THE LENGTH OF THE FULL HIERARCHY OF NORMS

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ABSTRACT. We give upper and lower bounds for the length of the Full Hierarchy of Norms.

1. Introduction

The Hierarchy of Norms goes back to Moschovakis' proof of the First Periodicity Theorem and has been investigated by van Engelen, Miller and Steel in [vEMiSt87], and more recently, by Chalons [Ch00] and Duparc [Du03] under the name "Steel hierarchy".

Duparc [Du03, Theorem 7] calculated the length of the hierarchy of Borel norms of length $\omega \cdot \delta < \omega_1$ to be $V_{\omega_1}(1+\delta)$. This should be compared to the height of the Borel Wadge hierarchy which is $V_{\omega_1}(2)$ by a theorem of William Wadge's [Wa83].[1]

In the context of the Axiom of Determinacy, both the Wadge hierarchy and the Hierarchy of Norms are wellfounded (almost) linear quasi-orderings. It is well known that the length of the Wadge hierarchy is exactly $\Theta = \sup\{\alpha\,;$ there is a surjection from $\mathbb{R}$ onto $\alpha\}$.

In this short paper, we comment on the length of the full Hierarchy of Norms which we shall call Σ. We can prove that $\Theta^2 \leq \Sigma < \Theta^+$.

2. Definitions & Basics

As usual in set theory, we identify the real numbers $\mathbb{R}$ with Baire space $\mathbb{N}^{\mathbb{N}}$ and use standard notation for Baire space. In particular, we write $x * y$ for the real defined by

$$x * y(n) := \begin{cases} x(k) & \text{if } n = 2k, \\ y(k) & \text{if } n = 2k+1, \end{cases}$$

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[1] The function V_α is the αth Veblen function to the base ω_1.

and use the symbol $s^\frown x$ for the concatenation of the finite sequence s with the infinite sequence x. We also fix a listing of all continuous functions $\{\mathbf{g}_x : x \in \mathbb{R}\}$.

2.1. **Set Theory without the Axiom of Choice.** Since the main results of this paper will be in the context of the Axiom of Determinacy which contradicts the Axiom of Choice, let us briefly comment on some features of choiceless set theory. (We will be giving the exact axiomatic system for all results in order to avoid confusion.)

It is the Axiom of Choice that guarantees the existence of lots of functions between ordinals and other sets, most notably, the real numbers $\mathbb{R}$, the powerset of the real numbers $\wp(\mathbb{R})$, and related sets. In ZF (without using the Axiom of Choice), the following are equivalent for a set X:

(1) X is wellorderable (*i.e.*, X is in bijection with some ordinal),
(2) there is an ordinal α and a surjection $f : \alpha \to X$, and
(3) there is an ordinal β and an injection $g : X \to \beta$.

The question of existence of injections from ordinals into arbitrary sets and surjections from arbitrary sets into ordinals is much more subtle. Without the Axiom of Choice, the ordinals

$$\begin{aligned} \Omega &:= \sup\{\alpha\,;\text{ there is an injection } f : \alpha \to \mathbb{R}\} \text{ and} \\ \Theta &:= \sup\{\alpha\,;\text{ there is a surjection } f : \mathbb{R} \to \alpha\} \end{aligned}$$

can very well differ.

If the set of real numbers is not wellorderable, then $\Omega > \omega_1$ implies that there is an uncountable set of reals without the perfect set property;[2] in particular, $\mathsf{ZF} + \mathsf{AD}$ implies that $\Omega = \omega_1$. On the other hand, Θ can be rather large. In the literature on the Axiom of Determinacy, Θ plays an important rôle, and the following is known about it:[3]

Proposition 2.1. (ZF) There is no surjection from $\mathbb{R}$ onto Θ. If AD holds, then Θ is a fixed point of the $\aleph$-function, *i.e.*, $\Theta = \aleph_\Theta$. If in addition $\mathbf{V} = \mathbf{L}(\mathbb{R})$, then Θ is regular.

