MODAL LOGICS FOR PRODUCTS OF TOPOLOGIES

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Abstract. We introduce the horizontal and vertical topologies on the product of topological spaces, and study their relationship with the standard product topology. We show that the modal logic of products of topological spaces with horizontal and vertical topologies is the fusion $\mathbf{S4} \oplus \mathbf{S4}$. We axiomatize the modal logic of products of topological spaces with horizontal, vertical, and standard product topologies. We prove that both of these logics are complete for the product of rational numbers $\mathbb{Q} \times \mathbb{Q}$ with the appropriate topologies.

1. Introduction

The study of products of Kripke frames and their modal logics was initiated by Shehtman [16]. A systematic study of multi-dimensional modal logics of products of Kripke frames can be found in Gabbay and Shehtman [8], and for the up to date account of the most important results in the field we refer to Gabbay et al. [9]. We recall that for given two frames $\mathcal{F} = \langle W, S\rangle$ and $\mathcal{G} = \langle V, T\rangle$, the 'horizontal' and 'vertical' relations on the product $W \times V$ are defined as follows.

$$(w, v)R_1(w', v') \text{ iff } wSw' \text{ and } v = v'$$
$$(w, v)R_2(w', v') \text{ iff } w = w' \text{ and } vTv'$$

Amongst many other results, Gabbay and Shehtman proved that if L_1 and L_2 are modal logics complete with respect to frame classes $\mathbb{F}_1$ and $\mathbb{F}_2$ defined by universal Horn conditions, then the logic $L_1 \times L_2$ of the class of products

$$\mathbb{F}_1 \times \mathbb{F}_2 = \{\langle W \times V, R_1, R_2\rangle : \langle W, S\rangle \in \mathbb{F}_1 \text{ and } \langle V, T\rangle \in \mathbb{F}_2\}$$

is axiomatized by the fusion $L_1 \oplus L_2$ plus the two additional principles of commutation $com = \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$ and convergence (also known as the Church-Rosser principle) $chr = \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$. In particular, since $\mathbf{S4}$ is complete with respect to the universal Horn class of reflexive and transitive frames, the product $\mathbf{S4} \times \mathbf{S4}$ is axiomatized as $\mathbf{S4} \oplus \mathbf{S4}$ plus *com* and *chr*.

It is known that topological semantics generalizes Kripke semantics for $\mathbf{S4}$. In this paper we consider products of topological spaces. We generalize the notions of horizontal and vertical relations to horizontal and vertical topologies and study their relationship with the standard product topology. We show that the modal logic of products of topological spaces with horizontal and vertical topologies is $\mathbf{S4} \oplus \mathbf{S4}$, and thus much weaker than $\mathbf{S4} \times \mathbf{S4}$.

Since the topological setting strongly suggests adding the 'true product topology', we also investigate the modal logic of products of topological spaces with all three topologies: horizontal, vertical, and the standard product topology. We show

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that the modal operator associated with the product topology is not definable in terms of the modal operators associated with the horizontal and vertical topologies, and we axiomatize the modal logic of products of topological spaces with all three topologies.

The paper is organized as follows. In Section 2 we recall some basic facts about topological semantics of **S4** and present a new proof of completeness of **S4** with respect to the rationals. We also recall the fusion **S4**$\oplus$**S4** and the product **S4**$\times$**S4**. In Section 3 we introduce the horizontal and vertical topologies, and investigate their relationship with the standard product topology. Section 4 is concerned with the commutation and convergence principles in the topological setting, while Sections 5 and 6 contain completeness results for modal languages with operators corresponding to the horizontal, vertical, and standard product topologies. In the concluding Section 7 we point out some of the remaining open questions.

2. Preliminaries

2.1. **Topological completeness of S4.** If we interpret the modal operators $\Box$ and $\Diamond$ in topological spaces as the interior and closure operators, then the complete modal logic of all topological spaces is **S4** (McKinsey and Tarski [13]). A much stronger result, also due to McKinsey and Tarski, states that **S4** is in fact the complete modal logic of any metric separable dense-in-itself space. In particular, **S4** is the complete modal logic of the real line $\mathbb{R}$, the rational line $\mathbb{Q}$, or the Cantor space **C**. An alternative proof of completeness of **S4** with respect to **C** can be found in [14], and that with respect to $\mathbb{R}$ in [2]. In the subsequent sections we will need completeness of **S4** with respect to $\mathbb{Q}$. In order to make the paper self-contained, we present here an alternative proof of this fact, which might be of an independent interest.

To this end we recall that a topological model is a structure $M = \langle X, \tau, \nu \rangle$, where $\langle X, \tau \rangle$ is a topological space, and ν is a valuation assigning subsets of X to propositional variables of the modal language. Then for $x \in X$, the modal operators $\Box$ and $\Diamond$ are interpreted as follows.

$$\begin{array}{lll} x \models \Box\varphi & \text{iff} & (\exists U \in \tau)(x \in U \text{ and } (\forall y \in U)(y \models \varphi)) \\ x \models \Diamond\varphi & \text{iff} & (\forall U \in \tau)(x \in U \Rightarrow (\exists y \in U)(y \models \varphi)) \end{array}$$

A *topo-bisimulation* between two topological models $M = \langle X, \tau, \nu \rangle$ and $M' = \langle X', \tau', \nu' \rangle$ is a non-empty relation $\leftrightharpoons \subseteq X \times X'$ such that if $x \leftrightharpoons x'$ then

(I) Base: $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any propositional variable p

(II) Forth condition: $x \in U \in \tau$ implies that there exists $U' \in \tau'$ such that $x' \in U'$ and for every $y' \in U'$ there is $y \in U$ with $y \leftrightharpoons y'$

(III) Back condition: $x' \in U' \in \tau'$ implies that there exists $U \in \tau$ such that $x \in U$ and for every $y \in U$ there is $y' \in U'$ with $y \leftrightharpoons y'$

An important feature of topo-bisimulations that will be used throughout is that they preserve truth of modal formulas [1].

Let T_2 be the infinite binary tree with the (reflexive and transitive) descendant relation. Formally, T_2 can be defined as $\langle W, R \rangle$, where $W = \{0, 1\}^*$ is the set of strings (including the empty string) over $\{0, 1\}$ and sRt iff $\exists u : s \cdot u = t$.

In our proof of completeness we will rely on the following two well-known results.

Theorem 2.1. *(van Benthem-Gabbay)* **S4** *is complete with respect to* T_2.

Proof. For a proof see, e.g., [10, Theorem 1 and the subsequent discussion]. □

Theorem 2.2. *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to* $\mathbb{Q}$.

Proof. For a proof see, e.g., [12, Page 217, Theorem 2]. □

Remark 2.3. We recall that if $\langle X, < \rangle$ is a linearly ordered set and $x, y \in X$ with $x < y$, then the *open interval* (x, y) is defined as the set $\{z \in X : x < z < y\}$. If we view linearly ordered sets as topological spaces using the set of open intervals as a basis for the topology, then it follows from Cantor's theorem that every countable dense unbounded linear ordering is (as a topological space) homeomorphic to $\mathbb{Q}$.

We are now ready to proceed with the proof.

Theorem 2.4. S4 *is complete with respect to* $\mathbb{Q}$.

Proof. This result is well-known and can be proved in many different ways. Here we give a proof that later will be extended to prove our two main completeness theorems. Our strategy is as follows. We use completeness of **S4** with respect to T_2; view T_2 as a topological space with the topology defined from the order of T_2; label a dense unbounded subset L of $\mathbb{Q}$ with nodes of T_2; establish a topo-bisimulation between L and T_2. This will allow to transfer counterexamples from T_2 to L, which by Cantor's theorem is homeomorphic to $\mathbb{Q}$.

Our labelling is defined as follows.

Stage 0: Label 0 with the root r of the tree T_2.

Stage 1: Label -1 with the immediate left R-successor, and 1 with the immediate right R-successor of r. Call these two numbers *environmental numbers at distance* $\frac{1}{3^0}$ from 0.

Stage n: The environmental numbers labelled at stage $n-1$ are no nearer to each other than $\frac{1}{3^{n-1}}$. Now for each of labelled numbers we create two environmental numbers at distance $\frac{1}{3^n}$ and label them with the immediate R-successors in the tree.

Define the partial function $l : \mathbb{Q} \to T_2$ by letting $l(q)$ to be the node in T_2 assigned to q by some stage of the labelling. Subsequently we will refer to the stage at which an element of $dom(l)$ is first labelled. For example, 0 is labelled at stage 0; -1 and 1 are labelled at stage 1; $-\frac{4}{3}, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$ are labelled at stage 2, and so on.

There are several properties of l that need to be established. We need to show that (i) l is well-defined, (ii) the domain of l is homeomorphic to $\mathbb{Q}$, (iii) l is continuous and open.

