found a_j for all $j{<}i$. Using D and C and $a_0,..,a_{i-1}$ we can compute the set $\mathrm{dom}(R_0^\frown..^\frown R_{i-1})$ and determine the condition $P^\frown R_i^\frown..^\frown R_{m-1}$. Look for the minimal a such that $\neg\varphi(a, X)$ is forced by some condition that extends $P^\frown R_i^\frown..^\frown R_{m-1}$ to A_a, B_a. Clearly, this a must coincide with a_i.

Concluding, the conditions in $\mathcal{R}_{n-l}\setminus\mathcal{S}$ cannot exceed the number of conditions of length $n{-}l{+}s$ times the number of pairs of subsets of cardinality s of $[2s)\times[t)$. That is

$$\binom{n}{l-s}\binom{n+d}{l+d-s}(n-l+s)!\binom{2st}{s}^2.$$

Divide by the cardinality of $\mathcal{R}_{n-l}$,

$$\frac{\binom{n}{l-s}\binom{n+d}{l+d-s}\binom{2st}{s}^2(n-l+s)!}{\binom{n}{l}\binom{n+d}{l+d}(n-l)!} \quad\leq\quad \left(\frac{l(l+d)4t^2}{n-l}\right)^s.$$

This yields the first part of the lemma. The claim on uniformity is immediate. □

Theorem 4 Let N, M, n and d be as above and assume that $N, M < 2^{n^\epsilon}$ and $d < n^\epsilon$ for some standard h. Let $\varphi(\bar{x}, X)$ be a Σ_0^p formula with parameters $< \exp n^\epsilon$.

For some standard k there is a condition P of length $< n - n^{1/k}$ and a definable family $A_{\bar{x}}, B_{\bar{x}}$ of supports $< n^\epsilon$ such that for all $\bar{x} < \exp n^\epsilon$ the formula $\varphi(\bar{x}, X)$ has P-support $A_{\bar{x}}, B_{\bar{x}}$.

Proof. The theorem is obvious when φ is open. In fact, in this case both conditions and supports may be chosen to be finite. The inductive step for negation is trivial, so, it suffices to prove inductive step for the universal quantifier. All universal quantifiers may be assumed to be of the form $\forall x{<}a$ where a is some constant $< \exp n^\epsilon$. (Indeed, the hypothesis $N, M < \exp n^\epsilon$ is essential to deal with quantifiers like $\forall x{\in}X$.)

So, let $\varphi(\bar{x}, X)$ have the form $(\forall y{<}a)\psi(y, \bar{x}, X)$. As induction hypothesis we assume the existence of a definable support $A_{y,\bar{x}}$ of ψ that is $< n^\epsilon$. There is no loss of generality if we assume that the condition Q forcing this support is $\emptyset$. Otherwise, we can take $N' := N \setminus \mathrm{dom}(Q)$ as new domain, $M' := M \setminus \mathrm{range}(Q)$ as new codomain and $\psi'(y, \bar{x}, X) := \psi(y, \bar{x}, Q \cup X)$ as new ψ. Check that, since N' has cardinality $n' := n^{1/h}$ for some standard h, the difference between the cardinalities of N' and M' is still $< n'^\epsilon$. Also, the support of $\psi'(y, \bar{x}, X)$, that is,

$$A_{y,\bar{x}} \cap N', \ B_{y,\bar{x}} \cap M'$$

has cardinality $< n'^\epsilon$. Also, if we find a P' (with domain N' and codomain M') of length $n' - n'^{1/k}$ forcing $(\forall y{<}a)\psi'(y, \bar{x}, X)$ to have support $< n'^\epsilon$, the condition $P := Q \cup P'$ of length $n - n^{1/(hk)}$ will force φ to have a support $< n^\epsilon$.