It is not decided by $\mathsf{ZF} + \mathsf{AD}$ alone whether Θ is regular. It is consistent with both AD and the stronger $\mathsf{AD}_\mathbb{R}$ that Θ is singular (*cf.* [So78]).

Without the Axiom of Choice, successor cardinals are not necessarily regular. The following is a well-known weak analogue of the pigeon hole

[2]If $f : \omega_1 \to \mathbb{R}$ is an injection, then $X := \{f(\alpha)\,;\,\alpha < \omega_1\}$ is uncountable, but a perfect subset $P \subseteq X$ would give an injection from $\mathbb{R}$ into ω_1 making it wellorderable.

[3]*Cf.* [Ka94, p. 396 *sqq.*].

principle for successor cardinals in ZF. We give its simple proof for the benefit of the reader who is less familiar with the $\neg\mathsf{AC}$-context.

Lemma 2.2 (Pigeon Hole Principle for Successor Cardinals)**.** (ZF) If κ is an infinite cardinal, then $\kappa^+ \to (\kappa)^1_\kappa$, *i.e.*, for every function $f : \kappa^+ \to \kappa$ there is a set S of cardinality κ such that $f[S] = \{\alpha\}$ for some α.

Proof. For each $\alpha < \kappa$, define $S_\alpha := \{\xi\,;\, f(\xi) = \alpha\}$. If one of the sets S_α has cardinality κ, we are done. Otherwise, for all $\alpha < \kappa$, $\mathrm{o.t.}(S_\alpha) < \kappa$. Let $\pi_\alpha : S_\alpha \to \kappa$ be the Mostowski collapse of S_α. Clearly, $\kappa^+ = \bigcup_{\alpha<\kappa} S_\alpha$. We define

$$F : \begin{array}{lll} \kappa^+ & \to & \kappa \times \kappa \\ \xi & \mapsto & \langle f(\xi), \pi_{f(\xi)}(\xi)\rangle. \end{array}$$

Then F is an injection of κ^+ into $\kappa \times \kappa$ which is a contradiction to the definition of κ^+. q.e.d.

In the following, we will call a surjection $\varphi : \mathbb{R} \to \alpha$ a **norm**. We call $\mathrm{lh}(\varphi) := \alpha$ the **length of** φ. By Proposition 2.1,

$$\Theta = \{\alpha\,;\, \exists\varphi\,(\,\varphi \text{ is a norm } \ \&\ \alpha = \mathrm{lh}(\varphi)\,)\}.$$

For each norm φ, we can define a prewellordering $\leq_\varphi$ on $\mathbb{R}$, defined by

$$x \leq_\varphi y : \Longleftrightarrow \ \varphi(x) \leq \varphi(y),$$

and furthermore identify the norm with the set $X_\varphi := \{x * y\,;\, x \leq_\varphi y\} \subseteq \mathbb{R}$.

2.2. **The Wadge Hierarchy.** The Wadge ordering on sets of reals, defined by

$$A \leq_{\mathrm{W}} B : \Longleftrightarrow \text{ there is a continuous } f \text{ such that } f^{-1}[B] = A$$

defines one of the most fundamental complexity hierarchies of descriptive set theory. From $\leq_{\mathrm{W}}$, we derive the Wadge degrees

$$[A]_{\mathrm{W}} := \{B\,;\, A \leq_{\mathrm{W}} B \ \&\ B \leq_{\mathrm{W}} A\}.$$

If $\mathcal{D}_{\mathrm{W}}$ denotes the set of the Wadge degrees, we call the ordering $\langle \mathcal{D}_{\mathrm{W}}, \leq_{\mathrm{W}}\rangle$ the **Wadge hierarchy**.

The following facts about the Wadge hierarchy are well-known:

Proposition 2.3 (Wadge's Lemma)**.** ($\mathsf{ZF} + \mathsf{AD}$) For sets $A, B \subseteq \mathbb{R}$, we either have $A \leq_{\mathrm{W}} B$ or $\mathbb{R}\backslash B \leq_{\mathrm{W}} A$. Thus the Wadge hierarchy is almost linear (except for antichains of length two).