Claim 2.5. l is well-defined; that is, no $q \in \mathbb{Q}$ is labelled by two different nodes of T_2.

Proof. This is were we use the particular distances in the labelling. The proof is by induction. For stages 0 and 1 it is obvious that no q is labelled by two nodes. Since in the stage n the mutual distance between closest labelled rational numbers is no smaller than $\frac{1}{3^n}$ and the new numbers labelled in stage $n+1$ are no further than $\frac{1}{3^{n+1}}$ from the already labelled numbers whose environmental numbers they are, no q is labelled twice in stage $n+1$. □

Claim 2.6. The domain of l is a dense unbounded subset of $\mathbb{Q}$, hence homeomorphic to $\mathbb{Q}$.

Proof. To show that the domain of l is dense let $q < q'$ be in the domain of l. We show that there is q'' in the domain of l such that $q < q'' < q'$. Let n be the earliest stage at which q and q' are labelled. Then at the next stage one of the successors of q labels $q + \frac{1}{3^{n+1}}$. But since the distance between q and q' is at least $\frac{1}{3^n}$, we have that $q < q + \frac{1}{3^{n+1}} < q'$.

To show that the domain of l is unbounded, assume that there is a bound q. Let n be the stage at which q is labelled. Then at stage $n + 1$ both $q - \frac{1}{3^{n+1}}$ and $q + \frac{1}{3^{n+1}}$ are labelled, contradicting our assumption. $\square$

Let L denote the domain of l. It remains to check that $l : L \to T_2$ is continuous and open. To prove this, we will use the facts that the open intervals

$$\{(q - \frac{1}{2^n}, q + \frac{1}{2^n}) \, : \, q \text{ is labelled at stage } m < n\}$$

form a basis for the subspace topology on L, and that a basis for the topology on T_2 is $\mathcal{B} = \{B_t\}_{t \in T_2}$ where $B_t = \{s \in T_2 \, : \, tRs\}$.

Claim 2.7. l is an open continuous map from L onto T_2.

Proof. That $l : L \to T_2$ is onto is obvious.

(i) (Openness.) Given a basic open interval $(q - \frac{1}{2^n}, q + \frac{1}{2^n})$, where q is labelled at some stage $m < n$, we will show that $l(q - \frac{1}{2^n}, q + \frac{1}{2^n}) = B_{l(q)}$.

($\subseteq$) Let $q' \in (q - \frac{1}{2^n}, q + \frac{1}{2^n})$. Clearly q' was not labelled before stage n (because of its small distance to q). Suppose that q' was labelled at stage $n + k$. Then an easy induction on k shows that $l(q')$ is a descendant of $l(q)$, i.e., $l(q') \in B_{l(q)}$.

($\supseteq$) Let $t \in B_{l(q)}$, and the length of the path between $l(q)$ and t be k. Then an easy induction shows that at stage $n + k$ some point in the interval $(q - \frac{1}{2^n}, q + \frac{1}{2^n})$ is labelled with t.

(ii) (Continuity.) It suffices to show that for each $t \in T_2$, the inverse image of B_t is open. Let $q \in l^{-1}(B_t)$. Consider $(q - \frac{1}{2^n}, q + \frac{1}{2^n})$, where n is greater than the stage at which q was labelled. Then the same reasoning as above guarantees that $l(q - \frac{1}{2^n}, q + \frac{1}{2^n}) = B_{l(q)} \subseteq B_t$. $\square$

To complete the proof, if $\mathbf{S4} \not\vdash \varphi$, then by Theorem 2.1, there is a valuation ν on T_2 such that $\langle T_2, \nu \rangle, r \not\models \varphi$. Define a valuation ν' on L by $\nu'(p) = l^{-1}(\nu(p))$. Since l is continuous and open and $l(0) = r$, we have that 0 and r are topo-bisimilar. Therefore, $\langle L, \nu' \rangle, 0 \not\models \varphi$. Now since L is homeomorphic to $\mathbb{Q}$, we obtain that φ is also refutable on $\mathbb{Q}$. $\square$

2.2. **The fusion S4$\oplus$S4.** Let $\mathcal{L}_{\Box_1\Box_2}$ be a bimodal language with modal operators $\Box_1$ and $\Box_2$. We recall that the *fusion* of **S4** with itself, denoted by **S4** $\oplus$ **S4**, is the least set of formulas containing **S4**-axioms for both $\Box_1$ and $\Box_2$, and closed under modus ponens, substitution, $\Box_1$-necessitation, and $\Box_2$-necessitation.

S4 $\oplus$ **S4**-frames are triples $\langle W, R_1, R_2 \rangle$, where W is a nonempty set and R_1 and R_2 are reflexive and transitive. Define a new relation R on W by putting wRv if there exists a sequence $w_1, \ldots, w_n$ of elements of W such that

$$w = w_1 R_1 w_2 R_2 \ldots R_1 w_{n-1} R_2 w_n = v.$$

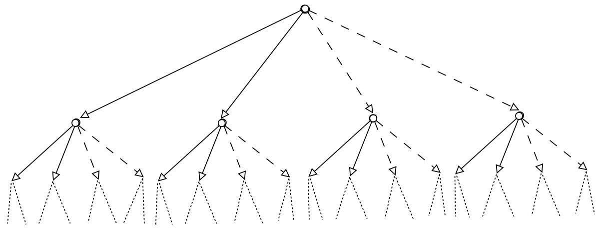

FIGURE 1. $T_{2,2}$. The solid lines represent R_1 and the dashed lines represent R_2. The dotted lines at the final nodes indicate that the pattern repeats on infinitely.

It is obvious that R is reflexive and transitive. Call $\langle W, R_1, R_2\rangle$ *rooted* if $\langle W, R\rangle$ is rooted.

Theorem 2.8. *(Kracht-Wolter and Fine-Shurz)* $\mathbf{S4}\oplus\mathbf{S4}$ *has the finite model property; in fact,* $\mathbf{S4}\oplus\mathbf{S4}$ *is complete with respect to finite rooted* $\mathbf{S4}\oplus\mathbf{S4}$*-frames.*

Proof. For a proof see, e.g., [9, Page 196, Theorem 4.2]. □

Let $T_{2,2}$ denote the infinite quaternary tree such that each node of $T_{2,2}$ is R_1-related to two of its four immediate successors and R_2-related to the other two; both R_1 and R_2 are taken to be reflexive and transitive. Formally $T_{2,2}$ can be defined as $\langle W, R_1, R_2\rangle$, where $W = \{0,1,2,3\}^*$, sR_1t iff $\exists u \in \{0,1\}^* : s \cdot u = t$, and sR_2t iff $\exists u \in \{2,3\}^* : s \cdot u = t$ (see Figure 1). Clearly $T_{2,2}$ is a rooted $\mathbf{S4}\oplus\mathbf{S4}$-frame. In fact, $\mathbf{S4}\oplus\mathbf{S4}$ is complete with respect to $T_{2,2}$.

Proposition 2.9. $\mathbf{S4}\oplus\mathbf{S4}$ *is complete with respect to* $T_{2,2}$.

Proof. A straightforward generalization of the standard unravelling procedure for $\mathbf{S4}$ (cf., e.g., [10] or [2]) unravels an arbitrary finite rooted $\mathbf{S4}\oplus\mathbf{S4}$-frame into a bisimilar branching tree of the form $T_{2,2}$. For details see the forthcoming [15]. □

2.3. **The product $\mathbf{S4}\times\mathbf{S4}$.** For two $\mathbf{S4}$-frames $\mathcal{F} = \langle W, S\rangle$ and $\mathcal{G} = \langle V, T\rangle$, define the *product frame* $\mathcal{F}\times\mathcal{G}$ to be the frame $\langle W\times V, R_1, R_2\rangle$, where for $w, w' \in W$ and $v, v' \in V$,

$$\begin{aligned}(w,v)R_1(w',v') &\text{ iff } wSw' \text{ and } v = v'\\ (w,v)R_2(w',v') &\text{ iff } w = w' \text{ and } vTv'\end{aligned}$$

The frame $\mathcal{F}\times\mathcal{G}$ can be viewed as an $\mathbf{S4}\oplus\mathbf{S4}$-frame by interpreting the modalities $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ as follows.

$$\begin{aligned}(w,v)\models\Box_1\varphi \quad&\text{iff}\quad \forall(w',v')\ .\ (w,v)R_1(w',v') \Rightarrow (w',v')\models\varphi\\ (w,v)\models\Box_2\varphi \quad&\text{iff}\quad \forall(w',v')\ .\ (w,v)R_2(w',v') \Rightarrow (w',v')\models\varphi\end{aligned}$$

Let $\mathbf{S4}\times\mathbf{S4}$ denote the logic of products of $\mathbf{S4}$-frames. As we pointed out in the introduction, the product logic $\mathbf{S4}\times\mathbf{S4}$ is axiomatized by adding the following two axioms to the fusion $\mathbf{S4}\oplus\mathbf{S4}$:

$$\begin{aligned}com &= \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p\\ chr &= \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p\end{aligned}$$

