

# Guarded fragments with constants

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## Abstract

We prove EXPTIME-completeness of the satisfiability problem for (loosely) guarded first-order formulas with a bounded number of variables and an unbounded number of constants. Guarded fragments with constants are interesting because of their connection to hybrid logic.

## 1 Introduction

The guarded fragment of first-order logic was first introduced by Andreka, Van Benthem and Nemeti [1], who proved that it is decidable and that it has a number of other desirable properties. Van Benthem [3] improved on this by generalizing the guarded fragment to the loosely guarded fragment and showing that the latter is still decidable.

Grädel [7] further improved on these results of Andreka, Van Benthem and Nemeti in a number of ways. He generalized the guarded and loosely guarded fragments by allowing constants to occur in the formulas (but not function symbols of positive arity), and by allowing identity statements of the form  $x = x$  or  $x = y$  as guards, and subsequently proved the following:

**Theorem 1 (Grädel [7])** *The satisfiability problem for loosely guarded formulas is 2EXPTIME-complete. The same problem is only EXPTIME-complete for loosely guarded relational formulas with a bounded number of variables, and for guarded relational formulas with a bound on the arity of the relation symbols.*

With a relational formula, we mean a formula that contains no constants (function symbols of positive arity were already excluded).

Furthermore, Grädel suggests in his paper that his results also work for (loosely)  $\forall$ -guarded formulas, i.e., formulas of which only the universal quantifiers are (loosely) guarded. However, the details are not spelled out.<sup>1</sup>

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<sup>1</sup>Marx [9] does explicitly state and prove the *decidability* of the satisfiability problem for loosely  $\forall$ -guarded formulas.

This note serves two purposes. Firstly, it formulates the precise results for universally guarded formulas that may be obtained with Grädel’s techniques, and it contains the details of the proofs. Secondly, and more importantly, we improve Grädel’s results by showing that the qualification ‘*relational*’ in the above theorem may be dropped.

Concretely, we prove the following.

**Theorem 2** *The satisfiability problem for loosely  $\forall$ -guarded formulas is  $2\text{EXPTIME}$ -complete. The same problem is only  $\text{EXPTIME}$ -complete for loosely  $\forall$ -guarded formulas with a bounded number of variables and for guarded formulas with a bounded arity.*

To appreciate the additional value of Theorem 2, we must return to the original motivation behind the guarded fragment. The guarded fragment was invented in order to explain and generalize the large number of decidability and low complexity results in modal logic. The key observation is that modal operators express a guarded form of quantification, where the accessibility relations are the guards.

For explaining *decidability* results in modal logic, the first part of Theorem 1 often suffices. However, in order to explain *low complexity*, a more refined analysis is needed. Consider for instance the global consequence problem for modal formulas (*does every model that globally satisfies  $\phi$  globally satisfy  $\psi$ ?*). This is an  $\text{EXPTIME}$ -complete problem. To understand why this problem is in  $\text{EXPTIME}$ , it suffices to observe that global truth of a modal formula  $\phi$  can be expressed by means of a guarded first-order formula with only two variables, namely  $\forall x.(x = x \rightarrow ST_x(\phi))$ .<sup>2</sup> This shows the importance of bounded variable guarded fragments.

Recently there has been much interest in modal languages with *nominals* [4, 10]. Nominals are special proposition letters that denote singleton sets. From a first-order perspective, they correspond to constants. Thus, if we translate formulas containing nominals into first order logic, then we arrive in the two-variable guarded fragment with an unlimited number of constants. Clearly, Theorem 1 will not allow us to prove, say, that the global consequence problem for modal formulas with nominals is in  $\text{EXPTIME}$ . Theorem 2 does, and it thereby broadens the application of guarded fragments to the field of hybrid logic (this is the common name for the family of modal languages with nominals and related machinery). A concrete example of a complexity result from the literature that follows immediately from Theorem 2 is the  $\text{EXPTIME}$ -membership of the satisfiability problem for the hybrid language  $\mathcal{H}(@, \diamond^{-1}, \text{E})$  [2].

