Hybrid Definability in Topological Spaces

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Abstract

We present some results concerning definability of classes of topological spaces in hybrid languages. We use language L_t described in [9] to establish notion of "elementarity" for classes of topological spaces. We use it to prove the analogue of Goldblatt-Thomason theorem in topological spaces for hybrid languages $H(E)$ and $H(@)$. We also prove a theorem that allows to reformulate definability result of Gabelaia ([10]) for modal logic in terms of elementary topological space classes.

1 Introduction

In the history of modal logic, the topological semantics was introduced before Kripke semantics. The pioneering work of McKinsey and Tarsky in 1944 [13] offered proofs of completeness for $S4$ with respect to topological spaces. With the advent of frame semantics, however, the topological semantics lost popularity. Yet the ideas behind topological semantics are probably just as intuitive as the idea of possible worlds. In topological semantics the box ($\Box$) means "the property is true in every point of some neighborhood of the current point", which closely resembles the kripkean meaning of the box: "the property is true in every point accessible from the current point". The spacial interpretation of the box allows us to look at the modal logic as a language for talking about topological spaces — figures (rather than graphs in relational semantics).

Hybrid languages (see [2]) are extensions of modal languages with nominals and other constructions that augment their expressive power. In fact some hybrid languages are as expressive as first-order logic. Nominals are basically propositional variables that depict singleton sets, thus, if modal logic allows naming of regions, hybrid logic allows naming of points as well. Hybrid languages can have a global satisfaction operator that internalizes the notion of satisfiability in the model into the language. Hybrid logic was initially presented along with traditional relational semantics, but ideas proposed by it can be adopted in a natural way to the topological semantics case.

In this paper we will talk about definability. The definability of some property is a possibility to express it (*define* it) in some language. Formally speaking, a property corresponds to the class of objects that satisfy it and the property is definable in some language L if there exists a sentence ϕ of that language such that ϕ is true precisely for the objects belonging to the class, i.e. ϕ is true precisely for the objects satisfying the property. In the relational case the basic modal language interpreted on frames behaves like second-order logic: this is due to the fact that the quantification over sets is built into the definition of truth of a modal formula on a frame. If we restrict ourselves to the frame classes that can be defined in the first-order language then the Goldblatt-Thomason theorem (see [3]) can give an answer to the question: "Is this frame class definable in basic modal language?" The work of David Gabelaia [10] proposes a variant of that theorem for topological semantics. Balder ten Cate's work [4] proposes another variant for hybrid languages in relational semantics. In this paper we will give the missing piece of the puzzle and will present a third variant for hybrid languages in topological spaces.

We assume acquaintance with the syntax and relational semantics of modal [3] and hybrid logic [2] as well as with basic notions of general topology [7]. We will not deal with the binding operator; only languages $H(@)$ and $H(E)$ will be considered. This paper builds on fundamental notions elaborated upon in [10, 9, 4].

2 Topological semantics for modal logic

First of all let us introduce the basic definitions.

Definition 1. The *basic modal language* consists of a countable set of propositional letters $p, q, r, \dots$ and a unary modal operator $\Box$. The well-formed formulas ϕ of the basic modal language are built as follows:

$$\phi ::= p \mid \bot \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi$$

$\Diamond\phi$ is an abbreviation for $\neg\Box\neg\phi$, $\phi \to \psi$ is an abbreviation for $\neg\phi \vee \psi$, $\phi \equiv \psi$ is an abbreviation for $(\neg\phi \vee \psi) \wedge (\phi \vee \neg\psi)$.

Definition 2. A *topological model* $\mathfrak{M}$ is a triple (T, τ, V) where (T, τ) is a topological space and the valuation V sends propositional letters to subsets of T. We inductively define a formula ϕ to be true at point x in a model $\mathfrak{M}$ (noted $\mathfrak{M}, x \models \phi$) as following:

$$\begin{array}{rll} \mathfrak{M}, x \models p & \text{iff} & x \in V(p) \\ \mathfrak{M}, x \models \bot & & \text{never} \\ \mathfrak{M}, x \models \phi \vee \psi & \text{iff} & \mathfrak{M}, x \models \phi \text{ or } \mathfrak{M}, x \models \psi \\ \mathfrak{M}, x \models \neg\phi & \text{iff} & \mathfrak{M}, x \not\models \phi \\ \mathfrak{M}, x \models \Box\phi & \text{iff} & \exists O \in \tau \text{ such that } \forall y \in O\, \mathfrak{M}, y \models \phi \end{array}$$

Sometimes, for topological space $\mathcal{T} = (T, \tau)$ when it is clear from the context, we will use $\mathcal{T}$ to refer to T.

If $\mathfrak{M}, x \models \phi$ for all $x \in O$ where $O \subseteq T$ we write $O \models \phi$. Similarly $\mathfrak{M} \models \phi$ means that $\mathfrak{M}, x \models \phi$ for any $x \in T$. If $\mathcal{T} = (T, \tau)$ is a topological space we write $\mathcal{T} \models \phi$ when $(T, \tau, V) \models \phi$ for any valuation V. If K is a class of topological spaces we write $\mathsf{K} \models \phi$ when $\mathcal{T} \models \phi$ for any $\mathcal{T} \in \mathsf{K}$.

Definition 3. We say that a set of formulas Γ *defines* a class of structures (frames or topological spaces) K if for any structure $\mathcal{T}$ we have that $\mathcal{T} \in \mathsf{K}$ iff $\forall \phi \in \Gamma\, \mathcal{T} \models \phi$ ($\mathcal{T} \models \Gamma$).

In frame semantics we have a result that allows us to determine whether a given class of frames is definable in modal language:

Theorem 1 (Goldblatt-Thomason theorem). *A first-order definable class of frames is modally definable if and only if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.*

In order to translate the definability result for frames into definability result for topological spaces we need to specify the topological analogues of closure conditions. [10] describes such analogues to introduce definability result for modal logic in topological semantics. We will reproduce the definitions that we will need here, in a slightly modified form.

Definition 4. Let $\mathcal{T} = (T, \tau)$ and $\mathcal{S} = (S, \sigma)$ be topological spaces, a map $f : T \to S$ is called *interior* if for any $O \in \tau$, $f(O) \in \sigma$ and for any $U \in \sigma$, $f^{-1}(U) \in \tau$ (it is equivalent to saying that f is open and continuous at the same time).