By the Sahlqvist theory, *com* (for commutativity) and *chr* (for Church-Rosser) have the following first-order correspondents:

$$\forall x \forall y (\exists z (x R_1 z \wedge z R_2 y) \leftrightarrow \exists z (x R_2 z \wedge z R_1 y))$$

$$\forall x \forall y \forall z ((x R_1 y \wedge x R_2 z) \rightarrow \exists w (y R_2 w \wedge z R_1 w))$$

Besides R_1 and R_2, there is yet another (reflexive and transitive) relation on the product $W \times V$ defined componentwise:

$$(w, v) R (w', v') \text{ iff } w S w' \text{ and } v T v'$$

This allows us to interpret yet another modal operator $\Box$ in $\mathcal{F} \times \mathcal{G}$:

$$(w, v) \models \Box\varphi \quad \text{iff} \quad \forall (w', v') \, . \, (w, v) R (w', v') \Rightarrow (w', v') \models \varphi$$

However, since in product frames we have that $R = R_1 \circ R_2$, $\Box\varphi$ becomes equivalent to $\Box_1 \Box_2 \varphi$, and so $\Box$ turns out to be definable in terms of $\Box_1$ and $\Box_2$. As we will see shortly, in the subtler setting of topological products, the analogue of $\Box$ is independent of the analogues of $\Box_1$ and $\Box_2$.

3. Product spaces and product topo-bisimulations

3.1. **Horizontal and vertical topologies.** Let $\mathcal{X} = \langle X, \eta \rangle$ and $\mathcal{Y} = \langle Y, \theta \rangle$ be two topological spaces. We recall that the *standard product topology* τ on $X \times Y$ is defined by letting the sets $U \times V$ form a basis for τ, where U is open in $\mathcal{X}$ and V is open in $\mathcal{Y}$. Let I denote the interior operator and C denote the closure operator of τ.

We will define two additional one-dimensional topologies on $X \times Y$ by 'lifting' the topologies of the components.

Suppose $A \subseteq X \times Y$. We say that A is *horizontally open (H-open)* if for any $(x, y) \in A$ there exists $U \in \eta$ such that $x \in U$ and $U \times \{y\} \subseteq A$. Similarly, we say that A is *vertically open (V-open)* if for any $(x, y) \in A$ there exists $V \in \theta$ such that $y \in V$ and $\{x\} \times V \subseteq A$. If A is both H- and V-open, then we call it *HV-open*. The complements of horizontally open sets are called *horizontally closed (H-closed)*, the complements of vertically open sets are called *vertically closed (V-closed)*, and the complements of HV-open sets are called *HV-closed*.

Let τ_1 denote the set of all H-open subsets of $X \times Y$ and τ_2 denote the set of all V-open subsets of $X \times Y$. It is easy to verify that both τ_1 and τ_2 form topologies on $X \times Y$. We call τ_1 the *horizontal topology* and τ_2 the *vertical topology*.

We point out that the set $\{U \times \{y\} : U \in \eta \ \& \ y \in Y\}$ forms a basis for the horizontal topology, and the set $\{\{x\} \times V : x \in X \ \& \ V \in \theta\}$ forms a basis for the vertical topology. Moreover, a point (x, y) is a *horizontal interior point* of $A \subseteq X \times Y$ if there exists a neighborhood U_x of x such that $U_x \times \{y\} \subseteq A$. Similarly, (x, y) is a *vertical interior point* of A if there exists a neighborhood V_y of y such that $\{x\} \times V_y \subseteq A$. Let $I_1(A)$ denote the *horizontal interior* of A and $I_2(A)$ denote the *vertical interior* of A. Then A is H-open iff $A = I_1(A)$, and A is V-open iff $A = I_2(A)$.

We also point out that a point (x, y) belongs to the *horizontal closure* of $A \subseteq X \times Y$ iff for any neighborhood U_x of x, $(U_x \times \{y\}) \cap A \neq \emptyset$. Similarly, (x, y) belongs to the *vertical closure* of A iff for any neighborhood V_y of y, $(\{x\} \times V_y) \cap A \neq \emptyset$. Let $C_1(A)$ denote the *horizontal closure* of A and $C_2(A)$ denote the *vertical closure* of A. Then A is H-closed iff $A = C_1(A)$, and A is V-closed iff $A = C_2(A)$.

Remark 3.1. Let I_η and C_η denote the interior and closure operators of $\mathcal{X}$, and I_θ and C_θ denote the interior and closure operators of $\mathcal{Y}$. Let $A \subseteq X \times Y$. For $x \in X$ let $A_x = \{y \in Y : (x, y) \in A\}$, and for $y \in Y$ let $A_y = \{x \in X : (x, y) \in A\}$. Then we can represent A horizontally as $A = \bigcup_{y \in Y}(A_y \times \{y\})$ or vertically as $A = \bigcup_{x \in X}(\{x\} \times A_x)$. Using the horizontal representation of A we obtain that

$$I_1(A) = \bigcup_{y \in Y}(I_\eta(A_y) \times \{y\}) \text{ and } C_1(A) = \bigcup_{y \in Y}(C_\eta(A_y) \times \{y\})$$

and using the vertical representation we obtain that

$$I_2(A) = \bigcup_{x \in X}(\{x\} \times I_\theta(A_x)) \text{ and } C_2(A) = \bigcup_{x \in X}(\{x\} \times C_\theta(A_x))$$

Remark 3.2. It is obvious that a set open in the standard product topology is both horizontally and vertically open. That is $\tau \subseteq \tau_1$ and $\tau \subseteq \tau_2$. However, the converse inclusions don't hold in general. In fact, we will show below that I is not modally definable by means of I_1 and I_2.

We interpret the modal operators $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ in $\langle X \times Y, \tau_1, \tau_2\rangle$ as follows.

$$(x, y) \models \Box_1\varphi \text{ iff } (\exists U \in \tau_1)((x, y) \in U \text{ and } \forall (x', y') \in U \,.\, (x', y') \models \varphi)$$

$$(x, y) \models \Box_2\varphi \text{ iff } (\exists V \in \tau_2)((x, y) \in V \text{ and } \forall (x', y') \in V \,.\, (x', y') \models \varphi)$$

Dually,

$$(x, y) \models \Diamond_1\varphi \text{ iff } (\forall U \in \tau_1)((x, y) \in U \Rightarrow \exists (x', y') \in U : (x', y') \models \varphi)$$

$$(x, y) \models \Diamond_2\varphi \text{ iff } (\forall V \in \tau_2)((x, y) \in V \Rightarrow \exists (x', y') \in V : (x', y') \models \varphi)$$

We say that a formula φ (of the language $\mathcal{L}_{\Box_1\Box_2}$) is *valid at* (x, y) if for every valuation on $X \times Y$ we have $(x, y) \models \varphi$. We say that φ is *valid in* $\langle X \times Y, \tau_1, \tau_2\rangle$ if φ is valid at every $(x, y) \in X \times Y$.

The one-dimensional nature of the horizontal and vertical topologies is emphasized by the following proposition.

Proposition 3.3.
(1) *A formula φ constructed from the Booleans and the modal operator $\Box_1$ is valid in $\langle X \times Y, \tau_1, \tau_2\rangle$ iff φ is valid in $\langle X, \eta\rangle$.*
(2) *A formula φ constructed from the Booleans and the modal operator $\Box_2$ is valid in $\langle X \times Y, \tau_1, \tau_2\rangle$ iff φ is valid in $\langle Y, \theta\rangle$.*

Proof. See the forthcoming [15] for details on this and similar results. □

3.2. **Failure of** ***com*** **and** ***chr*** **on $\mathbb{R} \times \mathbb{R}$.** Whenever topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are Alexandroff, i.e., defined from **S4**-frames, the horizontal and vertical topologies on their product $X \times Y$ can be defined from the horizontal and vertical relations on the product of these frames. Hence, our topological setting generalizes the case for products of Kripke frames. Nevertheless, there are crucial differences between these two settings. In particular, both *com* and *chr*, while valid on products of Kripke frames, can be refuted on topological products. To stimulate intuitions before plunging into general theory, we exhibit their failure on $\mathbb{R} \times \mathbb{R}$.