## 2 Normal forms for (loosely) guarded formulas

We will consider first-order languages with arbitrarily many relation symbols of any arity, constants and equality, but without function symbols of arity greater

<sup>2</sup>Here,  $ST$  refers to the well-known Vardi-style standard translation of modal formulas to first-order formulas, that uses only two variables.

than zero. A first-order formula  $\phi$  of such a language is called *guarded* if it is built up from atomic formulas using the Boolean connectives and guarded quantifiers of the form  $\exists x_1 \dots x_n.(\pi \wedge \psi)$  or  $\forall x_1 \dots x_n.(\pi \rightarrow \psi)$ , where  $\pi$  is an atomic formula and the free variables of  $\psi$  all occur in  $\pi$ . A formula is called  *$\forall$ -guarded* if it is built up from atomic formulas and negated atomic formulas using conjunction, disjunction, ordinary existential quantifiers and guarded universal quantifiers. Note that the guards  $\pi$  may be atomic equality statements. In particular, if a guarded formula  $\phi$  has only one free variable  $x$ , then  $\exists x.(x = x \wedge \phi)$  and  $\forall x.(x = x \rightarrow \phi)$  are guarded formulas. These formulas are equivalent to  $\exists x.\phi$  and  $\forall x.\phi$ , respectively.

The loosely guarded fragment is an extension of the guarded fragment. A first-order formula  $\phi$  is called *loosely guarded* if it is built up from atomic formulas using the Boolean connectives and loosely guarded quantifiers of the form  $\exists x_1 \dots x_n.(\pi \wedge \psi)$  or  $\forall x_1 \dots x_n.(\pi \rightarrow \psi)$ , where  $\pi$  is conjunction of atomic formulas, such that every quantified variable  $x_i$  co-occurs with every free variable  $y \neq x_i$  of  $\psi$  in some conjunct of  $\pi$ . A formula is called *loosely  $\forall$ -guarded* if it is built up from atomic formulas and negated atomic formulas using conjunction, disjunction, ordinary existential quantifiers and loosely guarded universal quantifiers. Note that if a loosely guarded formula  $\phi$  has only one free variable  $x$ , then  $\exists x.(\top \wedge \phi)$  and  $\forall x.(\top \rightarrow \phi)$  are loosely guarded.

Grädel [7] proved his main decidability and complexity results for guarded formulas using the following normal form.

**Definition 1** *A (loosely)  $\forall$ -guarded formula is in normal form if it is of the form*

$$\exists \vec{x}. P\vec{x} \wedge \bigwedge_{i \in I} \forall \vec{x}. (\pi_i(\vec{x}) \rightarrow \exists \vec{y}. \phi_i(\vec{x}, \vec{y}))$$

where, for each  $i \in I$ , the variables  $\vec{x}, \vec{y}$  are distinct,  $\pi_i$  is a (loose) guard and  $\phi_i(\vec{x}, \vec{y})$  is a quantifier-free formula.

Grädel showed that every (loosely) guarded formula can be translated in polynomial time into an equisatisfiable (loosely)  $\forall$ -guarded formula in normal form. A slight variation of Grädel's proof works for (loosely)  $\forall$ -guarded sentences, thus turning it into a true normal form theorem for (loosely)  $\forall$ -guarded formulas. To be sure, we will spell out the proof here for the case of (loosely)  $\forall$ -guarded formulas.

For any formula  $\phi$ , let  $\text{WIDTH}(\phi)$  be the maximal number of free variables of a subformula of  $\phi$ , i.e.,  $\text{WIDTH}(\phi)$  is the largest natural number  $n$  such that  $\phi$  has a subformula with  $n$  free variables.

**Proposition 1** *Every (loosely)  $\forall$ -guarded formula  $\phi$  can be transformed in polynomial time into an equisatisfiable (loosely)  $\forall$ -guarded sentence  $\chi$  in normal form. Moreover,  $\text{WIDTH}(\chi) \leq \text{WIDTH}(\phi)$ .*

**Proof:** We first give the proof for  $\forall$ -guarded formulas, and then show how the proof generalizes to loosely  $\forall$ -guarded formulas. Let  $\phi$  be  $\forall$ -guarded, and assume

without loss of generality that no equality sign occurs inside a guard in  $\phi$ . If  $\phi$  is quantifier-free, then we are already done. Otherwise, there are two possibilities.