So, assume the induction hypothesis holds for $\psi(y, \bar{x}, X)$ with the empty condition. Fix a tuple $\bar{x}$. We apply the lemma with $l = n^{1/4}$ to $\varphi(\bar{x}, X) = (\forall y{<}a)\psi(y, \bar{x}, X)$. There is a definable set $\mathcal{S}_{\bar{x}}$ containing conditions of length $n - n^{1/4}$ that force $\varphi(\bar{x}, X)$ to have a support $< n^\epsilon$. The cardinality of $\mathcal{S}_{\bar{x}}$ is at least

$$\left(1 - 2^{-n^\epsilon}\right) \cdot \|\mathcal{R}_{n - n^{1/4}}\|.$$

Now, take the intersection of the $\mathcal{S}_{\bar{x}}$ for $\bar{x} < 2^{n^\epsilon}$. By elementary counting the intersection is not empty. By the uniformity of the function defined in the lemma, we have that the supports $A_{\bar{x}}, B_{\bar{x}}$ of $\varphi(\bar{x}, X)$ are definable in terms of $\bar{x}$. □

4.2 The construction of the model

At this point the proof proceeds with a standard argument of compactness. We state the main theorem.

Theorem 5 Fix $N, M \in \mathcal{M}$ of cardinality n and $n+d$ such that $N, M < 2^{n^\epsilon}$ and $d < n^\epsilon$. There is a model $\mathcal{N}$ of Σ_0^p-*comp* that contains all (first- and second-order) elements of $\mathcal{M}$ that are $< \exp n^\epsilon$ and contains the graph of a bijection between N and M.

Proof. Let $\{\varphi_i(x, X)\}_{i\in\omega}$ enumerate all Σ_0^p-formulas and parameters $< 2^{n^\epsilon}$ with at most the variables x and X free. We construct a chain $\{P_i\}_{i\in\omega}$ of conditions each of length $n - n^{1/k}$ for some standard k. Begin with $P_0 := \emptyset$. At stage $2i+1$ find a condition $P \prec P_{2i}$ that is of length $n - n^{1/k}$ for some k standard and such that $\varphi_i(x, X)$ has a (definable) P-support A_x, B_x of cardinality $< n^\epsilon$. The theorem above guarantees the existence of this support. Now, find the minimal x such that some condition Q extending P to A_x, B_x forces $\varphi_i(x, X)$ (this is legitimate by the remark above the lemma). Let P_{2i+1} be this Q. At even stages we prolong P_{2i+1} in order to ensure that eventually all elements of M fall in the range of $G := \bigcup_{i\in\omega} P_i$ and that G is locally internal.

We stipulate that the model $\mathcal{N}$ contains all and only the first-order elements of $\mathcal{M}$ that are $< 2^{n^\epsilon}$ and exactly all the sets of the form $\{x < a : \varphi_i(x, G)\}$ where a is $< \exp n^\epsilon$ and φ is a Σ_0^p formula with parameters $< \exp n^\epsilon$. By construction, G is a bijection between N and M. Evidently, $\mathcal{N}$ is a model of the schema of Σ_0^p-comprehension. Let us check that the least number principle is satisfied since all other axioms are evident. Every set in $\mathcal{N}$ is definable by a formula $\varphi_i(x, G)$. So, it is sufficient that the least number principle holds for these formulas. At stage $2i+1$ we forced $\varphi_i(\hat{x}, X)$ true for the least of the possible $\hat{x}$. Since G is locally internal, $\varphi_i(\hat{x}, G)$ holds. Since the chain $\{P_i\}_{i\in\omega}$ will eventually decide all $\varphi(c, X)$ for $c < \hat{x}$, these are all eventually forced to be false. Again by the local definability of G, we have that $\neg\varphi(c, G)$ for all $c < \hat{x}$. □

5 Note

We have presented a model-theoretical method that simplifies the forcing set up in bounded arithmetic. A particular result has been chosen to serve our expository purposes. More elaborated constructions (corresponding to those of [3], [4] and [13]) can be build on the same forcing set up.

When this paper was completed we obtained the note "A switching lemma primer" by P.Beame. There, techniques inspired by [9] are used to shorten the proof of the main lemma of [12] and [8]. A complexity/proof-theoretical approach is used there while we proceed in a model-theoretical setting.

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