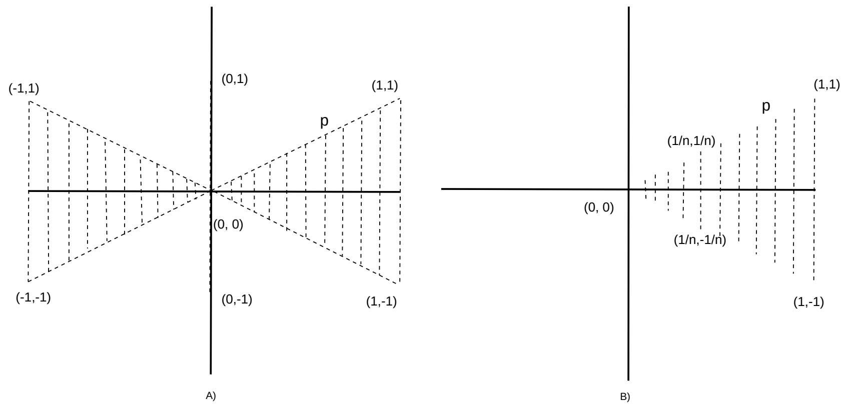

FIGURE 2. Counterexamples of *com* and *chr* on $\mathbb{R} \times \mathbb{R}$.

(a) Failure of *com*: Let

$$\nu(p) = (\bigcup_{x \in (-1,0)} \{x\} \times (x, -x)) \cup (\{0\} \times (-1,1)) \cup (\bigcup_{x \in (0,1)} \{x\} \times (-x, x))$$

(see Figure 2a). Then there is a basic horizontal open $(-1,1) \times \{0\}$ such that $(0,0)$ is in it and every point in $(-1,1) \times \{0\}$ sits in a vertically open subset of p. Thus, $\Box_1 \Box_2 p$ is true at $(0,0)$. On the other hand, there is no vertical open containing $(0,0)$ in which every point sits inside a horizontally open subset of p, implying that $\Box_2 \Box_1 p$ is false at $(0,0)$.

(b) Failure of *chr*: Let $\nu(p) = \bigcup\{\{\frac{1}{n}\} \times (-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ (see Figure 2b). Then in any basic horizontal open around $(0,0)$ there is a point that sits in a basic vertical open in which p is true everywhere. Thus, $\Diamond_1 \Box_2 p$ is true at $(0,0)$. On the other hand, since the horizontal closure of $\nu(p)$ is $\nu(p) \cup \{(0,0)\}$ and since the vertical interior of $\nu(p) \cup \{(0,0)\}$ is $\nu(p)$, we have that $(0,0)$ is not in $I_2(C_1(\nu(p)))$, implying that $\Box_2 \Diamond_1 p$ is false at $(0,0)$.

As we will see in Section 4, the structure of these counterexamples on $\mathbb{R} \times \mathbb{R}$ is not accidental. We will show under which circumstances they can be reproduced in other products of topological spaces.

3.3. **Product topo-bisimulations.** As in Kripke semantics, an appropriate notion of bisimulation plays crucial role in understanding and developing topological semantics. In this subsection we generalize the notion of topo-bisimulation introduced in Section 2.1 to topological models equipped with several topologies. We will use it to show that the standard product interior is not definable in terms of the horizontal and vertical interiors. Another important application of multi-dimensional topo-bisimulations will come in the completeness proofs below.

We exhibit the case of two topologies, but the generalization to any number of topologies is straightforward.

Definition 3.4. Let $M = \langle X, \tau_1, \tau_2, \nu \rangle$ and $M' = \langle X', \tau_1', \tau_2', \nu' \rangle$ be topological models equipped with two topologies each. A 2-*topo-bisimulation* is a nonempty relation $\leftrightharpoons \subseteq X \times X'$ such that if $x \leftrightharpoons x'$ then:

(I) BASE: $x \in \nu(p)$ iff $x' \in \nu'(p)$, for any proposition variable p

(II) (FORTH CONDITION):
1. $x \in U \in \tau_1$ implies that there exists $U' \in \tau_1'$ such that $x' \in U'$ and for all $z' \in U'$ there exists $z \in U$ with $z \rightleftharpoons z'$
2. $x \in V \in \tau_2$ implies that there exists $V' \in \tau_2'$ such that $x' \in V'$ and for all $z' \in V'$ there exists $z \in V$ with $z \rightleftharpoons z'$

(III) (BACK CONDITION):
1. $x' \in U' \in \tau_1'$ implies that there exists $U \in \tau_1$ such that $x \in U$ and for all $z \in U$ there exists $z' \in U'$ with $z \rightleftharpoons z'$
2. $y' \in V' \in \tau_2'$ implies that there exists $V \in \tau_2$ such that $y \in V$ and for all $z \in V$ there exists $z' \in V'$ with $z \rightleftharpoons z'$

The 2-topo-bisimulation $\rightleftharpoons$ is called *total* if it is defined for all elements of X and X', i.e., $dom(\rightleftharpoons) = X$ and $rng(\rightleftharpoons) = X'$. The fundamental invariance property of 2-topo-bisimulations is given by the following proposition.

Proposition 3.5. *Let $M = \langle X, \tau_1, \tau_2, \nu\rangle$ and $M' = \langle X', \tau_1', \tau_2', \nu'\rangle$ be topological models equipped with two topologies each, and let $x \rightleftharpoons x'$ for some 2-topo-bisimulation $\rightleftharpoons \subseteq X \times X'$. Then for every modal formula φ in $\mathcal{L}_{\Box_1\Box_2}$ we have that $M, x \models \varphi$ iff $M', x' \models \varphi$.*

Proof. The proof is a straightforward generalization of the 1-topo-bisimulation version found in [1] and we omit the details of the induction. $\Box$

Definition 3.4 and Proposition 3.5 apply to arbitrary topological models M, N with two (or more) topologies each.[1] In the special case when M and N consist of product spaces with the horizontal and vertical topologies, the 2-topo-bisimulation $\rightleftharpoons$ is called a *product topo-bisimulation*.

To give an example of a 2-topo-bisimulation, let $M = \langle X, \tau_1, \tau_2, \nu\rangle$ be a topological model, and let $U \in \tau_1 \cap \tau_2$. Let also τ_1' and τ_2' denote the restrictions of τ_1 and τ_2 to U. For a valuation ν' on $\langle U, \tau_1', \tau_2'\rangle$, define a valuation ν on X by putting $\nu(p) = \nu'(p)$. Then it is routine to check that the identity map $i : U \to X$ is a 2-topo-bisimulation between the topological models $M' = \langle U, \tau_1', \tau_2', \nu'\rangle$ and M. In particular, if $X = Y \times Z$ for some topological spaces Y and Z, and if $U = Y' \times Z'$ for some $Y' \subseteq Y$ and $Z' \subseteq Z$, then i is a product topo-bisimulation.

For another example, let $\langle X, \tau_1, \tau_2\rangle$ and $\langle X', \tau_1', \tau_2'\rangle$ be given. Let also a map $f : X \to X'$ be continuous and open with respect to both topologies. For a valuation ν' on X' define a valuation ν on X by putting $\nu(p) = f^{-1}(\nu'(p))$. Then it is easy to verify that f is a 2-topo-bisimulation between the models $M = \langle X, \tau_1, \tau_2, \nu\rangle$ and $M' = \langle X', \tau_1', \tau_2', \nu'\rangle$. In particular, if $X = Y \times Z$ and $X' = Y' \times Z'$ with τ_1 and τ_1' being the horizontal topologies, and τ_2 and τ_2' being the vertical topologies, then f is a product topo-bisimulation. In this case we call f *HV-continuous* and *HV-open*. If in addition f is a bijection, then we call f a *HV-homeomorphism*.

We now have a technique that can be put to various uses such as establishing undefinability or transfer from one model to another. Our first illustration concerns the undefinability of the interior operator of the standard product topology in terms of the interiors of the horizontal and vertical topologies.

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[1] By analogy with Kripke semantics, one can think of such models as fusion models.

For two topological spaces $\mathcal{X}$ and $\mathcal{Y}$, consider the product space $\langle X \times Y, \tau, \tau_1, \tau_2\rangle$, where τ stands for the standard product topology, τ_1 for the horizontal topology, and τ_2 for the vertical topology. We recall that $\Box_1$ and $\Box_2$ are interpreted via the horizontal and vertical topologies, while $\Box$ is interpreted via the standard product topology.

Proposition 3.6. *$\Box$ is not definable in the language $\mathcal{L}_{\Box_1\Box_2}$.*

Proof. It is sufficient to find two product models $M = \langle X \times Y, \tau_1, \tau_2, \nu\rangle$ and $M' = \langle X' \times Y', \tau_1', \tau_2', \nu'\rangle$ with $(x, y) \in X \times Y$ and $(x', y') \in X' \times Y'$, and a product topo-bisimulation $\leftrightharpoons \subseteq (X \times Y) \times (X' \times Y')$ such that $(x, y) \leftrightharpoons (x', y')$, that $M, (x, y) \models \Diamond p$, and that $M', (x, y) \not\models \Diamond p$. Since all formulae in the language $\mathcal{L}_{\Box_1\Box_2}$ are preserved by product topo-bisimulations and $\Diamond p$ is not, we conclude that $\Diamond p$ is not equivalent to any formula of $\mathcal{L}_{\Box_1\Box_2}$ (or to any infinite set of such formulae for that matter). It follows that neither is $\Box p$.