1.  $\phi$  contains a subformula of the form  $\chi(\vec{x}) = \exists y.\psi(\vec{x}, y)$ , where  $\psi$  is quantifier-free. Pick a new predicate  $R_\chi$  of the appropriate arity, and let  $\phi[\chi/R_\chi]$  be the result of replacing  $\chi(\vec{x})$  in  $\phi$  by  $R_\chi(\vec{x})$ . Finally, let

$$\phi' = \phi[\chi/R_\chi] \wedge \forall \vec{x}.(R_\chi(\vec{x}) \rightarrow \exists y.\psi(\vec{x}, y))$$

Then  $\phi'$  is equi-satisfiable to  $\phi$ , and one step closer to being of the required form.

2.  $\phi$  contains a subformula of the form  $\chi(\vec{x}) = \forall \vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow \psi(\vec{x}, \vec{y}))$ , where  $\psi$  is quantifier-free. Pick a new relation symbol  $R_\chi$  with the appropriate arity, and let  $\phi[\chi/R_\chi]$  be the result of replacing  $\chi(\vec{x})$  in  $\phi$  by  $R_\chi(\vec{x})$ . Finally, let

$$\phi' = \phi[\chi/R_\chi] \wedge \forall \vec{x}\vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow (R_\chi(\vec{x}) \rightarrow \psi(\vec{x}, \vec{y})))$$

Then  $\phi'$  is equi-satisfiable to  $\phi$ , and one step closer to being of the required form.

Repeating these steps, we eventually obtain a formula of the form  $\phi''(\vec{x}) \wedge \eta$ , where  $\phi''(\vec{x})$  is quantifier-free, and  $\eta$  is a conjunction of formulas of the form  $\forall \vec{x}(\pi(\vec{x}) \rightarrow \exists y.\psi(\vec{x}, y))$ . As a final step, pick a new predicate  $P$  and let  $\theta = \exists \vec{x}.P(\vec{x}) \wedge \forall \vec{x}(P(\vec{x}) \rightarrow \phi''(\vec{x})) \wedge \eta$ . Then  $\theta$  is in normal form and equi-satisfiable to the original formula  $\phi$ .

A slight variation of this argument works for loosely  $\forall$ -guarded formulas. Suppose  $\phi$  is loosely  $\forall$ -guarded and contains a subformula of the form  $\chi(\vec{x}) = \forall \vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow \psi(\vec{x}, \vec{y}))$ , where  $\psi$  is quantifier-free. As before, we pick a new relation symbol  $R_\chi$  with the appropriate arity, but now we also pick a new binary relation symbol  $Z$ . Also, the conjunct we add to  $\phi$  is slightly different: instead of  $\forall \vec{x}\vec{y}.(\pi(\vec{x}, \vec{y}) \rightarrow (R_\chi(\vec{x}) \rightarrow \psi(\vec{x}, \vec{y})))$ , we add  $\forall \vec{x}\vec{y}.((\pi(\vec{x}, \vec{y}) \wedge \bigwedge_{z, z' \in \{\vec{x}\}} Zzz') \rightarrow (R_\chi(\vec{x}) \rightarrow \psi(\vec{x}, \vec{y})))$ . This ensures that each two variables in  $\vec{x}$  co-occur in some atom of the guard, to guarantee that the universal quantifier is properly loosely guarded. Finally, to ensure that the new formula is equi-satisfiable to the original one, instead of replacing  $\chi(\vec{x})$  in  $\phi$  by  $R_\chi(\vec{x})$ , we do it by  $R_\chi(\vec{x}) \wedge \bigwedge_{z, z' \in \{\vec{x}\}} Zzz'$ . The rest of the proof remains the same.  $\square$

In the case of loosely guarded formulas, one can furthermore ensure that the arity of the relation symbols occurring in the formula is bounded by the width. For any formula  $\phi$ , let  $\text{MAXARITY}(\phi)$  denote the highest arity of a relation symbol occurring in  $\phi$ .

**Proposition 2** *Every loosely  $\forall$ -guarded formula  $\phi$  can be transformed in polynomial time into an equisatisfiable loosely  $\forall$ -guarded formula  $\chi$  in normal form, such that  $\text{WIDTH}(\chi) \leq \max\{\text{WIDTH}(\phi), 2\}$  and  $\text{MAXARITY}(\chi) \leq \max\{\text{WIDTH}(\phi), 2\}$ .*

**Proof:** The proof proceeds in two steps. First, we will reduce the arity of the relation symbols occurring in  $\phi$  to two. Then, we will write the resulting formula in normal form. The latter step might increase the arity of the relation symbols again, but it will still be bounded by the width of the formula.

