# A parametrised choice principle and Martin's Conjecture on Blackwell determinacy

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We de£ne a parametrised choice principle PCP which (under the assumption of the Axiom of Blackwell Determinacy) is equivalent to the Axiom of Determinacy. PCP describes the difference between these two axioms and could serve as a means of proving Martin's conjecture on the equivalence of these axioms.

## 1 Blackwell Determinacy and Martin's Conjecture

Tony Martin proved in [Mar98] that the Axiom of Determinacy AD implies the Axiom of Blackwell Determinacy Bl-AD, and conjectured that the converse holds as well:

**Conjecture 1** The Axiom of Blackwell Determinacy Bl-AD implies the Axiom of Determinacy AD.

Although Martin, Neeman and Vervoort in their [MarNeeVer03] have made considerable progress on Martin's Conjecture 1, the full conjecture remains open. In this note, we investigate a fragment of the Axiom of Choice that we shall call the Parametrised Choice Principle PCP which captures the difference between the two axioms (if there is one). Our result is that (in the base theory $\mathsf{ZF}+\mathsf{Bl}\text{-}\mathsf{AD}$), AD and PCP are equivalent (Theorem 3). So, showing Martin's Conjecture 1 is tantamount to proving the choice principle from Bl-AD.

We will not go into details of the motivation and de£nition of the Axiom of Blackwell Determinacy Bl-AD here and refer the reader to [Mar98], [Lö02a], [Lö02b], and [Lö04]. Our de£nition is equivalent to Vervoort's original de£nition in [Ver95]; *cf.* [Lö04, Theorem 2.5 (b)].

Let us denote by $\omega^{\mathrm{Even}}$ the set of £nite sequences of natural numbers of even length, by $\omega^{\mathrm{Odd}}$ the set of such sequences of odd length, and by $\mathrm{Prob}(\omega)$ the set of probability measures on $\omega$.

We call a function $\sigma : \omega^{\mathrm{Even}} \to \mathrm{Prob}(\omega)$ a **(mixed) strategy for player I** and a function $\sigma : \omega^{\mathrm{Odd}} \to \mathrm{Prob}(\omega)$ a **(mixed) strategy for player II**. A mixed strategy $\sigma$ is called **pure** if for all $s \in \mathrm{dom}(\sigma)$ the measure $\sigma(s)$ is a Dirac measure, *i.e.*, there is a natural number $n$ such that $\sigma(s)(\{n\}) = 1$. This is of course equivalent to being a strategy in the usual (perfect information) sense. We denote the class of all mixed strategies with $\mathcal{S}_{\mathrm{mix}}$.

If $\sigma$ and $\tau$ are strategies for player I and II, respectively, then they completely describe a play of the game between these two players: Player I randomizes to choose his £rst move $a_0$ according to the probability measure $\sigma(\varnothing)$, then player II looks at $a_0$, consults his strategy about the measure $\tau(\langle a_0\rangle)$ and plays according to that probability measure.

Let

$$\nu(\sigma,\tau)(s) := \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even, and} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then for any $s \in \omega^{<\omega}$, we can de£ne

$$\mu_{\sigma,\tau}([s]) := \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s{\restriction}i)(\{s_i\}).$$

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This generates a Borel probability measure on $\omega^\omega$ which can be seen as a measure of how well the strategies $\sigma$ and $\tau$ performs against each other. If $B$ is a Borel set, $\mu_{\sigma,\tau}(B)$ is interpreted as the probability that the result of the game ends up in the set $B$ when player I randomizes according to $\sigma$ and player II according to $\tau$.

Let $\mathcal{S}$ be a class of strategies, $\sigma$ a mixed strategy for player I, and $\tau$ a mixed strategy for player II. We say that $\sigma$ is **$\mathcal{S}$-optimal** for the payoff set $A \subseteq \omega^\omega$ if for all $\tau_* \in \mathcal{S}$ for player II, $\mu^-_{\sigma,\tau_*}(A) = 1$, and similarly, we say that $\tau$ is **$\mathcal{S}$-optimal** for the payoff set $A \subseteq \omega^\omega$ if for all $\sigma_* \in \mathcal{S}$ for player I, $\mu^+_{\sigma_*,\tau}(A) = 0$.[1]

We call a set $A \subseteq \omega^\omega$ **Blackwell determined** if either player I or player II has an $\mathcal{S}_{\rm mix}$-optimal strategy, and we call a pointclass $\Gamma$ Blackwell determined if all sets $A \in \Gamma$ are Blackwell determined. We write $\mathsf{Bl\text{-}Det}(\Gamma)$ for this statement and $\mathsf{Bl\text{-}AD}$ for full axiom claiming Blackwell determinacy for all sets.