Theorem 2.4 (Martin-Monk Theorem)**.** ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}(\mathbb{R})$) The Wadge hierarchy is wellfounded.

Now, using Theorem 2.4 under the assumption of $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$, we can assign ordinals called the **Wadge rank** to sets of reals by

$$|A|_{\mathrm{W}} := \mathrm{height}(\langle\{B\,;\,B <_{\mathrm{W}} A\}, \leq_{\mathrm{W}}\rangle).$$

Theorem 2.5 (Wadge)**.** The height of the Wadge hierarchy is Θ.

For each $\alpha < \Theta$, we define

$$\begin{aligned} \wp_\alpha &:= \{A\,;\,|A|_{\mathrm{W}} = \alpha\}, \text{ and} \\ \wp_{\leq\alpha} &:= \{A\,;\,|A|_{\mathrm{W}} \leq \alpha\}. \end{aligned}$$

Proposition 2.6. ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$) For each $\alpha < \Theta$, there is a surjection $f:\mathbb{R}\to\wp_{\leq\alpha}$.

Proof. Fix $A\in\wp_\alpha$. Then the function $x\mapsto \mathbf{g}_x^{-1}[A]$ is a surjection from $\mathbb{R}$ onto $\wp_{\leq\alpha}$. q.e.d.

Corollary 2.7. ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}(\mathbb{R})$) Let $\alpha<\Theta$ be fixed. Suppose that $\langle\mathcal{A}_\gamma\,;\,\gamma<\Theta\rangle$ is a sequence such that $\mathcal{A}_\gamma\subseteq\wp_{\leq\alpha}$ for all $\gamma<\Theta$. Then there are $\gamma_0\neq\gamma_1$ such that $\mathcal{A}_{\gamma_0}\cap\mathcal{A}_{\gamma_1}\neq\varnothing$.

Proof. If not, the function

$$A\mapsto\min\{\gamma\,;\,A\in\mathcal{A}_\gamma\}$$

is a surjection from $\wp_{\leq\alpha}$ onto Θ. Together with the surjection from Proposition 2.6, this yields a surjection from $\mathbb{R}$ onto Θ which contradicts Proposition 2.1. q.e.d.

3. The Hierarchy of Norms

For two norms φ and ψ, we say that φ is **FPT-reducible to** ψ (for "**F**irst **P**eriodicity **T**heorem"; in symbols: $\varphi\leq_{\mathrm{FPT}}\psi$) if there is a continuous function $F:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}$, we have

$$\varphi(x)\leq\psi(F(x)).$$

FPT-reducibility can be expressed in game terms: Look at the two-player perfect information game where player I plays x, player II plays y and is allowed to pass provided he plays infinitely often, and player II wins if and only if $\varphi(x)\leq\psi(y)$. We call this game $\mathbf{G}_\leq(\varphi,\psi)$; then $\varphi\leq_{\mathrm{FPT}}\psi$ if and only if player II has a winning strategy in $\mathbf{G}_\leq(\varphi,\psi)$.

We write $\varphi\equiv_{\mathrm{FPT}}\psi$ for $\varphi\leq_{\mathrm{FPT}}\psi$ & $\psi\leq_{\mathrm{FPT}}\varphi$, define FPT-degrees by

$$[\varphi]_{\mathrm{FPT}} := \{\psi\,;\,\psi\equiv_{\mathrm{FPT}}\varphi\},$$

denote the set of FPT-degrees by $\mathcal{D}_{\mathrm{FPT}}$, and call the structure

$$\langle\mathcal{D}_{\mathrm{FPT}},\leq_{\mathrm{FPT}}\rangle$$

the **Full Hierarchy of Norms**.[4]

Lemma 3.1. (ZF) If φ and ψ are norms and $\mathrm{lh}(\psi) < \mathrm{lh}(\varphi)$, then $\psi <_{\mathrm{FPT}} \varphi$.