Definition 5. Let $\mathcal{T} = (T, \tau)$ be a topological space and S its open subset. A topological space $\mathcal{S} = (S, \sigma)$ where $\sigma = \{O \cap S \mid O \in \tau\}$ is an *open subspace* of $\mathcal{T}$.

Definition 6. A collection F of subsets of some set X is called a *filter* if it satisfies the folowing conditions:

1. if $A, B \in F$ then $A \cap B \in F$
2. if $B \supset A$ and $A \in F$ then $B \in F$
3. $X \in F$

A filter F is called *proper*, if $\emptyset \notin F$.

Let $\mathcal{T} = (T, \tau)$ be a topological space. A filter $F \subseteq \mathcal{P}(T)$ is called *open* if for any $O \in F$, $I(O) \in F$ where $I(O) = \bigcup_{U \in \tau, U \subseteq O} U$ stands for O interior. This is equivalent to the existence of a base of the filter containing only open sets.

We say that a set of ultrafilters O is an *extension* of a filter f if $O = \{u \in X | f \subseteq u\}$

We call an *Alexandroff extension* of a topological space $\mathcal{T} = (T, \tau)$ the following topological space $T^* = (\mathit{Uf}\ (T), \tau^*)$. τ^* topology is generated by the collection of all sets of the form $\{u \in \mathit{Uf}\ (T) \mid F \subseteq u\}$ where F is an open filter (i.e. the set is open iff it is obtained by unions and finite intersections from sets of a mention form).

If one wants to draw parallels between relational and topological semantics, one can say that interior maps are to topological spaces what bounded morphisms are to Kripke frames; open subspaces are like generated submodels and Alexandroff extensions are like ultrafilter extensions. Alexandroff extension is in some sense a completion of a topological space: it introduces enough extra points so that arbitrary intersections of open sets would be an open set (topological spaces with such a property are called Alexandroff spaces).

We will need some machinery to work with Alexandroff extensions: the $*$-map. Following [10] we give the following

Definition 7. Let T be a topological space, $* : \mathcal{P}(\mathcal{T}) \to \mathcal{P}(\mathcal{T}^*)$ is a map which is defined as follows:

$$O^* = \{u \in \mathit{Uf}\ (T) | O \in U\}$$

We will use notation a^* instead of $\{a\}^*$ for singleton sets. Here are some properties of $*$-map without proofs (they can be found in [10])

Proposition 2.

1. *x^* is principal ultrafilter of x*

2. $(A \cap B)^* = A^* \cap B^*$, $(A \cup B)^* = A^* \cup B^*$

3. $*$ *sends open sets to open sets*

And here are some properties we will need later.

Proposition 3. $\{x^* | x \in O\} \subseteq O^*$. *If* O *is finite then* $\{x^* | x \in O\} = O^*$.

Proof. Take any $x \in O$, hence $O \in x^*$ and by definition of $*$-map $x^* \in O^*$. Now let O be finite and prove the reverse inclusion. Take any $u \in O^*$. We need to prove that there exists $x \in O$ s.t. $u = x^*$.We will argue by contradiction. Suppose that for every $x \in O$ there exists $A_x \in u$ s.t. $x \notin A_x$. Then $\cap_{x \in O} A_x \subseteq X - O$, hence, by filter definition $X - O \in u$, which entails $O \notin u$, which in its turn contradicts $u \in O^*$. ⊣

Proposition 4. *Every open set of Alexandroff extension of some topological space* T *contains an open set that is either* $*$*-image of an open set of* T *or an infinite intersection of* $*$*-images of open sets of* T.

Proof. The base of Alexandroff extension topology are sets of ultrafilters which are extensions of open filters. So, every open set A of Alexandroff extension of T contains a set O, that contains ultrafilters that are extensions of an open filter F. It remains to be proved that

$$O = \bigcap_{X \in F} X^*$$

Indeed, take $x \in O$, then $X \in x$ for all $X \in F$, hence $x \in X^*$ for all $X \in F$. Next, take $x \in \bigcap_{X \in F} X^*$ then $X \in X$ for all $X \in F$ or in other words $F \subset x$, hence $x \in O$.

Particularly, in case when $F = \{X \mid X \supset O\}$ for some open set O, $A = O^*$. ⊣

With these definitions we can formulate the theorem of David Gabelaia that is an analogue of Goldblatt-Thomason theorem for topological semantics.

Theorem 5. *The class* K *of topological spaces which is closed under formation of Alexandroff extensions is modally definable iff it is closed under taking opens subspaces, interior images, topological sums and it reflects Alexandroff extensions*

3 Hybrid definability on frames

Hybrid languages are extensions of modal languages. Let us briefly recall their syntax and relational semantics.

Definition 8. *Hybrid languages* have nominals (which we denote by letters $i, j, k, \ldots$) in addition to propositional letters. Language $H(@)$ is given by the grammar:

$$\phi ::= p \mid i \mid \perp \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi \mid @_i\phi$$

Language $H(E)$ is given by the grammar:

$$\phi ::= p \mid i \mid \perp \mid \neg\phi \mid \phi \vee \phi \mid \Box\phi \mid @_i\phi \mid E\phi$$

$A\phi$ is an abbreviation for $\neg E \neg\phi$.

Definition 9. A *Kripke frame* is a pair $\mathfrak{F} = (W, R)$ where W is a set called the domain (support) of $\mathfrak{F}$ and R is a binary relation over W. A *Kripke model* $\mathfrak{M}$ is a pair $(\mathfrak{F}, V)$ where $\mathfrak{F}$ is a Kripke frame and V is a function (called *valuation*) that maps propositional letters to subsets of W and nominals to singleton subsets of W. Sometimes, when it is clear from context, we will use the name of frame ($\mathfrak{F}$) to refer to its domain.

The Kripke semantics of hybrid languages is defined as follows. For model $\mathfrak{M} = (W, R, V)$:

$$\begin{array}{rll}
\mathfrak{M}, w \models p & \text{iff} & x \in V(p) \\
\mathfrak{M}, w \models i & \text{iff} & x \in V(i) \\
\mathfrak{M}, w \models \perp & & \text{never} \\
\mathfrak{M}, w \models \neg\phi & \text{iff} & \mathfrak{M}, w \not\models \phi \\
\mathfrak{M}, w \models \phi \vee \psi & \text{iff} & \mathfrak{M}, w \models \phi \text{ or } \mathfrak{M}, w \models \psi \\
\mathfrak{M}, w \models \Diamond\phi & \text{iff} & \exists v \in W \text{ such that } Rwv, \mathfrak{M} \text{ and } v \models \phi \\
\mathfrak{M}, w \models @_i\phi & \text{iff} & \mathfrak{M}, v \models \phi \text{ where } V(i) = \{v\} \\
\mathfrak{M}, w \models E\phi & \text{iff} & \exists v \in W \text{ such that } \mathfrak{M}, v \models \phi
\end{array}$$

$@_i$ and E are interpreted in the same fashion in topological semantics.