Let  $\phi$  be any loosely  $\forall$ -guarded formula. For each  $n$ -ary relation symbol  $R$  occurring in  $\phi$ , with  $n > 2$ , introduce  $n + 1$  new binary relation symbols,  $R_0, \dots, R_n$ . These relation symbols will be used to encode the tuples that stand in the relation  $R$ : a tuple  $\langle d_1, \dots, d_n \rangle$  will be thought to stand in the relation if each pair  $\langle d_\ell, d_m \rangle$  stands in the  $R_0$  relation ( $1 \leq \ell, m \leq n$ ), and there exists an element  $e$  such that  $\langle e, d_\ell \rangle \in R_\ell$  for  $1 \leq \ell \leq n$ .

Replace each subformula of  $\phi$  of the form  $R(t_1, \dots, t_n)$  that is not inside a guard by

$$\bigwedge_{1 \leq \ell, m \leq n} R_0(t_\ell, t_m) \wedge \exists u. \bigwedge_{1 \leq \ell \leq n} R_\ell(u, x_\ell)$$

If  $\phi$  has a subformula of the form  $\forall \vec{x}(\pi \rightarrow \psi)$ , where the guard  $\pi$  contains a conjunct of the form  $R(t_1, \dots, t_n)$ , then replace that conjunct by  $\bigwedge_{1 \leq \ell, m \leq n} R_0(t_\ell, t_m)$ , and replace  $\psi$  by  $\exists u. (\bigwedge_{1 \leq \ell \leq n} R_\ell(u, x_\ell) \wedge \top) \rightarrow \psi$ .

The resulting formula contains no relation symbols of arity greater than 2, and it is satisfiable iff the original formula  $\phi$  is satisfiable. Furthermore, the width of the resulting formula is at most  $\max(\text{WIDTH}(\phi), 2)$ .

Finally, we apply Proposition 1 to bring the resulting formula into normal form. Inspection of the proof of Proposition 1 shows that the arity of the relation symbols added during the normal form translation is bounded by the width of the input formula. Hence, we end up with a formula with the desired properties.  $\square$

Incidentally, the constraints of bounded width and of bounded number of variables in a first-order formula are equivalent, as proved in the following theorem.

**Proposition 3** *For  $k \in \mathbb{N}$ , every first-order formula  $\phi$  of width  $k$  can be transformed in polynomial time into an equivalent formula containing  $k$  variables.*

**Proof:** The proof is by structural induction on the input formula  $\phi$ . If  $\phi$  is an atomic formula, then its width equals the number of variables occurring in it, hence the claim holds. If  $\phi$  is of the form  $\neg\psi$  or  $\exists x.\psi$ , then the claim follows immediately from the induction hypothesis (note that, in the second case, we may assume that  $x$  occurs in  $\psi$ ). This leaves us with the case in which  $\phi$  is a conjunction.

Let  $\phi$  be of the form  $\psi \wedge \chi$ . By induction hypothesis, we may assume that  $\psi$  and  $\chi$  each have at most  $k$  variables. We may also assume that the only variables occurring both in  $\psi$  and in  $\chi$  are the ones that occur freely in  $\psi$  and in  $\chi$ . It follows that the set of all variables occurring in  $\phi$  can be partitioned into disjoint subsets  $X, Y, Z, U, V$  such that  $\text{free}(\psi) = X \cup Y$ ,  $\text{free}(\chi) = Y \cup Z$ ,

$\text{bound}(\psi) \setminus \text{free}(\psi) = U$  and  $\text{bound}(\chi) \setminus \text{free}(\chi) = V$ . In other words,

$$\begin{array}{ccc} \phi(X, Y, Z) & = & \psi(X, Y) \quad \wedge \quad \chi(Y, Z) \\ & & \text{(additional bound variables } U) \quad \quad \quad \text{(additional bound variables } V) \end{array}$$

Let  $W$  be a new set of variables, disjoint from  $X, Y, Z, U, V$ , such that  $|W| = k - |X \cup Y \cup Z|$ . By disjointness of the sets involved,  $|W \cup Z| = k - |X \cup Y| \leq |U|$  and  $|W \cup X| = k - |Y \cup Z| \leq |V|$ . This means that we can safely replace the (bound) variables  $U$  in  $\psi$  by the variables  $W \cup Z$ , and replace the (bound) variables  $V$  in  $\chi$  by the variables  $X \cup W$ . The resulting formula is equivalent to the original, but only contains variables in  $X \cup Y \cup Z \cup W$ , of which there are only  $k$  many.  $\square$