For the product space we take $\mathbb{Q} \times \mathbb{Q}$. Let $\nu(p) = \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ and $\nu'(p) = \emptyset$. Let also $\leftrightharpoons$ be the identity relation on $(\mathbb{Q} \times \mathbb{Q})\backslash\{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$. It is not hard to see that $\leftrightharpoons$ is a product topo-bisimulation between the models $\langle\mathbb{Q}\times\mathbb{Q}, \nu\rangle$ and $\langle\mathbb{Q}\times\mathbb{Q}, \nu'\rangle$ that connects $(0, 0)$ to $(0, 0)$. Since $(0, 0)$ is in the closure of $\nu(p)$, we have that $\langle\mathbb{Q} \times \mathbb{Q}, \nu\rangle, (0, 0) \models \Diamond p$. On the other hand, it is obvious that $\langle\mathbb{Q} \times \mathbb{Q}, \nu'\rangle, (0, 0) \models \Box\neg p$. □

4. Correspondence for *com* and *chr*

As we have seen above, unlike products of Kripke frames, products of topological spaces do not always validate *com* and *chr*. In this section we specify classes of products of topological spaces in which *com* and *chr* hold. We start by investigating the validity of *com*. It is useful to split *com* into $com_{\rightarrow} = \Box_1\Box_2 p \rightarrow \Box_2\Box_1 p$ and $com_{\leftarrow} = \Box_2\Box_1 p \rightarrow \Box_1\Box_2 p$.

Let $\mathcal{X} = \langle X, \eta\rangle$ be a topological space. We recall that $\mathcal{X}$ is Alexandroff if the intersection of any family of open sets is again open. We call $\mathcal{X}$ *κ-Alexandroff* if the intersection of any family of open sets of cardinality κ is again open; that is, $\eta' \subseteq \eta$ and $|\eta'| \leq \kappa$ imply $\bigcap \eta' \in \eta$.

Proposition 4.1. *If $\mathcal{X} = \langle X, \eta\rangle$ is κ-Alexandroff and $|Y| \leq \kappa$, then $\mathcal{X}\times\mathcal{Y} \models com_{\leftarrow}$ and $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$.*

Proof. We show that $\mathcal{X}\times\mathcal{Y} \models com_{\leftarrow}$. That $\mathcal{Y}\times\mathcal{X} \models com_{\rightarrow}$ is proved symmetrically. Suppose for a point $(x, y) \in X \times Y$ and a valuation ν on $\mathcal{X} \times \mathcal{Y}$ we have that $(x, y) \models \Box_2\Box_1 p$. Then there exists a neighborhood V of y such that for each $z \in V$ there is a neighborhood U_z of z with $U_z \times \{z\} \subseteq \nu(p)$. Since $|V| \leq \kappa$ and $\mathcal{X}$ is κ-Alexandroff, we have that $U = \bigcap\{U_z : z \in V\} \in \eta$. But then $U \times V \subseteq \nu(p)$, implying that $(x, y) \models \Box_1\Box_2 p$. □

Corollary 4.2. *If $\mathcal{X}$ is Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$ and $\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$ for any topological space $\mathcal{Y}$.*

Proof. It is sufficient to observe that every Alexandroff space is κ-Alexandroff for every cardinal κ, and apply Proposition 4.1. □

It follows that if both $\mathcal{X}$ and $\mathcal{Y}$ are Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models com$. Given the well-known correspondence between Kripke frames for **S4** and Alexandroff topologies, the above corollary sheds some topological light on the validity of *com* on products of Kripke frames.

The converse of Corollary 4.2 does not hold. For instance, every topology commutes with the discrete topology of any cardinality. Thus, it can happen that $\mathcal{X}$ or $\mathcal{Y}$ are not Alexandroff and yet $\mathcal{X} \times \mathcal{Y} \models com$. However, if $\mathcal{X}$ and $\mathcal{Y}$ coincide, then the converse of Corollary 4.2 holds. To see this, for $x \in X$, let η_x denote the set of all neighborhoods of x.

Lemma 4.3. *If $\mathcal{X}$ is not Alexandroff, then there is a point $x \in X$ such that $\bigcap \eta_x \notin \eta$.*

Proof. Since $\mathcal{X}$ is not Alexandroff, there exists a set B of opens such that $\bigcap B \notin \eta$. Let $x \in \bigcap B$. Obviously $\bigcap \eta_x \subseteq \bigcap B$ and $\bigcap B = \bigcup\{\bigcap \eta_x : x \in \bigcap B\}$. If $\bigcap \eta_x$ were open for every $x \in \bigcap B$, then $\bigcap B$ would be open. Therefore, there exists $x \in \bigcap B$ such that $\bigcap \eta_x$ is not open. $\square$

Proposition 4.4. *If $\mathcal{X}$ is not Alexandroff, then $\mathcal{X} \times \mathcal{X} \not\models com_{\leftarrow}$ and $\mathcal{X} \times \mathcal{X} \not\models com_{\rightarrow}$.*

Proof. We show that $\mathcal{X} \times \mathcal{X} \not\models com_{\leftarrow}$. The case for $\mathcal{X} \times \mathcal{X} \not\models com_{\rightarrow}$ is symmetric. Since $com_{\leftarrow}$ is equivalent to $\Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$, it is enough to show that $\mathcal{X} \times \mathcal{X} \not\models \Diamond_1\Diamond_2 p \to \Diamond_2\Diamond_1 p$. As $\mathcal{X}$ is not Alexandroff, by Lemma 4.3 there exists $x \in X$ such that $\bigcap \eta_x \notin \eta$. Let $\eta_x = \{U_i\}_{i \in I}$. We order I by putting $i \leq j$ iff $U_i \supseteq U_j$. Since $U_i, U_j \in \eta_x$ implies $U_i \cap U_j \in \eta_x$, it follows that $(I, \leq)$ is a directed partial order. Let $J = \{i \in I : \exists j \geq i \text{ with } U_i - U_j \neq \emptyset\}$. We show that J is cofinal in I. If not, then there exists $i \in I$ such that for any $j \geq i$ we have $U_i - U_j = \emptyset$. Therefore, $U_i = U_j$ for any $j \geq i$. Thus, $\bigcap \eta_x = \bigcap_{i \in I} U_i = \bigcap_{j \geq i} U_i = U_i \in \eta$, a contradiction. For $i \in J$ let $j \geq i$ be such that $U_i - U_j \neq \emptyset$ and pick $x_i \in U_i - U_j$. Then $\{x_i\}_{i \in J}$ is a net converging to x. Let ν be a valuation on $\mathcal{X} \times \mathcal{X}$ such that $\nu(p) = \{(x_i, x_j) : i, j \in J \text{ and } i \leq j\}$. For $U \in \eta_x$ and $i \in J$, let $U_j = U \cap U_i$. Then $i \leq j$. Since J is cofinal in I we can assume that $j \in J$. Therefore, $(x_i, x_j) \in \nu(p)$. It follows that $(x_i, x) \models \Diamond_2 p$. Thus, $(x, x) \models \Diamond_1\Diamond_2 p$. On the other hand, for any $U \in \eta_x$ and for any $x_j \in U$ we have $(U_i \times \{x_j\}) \cap \nu(p) = \emptyset$ for any $i \in J$ with $i > j$. Therefore, $(x, x) \not\models \Diamond_2\Diamond_1 p$. $\square$

From Corollary 4.2 and Proposition 4.4 we obtain the following characterization of Alexandroff spaces.

Corollary 4.5. *The following conditions are equivalent:*

(1) *$\mathcal{X}$ is Alexandroff.*
(2) *$\mathcal{X} \times \mathcal{X} \models com$.*
(3) *$\mathcal{X} \times \mathcal{Y} \models com_{\leftarrow}$ for every topological space $\mathcal{Y}$.*
(4) *$\mathcal{Y} \times \mathcal{X} \models com_{\rightarrow}$ for every topological space $\mathcal{Y}$.*

We end this section by investigating validity of *chr* in the products of topological spaces.

Proposition 4.6. *If either $\mathcal{X}$ or $\mathcal{Y}$ is Alexandroff, then $\mathcal{X} \times \mathcal{Y} \models chr$.*

Proof. Let $\mathcal{X} = \langle X, \eta\rangle$ and $\mathcal{Y} = \langle Y, \theta\rangle$. First suppose that $\mathcal{X}$ is Alexandroff. So every $x \in X$ has a least neighborhood U_x. If for a valuation ν on $\mathcal{X} \times \mathcal{Y}$ and a point $(x, y) \in X \times Y$ we have that $(x, y) \models \Diamond_1\Box_2 p$, then there exists $z \in U_x$ such that $(z, y) \models \Box_2 p$. Therefore, there exists $V \in \theta_y$ such that $\{z\} \times V \subseteq \nu(p)$. But then for every $u \in V$ we have $(x, u) \models \Diamond_1 p$, implying that $(x, y) \models \Box_2\Diamond_1 p$.