In [4] it is proven that a certain elementary class of frames can be defined in hybrid logic if it obeys some closure conditions. To formulate this result we will need the following notion:

Definition 10. A frame $\mathfrak{G}$ is an *ultrafilter morphic image* of frame $\mathfrak{F}$ if there exists a bounded morphism $f : \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{G}$, such that $|f^{-1}(x)| = 1$ for every principal ultrafilter $x \in \mathfrak{ue}\,\mathfrak{G}$ (that is to say, f is injective on principal ultrafilters).

Our aim in this paper is to introduce a topological analogue of the following theorems:

Theorem 6 (Hybrid definability for $H(@)$). *An elementary class of frames* K *is definable using basic hybrid language* $H(@)$ *if it is closed under ultrafilter morphic images and under generated submodels.*

Theorem 7 (Hybrid definability for $H(E)$). *An elementary class of frames* K *is definable using basic hybrid language* $H(E)$ *if it is closed under ultrafilter morphic images.*

To illustrate these results let us consider the following property:

$$\text{There exist two distinct linked points (i.e. } \exists x, y\, x \neq y \text{ such that } Rxy) \quad (1)$$

and apply the mentioned theorems to find out whether it is definable in hybrid language.

Consider the set of natural numbers equipped with minimal relation R s.t. Rnn for every natural number n. Let us call this frame $\mathbb{N}_R$. Now let us construct the frame F whose support is a set of all ultrafilters over $\mathbb{N}_R$ and the accessibility relation Z is the following (we write π_n to denote a principal ultrafilter of n): for every point u except π_1 and π_2, Zuu, $Z\pi_1\pi_2$ and $Z\pi_2\pi_1$. The frame F satisfy the given property.

Then take arbitrary non-principal ultrafilter $u \in \mathbb{N}_R$ and construct a surjective bounded morphism $f : F \to \mathfrak{ue}\,\mathbb{N}_R$ as follows: for every non-principal ultrafilter $x \in F$, $f(x) = x$, $f(\pi_1) = u$, $f(\pi_2) = u$ and $\forall n > 2 f(\pi_n) = f(\pi_{n-2})$. f is injective on principal ultrafilters (it is non-injective only on u in fact). f maps every isolated reflexive point of F except u, π_1 and π_2 to isolated reflexive point of $\mathfrak{ue}\,\mathbb{N}_R$, so back and forth conditions for mentioned points can easily checked, it is easy then to verify "manually" the back and forth conditions for u,π_1 and π_2 as well.

But $\mathbb{N}_R$ does not satisfy (1). Using theorems 6 and 7 we conclude that (1) is not definable neither in $H(@)$, nor in $H(E)$.

4 The elementarity notion for topological spaces

The original Goldblatt-Thomason theorem uses the notion of elementarity of a frame class. It is not evident, however, which topological space classes

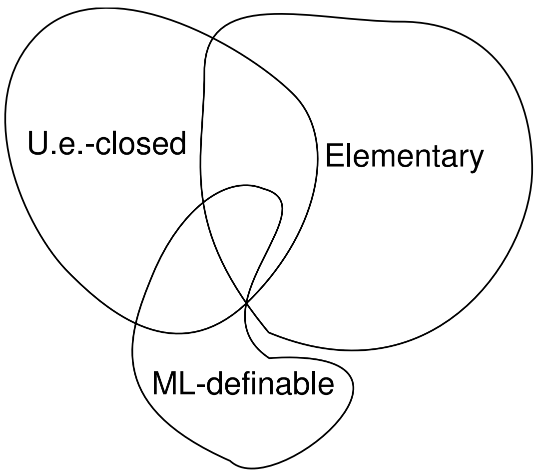


Figure 1: Modal logic situation

should be considered elementary.

The notion of elementarity is indeed very important. The definability result in [10] imposes an extra constraint on a class of frames in question: it should be closed under Alexandroff extensions, which corresponds to requiring closedness under ultrafilter extensions in relational semantics. It does not make a serious problem in modal logic, because every modally definable elementary class of frames is closed under ultrafilter extensions.

Now consider a class K of all frames with irreflexive accessibility relation. It is elementary and hybrid definable (with a formula $i \to \Box\neg i$, for example). The frame of natural numbers with natural strict order belongs to K. Its ultrafilter extension contains reflexive points and thus does not belong to K. That elementarity plus hybrid definability does not imply closedness under ultrafilter extensions (see figures 1 and 2)! In hybrid logic conditions of being closed under ultrafilter extension and elementarity are “symmetric”. We would like to have elementarity requirement in our topological version of the theorem because it looks more natural. Another argument for finding appropriate notion of elementarity is the fact that we could then reuse proof techniques from the proof of theorems 6 and 7.

To address this problem we propose to use a language L_t described in [9] and consider a class of topological spaces “elementary” if it can be defined by a formula of that language.

We start with a monadic second-order language L_2, which is interpreted over so-called weak structures.

Definition 11. Take first-order similarity type L, which consists of set of predicate symbols and function symbols. Consider then a first order-language

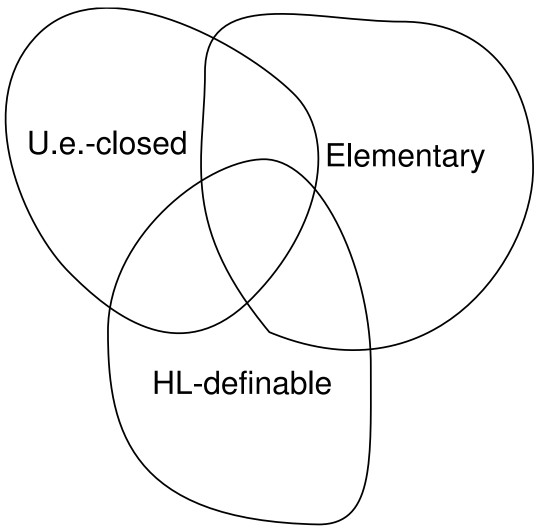


Figure 2: Hybrid logic situation

$L_{\omega\omega}$ associated L which is defined as usual (with only negation and disjunction as primitive connectives). A *monadic second-order language* L_2 is obtained from $L_{\omega\omega}$ by adding a symbol $\in$ and set variables (we will denote them with capital letters: X,Y,Z,W,...) and allowing new atomic formulas of the form $t \in X$, where t is the term and X is the set variable. Moreover, if ϕ is a formula, then $\forall X \phi$ and $\exists X \phi$ are also formulas.