**Another proof of Proposition 3:** The proof is by induction on the quantifier depth of the input formula  $\alpha$ . We prove the existence of an equivalence-preserving renaming translation that given a formula  $\alpha$  of width at most  $w$  returns a formula  $\alpha'$  using variables in  $\{z_1, \dots, z_w\}$ , where  $z_1, \dots, z_w$  do not appear in  $\alpha$ . By *renaming translation* we mean a translation that only renames the variables of the input formula. We do the proof by induction on the quantifier depth  $qd(\alpha)$  in the input formula  $\alpha$ .

If  $qd(\alpha) = 0$ , then  $\alpha$  does not contain any quantifier. Hence, all the variables of  $\alpha$  are free. Since  $\alpha$  has width at most  $w$ , the free variables of  $\alpha$  are at most  $w$ , and thus  $\alpha$  uses at most  $w$  variables. We may replace the free variables of  $\alpha$  with variables in  $\{z_1, \dots, z_w\}$  obtaining an equivalent formula using variables in  $\{z_1, \dots, z_w\}$ .

Suppose that  $qd(\alpha) = n > 0$ . Consider all subformulas of  $\alpha$  of the form  $\beta = \exists x.\gamma$ , where  $\gamma$  does not contain any quantifier. Without loss of generality, we assume that  $x$  is free in  $\gamma$  (otherwise, the existential quantifier is useless and hence it can be removed). We have that  $\text{free}(\beta) = \text{free}(\gamma) \setminus \{x\}$ . Since  $|\text{free}(\gamma)| \leq w$ , we have that  $|\text{free}(\beta)| < w$ . Replace each free occurrence of  $x$  in  $\gamma$  by a fresh constant  $c_x$  and drop the quantifier  $\exists x$  from  $\beta$ . Call  $\alpha_1$  the resulting formula after this operation has been done for all the  $\beta$ 's. We have that: (i) for each  $\beta$  we have  $|\text{free}(\beta)| < w$ , (ii) the width of  $\alpha_1$  is at most  $w$ , and (iii) the quantifier depth of  $\alpha_1$  is  $n - 1$ . By induction hypothesis, there exists an equivalence-preserving renaming translation that transforms  $\alpha_1$  into a formula  $\alpha_2$  using variables in  $\{z_1, \dots, z_w\}$ . Use this translation to get  $\alpha'$ , restore in  $\alpha'$  the removed quantifiers  $\exists x$ , and replace constants  $c_x$  by  $x$ . The resulting formula, say  $\alpha_3$ , is equivalent to  $\alpha$  and, for each  $\beta$ , it holds that: (i) the variables of  $\beta$  are in  $\{x, z_1, \dots, z_w\}$ , where  $x$  is some variable not occurring in  $\{z_1, \dots, z_w\}$ , (ii) the free variables of  $\beta$  are in  $\{z_1, \dots, z_w\}$ , and (iii)  $|\text{free}(\beta)| < w$ . Hence, for each  $\beta$ , there must be some  $z_i \in \{z_1, \dots, z_w\}$  not occurring free in  $\beta$ . Replace in  $\beta$  the quantifier  $\exists x$  by  $\exists z_i$  and each free occurrence of  $x$  by  $z_i$ . Let  $\alpha_4$  be the resulting formula after these renaming has been performed for all  $\beta$ 's. Notice that  $\alpha_4$  is equivalent to  $\alpha_3$ , and hence it is equivalent to  $\alpha$ . Moreover, it uses variables in  $\{z_1, \dots, z_w\}$ . Finally,  $\alpha_4$  and  $\alpha$  have the same length.  $\square$

Both proofs give rise to strategies that can be implemented as a recursive procedure whose time complexity is quadratic in the length of the input formula.

We make an example illustrating the two strategies used in the proof of the previous theorem. Let

$$\alpha = \exists x_1.(Rx_1x_2) \wedge \exists x_3.(Rx_3x_4)$$

The width of  $\alpha$  is two, but it uses four variables.