If the payoff set $A$ is universally measurable, every winning strategy is $\mathcal{S}_{\rm mix}$-optimal (by an easy measure theoretic argument due to Vervoort [Ver95]) and at most one of the two players can have an $\mathcal{S}_{\rm mix}$-optimal strategy.

The following is a distinctive mathematical difference between pure and mixed strategies in in£nite games: A winning strategy $\sigma$ in the game with payoff set $A$ (say, for player I) gives us a function $f_\sigma : \omega^\omega \to \omega^\omega$ that guarantees that for all $x \in \omega^\omega$, we have $f_\sigma(x) \in A$ thus essentially picking an element of $A$.

On the other hand, the analogous de£nition using an $\mathcal{S}_{\rm mix}$-optimal strategy $\sigma$ gives only a function assigning a measure $\mu_{\sigma,x}$ that gives the set $A$ measure one, but we seem to be lacking a way of picking an element of $A$ using that measure as input data.

The result of this note is that this technical difference between the two axioms is actually the whole story: Choosing correctly elements from measure 1 sets is all we need in order to get from $\mathsf{Bl\text{-}AD}$ to $\mathsf{AD}$. In this note, we shall introduce a principle which we shall call the "parametrised choice principle" $\mathsf{PCP}$. It is conceivable that this principle is provable from $\mathsf{Bl\text{-}AD}$ and it is a worthwhile project to try and show $\mathsf{PCP}$ from $\mathsf{Bl\text{-}AD}$ thereby proving Martin's Conjecture 1.

## 2 Parametrised Choice Principles

Given a set of reals $A \subseteq \omega^\omega$, de£ne two perfect information games, the I-puri£cation game on $A$ and the II-puri£cation game on $A$. In order not to confuse players in these games with the usual players in the standard game with payoff set $A$, we call the players in the puri£cation games "player 1" and "player 2" (as opposed to "player I" and "player II").

We de£ne the the **I-puri£cation game** for $A$: Player 1 plays a sequence $\langle \varrho_0, \langle \varrho_1, y_0\rangle, \langle \varrho_2, y_1\rangle, \ldots\rangle$ where $\varrho_i$ is a probability measure on $\omega$ and $y_i \in \omega$. Player 2 plays a sequence $\langle x_i\ ;\ i \in \omega\rangle$. Set $x := \langle x_i\ ;\ i \in \omega\rangle$ and $y := \langle y_i\ ;\ i \in \omega\rangle$. Then player 2 wins if $x * y \in A$.

In the I-puri£cation game, we interpret player 1 as playing both players I and II in a Blackwell game on $A$, providing actual moves for player II and playing probability measures for player I. Player 2 on the other hand tries to pick moves in the same game for player I according to the probability measures that player 1 provided.

We say that **player 1 chooses his measure moves $\mathcal{S}_{\rm mix}$-optimally in the I-puri£cation game** if there is an $\mathcal{S}_{\rm mix}$-optimal strategy $\sigma$ for player I in the game with payoff set $A$ such that $\varrho_i = \sigma(x{\restriction}i * y{\restriction}i)$.

A strategy $\pi$ for player 2 in the I-puri£cation game is called a **I-puri£cation** if $\pi$ wins against all plays in which player 1 chooses his measure moves $\mathcal{S}_{\rm mix}$-optimally.

Now de£ne the **II-puri£cation game** for $A$: Player 1 plays a sequence $\langle\langle \varrho_i, x_i\rangle\ ;\ i \in \omega\rangle$ where $\varrho_i$ is a probability measure on $\omega$ and $x_i \in \omega$.