Definition 12. (T, τ), where T is a first-order structure and $\tau \subseteq \mathcal{P}(T)$ is called a *weak structure*. If τ is a topology on T then (T, τ) is called a *topological structure*.

The interpretation of L_2 formulas over weak structure (T, τ) is defined naturally: $\in$ is interpreted as "a member of" and set variables can be only instantiated with values from τ. Language L_2 (over weak structures) is reducible to a two-sorted first-order language: one sort corresponds to individuals, another corresponds to sets. The idea is to define formula translations from L_2 to two-sorted first-order language and vice versa and describe how to construct a model for two-sorted first-order language out of topological model in a satisfiability-preserving fashion. This is done for full second-order language and omega-sorted language first-order language in the chapter 4 of [6], the same techniques can be applied in our case.

This allows to prove all usual first-order theorems: the Compactness, the Completeness and the Löwenheim-Skolem theorems to name a few.

Proposition 8 (Compactness theorem). *A set of L_2 sentences has a weak model if every finite subset does.*

This is true, however, only for the class of weak models. Since we are working with some special kind of models, that is weak modes where τ is a topology, we need to somehow "tame" L_2 to keep nice first-order properties. And that is why we introduce L_t.

Definition 13.

> An L_2 formula ϕ is *positive in set variable* X if all free occurrences of X are under an even number of negation signs. An L_2 formula ϕ is em negative in set variable X if all free occurrences of X are under an odd number of negation signs
>
> L_t contains all atomic L_2 formulas and is closed under conjunction, disjunction, negation, first-order quantification and the following restricted form of second-order quantification:
>
> - if formula ϕ is positive in set variable X and t is a term then $\forall X\ (t \epsilon X \rightarrow \phi)$ is a formula
> - if formula ϕ is negative in set variable X and t is a term then $\exists X\ (t \epsilon X \wedge \phi)$ is a formula

L_t possesses many regular properties of first order-languages with respect to *topological structures* : Compactness, Löwenheim-Skolem and Completeness. We can use usual first-order model theory constructions, for example, if we are working with ultraproducts we get "for free" Łoś theorem for L_t (in fact, for L_2 as well). L_t is nice also because it is precisely invariant for topologies fragment of L_2, i.e. every formula of L_t is equivalent to L_2 formula invariant for topologies:

Definition 14. A collection τ of subsets of some set T is said to *qualify for open base* if for any two sets $X, Y \in \tau$ their intersection $X \cap Y$ is a union of elements from τ. If τ qualify for open base then topology generated by τ is noted $\tilde{\tau}$. Putting it the other way, $\tilde{\tau}$ contains precisely all possible unions of elements of τ. If for some topology σ, $\sigma = \tilde{\tau}$ then τ is called an *open base* of σ.

Definition 15. Formula ϕ is called *invariant for topologies* if

$$(T, \tau) \models \phi \text{ iff } (T, \tilde{\tau}) \models \phi$$

for every weak model (T, τ) where τ is an open base.

L_2 being actually a "first-order-like" language for talking about weak models, we want to use all usual first-order machinery with it. For example, the notion of ω-saturated model can be introduced for L_2 with the reservation that when considering a set of formulas we can only use the formulas with free variable of the same sort — point or set. All usual model-theoretic results concerning ω-saturated models can thus be translated for L_2.

Definition 16. We say that an L_2-structure (A, σ) based on a space A is *2-saturated* (*ω-saturated*) if any finitely realizable type $\Gamma(x)$ (or $\Gamma(X)$) with a parameter $b \in A$ or $B \in \sigma$ (with countable set of parameters from A or σ) is realizable.

Another construction we will need for L_t is ultraproduct. Following [1] we use the following definition of it:

Definition 17. Let us take a collection of arbitrary topological models $(T_i, \tau_i)_{i \in I}$. Let U be an ultrafilter over I. The sets of the form $\prod_U O_i$ where $O_i \in \tau_i$ are called *open ultraboxes*. A topological space with the support $\prod_U T_i$ and topology generated by open ultraboxes is called a *topological ultraproduct* of (T_i, τ_i) and is denoted by $\prod_U (T_i, \tau_i)$. If all topological spaces (T_i, τ_i) are the same, the structure is called *topological ultrapower*.

Topological ultraproduct is an operation on topological spaces which has the same properties with respect to L_t as ultraproduct to first-order languages:

Theorem 9. *Let $(T_i, \tau_i)_{i \in I}$ be a set of topological models, ϕ an L_t formula, U an ultrafilter over an index set I. Then*

$$\{i \mid (T_i, \tau_i) \models \phi\} \in U \textit{ iff } \prod_U (T_i, \tau_i) \models \phi$$

Proof. It suffices to use the fact that Łoś theorem holds for L_2 and that L_t is invariant for topologies. ⊣

Ultrapowers are in some sense "completions" of original models that enrich them with lots of points but preserve the properties that can be described in L_t language. They are analogous in some sense to Alexandroff extensions, we will make this statement more precise in the next section.

Last notice we should make is about the way a saturated model can be obtained. From model theory we can adapt various means of constructing 2- or ω-saturated weak structures. What if we want a saturated structure which is a topological space? The simplest idea is if in weak structure (S, σ), σ qualifies for open base then we can work with.