We show how to rewrite it into an equivalent formula with two variables using the first strategy. Since the two conjuncts of  $\alpha$  are already of the required form, we only need to show that their conjunction can be written with only two variables. To this end, we first collect the free variables occurring in the conjuncts. These are  $x_2$  and  $x_4$ . Since we cannot use more than two variables in the final formula, we will only use these two. Finally, each of the quantified variables occurring in the two conjuncts is renamed to one of the target variables  $x_2, x_4$ , and more in particular, one that does not already occur freely in that conjunct. Following this strategy, we obtain  $\exists x_4.(Rx_4x_2) \wedge \exists x_2.(Rx_2x_4)$ , which is indeed of the required form and equivalent to the original formula.

We now show how to rewrite the above formula into an equivalent formula using variables  $z_1$  and  $z_2$  only by taking advantage of the second strategy. First, we fold the innermost quantifiers of  $\alpha$  obtaining  $\alpha_1 = Rc_1x_2 \wedge Rc_3x_4$ , where  $c_1$  and  $c_3$  are constants. The formula  $\alpha_1$  has nesting degree zero and width two. It is equivalent to  $\alpha_2 = Rc_1z_1 \wedge Rc_3z_2$ . If we unfold the quantifiers we get  $\alpha_3 = \exists x_1.(Rx_1z_1) \wedge \exists x_3.(Rx_3z_2)$ . By properly renaming variables  $x_1$  and  $x_2$  we get the formula

$$\alpha_4 = \exists z_2.(Rz_2z_1) \wedge \exists z_1.(Rz_1z_2)$$

which is equivalent to  $\alpha$  and uses only two variables.

### 3 Eliminating constants

Most results on guarded formulas have been stated only for relational first-order languages, i.e., languages without constants. The results discussed in this section show how the same techniques can be applied to formulas containing constants.

Let  $\text{NCONS}(\phi)$  be the number of constants occurring in  $\phi$ . Grädel [7] proved the following.<sup>3</sup>

**Proposition 4** *Every (loosely)  $\forall$ -guarded formula  $\phi$  can be transformed in polynomial time into an equisatisfiable relational (loosely)  $\forall$ -guarded formula  $\chi$ , such that  $\text{WIDTH}(\chi) \leq \text{WIDTH}(\phi) + \text{NCONS}(\phi)$ .*

For complexity reasons, we have a particular interest in formulas with a bounded width. Unfortunately, for such formulas  $\phi$ , Proposition 4 does not imply a bound

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<sup>3</sup>Strictly speaking, Grädel's proof for this proposition is flawed, since his translation does not correctly handle formulas containing equality. However, this problem can easily be fixed.

on the width of  $\chi$ . We will now present another method to eliminate constants, that allows us to circumvent this problem.

**Proposition 5** *Fix a natural number  $k \geq 2$ . Every loosely  $\forall$ -guarded formula  $\phi$  of width at most  $k$  can be transformed in polynomial time into an equisatisfiable relational loosely  $\forall$ -guarded formula  $\chi$  of width at most  $k$ .*

**Proof:** Consider any loosely  $\forall$ -guarded formula  $\phi$  of width at most  $k$ . By Proposition 2, we may assume that  $\phi$  is in normal form and that  $\text{MAXARITY}(\phi) \leq k$ .

Let  $\text{CONS}$  be the set of constants occurring in  $\phi$ . For each  $n$ -place relation symbol  $R$  occurring in  $\phi$ , except for equality, and for each partial function  $f : \{1, \dots, n\} \hookrightarrow \text{CONS}$ , introduce a new relation symbol  $R_f$  with arity  $n - |\text{dom}(f)|$ , where  $\text{dom}(f)$  is the set of all  $k \in \{1, \dots, n\}$  for which  $f(k)$  is defined. For example, if  $R$  is a ternary relation symbol and  $f = \{(1, c), (3, d)\}$ , then  $R_f$  is a unary relation symbol, which we will also denote by  $R_{c \bullet d}$ . The intended interpretation of  $R_{c \bullet d}(x)$  will be the same as  $R(c, x, d)$ . Also, for each pair of constants  $c, d$ , introduce a nullary relation symbol  $E_{cd}$ .

We will now eliminate all constants, with the help of these new relation symbols. For any sequence of variables  $\vec{x}$ , let  $T(\vec{x})$  be the set of all partial functions from  $\{\vec{x}\}$  to  $\text{CONS}$  (including the empty function). Note that there are  $(\text{NCONS} + 1)^{|\vec{x}|}$  such functions. For each  $\tau \in T(\vec{x})$  and formula  $\psi$ , let  $\psi^\tau$  be the result of replacing each occurrence of a variable  $x \in \text{dom}(\tau)$  by  $\tau(x)$ . Finally, let  $\phi^*$  be obtained from  $\phi$  by means of the following procedure.