Now suppose that $\mathcal{Y}$ is Alexandroff. So every $y \in Y$ has a least neighborhood V_y. If for a valuation ν on $\mathcal{X}\times\mathcal{Y}$ and a point $(x, y) \in X\times Y$ we have that $(x, y) \models \Diamond_1\Box_2 p$, then for every $U \in \eta_x$ there exists $z \in U$ such that $\{z\} \times V_y \subseteq \nu(p)$. But then for every $u \in V_y$ and for every $U \in \eta_x$ there exists $z \in U$ such that $(z, u) \in \nu(p)$. Thus, $(x, y) \models \Box_2\Diamond_1 p$. $\Box$

Since Kripke frames for **S4** correspond to Alexandroff topologies, the above proposition gives a topological insight into the soundness of *chr* with respect to products of Kripke frames. Even though the converse of Proposition 4.6 is not in general true, similar to the case with *com*, we have that if $\mathcal{X}$ and $\mathcal{Y}$ coincide, then the converse does indeed hold.

Proposition 4.7. *If $\mathcal{X}$ is not Alexandroff, then $\mathcal{X} \times \mathcal{X} \not\models$ chr.*

Proof. Let $x \in X$, $\eta_x = \{U_i\}_{i\in I}$, $J \subseteq I$, and the net $\{x_i\}_{i\in J}$ be chosen as in the proof of Proposition 4.4. We define a valuation ν on $\mathcal{X} \times \mathcal{X}$ by putting $\nu(p) = \bigcup_{i\in J}(\{x_i\} \times U_i)$. Then it is easy to verify that $(x, x) \models \Diamond_1\Box_2 p$ but $(x, x) \not\models \Box_2\Diamond_1 p$. $\Box$

Propositions 4.6 and 4.7 lead to yet another characterization of Alexandroff spaces.

Corollary 4.8. *The four equivalent conditions in Corollary 4.5 are equivalent to the following one:*

(5) $\mathcal{X} \times \mathcal{X} \models$ *chr*.

For more results in this direction we refer to the forthcoming [11].

5. The logic of product spaces

As we saw in the previous section, both *com* and *chr* can be refuted on products of topological spaces. This suggests that the complete logic of all products of topological spaces is weaker than **S4** $\times$ **S4**. The main goal of this section is to show that this logic is **S4** $\oplus$ **S4**. In fact, we will show that **S4** $\oplus$ **S4** is complete with respect to $\mathbb{Q} \times \mathbb{Q}$.

Theorem 5.1. **S4** $\oplus$ **S4** *is complete with respect to* $\mathbb{Q} \times \mathbb{Q}$.

Proof. By Proposition 2.9, **S4** $\oplus$ **S4** is complete with respect to the infinite quaternary tree $T_{2,2} = \langle W, R_1, R_2\rangle$. We view $T_{2,2}$ as equipped with two Alexandroff topologies defined from R_1 and R_2. So for completeness of **S4** $\oplus$ **S4** with respect to $\mathbb{Q} \times \mathbb{Q}$ it is sufficient to find a HV-open subspace L of (a HV-homeomorphic copy of) $\mathbb{Q} \times \mathbb{Q}$ and a continuous open map from L onto $T_{2,2}$ with respect to both topologies: this will allow us to transfer counterexamples on $T_{2,2}$ to L, and hence to (a HV-homeomorphic copy of) $\mathbb{Q} \times \mathbb{Q}$. To achieve this, we label a subset of $\mathbb{Q} \times \mathbb{Q}$ recursively with the nodes of $T_{2,2}$ as follows (see Figure 3):

Stage 0: Label $(0, 0)$ with the root r of the tree $T_{2,2}$.

Stage 1: Label $(-1, 0)$ with the immediate left R_1-successor, and $(1, 0)$ with the immediate right R_1-successor of r; also label $(0, -1)$ with the immediate left R_2-successor, and $(0, 1)$ with the immediate right R_2-successor of r. Call these four points *environmental points at distance* $\frac{1}{3^0}$ from $(0, 0)$.

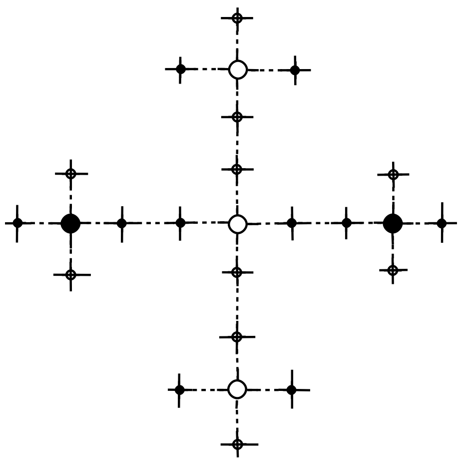

FIGURE 3. The labelling of $\mathbb{Q} \times \mathbb{Q}$. Points labelled with the R_1-successors are black, while the ones labelled with the R_2-successors are white. The crosses around each point suggest that the pattern repeats at further stages. At each stage, given a point labelled in the previous stage, four new points are labelled; two are labelled with the respective R_1-successors, and the other two with the respective R_2-successors.

Stage n: The environmental points labelled at stage $n-1$ are no nearer to each other than $\frac{1}{3^{n-1}}$. Now for each of labelled points we create four environmental points at the distance $\frac{1}{3^n}$–two at the horizontal distance $\frac{1}{3^n}$ and two at the vertical distance $\frac{1}{3^n}$–and label them with respective immediate R_1- and R_2-successors in the tree.

Define the partial function $l : \mathbb{Q} \times \mathbb{Q} \to T_{2,2}$ by letting $l(p, q)$ to be the node in $T_{2,2}$ assigned to (p, q) by some stage of the labelling. The same argument as in Claim 2.5 guarantees that the distance invariant gets maintained, and no point ever gets labelled twice. Thus, l is a well-defined partial function.

Let L denote the subset of $\mathbb{Q} \times \mathbb{Q}$ of all labelled points. Let also $P = \{p : (p, 0) \in L\}$ and $Q = \{q : (0, q) \in L\}$. From the same argument as in Claim 2.6 it follows that both P and Q are dense unbounded subsets of $\mathbb{Q}$, thus by Cantor's theorem homeomorphic to $\mathbb{Q}$. Therefore, $P \times Q$ is HV-homeomorphic to $\mathbb{Q} \times \mathbb{Q}$. Moreover, it follows from the labelling that L is a HV-open subset of $P \times Q$.[2]

Claim 5.2. l is an open and continuous map from L onto $T_{2,2}$ with respect to both topologies.

Proof. That $l : L \to T_{2,2}$ is onto is obvious. Let τ_1 and τ_2 denote the restrictions of the horizontal and vertical topologies on $\mathbb{Q} \times \mathbb{Q}$ to L, respectively. We prove that l is open and continuous with respect to τ_1. That it is open and continuous with

[2] Note that not all of $P \times Q$ gets labelled. In fact, the only point on the diagonal of $P \times Q$ that gets labelled is $(0, 0)$.

respect to τ_2 is proved symmetrically. We observe that

$$\{(p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times \{q\} \, : \, (p,q) \text{ is labelled at stage } m < n\}$$

forms a basis for τ_1. We also recall that a basis for the topology on $T_{2,2}$ defined from R_1 is $\mathcal{B}_1 = \{B^1_t\}_{t \in T_{2,2}}$ where $B^1_t = \{s \in T_{2,2} \, : \, tR_1 s\}$.

(i) (Openness.) Let $(p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times \{q\}$ be a basic open for τ_1. Then the same argument as in Claim 2.7 guarantees that $l((p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times \{q\}) = B^1_{l(p,q)}$.

(ii) (Continuity.) It suffices to show that for each $t \in T_{2,2}$, the inverse image of B^1_t belongs to τ_1. Let $(p,q) \in l^{-1}(B^1_t)$. Consider $(p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times \{q\} \in \tau_1$, where n is greater than the stage at which (p,q) was labelled. Then by the above reasoning $l((p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times \{q\}) = B^1_{l(p,q)} \subseteq B^1_t$. $\square$

To complete the proof, if $\mathbf{S4} \oplus \mathbf{S4} \not\vdash \varphi$, then by Proposition 2.9, there is a valuation ν on $T_{2,2}$ such that $\langle T_{2,2}, \nu \rangle, r \not\models \varphi$. Define a valuation ν' on L by $\nu'(p) = l^{-1}(\nu(p))$. Since l is continuous and open with respect to both topologies and $l(0,0) = r$, we have that $(0,0)$ and r are 2-topo-bisimilar. Therefore, $\langle L, \nu' \rangle, (0,0) \not\models \varphi$. Now since L is HV-open subset of $P \times Q$, we obtain that φ is refutable on $P \times Q$. Finally, since $P \times Q$ is HV-homeomorphic to $\mathbb{Q} \times \mathbb{Q}$, it follows that φ is also refutable on $\mathbb{Q} \times \mathbb{Q}$.