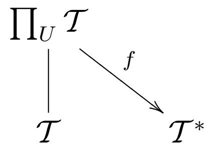
Theorem 10. *Let (S, σ) be an α-saturated weak structure and let σ qualify for an open base. Let $\Sigma(X)$ be a type (the free variable being a set) such that for any formula $\phi \in \Sigma$, $\exists X \phi$ is equivalent to an L_t sentence. If $\Sigma(X)$ is finitely realizable in $(S, \tilde{\sigma})$ then it is realizable in $(S, \tilde{\sigma})$. Also, for any type $\Sigma(x)$ if it is finitely realizable in $(S, \tilde{\sigma})$ then it is realizable in $(S, \tilde{\sigma})$.*

Proof. Straightforward, to prove the first claim one should use the fact that any formula that is equivalent to an L_t formula and L_t formulas are invariant for topologies. $\dashv$

5 Topological ultrapowers and Alexandroff extensions

We will present here a topological analogue of theorem 3.17 from [3]. The proof techniques used are essential for the main result of the paper. Moreover, the following theorem will allow us to extend the Goldblatt-Thomason theorem variant presented in [10] to the class of L_t-definable frames.

Theorem 11. *For every topological space $\mathcal{T} = (T, \tau)$ there exists an ultrafilter U and ultrapower $\prod_U \mathcal{T}$ and an onto interior map $f : \prod_U \mathcal{T} \to \mathcal{T}^*$.*



Proof. Let us consider a L_2-based language containing unary predicates $P_X(x)$ for every $X \subseteq T$, interpreted naturally on $\mathcal{T}$: $P_X(x)$ holds iff $x \in X$. The obvious properties of these predicates are:

- $\mathcal{T} \models \forall x (P_X(x) \wedge P_Y(x) \equiv P_{X \cap Y}(x))$
- for every $X, Y \subseteq T$ such that $X \subseteq Y$, $\mathcal{T} \models \forall x (P_X(x) \to P_Y(x))$
- $\mathcal{T} \models \forall x (\neg P_X(x) \equiv P_{T-X}(x))$

It is known from model theory (see [5], theorem 6.1.4 and 6.1.8 or [12], theorem 9.5.4) that there exists an ultrafilter U such that ultrapower $\prod_U T$ is a countably saturated model (as a weak structure). From now on we will consider $\prod_U \mathcal{T}$ as a topological structure, keeping in mind that it is generated by a countably-saturated weak structure. Let us define the map $f : \prod_U T \to Uf\,(T)$ in the following way:

$$f(x) = \{X \subseteq T \mid \prod_U \mathcal{T} \models P_X(x)\}$$

Claim 11.1. *f is well-defined, i.e. for any $x \in \prod_U T$, $f(x)$ is an ultrafilter.*

We will use the properties of predicates $P_X(x)$ to prove this. Take any two members of $f(x)$, say X and Y, since $\prod_U \mathcal{T} \models P_X(x)$ and $\prod_U \mathcal{T} \models P_Y(x)$ then $\prod_U \mathcal{T} \models P_X(x) \wedge P_Y(x)$ which is equal to $\prod_U \mathcal{T} \models P_{X \cap Y}(x)$, hence $X \cap Y \in f(x)$. Then suppose again, that X is a member of $f(x)$ and $X \subset Y$, we get $\prod_U \mathcal{T} \models P_X(x)$, but since $X \subset Y$, $\prod_U \mathcal{T} \models P_Y(x)$ and $Y \in f(x)$. Finally, taking arbitrary X we observe that either $\mathcal{T} \models P_X(x)$ or $\mathcal{T} \not\models P_X(x)$ what actually means that either $T - X \in f(x)$ or $X \in f(x)$.

Let us denote by $V(X)$ the set of $x \in \prod_U T$ such that $\prod_U \mathcal{T} \models P_X(x)$.

Claim 11.2. *f is open*

We will prove openness of f in two steps. First, we will prove the following equation. For any open ultrabox $O = \prod_U O_i$ where $O_i \in \tau$ it is true that

$$f(O) = \{u \in Uf\,(T)\} \mid \{X \subseteq T \mid V(X) \supseteq O\} \subset u\}$$

Second, we will show that $\{X \subseteq T \mid V(X) \supseteq O\}$ is an open ultrafilter. Since open ultraboxes form the base of topology of $\prod_U \mathcal{T}$, the equation implies that open sets are mapped to open sets.

$f(O) \subseteq \{u \in Uf\,(T) \mid \{X \subseteq T \mid V(X) \supseteq O\} \subset u\}$

Take $x \in O$ and let $y = f(x)$. For any X such that $V(X) \supseteq O$, it is true that $x \in V(X)$ and hence $X \in y$, or in other words $y \in \{u \in Uf\,(T)\} \mid \{X \subseteq T \mid V(X) \supseteq O\} \subseteq u\}$.

$f(O) \supseteq \{u \in Uf\,(T) \mid \{X \subseteq T \mid V(X) \supseteq O\} \subseteq u\}$

Take $v \in \{u \in Uf\,(T) \mid \{X \subseteq T \mid V(X) \supseteq O\} \subset u\}$, i.e. $v \supseteq \{X \subseteq T \mid V(X) \supseteq O\}$. We claim that for every $Y \in v$ we have that $V(Y) \cap O \neq \emptyset$. Suppose for the sake of contradiction that there exists some $Y \in v$ such that $V(Y) \cap O = \emptyset$. It follows that $Uf\,(T) - V(Y) = V(T - Y) \supseteq O$, hence $T - Y \in v$, which contradicts the fact that v is an ultrafilter.

Now consider a set of formulas

$$\Sigma(x) = \{x \in O\} \cup \{P_X(x) \mid X \in v\}$$

It is finitely realizable on $\prod_U \mathcal{T}$ and since $\prod_U \mathcal{T}$ is countably saturated, it is realizable and there exist some $x \in \bigcap_{X \in v} V(X) \cap O$; it is easy to check that $f(x) = v$.

Finally, we need to show that $F = \{X \subseteq T \mid V(X) \supseteq O\}$ is an open filter. It is easy to see that it is really a filter, so we only prove that if $Y \in F$ then $I(Y) \in F$ where $I(Y)$ is the interior of Y. Indeed, for any Y such that $V(Y) \supseteq O$ there exists $K \in U$ such that $O_i \subseteq Y$ for all $i \in K$. Hence, $Y \supseteq \bigcup_{i \in K} O_i$ and since $\bigcup_{i \in K} O_i \subset I(Y)$ we have that $O \subseteq V(\bigcup_{i \in K} O_i) \subseteq V(I(Y))$. It follows that $I(Y) \in F$.