Player 2 plays a sequence $\langle y_i\ ;\ i \in \omega\rangle$. Again, set $x := \langle x_i\ ;\ i \in \omega\rangle$ and $y := \langle y_i\ ;\ i \in \omega\rangle$. Then player 2 wins if $x * y \notin A$.

This time, player 1 provides moves for player I and probability measures for player II in the Blackwell game with payoff set $A$ and player 2 tries to pick the moves for player II according to the measures that player 1 played.

In this game, we say that **player 1 chooses his measure moves $\mathcal{S}_{\rm mix}$-optimally in the II-puri£cation game** if there is am $\mathcal{S}_{\rm mix}$-optimal strategy $\tau$ for player II in the game with payoff set $A$ such that $\varrho_i = \tau(x{\restriction}(i{+}1) * y{\restriction}i)$.

[1] Here, $\mu^+$ denotes outer measure and $\mu^-$ denotes inner measure with respect to $\mu$ in the usual sense of measure theory. If $A$ is Borel, then $\mu^+(A) = \mu^-(A) = \mu(A)$ for Borel measures $\mu$.

A strategy $\pi$ for player 2 is called a **II-puri£cation** if $\pi$ wins against all plays in the II-puri£cation game in which player 1 chooses his measure moves $\mathcal{S}_{\rm mix}$-optimally.

Using these notions, we de£ne the **Parametrised Choice Principle** PCP:

"For each $A \subseteq \omega^\omega$ there are I-puri£cations and II-puri£cations for $A$."[2]

Let us look at the following variant of the two puri£cation games: Instead of playing the measures move by move, player 1 has to present the entire Blackwell strategy before the game starts and then only play the $y_i$ moves. Obviously, it is easier to have a puri£cation in this type of game since the puri£cation needn't be uniform for all possible mixed strategies but can depend on the strategy. We shall call this version the **non-uniform Parametrised Choice Principle** nuPCP.

Suppose that $\sigma$ and $\tau$ are strategies for player I or II, respectively, in the game on $A$, that $\pi$ is a strategy for player 2 in the I-puri£cation game, and that $\pi^*$ is a strategy for player 2 in the II-puri£cation game (or their non-uniform variants). Let the function ${\rm pur}_{\rm I}(\pi,\sigma) : \omega^\omega \to \omega^\omega$ be de£ned by mapping a real $y$ to the answer of player 2 in the I-puri£cation game if player 1 plays $y$ and chooses his measure moves according to $\sigma$. Clearly, ${\rm pur}_{\rm I}(\pi,\sigma)$ is a Lipschitz function. Analogously, we can de£ne a Lipschitz map ${\rm pur}_{\rm II}(\pi^*,\tau)$.

If now $\sigma$ and $\tau$ are $\mathcal{S}_{\rm mix}$-optimal, $\pi$ is a I-puri£cation, and $\pi^*$ is a II-puri£cation (or their non-uniform variants)[3], then we have

$$
{\rm pur}_{\rm I}(\pi,\sigma)(y) * y \in A \text{ for all } y \in \omega^\omega\text{, and}
$$

$$
x * {\rm pur}_{\rm II}(\pi^*,\tau)(x) \notin A \text{ for all } x \in \omega^\omega.
$$

Thus, a different way of describing PCP is the following:

Suppose that $A$ is a set of reals, set $A_x := \{y\,;\, y * x \in A\}$ for every real $x$, and suppose that we have a Lipschitz function (in the codes) $\mu : x \mapsto \mu_x$ assigning Borel probability measures to reals with the property that for each $x$, $\mu_x(A_x) = 1$. Then we can construct a Lipschitz function that computes from the function $\mu$ and $x$ an element of $A_x$.

This principle is called "Parametrised Choice Principle" because it is an effective variant of the following fragment of the Axiom of Choice (which follows from the Axiom of Real Determinacy $\mathsf{AD}_\mathbb{R}$):

Let $\langle A_x\,;\, x \in \omega^\omega\rangle$ be any family of nonempty sets of reals. Then $\prod_{x\in\omega^\omega} A_x \neq \emptyset$.

**Observation 2** Assume AD. Then PCP holds.

P r o o f. Note that by AD every set of reals is universally measurable, hence all winning strategies are $\mathcal{S}_{\rm mix}$-optimal, and furthermore if one player has a winning strategy, the other player can have no $\mathcal{S}_{\rm mix}$-optimal strategy.