$\square$

Corollary 5.3. (1) $\mathbf{S4} \oplus \mathbf{S4}$ *is the logic of products of arbitrary topologies.*
(2) *The logic of products of arbitrary topologies is decidable; in fact, its satisfiability problem is $PSPACE$-complete.*

Proof. (i) follows from Theorem 5.1; for (ii) recall that the satisfiability problem for the fusion $\mathbf{S4} \oplus \mathbf{S4}$ is $PSPACE$-complete (see [17]). $\square$

Let us say that a logic L in the language $\mathcal{L}_{\Box_1 \Box_2}$ has the *finite topo-product model property* if any non-theorem of L is refuted on a finite product space. Then the logic of products of arbitrary topologies does not have the finite topo-product model property as finite spaces are Alexandroff, and hence validate *com* and *chr*.[3] This remark is not to be confused with the non existence of finite Kripke models: it follows from Theorem 2.8 that every non-theorem of $\mathbf{S4} \oplus \mathbf{S4}$ does indeed fail on a finite model.

To summarize, we showed that in the language $\mathcal{L}_{\Box_1 \Box_2}$ the logic of products of arbitrary topologies coincides with the logic of $\mathbb{Q} \times \mathbb{Q}$ and is the fusion $\mathbf{S4} \oplus \mathbf{S4}$. It follows that the logic has the finite model property, is decidable, and the satisfiability problem for it is $PSPACE$-complete. However, it does not have the finite topo-product model property.

6. Adding the true product interior

In this section we investigate the modal logic of products of topological spaces with all three horizontal, vertical, and standard product topologies. We add to the language $\mathcal{L}_{\Box_1 \Box_2}$ an extra modal operator $\Box$ with the intended interpretation as the interior operator of the standard product topology.

[3] In fact, the same argument implies that no logic in the interval $[\mathbf{S4} \oplus \mathbf{S4}, \mathbf{S4} \times \mathbf{S4}[$ has the finite topo-product model property.

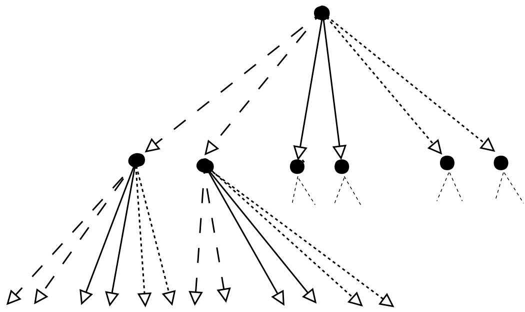

FIGURE 4. $T_{6,2,2}$. The solid lines represent R, the dashed lines represent R_1, and the dotted lines represent R_2. We assume that all dashed and dotted lines are also solid.

For two topological spaces $\mathcal{X} = \langle X, \eta \rangle$ and $\mathcal{Y} = \langle Y, \theta \rangle$, we will consider the product $\mathcal{X} \times \mathcal{Y} = \langle X \times Y, \tau, \tau_1, \tau_2 \rangle$ with three topologies, where τ is the standard product topology, τ_1 is the horizontal topology, and τ_2 is the vertical topology. Then $\Box$ is interpreted in $\mathcal{X} \times \mathcal{Y}$ as follows.

$$(x, y) \models \Box\varphi \quad \text{iff} \quad \exists U \in \eta \text{ and } \exists V \in \theta : U \times V \models \varphi$$

Since $\tau \subseteq \tau_1 \cap \tau_2$, we obtain that the modal principle

$$\Box p \to \Box_1 p \wedge \Box_2 p$$

is valid in product spaces. Our main goal in this section is to show that adding this principle to the fusion of three copies of **S4** axiomatizes the logic of products of topological spaces (with three topologies).

Definition 6.1. Let $\mathcal{L}_{\Box,\Box_1,\Box_2}$ be a modal language with three modal operators $\Box$, $\Box_1$, and $\Box_2$. We define the *topological product logic* **TPL** as the least set of formulas in $\mathcal{L}_{\Box,\Box_1,\Box_2}$ containing all axioms of $\mathbf{S4} \oplus \mathbf{S4} \oplus \mathbf{S4}$ plus the axiom $\Box p \to \Box_1 p \wedge \Box_2 p$, and closed under modus ponens, substitution, and $\Box$-, $\Box_1$-, and $\Box_2$-necessitation.

Let $T_{6,2,2}$ denote the infinite six branching tree such that each node of $T_{6,2,2}$ is R-related to all six of its immediate successors, R_1-related to the first two, and R_2-related to the last two; R, R_1, and R_2 are taken to be reflexive and transitive. Formally $T_{6,2,2}$ can be defined as $\langle W, R, R_1, R_2 \rangle$, where $W = \{0, 1, 2, 3, 4, 5\}^*$,

sRt iff $\exists u \in \{0, 1, 2, 3, 4, 5\}^* : s \cdot u = t$
sR_1t iff $\exists u \in \{0, 1\}^* : s \cdot u = t$
sR_2t iff $\exists u \in \{4, 5\}^* : s \cdot u = t$ (see Figure 4)

Theorem 6.2. **TPL** *is complete with respect to* $T_{6,2,2}$.

Proof. A straightforward generalization of the proofs of Theorem 2.1 and Proposition 2.9. For details see the forthcoming [15]. $\Box$

Theorem 6.3. **TPL** *is complete with respect to* $\mathbb{Q} \times \mathbb{Q}$.

Proof. Our strategy is similar to that of the proof of Theorem 5.1. By Theorem 6.2, **TPL** is complete with respect to $T_{6,2,2} = \langle W, R, R_1, R_2 \rangle$. We view $T_{6,2,2}$ as equipped with three Alexandroff topologies defined from R, R_1, and R_2. So for completeness of **TPL** with respect to $\mathbb{Q} \times \mathbb{Q}$ it is sufficient to show that there exists a total 3-topo-bisimulation between (a homeomorphic copy of) $\mathbb{Q} \times \mathbb{Q}$ and $T_{6,2,2}$. To achieve this, we label a subset of $\mathbb{Q} \times \mathbb{Q}$ recursively with the nodes of $T_{6,2,2}$ as follows (see Figure 6):

Stage 0: Label $(0, 0)$ with the root r of the tree $T_{6,2,2}$.

Stage 1: Label $(-1, 0)$ with the immediate left R_1-successor, and $(1, 0)$ with the immediate right R_1-successor of r; also label $(0, -1)$ with the immediate left R_2-successor, and $(0, 1)$ with the immediate right R_2-successor of r. With one of the remaining immediate R-successors of r we label the corners $(-1, -1)$ and $(1, 1)$; and with the other we label the corners $(-1, 1)$ and $(1, -1)$. Call these eight points *environmental points at distance* $\frac{1}{3^0}$ from $(0, 0)$.

Stage n: The environmental points labelled at stage $n - 1$ are at distance no smaller than $\frac{1}{3^{n-1}}$. Now for each of labelled points, (p, q), we create four environmental points at the distance $\frac{1}{3^n}$–two at the vertical distance $\frac{1}{3^n}$ and two at the horizontal distance $\frac{1}{3^n}$–and label them with respective immediate R_1- and R_2-successors in the tree. In addition, we create four corner points $(p+\frac{1}{3^n}, q+\frac{1}{3^n})$, $(p-\frac{1}{3^n}, q-\frac{1}{3^n})$, $(p+\frac{1}{3^n}, q-\frac{1}{3^n})$, and $(p-\frac{1}{3^n}, q+\frac{1}{3^n})$. The first two we label with one of the remaining immediate R-successors, and the last two with the other.

Define the partial function $l : \mathbb{Q} \times \mathbb{Q} \to T_{6,2,2}$ by letting $l(p, q)$ to be the node in $T_{6,2,2}$ assigned to (p, q) by some stage of the labelling. The same argument as in the proof of Theorem 5.1 guarantees that no point ever gets labelled twice. Thus, l is a well-defined partial function.

We recall that $P = \{p \in \mathbb{Q} : (p, 0) \text{ is labelled at some stage}\}$ and $Q = \{q \in \mathbb{Q} : (0, q) \text{ is labelled at some stage}\}$ are both dense unbounded subsets of $\mathbb{Q}$. Moreover, it follows from the above labelling that $P \times Q$ is the set of all labelled points of $\mathbb{Q} \times \mathbb{Q}$. We view $P \times Q$ as equipped with the three subspace topologies, which we also denote by τ, τ_1, and τ_2. Then $P \times Q$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ with respect to all three topologies.

Claim 6.4. l is a open and continuous map from $P \times Q$ onto $T_{6,2,2}$ with respect to all three topologies τ, τ_1, and τ_2.