Claim 11.3. *f is continuous*

Take arbitrary open set $O \subseteq \mathcal{T}^*$. When proving continuity we can assume without loss of generality that O belongs to $\mathcal{T}^*$ open base. In other words, there exists open filter F, such that O consists precisely of ultrafilters extending F. According to proposition 4, O is an intersection of O_i^* where $O_i \in F$ are open sets.

We have that

$$f(\bigcap_{X \in F} V(X)) = O$$

Indeed, take $x \in \bigcap\limits_{X \in F} V(X)$, then $X \in f(x)$ for all $X \in F$ which means that $f(x) \in O$. Now take $x \in O$, considering the set of formulas

$$\Sigma(y) = \{P_X(y) \mid X \in x\}$$

and using countable saturatedness of $\prod_U \mathcal{T}$ we find a point $y \in \prod_U T$ such that $f(y) = x$. Using a similar argument it can be shown that $\bigcap\limits_{X \in F} V(X)$ is not empty.

There remains to prove that $\bigcap\limits_{X \in F} V(X)$ is open. Take some $x \in \bigcap\limits_{X \in F} V(X)$ and use countable saturatedness for the set of formulas (note that x is a parameter here)

$$\Sigma(X) = \{x \in X\} \cup \{\forall y (y \in X \rightarrow y \in P_O(y) \mid O \in F\}$$

Note that we have right to use this set of formulas (according to theorem 10), since every formula it contains is equivalent to a formula in L_t. We have proved that for any $x \in \bigcap\limits_{X \in F} V(X)$ there exists an open set containing x that is contained in $\bigcap\limits_{X \in F} V(X)$. We deduce that $\bigcap\limits_{X \in F} V(X)$ is open.

Claim 11.4. *f is onto*

Take $u \in \mathcal{T}^*$. Consider the set of formulas $\Sigma(x) = \{P_X(x) | X \in u\}$. $\Sigma(x)$ is finitely satisfiable in $\prod_U \mathcal{T}$. Indeed, for every finite $\delta = \{P_{X_i}(x)\} \subset \sigma$, since u has a finite intersection property, satisfiability of $P_{\cap X_i}$ entails satisfiability of δ. But since $\prod_U \mathcal{T}$ is countably saturated then there exists $x \in \prod_U T$ such that $\prod_U \mathcal{T} \models \Sigma(w)$, hence $f(w) = u$.

$\dashv$

As a nice byproduct we get a reformulation of Gabelaia's theorem about modal definability in topological spaces (theorem 2.3.4 in [10]). Theorem 11 allows us to state the theorem 5 for topological space classes definable in L_t:

Theorem 12. *The class* K *of topological spaces which is definable in* L_t *is modally definable iff it is closed under taking opens subspaces, interior images, topological sums and it reflects Alexandroff extensions*

Proof. The proof proceeds the same way as in original theorem up to the following moment:

> Note that $(P_*)^*$ is nothing else but the Alexandroff extension of P and thus belongs to K by the conditions we imposed on this class; but then so is H, being the interior image of the space from K. So $H^* \in K$.

Instead we use the fact that K is closed under ultrapowers and interior images, thus by theorem 11 $P \in \mathsf{K}$ entails $(P_*)^* \in \mathsf{K}$. ⊣

6 Hybrid definability on topological spaces

Let us finally turn back to our main objective — providing topological analogue of theorems 6 and 7. Elementarity notion being reintroduced for topological spaces, we now need to spell out the ultrafilter morphic images notion.

Definition 18. Let $\mathcal{T}$ and $\mathcal{S}$ be topological spaces. $\mathcal{S}$ is called *topological ultrafilter morphic image* of $\mathcal{T}$ if there is a surjective interior map $f : \mathcal{T} \to \mathcal{S}^*$ such that $|f^{-1}(u)| = 1$ for every principal ultrafilter $u \in \mathcal{S}^*$ (one can say figuratively "f is injective on principal ultrafilters").

The following lemmas make sure that topological ultrafilter morphic images are compatible with hybrid definability.

Lemma 13. $H(E)$ *formulas validity is preserved under topological ultrafilter morphic images.*

Proof. Let $\mathcal{T} = (T, \tau)$ and $\mathcal{S} = (S, \sigma)$ be topological spaces and $f : \mathcal{S} \to \mathcal{T}^*$ an interior map. We need to prove that if $\mathcal{T} \not\models \phi$ then $\mathcal{S} \not\models \phi$. To show that it suffices to consider standard Alexandroff extension valuation on $\mathcal{T}^*$ defined as $V^*(p) = \{x \in Uf(T) | (\mathcal{T}, V), x \models p\}$, where p can be both proposition letter and nominal, the valuation V' on $\mathcal{S}$ defined as $V'(p) = \{x \in S | (\mathcal{T}^*, V^*), f(x) \models p\}$ and then apply inductive argument on the structure of ϕ. ⊣

We now have all the definitions to formulate the main result. The proof is inspired by Balder ten Cate's proof for relational case in [4].

Theorem 14. *A class of topological spaces K which is definable by L_t sentence, is definable in $H(E)$ iff it is closed under topological ultrafilter morphic images.*

Proof. Lemma 13 constitutes the proof of the left-to-right conjecture. It is left to prove the right-to-left direction.

Take a set $Th(\mathsf{K})$ of $H(E)$ formulas valid on K. Suppose that $T \models Th(\mathsf{K})$. If we can show that $\mathcal{T} \in \mathsf{K}$, the theorem is proven. Just like in theorem 11 we introduce propositional letters p_A for every subset $A \subset T$ and nominals i_w for every $w \in T$. p_A and i_w are interpreted naturally on $\mathcal{T}$. Let Δ be a set of the following formulas, A and B ranging over all subsets of T and w ranging over all points of T:

$$\begin{array}{l} p_{T-A} \equiv \neg p_A \\ p_{A\cap B} \equiv p_A \wedge p_B \\ p_{Int(A)} \equiv \Box p_A \\ i_w \equiv p_{\{w\}} \end{array}$$

Let $\Delta_T = \{A\delta | \delta \in \Delta\}$. By definition, Δ_T is satisfiable on $\mathcal{T}$.

Then Δ_T is satisfiable on K. Since L_t has compactness theorem and hybrid formulas can be translated into L_t we can only show that every finite conjunction of formulas from Δ_T is satisfiable in K. δ is satisfiable on $\mathcal{T}$, it follows that $\neg\delta$ is not valid on K, hence δ is satisfiable on K.