Proof. Let x be such that $\varphi(x) \geq \mathrm{lh}(\psi)$. The strategy "play x regardless of what your opponent does" is winning for player I in $\mathbf{G}_{\leq}(\varphi, \psi)$ and for player II in $\mathbf{G}_{\leq}(\psi, \varphi)$. q.e.d.

Lemma 3.2. (ZF) If φ and ψ are norms and $\mathrm{lh}(\varphi) = \mathrm{lh}(\psi) = \alpha + 1$, then $\varphi \equiv_{\mathrm{FPT}} \psi$.

Proof. There are x and y such that $\varphi(x) = \psi(y) = \alpha$. Then "play x" is a winning strategy for player II in $\mathbf{G}_{\leq}(\psi, \varphi)$ and "play y" is a winning strategy for player II in $\mathbf{G}_{\leq}(\varphi, \psi)$. q.e.d.

The following theorem is implicitly contained in Moschovakis' proof of the First Periodicity Theorem (*cf.* [Mo80, 6B]):

Theorem 3.3 (Moschovakis)**.** (ZF + AD + DC($\mathbb{R}$)) The relation $\leq_{\mathrm{FPT}}$ is a prewellordering. Thus, $\langle \mathcal{D}_{\mathrm{FPT}}, \leq_{\mathrm{FPT}} \rangle$ is a wellordering.

We write

$$\begin{aligned} |\varphi|_{\mathrm{FPT}} &:= \mathrm{o.t.}(\langle \{\psi\,;\, \psi <_{\mathrm{FPT}} \varphi\}, \leq_{\mathrm{FPT}} \rangle), \text{ and} \\ \Sigma &:= \mathrm{o.t.}(\langle \mathcal{D}_{\mathrm{FPT}}, \leq_{\mathrm{FPT}} \rangle). \end{aligned}$$

It is the goal of this paper to give upper and lower bounds for Σ. In analogy to the classes $\wp_\alpha$ and $\wp_{\leq\alpha}$, we define for $\xi < \Sigma$:

$$\begin{aligned} \Phi_\xi &:= \{\varphi\,;\, |\varphi|_{\mathrm{FPT}} = \xi\}, \text{ and} \\ \Phi_{\leq\xi} &:= \{\varphi\,;\, |\varphi|_{\mathrm{FPT}} \leq \xi\}. \end{aligned}$$

The two hierarchies are related and yet notably different. First, let us note that the classes Φ_ξ are much larger than the classes $\wp_\alpha$:

Proposition 3.4. (ZF + AD + DC($\mathbb{R}$)) If $0 < \xi < \Sigma$, then there is a surjection from Φ_ξ onto $\wp(\mathbb{R})$.

Proof. Let $\varphi \in \Phi_\xi$. For $x \in \mathbb{R}$, let $x^+(n) := x(n+1)$. For each $A \in \wp(\mathbb{R})$, we define a norm

$$\varphi_A(x) := \begin{cases} \varphi(x^+) & \text{if } x(0) = 0, \\ 1 & \text{if } x(0) \neq 0 \text{ and } x^+ \in A, \text{ and} \\ 0 & \text{if } x(0) \neq 0 \text{ and } x^+ \notin A. \end{cases}$$

We note that "play 0 and after that copy" is a winning strategy for player II in $\mathbf{G}_{\leq}(\varphi, \varphi_A)$, and "if the first move is 0, copy; if the first move is not 0, then play an arbitrary real x such that $\varphi(x) \geq 1$" is a

[4] "Full" in order to distinguish it from Duparc's variants with bounded length and/or bounded complexity.

winning strategy for player II in $\mathbf{G}_{\leq}(\varphi_A, \varphi)$, so we have that $\varphi_A \in \Phi_\xi$. The function

$$\varphi \mapsto \begin{cases} A & \text{if } \varphi = \varphi_A, \\ \varnothing & \text{otherwise} \end{cases}$$

is a surjection from Φ_ξ onto $\wp(\mathbb{R})$. q.e.d.