Proof. That $l : P \times Q \to T_{6,2,2}$ is onto is obvious. The argument that l is open and continuous with respect to τ_1 and τ_2 carries over directly from Claim 5.2. The same technique can be used to show that l is open and continuous with respect to τ. To see this, we observe that

$$\{(p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times (q - \frac{1}{2^n}, q + \frac{1}{2^n}) : (p, q) \text{ is labelled at some stage } m < n\}$$

form a basis for τ on $P \times Q$. We also observe that a basis for the topology on $T_{6,2,2}$ defined from R is $\mathcal{B} = \{B_t\}_{t \in T_{6,2,2}}$ where $B_t = \{s \in T_{6,2,2} : tRs\}$.

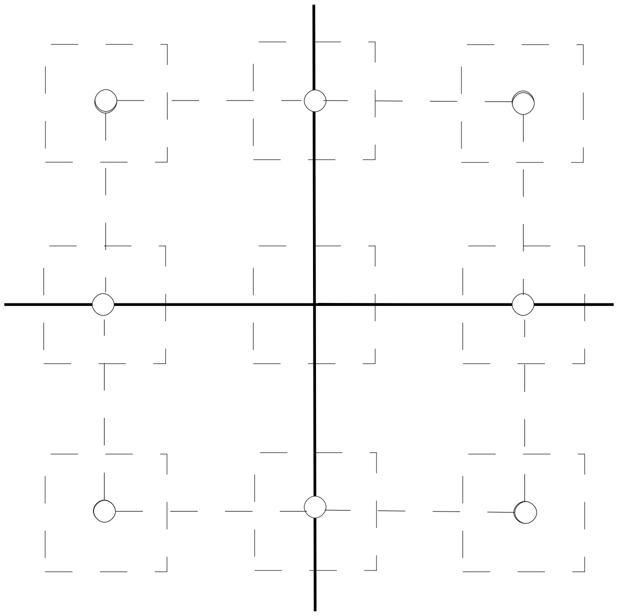

FIGURE 5. The second labelling of $\mathbb{Q} \times \mathbb{Q}$. If $w \in T_{6,2,2}$ labels $(p,q) \in \mathbb{Q} \times \mathbb{Q}$, then the points lying around the bigger dashed box are the points labelled at stage n, while the points lying around the smaller dashed boxes are the points labelled at stage $n+1$. There are eight environmental points of (p,q) labelled at stage n; two horizontal ones are labelled with the immediate R_1-successors of w, two vertical ones with the immediate R_2-successors of w, and the remaining four corner ones with the remaining two immediate R-successors of w. This pattern repeats itself.

(i) (Openness.) Let $(p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times (q - \frac{1}{2^n}, q + \frac{1}{2^n})$ be a basic open for τ. Then the same argument as in Claim 5.2 guarantees that

$$l((p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times (q - \frac{1}{2^n}, q + \frac{1}{2^n})) = B_{l(p,q)}.$$

(ii) (Continuity.) It suffices to show that for each $t \in T_{6,2,2}$, the inverse image of B_t belongs to τ. Let $(p,q) \in l^{-1}(B_t)$. Consider $(p-\frac{1}{2^n}, p+\frac{1}{2^n}) \times (q-\frac{1}{2^n}, q+\frac{1}{2^n}) \in \tau$, where n is greater than the stage at which (p,q) was labelled. Then by the above reasoning

$$l((p - \frac{1}{2^n}, p + \frac{1}{2^n}) \times (q - \frac{1}{2^n}, q + \frac{1}{2^n})) = B_{l(p,q)} \subseteq B_t.$$

$\square$

To complete the proof, if $\mathbf{TPL} \nvdash \varphi$, then by Theorem 6.2, there is a valuation ν on $T_{6,2,2}$ such that $\langle T_{6,2,2}, \nu \rangle, r \not\models \varphi$. Define a valuation ν' on $P \times Q$ by $\nu'(p) = l^{-1}(\nu(p))$. Since l is continuous and open with respect to all three topologies and $l(0,0) = r$, we have that $(0,0)$ and r are 3-topo-bisimilar. Therefore, $\langle P \times Q, \nu' \rangle, (0,0) \not\models \varphi$. Now since $P \times Q$ is homeomorphic to $\mathbb{Q} \times \mathbb{Q}$ with respect to all three topologies, it follows that φ is also refutable on $\mathbb{Q} \times \mathbb{Q}$. $\square$

Corollary 6.5. *In the language $\mathcal{L}_{\Box,\Box_1,\Box_2}$,* **TPL** *is the logic of products of arbitrary topologies.*

Incidentally, (using Kripke semantics) it is easy to show that **TPL** is a conservative extension of **S4** $\oplus$ **S4**, and that **S4** $\oplus$ **S4** is a conservative extension of **S4**. Therefore, Theorem 2.4 becomes a corollary of Theorem 5.1, while Theorem 5.1 becomes a corollary of Theorem 6.3.

7. Conclusions and further directions

We introduced the horizontal and vertical topologies on the product of two topological spaces and we showed that the modal logic of products of topological spaces with two horizontal and vertical topologies is the fusion **S4** $\oplus$ **S4**. In addition, we axiomatized the modal logic of products of topological spaces with three horizontal, vertical, and standard product topologies. We conclude by mentioning several open questions that arise naturally from this study.

7.1. **Special spaces.** Although we showed that **S4** $\oplus$ **S4** is complete with respect to $\langle \mathbb{Q}\times\mathbb{Q}, \tau_1, \tau_2\rangle$, and that **TPL** is complete with respect to $\langle \mathbb{Q}\times\mathbb{Q}, \tau, \tau_1, \tau_2\rangle$, it is still an open question what the logics of $\langle \mathbb{R}\times\mathbb{R}, \tau_1, \tau_2\rangle$ and $\langle \mathbb{R}\times\mathbb{R}, \tau, \tau_1, \tau_2\rangle$ are.

Since Alexandroff spaces correspond to reflexive and transitive frames, it follows from Gabbay at al. [9] that the modal logic of the class

$$\{\mathcal{X}\times\mathcal{Y} : \mathcal{X} \text{ and } \mathcal{Y} \text{ are Alexandroff}\}$$

is **S4** $\times$ **S4**. On the other hand, it is still unknown what the modal logics of the following two classes are:

$$K_1 = \{\mathcal{X}\times\mathcal{Y} : \mathcal{X} \text{ is Alexandroff}\} \quad \text{and} \quad K_2 = \{\mathcal{X}\times\mathcal{Y} : \mathcal{Y} \text{ is Alexandroff}\}.$$

We conjecture that the modal logic of K_1 is **S4**$\oplus$**S4**$+com_{\leftarrow}+chr$, and that the modal logic of K_2 is **S4**$\oplus$**S4**$+com_{\rightarrow}+chr$. We also conjecture that in the language enriched with the third modality (for the standard product topology), the logics of these two classes are **TPL** $+ com_{\leftarrow} + chr$ and **TPL** $+ com_{\rightarrow} + chr$, respectively.

7.2. **Enriching the language.** It is only natural to look at various language extensions of $\mathcal{L}_{\Box_1\Box_2}$. In adding $\Box$ we have made the first step in this direction, but there are several others that can be taken. For instance, adding the universal modality or nominals.

A very natural extension of the language would be with the common knowledge operator. In the standard Kripke setting, there are several ways of defining common knowledge that all turn out to be equivalent (see [3]). In [4] we examine two most prominent such ways and show that in the topological setting the two are in fact distinct. The first defines the common knowledge as an infinite conjunction of claims in the original language, and the second takes common knowledge to be the greatest fixed point of an operator. Thus in our setting the two are:

(1) $C_{1,2}\varphi :=$ an infinite conjunction of all finite nestings of $\Box_1$ and $\Box_2$:

$$\varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_2\varphi \wedge \ldots$$

(2) $K_{1,2}\varphi :=$ the greatest fixed point of the operator $\lambda X.([|\phi|] \cap I_1X \cap I_2X)$, as in the following formula of the modal μ-calculus:

$$\nu p.(\varphi \wedge \Box_1 p \wedge \Box_2 p)$$

We argue in [4] that the common knowledge as the greatest fixed point is most interesting from the topological perspective.

7.3. **Further exploration of the connection with Kripke semantics.** We have shown that the topological setting has greater power of discrimination than the relational setting. In particular, topological products validate less principles than products of Kripke frames, and the true product interior modality is not definable in terms of the horizontal and vertical modalities. However, we can generalize products of Kripke frames by restricting the universe of admissible product subsets (see, e.g., [5]). The latter is a well-known strategy in relational algebra and arrow logic (see Chapter 7 of [6]). In particular, over such generalized relational products we have:

(1) *com* and *chr* are no longer valid.
(2) The product $\Box$ is no longer definable as $\Box_1\Box_2$.

This similarity suggests a connection between topological products and generalized relational products.

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