1. Replace each subformula of the form  $\forall \vec{x}. \psi$  by  $\bigwedge_{\tau \in T(\vec{x})} \forall \vec{x}. \psi^\tau$ , and replace each subformula of the form  $\exists \vec{y}. \psi$  by  $\bigvee_{\tau \in T(\vec{y})} \exists \vec{y}. \psi^\tau$ .<sup>4</sup>
2. Replace each atomic formula of the form  $R(c_1, \dots, c_n, x_1, \dots, x_m)$  by  $R_{c_1 \dots c_n \bullet \dots \bullet}(x_1, \dots, x_m)$  (and similarly for other permutations)
3. Replace each atomic formulas of the form  $c = d$  by  $E_{cd}$ , and replace each atomic formula of the form  $x = c$  or  $c = x$  by  $\perp$ .

Let  $\chi$  be the conjunction of  $\phi^*$  with

$$\bigwedge_{c \in \text{CONS}} E_{cc} \wedge \bigwedge_{c, d \in \text{CONS}} E_{cd} \rightarrow E_{dc} \wedge \bigwedge_{c, d, e \in \text{CONS}} E_{cd} \wedge E_{de} \rightarrow E_{ce}$$

and all formulas of the form

$$\forall x_1 \dots x_m. (R_{c_1 \dots c_\ell \dots c_n \bullet \dots \bullet}(x_1, \dots, x_m) \rightarrow (E_{c_\ell d} \rightarrow R_{c_1 \dots d \dots c_n \bullet \dots \bullet}(x_1, \dots, x_m)))$$

(including all permutations the sequence  $c_1, \dots, c_n x_1, \dots, x_m$ ).<sup>5</sup>

<sup>4</sup>Note that this will only polynomially increase the length of the formula, due to the fact that both the width and the quantifier depth of  $\phi$  is bounded (keep in mind that  $\phi$  is in normal form).

<sup>5</sup>The number of such formulas is approximately  $\text{NREL}(\phi) \cdot (\text{NCONS}(\phi))^{\text{MAXARITY}(\phi)}$ , where  $\text{NREL}(\phi)$  is the number of relation symbols occurring in  $\phi$ . This is polynomial in the length of  $\phi$ , given that  $\text{MAXARITY}(\phi) \leq k$ .

Clearly,  $\chi$  does not contain any constants, and is loosely  $\forall$ -guarded. Furthermore, the length of  $\chi$  is polynomial in the length of  $\phi$ , and that  $\chi$  can be obtained from  $\phi'$  in polynomial time.

Finally, we claim that  $\chi$  is satisfiable iff  $\phi$  is satisfiable. One direction of this claim is easy: a model for  $\phi$  is easily turned into a model for  $\chi$ . As for the other direction, every model  $M$  satisfying  $\chi$  can be turned into a model  $M'$  for  $\phi$  in the following way: define an equivalence relation on the set  $\text{CONS}$  by putting  $c \sim d$  iff  $M \models E_{cd}$ , extend the domain of  $M$  with one element for each equivalence class, and extend the relations to the new elements in the obvious way:  $([c_1], \dots, [c_n], e_1, \dots, e_m) \in R$  iff  $(e_1, \dots, e_m) \in R_{c_1 \dots c_n \bullet \dots \bullet}$ , and likewise for other permutations. It is easily seen that the resulting model  $M'$  satisfies  $\phi$ .  $\square$

Note that the translation used in the above proof is polynomial only provided that the width of the input formula is bounded by a constant. Unlike Grädel's translation, it is in general exponential.

We will now proceed with the proof of Theorem 2, using the help of the above results. As we already mentioned, Grädel [7] states his main results only in terms of guarded or loosely guarded formulas. Nevertheless, the central argument on which these results are based is formulated in terms of relational loosely  $\forall$ -guarded formulas in normal form, cf. Definition 1. Specifically, Grädel shows that the satisfiability problem for such formulas is 2EXPTIME-complete, and that it becomes EXPTIME-complete if there is a bound on the width of the (normal form) formula. Together with our above results, this allows us to prove Theorem 2.