Corollary 3.5. (ZF + AD + DC($\mathbb{R}$)) If $0 < \xi\Sigma$ and $\alpha < \Theta$ arbitrary, then $\Phi_\xi \not\subseteq \wp_{\leq\alpha}$.

Proof. Suppose $\Phi_\xi \subseteq \wp_{\leq\alpha}$, then there is a surjection from $\wp_{\leq\alpha}$ onto Φ_ξ, hence onto $\wp(\mathbb{R})$ by Proposition 3.4. Now Proposition 2.6 gives us a surjection from $\mathbb{R}$ onto $\wp(\mathbb{R})$, which of course contradicts Cantor's Theorem. q.e.d.

4. Lower and upper bounds for Σ

Lemma 4.1 (Diagonal Lemma)**.** (ZF) If $\lambda < \Theta$ is a limit ordinal and φ is a norm of length λ, then there is a norm φ^+ of length λ such that $\varphi <_{\mathrm{FPT}} \varphi^+$.

Proof. For a function φ define φ^+ by

$$\varphi^+(x) := \begin{cases} \varphi(\mathbf{g}_{x^+}(x)) + 1 & \text{if } x(0) \neq 0, \text{ and} \\ \varphi(x^+) & \text{otherwise.} \end{cases}$$

Note that φ^+ is a norm with $\mathrm{lh}(\varphi^+) = \mathrm{lh}(\varphi) = \lambda$. Towards a contradiction, let F witness $\varphi \geq_{\mathrm{FPT}} \varphi^+$. Let z be a code for F, *i.e.*, $F = \mathbf{g}_z$. Then

$$\begin{aligned} \varphi(F(\langle 1\rangle^\frown z)) &\geq \varphi^+(\langle 1\rangle^\frown z) \\ &= \varphi(\mathbf{g}_z(\langle 1\rangle^\frown z)) + 1 \\ &= \varphi(F(\langle 1\rangle^\frown z)) + 1 \\ &> \varphi(F(\langle 1\rangle^\frown z)). \end{aligned}$$

Contradiction. q.e.d.

We say that φ is **embedded** in ψ if there is some x such that we have $\psi(x * y) = \varphi(y)$ for all y.

Lemma 4.2. (ZF) If φ is embedded in ψ, then $\varphi \leq_{\mathrm{FPT}} \psi$.

Proof. Let x witness that φ is embedded in ψ. Then "Play x on your even moves and copy the moves of player I on your odd moves" is a winning strategy for player II in $\mathbf{G}_{\leq}(\varphi, \psi)$: If player I plays y, player II answers $x * y$ and wins since $\varphi(y) = \psi(x * y)$. q.e.d.

Lemma 4.3. (ZF) If $\lambda < \Theta$ is a limit ordinal and $\alpha < \Theta$, then there is a $<_{\mathrm{FPT}}$-increasing sequence $\langle \varphi_\nu\,;\,\nu < \alpha \rangle$ of norms such that for all $\nu < \alpha$, we have $\mathrm{lh}(\varphi_\nu) = \lambda$.

Proof. Let $\alpha < \Theta$, and fix a surjection $f : \mathbb{R} \to \alpha$ and a norm $\varphi : \mathbb{R} \to \lambda$. We define norms φ_ν by induction, beginning with $\varphi_0 := \varphi$.

Assume that φ_ξ is defined for $\xi < \nu$ and let

$$\varphi^*_\nu(x * y) := \begin{cases} \varphi_{f(x)}(y) & \text{if } f(x) < \nu, \\ \varphi(y) & \text{otherwise.} \end{cases}$$

Note that for $\xi < \nu$, φ_ξ is embedded in φ^*_ν. Thus, by Lemma 4.2, $\varphi_\xi \leq_{\mathrm{FPT}} \varphi^*_\nu$.