That means that there exist some topological space $\mathcal{S} \in \mathsf{K}$ and valuation V such that $(\mathcal{S}, V) \models \Delta_T$. Then model $(\mathcal{S}, V)$ globally satisfies Δ.

We use again [5], theorem 6.1.4 and 6.1.8 (or [12], theorem 9.5.4) to construct an ω-saturated ultrapower of model $(\mathcal{S}, V)$, $\prod_U(\mathcal{S}, V')$.

The remaining part of the proof will be devoted to showing that $\mathcal{T}$ is the topological ultrafilter morphic image of $\prod_U \mathcal{S}$, which will allow us to deduce $\mathcal{T} \in \mathsf{K}$.

$$\begin{array}{ccc} \prod_U \mathcal{S} & \xrightarrow{f} & T^* \\ | & & | \\ \mathcal{S} & & \mathcal{T} \end{array}$$

We define interior map $f : \prod_U S \to Uf\ (T)$ as

$$f(x) = \{A \subseteq T | (\mathcal{S}^*, V'), v \models p_A\}$$

f is well-defined, i.e. for any given x, $f(x)$ is an ultrafilter. Consider $A, B \in f(x)$, we have by f definition that $(\prod_U \mathcal{S}, V'), x \models p_A$ and $(\prod_U \mathcal{S}, V'), x \models p_B$. It follows from the fact that Δ is globally true in $(\prod_U S, V')$ that $(\prod_U S, V'), x \models p_{A\cap B)}$ which means that $A \cap B \in f(x)$. Using the same technique it can be shown $A \in f(x)$, $A \subset B$ implies $B \in f(x)$ and that for any $A \subset T$, either $A \in f(x)$ or $T - A \in f(x)$.

Following the proof schema of theorem 11 it can be shown that f is an interior surjective map.

The last statement to prove is $|f^{-1}(u)| = 1$ for any $u \in Uf(T)$. Suppose there exist $x, y \in \prod_U \mathcal{S}$ and $f(x) = f(y) = \pi_w$ where $w \in T$ and π_w is a principal ultrafilter containing $\{w\}$. By definition, $(\prod_U \mathcal{S}, V'), x \models p_{\{w\}}$. By global truth of Δ, $(\prod_U \mathcal{S}, V'), x \models i_w$ and $(\prod_U \mathcal{S}, V'), y \models i_w$, hence $x = y$.
$\dashv$

To characterize the $H(@)$ definability we need the following easy result:

Lemma 15. *$H(@)$ formulas validity is preserved under taking open subspaces.*

Proof. There is no significant differences from the proof of corresponding modal result (see [10]). $\dashv$

The following theorem is a modified version of theorem 14 dealing with $H(@)$.

Theorem 16. *A class of topological spaces K which is definable by L_t sentence, is definable in $H(@)$ iff it is closed under topological ultrafilter morphic images and taking open subspaces.*

Proof. The proof mostly repeats the proof of theorem 14.

The main difference is that the set of formulas Δ_T is defined differently:

$$\Delta_T = \{@_{i_w}\Box\delta | w \in T, \delta \in \Delta\}$$

Later, when we consider a topological subspace $\mathcal{S}$ with valuation V which satisfies Δ_T we need an extra intermediate construction to continue the proof the way it is done in theorem 14.

We build a topological space $\tilde{\mathcal{S}}$ as follows. From the fact, that for some $w \in \mathcal{S}$, $(\mathcal{S}, V), w \models \Delta_T$ it follows that for every point in $\mathcal{S}$ named by a nominal i_w there exist an open neighborhood O_w where Δ holds. We define $\tilde{\mathcal{S}}$ as $\mathcal{S}$ restriction on $\bigcup_{w \in T} O_w$ (which is an open set). Since $\mathcal{S} \in \mathsf{K}$ and K is closed under taking open subspaces, $\tilde{\mathcal{S}} \in \mathsf{K}$ as well. We then build $\prod_U \tilde{\mathcal{S}}$ and the rest of the proof goes as usual.

$\dashv$

The topological definability results obtained in this section look even more analogous to the relational definability results if we take into consideration the fact that $H(E)$ (and hence $H(@)$) can be embedded into L_t:

$$
\begin{aligned}
ST_x(i) &= x = x_i && (2)\\
ST_x(q) &= x \,\epsilon\, Q && (3)\\
ST_x(\neg\phi) &= \neg ST_x(\phi) && (4)\\
ST_x(\phi \wedge \psi) &= ST_x(\phi) \wedge ST_x(\psi) && (5)\\
ST_x(\phi \vee \psi) &= ST_x(\phi) \vee ST_x(\psi) && (6)\\
ST_x(\Box\phi) &= \exists O(x \,\epsilon\, O \wedge \forall y(y \,\epsilon\, O \to ST_y(\phi))) && (7)\\
ST_x(@_i\phi) &= ST_x(\phi)[x_i/x] && (8)\\
ST_x(A\phi) &= \forall y\, ST_y(\phi) && (9)
\end{aligned}
$$

where i is a nominal, x_i is a corresponding constant, q is propositional letter and Q is a corresponding set constant.

Theorem 17. *If $\mathfrak{M}$ is a topological model, then for all modal formulas φ*

$$\mathfrak{M}, a \models \varphi \quad \text{iff} \quad \mathfrak{M} \models ST_x(\varphi)[a]$$

Proof. By induction on the complexity of φ. ⊣

7 Separation axioms

It is interesting to apply the obtained results to classes of topological spaces satisfying separation axioms. Recall the definition of the first three of them:

T_0 For every two distinct points x and y there exists either an open set $O_x \ni x$ s.t. $y \notin O_x$ or an open set $O_y \ni y$ s.t. $x \notin O_y$.

T_1 Every singleton set is closed.

T_2 For every two distinct points x and y there exists two disjoint open sets $O_x \ni x$ and $O_y \ni y$.

In fact, we were "lucky" to be able to prove theorems 14 and 16 for topological space classes definable in L_t and not for classes closed under taking Alexandroff extensions, because of the following simple observation: both T_0 and T_1 axioms are not preserved under Alexandroff extension. Take some topological space $\mathcal{T} = (T, \tau)$ with co-finite topology τ (i.e. a set is open iff it is co-finite). It is known that every non-principal ultrafilter contains every co-finite set, and since they are the only open sets in our topology, all non-principal ultrafilters of $\mathcal{T}^*$ are contained in one gigantic open set, and there is no smaller open set containing non-principal ultrafilters. $\mathcal{T}$ is T_0 and T_1, but $\mathcal{T}^*$ clearly does not satisfy T_0 and T_1.