**Proof of Theorem 2:** The 2EXPTIME-completeness of the satisfiability problem for loosely  $\forall$ -guarded formulas follows from Grädel's result by Proposition 4 and Proposition 1. The EXPTIME-completeness of the satisfiability problem for loosely  $\forall$ -guarded formulas with a bounded number of variables follows from Grädel's result by Proposition 5 and Proposition 1 (if a formula  $\phi$  contains at most  $k$  variables, then, trivially,  $\text{WIDTH}(\phi) \leq k$ ).

Finally, it is easy to see that the width of a guarded formula is bounded by the arity of the relation symbols occurring in it. Note that, in general, this does not hold for  $\forall$ -guarded formulas, nor for loosely guarded formulas. Indeed, by a similar argument as used in the proof of Proposition 2, the satisfiability problem for loosely guarded formulas with arity at most 2 is already as hard as the satisfiability problem for loosely guarded formulas in general, i.e., 2EXPTIME-complete.  $\square$

## 4 Connections with hybrid logic

As we already mentioned in the introduction, guarded fragments with constants have important applications in the area of hybrid logic (cf. [4, 10] for an introduction to hybrid logic). Many hybrid languages can be embedded into finite variable guarded fragments with constants, and Theorem 2 allows us to derive

EXPTIME-membership results for such languages. As concrete examples, we mentioned the global consequence relation for the hybrid language  $\mathcal{H}$  (i.e., the extension of the basic modal language with a countably infinite set of nominals) and the satisfiability problem for the hybrid language  $\mathcal{H}(@, \diamond^{-1}, E)$  [2].

Conversely, results from the hybrid logic literature may have applications to guarded fragments with constants. Here, we will discuss one such application, which concerns the interpolation property. When the guarded fragment was introduced in [1], it was hoped that it has interpolation. Unfortunately, it was shown in [8] that this is not the case. This negative interpolation result was strengthened for guarded fragments with constants in [6], where the following was shown:

**Theorem 3 ([6])** *Let  $F$  be any fragment of first-order logic with constants that contains all atomic formulas, is closed under the Boolean connectives and is closed under guarded quantification (i.e., if  $\phi(\vec{x}\vec{y}) \in F$  and  $\alpha(\vec{x}\vec{y})$  is atomic then  $\exists\vec{x}(\alpha(\vec{x}\vec{y}) \wedge \phi(\vec{x}\vec{y})) \in F$   $\forall\vec{x}(\alpha(\vec{x}\vec{y}) \rightarrow \phi(\vec{x}\vec{y})) \in F$ ). Furthermore suppose that  $F$  satisfies the following form of interpolation:*

*For all formulas  $\phi(x), \psi(x) \in F$  with at most one free variable  $x$ , if  $\models \phi(x) \rightarrow \psi(x)$  then there is a formula  $\vartheta(x) \in F$  such that  $\models \phi(x) \rightarrow \vartheta(x)$ ,  $\models \vartheta(x) \rightarrow \psi(x)$ , and all relation symbols and constants occurring in  $\vartheta$  occur both in  $\phi$  and in  $\psi$ .*

*Then every first-order sentence  $\phi$  is equivalent to a formula in  $F$ .*

In other words, it is not possible to repair interpolation for the guarded fragment by increasing its expressivity without ending up with full first-order logic. Note the modal character of interpolation property used in Theorem 3: it applies to formulas with at most one free variable. Also note that, while this result applies to the loosely guarded fragment, it does not cover the universally guarded fragment, or other fragments that are not closed under negation.

The proof given in [6] is based on results on hybrid logics.

## 5 Discussion

We finish by discussing two open questions. The first question is the following:

*What is the complexity of the satisfiability problem for  $\forall$ -guarded formulas with bounded arity?*

Note that the answer to this question does not depend on the presence of constants. Our conjecture is that this problem is EXPTIME-complete.

A second interesting question would be the following question:

*Classify, in the style of Börger et al. [5], the quantifier patterns  $\pi$  for which the satisfiability problem for sentences consisting of a sequence of quantifiers conform  $\pi$  followed by a guarded formula, is decidable.*

The satisfiability problem for  $\pi = \exists^*\forall$  is still decidable, as can be seen by replacing the existentially quantified variables by constants and guarding the universal quantifier by an identity statement of the form  $x = x$ . On the other hand,  $\pi = \forall^3$  is already a conservative reduction class, as follows from results of Grädel [7]. What about  $\pi = \exists^*\forall^2$ ?

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