Let $\varphi_\nu := (\varphi^*_\nu)^+$. Then for all $\xi < \nu$, we have $\varphi_\xi <_{\mathrm{FPT}} \varphi_\nu$ by the Diagonal Lemma 4.1. q.e.d.

In the following, let $\langle \lambda_\alpha\,;\,\alpha < \Theta \rangle$ be the strictly increasing enumeration of all limit ordinals in Θ (the inverse of the Mostowski collapse).

Theorem 4.4. ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}(\mathbb{R})$) Let $\alpha < \Theta$ and let φ be a norm of length $\lambda_\alpha + 1$. Then $|\varphi|_{\mathrm{FPT}} \geq \Theta \cdot \alpha$.

Proof. We prove the claim by induction on α. For $\alpha = 0$, the claim is trivial. Let α be the least counterexample as witnessed by φ (of length $\lambda_\alpha + 1$).

Case 1. Let $\alpha = \gamma + 1$, and let ψ be a norm of length $\lambda_\gamma + 1$. By minimality of α, $|\psi|_{\mathrm{FPT}} \geq \Theta \cdot \gamma$. Since α was a counterexample, $|\varphi|_{\mathrm{FPT}} = \Theta \cdot \gamma + \zeta$ for some $\zeta < \Theta$. We apply Lemma 4.3 to get a $<_{\mathrm{FPT}}$-increasing sequence $\langle \psi_\eta\,;\,\eta < \zeta + 2 \rangle$ such that all ψ_η have length λ_α. Lemma 3.1 yields that $\psi <_{\mathrm{FPT}} \psi_\eta <_{\mathrm{FPT}} \varphi$ (for all η), but $|\psi_{\zeta+1}|_{\mathrm{FPT}} \geq \Theta \cdot \gamma + \zeta + 1 > \Theta \cdot \gamma + \zeta = |\varphi|_{\mathrm{FPT}}$. Contradiction.

Case 2. If α is a limit ordinal, then $\Theta \cdot \alpha = \bigcup_{\gamma < \alpha} \Theta \cdot \gamma$. By induction hypothesis, we have $|\varphi|_{\mathrm{FPT}} \geq \Theta \cdot \gamma$ for all $\gamma < \alpha$, so $|\varphi|_{\mathrm{FPT}} \geq \Theta \cdot \alpha$, so α was no counterexample. q.e.d.

Corollary 4.5. ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}(\mathbb{R})$) $\Theta^2 \leq \Sigma$.

Theorem 4.6. ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}(\mathbb{R})$) $\Sigma < \Theta^+$.

Proof. Towards a contradiction, suppose that $\Phi_\xi \neq \varnothing$ for all $\xi < \Theta^+$. Define $w : \Theta^+ \to \Theta$ by

$$w(\xi) := \min\{\alpha\,;\,\exists \varphi \in \Phi_\xi\,(|X_\varphi|_{\mathrm{W}} = \alpha)\}.$$

By the Pigeon Hole Principle 2.2, we find $\alpha \in \Theta$, $S \subseteq \Theta^+$ and $b : \Theta \to S$ such that b is a bijection and for all $\xi \in S$, we have $w(\xi) = \alpha$.

For $\xi \in S$, we define

$$H_\xi := \{X_\varphi\,;\,\varphi \in \Phi_\xi \;\&\; |X_\varphi|_{\mathrm{W}} = \alpha\} \subseteq \wp_\alpha.$$

Then $\langle H_{b(\gamma)}\,;\,\gamma \in \Theta\rangle$ is a sequence of subsets of $\wp_\alpha$ as in Corollary 2.7, and so there are $\gamma_0 \neq \gamma_1$ such that $H_{b(\gamma_0)} \cap H_{b(\gamma_1)} \neq \varnothing$. But if $X_\varphi \in H_{b(\gamma_0)} \cap H_{b(\gamma_1)}$, then $|\varphi|_{\mathrm{FPT}} = b(\gamma_0) \neq b(\gamma_1) = |\varphi|_{\mathrm{FPT}}$ which is absurd. q.e.d.

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