Let $A \subseteq \omega^\omega$ be £xed. By AD, either player I or player II has a winning strategy in the game on $A$. Without loss of generality, let player I have a winning strategy $\sigma$. Then for any sequence $\varrho := \langle \varrho_i\,;\, i \in \omega\rangle$ and any $y \in \omega^\omega$, $\sigma$ wins as a strategy for player 2 in the I-puri£cation game against $\langle \varrho, y\rangle$ by just ignoring the measure moves, so $\sigma$ is a I-puri£cation. But if $\sigma$ is a winning strategy for player I there cannot be a $\mathcal{S}_{\rm mix}$-optimal strategy for player II, so every strategy in the II-puri£cation game is a II-puri£cation because the condition is vacuously true. □

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[2] Note that the symmetry of PCP is arti£cial: Whenever player I has an $\mathcal{S}_{\rm mix}$-optimal strategy in the game on $A$, player II can't have one, hence the premiss of the de£nition of a II-puri£cation (existence of an $\mathcal{S}_{\rm mix}$-optimal strategy for player II) is false and thus by *ex falso quodlibet* every strategy in the II-puri£cation game is a II-puri£cation. In the context of Bl-AD, an equivalent way of stating PCP is: "If player I has an $\mathcal{S}_{\rm mix}$-optimal strategy in the game on $A$, there is a I-puri£cation; if player II has an $\mathcal{S}_{\rm mix}$-optimal strategy in the game on $A$, there is a II-puri£cation."

[3] The difference between PCP and nuPCP is in the way the strategies $\sigma$ or $\tau$ get involved in the construction of the Lipschitz maps ${\rm pur}_{\rm I}(\pi,\sigma)$ and ${\rm pur}_{\rm II}(\pi^*,\tau)$: Using PCP, we get that the functions depends on the strategy continuously in the code. In our equivalence result Theorem 3 we show that under the assumption of Bl-AD, the difference doesn't really matter.

## 3 The Equivalence

**Theorem 3** Assume Bl-AD. Then the following are equivalent:

1. nuPCP,
2. PCP, and
3. AD.

Proof. The direction "(3.) $\Rightarrow$ (2.)" is just Observation 2 and "(2.) $\Rightarrow$ (1.)" is obvious. So let's prove "(1.) $\Rightarrow$ (3.)":

By a theorem of Blass [Bl73], it is enough to show that for every set $A \subseteq \omega^\omega \times \omega^\omega$ there is either a continuous function $f$ such that

$$\forall y(\langle f(y), y\rangle \in A),$$

or a continuous function $f$ such that

$$\forall x(\langle x, f(x)\rangle \notin A).$$

Take any set $A \subseteq \omega^\omega \times \omega^\omega$. Set $A^* := \{x * y \; ; \; \langle x, y\rangle \in A\}$. By nuPCP we have a I-puri£cation $\pi$ and a II-puri£cation $\pi^*$ for $A^*$. Using Bl-AD, we either get an $\mathcal{S}_{\rm mix}$-optimal strategy $\sigma$ for player I for the game on $A^*$ or an $\mathcal{S}_{\rm mix}$-optimal strategy $\tau$ for player II for the game on $A^*$.

If $\sigma$ is $\mathcal{S}_{\rm mix}$-optimal for player I, then $\mathrm{pur}_{\mathrm{I}}(\pi, \sigma)$ is the required continuous function; if $\tau$ is $\mathcal{S}_{\rm mix}$-optimal for player II, then $\mathrm{pur}_{\mathrm{II}}(\pi^*, \tau)$ is the required continuous function. □

Note that Theorem 3 can be localized: If $\mathbf{\Gamma}$ is a boldface pointclass that is closed under £nite unions and contains all Borel sets[4], then Blackwell Determinacy for all sets in $\mathbf{\Gamma}$ and the non-uniform Parametrised Choice Principle for all sets in $\mathbf{\Gamma}$ imply Determinacy for all sets in $\mathbf{\Gamma}$.

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[4] These closure properties are used in Blass' coding argument for the equivalence of [Bl73].