It is known from general topology, that classes of T_0-, T_1- and T_2-spaces are closed under open subspaces. We estblish that T_0- and T_1-spaces are also closed under taking ultrafilter morphic images.

Proposition 18. *The class of T_0-spaces is closed under topological ultrafilter morphic images and open subspaces.*

Proof. Let $\mathcal{T}$ be T_0-space and f be an interior map from $\mathcal{T}$ to $\mathcal{S}^*$ injective on principal ultrafilters. Take any two distinct points x and y from $\mathcal{S}$. Denote $a = f^{-1}(x^*)$, $b = f^{-1}(y^*)$. Since f is injective on principal ultrafilters, a and b are distinct and unique. Since T is a T_0 space, there exists either an open neighborhood O_a of a s.t. $b \notin O_a$ or an open neighborhood O_b of a s.t. $a \notin O_b$. We will consider the first case (the other one is completely analogous). Denote $O_{x^*} = f(O_a)$, it is clear that $y^* \notin O_{x^*}$. It follows that there exists O_x s.t. $x \in O_x$ and $y \notin O_x$.

To prove this a simple argument by contradiction suffices. First of all, without loss of generality we can consider O_{x^*} to be an intersection of $*$-images of open sets. Suppose every neighborhood of x contains y. Then from the $*$-map properties it follows that $y^* \in O_{x^*}$ which is contradiction. ⊣

Proposition 19. *The class of T_1-spaces is closed under topological ultrafilter morphic images and open subspaces.*

Proof. Let $\mathcal{T}$ be T_1-space and f be an interior map from $\mathcal{T}$ to $\mathcal{S}^*$ injective on principal ultrafilters. For every $x \in \mathcal{S}$, x^* is closed since it is interior image of some closed singleton set in $\mathcal{T}$. Then $\{x\}$ is closed too, for if it is open, then x^* would be open too (by proposition 2). ⊣

And indeed, T_0 and T_1 spaces can be defined by the following hybrid formulas (taken from [10]):

$$t_0 : @_i \neg j \to @_j \Box \neg i \vee @_i \Box \neg @_j$$
$$t_1 : i \equiv \Diamond i$$

Hybrid languages have been quite expressive so far, we could even express T_0 and T_1 with them, which is not possible with modal languages. But it turns out that neither $H(@)$, nor $H(E)$ are expressive enough to handle T_2.

Theorem 20. *The class of T_2 topological spaces in not definable in $H(@)$ and $H(E)$.*

Proof. We need to construct a counter-example to theorem 14 as follows: we should find a T_2 space $\mathcal{S}$ such that its topological ultrafilter morphic image $\mathcal{T}$ is not a T_2 space. Since T_1 can be expressed in $H(@)$ and hybrid formulas validity is preserved under topological ultrafilter morphic images, $\mathcal{T}$ should satisfy T_1 but should not satisfy T_2.

We suggest to put $\mathcal{T} = (\mathbb{N}, \tau)$ where τ is a co-finite topology.

Claim 20.1. *The Alexandroff extension $\mathcal{T}^*$ has the following topology: a set is open iff it contains the set F of all non-principal filters.*

Every open set that belongs to the base of Alexandroff topology is the set of all ultrafilters extending some proper open filter. But the only proper open filters possible in $\mathcal{T}$ topology are filters that contain only co-finite sets, because every set that is not co-finite has an empty interior that cannot belong to a proper filter. But then F is contained in any open set from the base of topology. Since any set in topology is a union of sets from the base, the same statement can be made for any open set.

Now let us prove the right-to-left direction. Consider an open filter O containing all co-finite sets that contain a. The set $\pi_a \cup F$ contains precisely ultrafilters that extend O. The claim is proved.

To construct $\mathcal{S}$ we appeal to the notion of resolvable topological space [11].

Definition 19. A topological space $\mathcal{T} = (T, \tau)$ is called *resolvable* if T has a pair of disjoint dense subset. More generally, X is said to be α-resolvable for a cardinal number α if X has α-many pairwise disjoint dense subsets.

We now apply an argument similar to that used in [10] to prove the undefinability of T_i axioms in modal language.

Let $\mathcal{R} = (R, \rho)$ be $2^{2^{\aleph_0}}$-resolvable topological space which satisfies T_2 (according to [8] such a space exists). We will denote dense subsets of R by R_i where $i \in F$ where F is the set of all non-principal ultrafilters over $\mathbb{N}$. Thus $R = \bigcup\limits_{i \in F} R_i \cup \bar{R}$. We can index dense subsets of R with points of F because the set of dense subsets of R has the same cardinality as F. Let $\mathcal{S} = (\mathbb{N} \cup R, \sigma)$ be a space with the following topology: the topology of R as a subspace coincides with the one of $\mathcal{R}$, R itself is an open set and sets of the form $X \cup O$ where $X \in \mathbb{N}$ and O is an open subset of R.

$\mathcal{S}$ is a T_2 space. Indeed, any two points that belong to R can be separated by two opens, since R is a T_2 space. Any two points $x, y \in \mathbb{N}$ can be separated by open sets of the form $\{x\} \cup O_x$ and $\{y\} \cup O_y$ where O_x and O_y are open sets from R such that $O_x \cap O_y = \emptyset$. Finally, two points x, y such that $x \in \mathbb{N}$ and $y \in R$ can be separated by the sets $\{x\} \cup O_x$ and O_y where again O_x and O_y are open sets from R such that $O_x \cap O_y = \emptyset$.

Fix some i^* and construct a function $f : \mathbb{N} \cup R \to \mathcal{T}$ as follows:

$$f(x) = \begin{cases} \pi_x & \text{if } x \in \mathbb{N} \\ i & \text{if } x \in R_i \\ i^* & \text{if } x \in \bar{R} \end{cases}$$

The f-preimage of any open set $O \in \mathcal{T}^*$ is of the form $R \cup X$ where $X \subseteq \mathbb{N}$, which is open. Any open set $O \in \mathbb{N} \cup R$ has non-empty intersection with R_i for all $i \in F$, hence $f(O)$ contains F and thus is open.

f is an interior mapping which is injective on principal ultrafilters by construction. The counter-example is built. $\dashv$

8 Conclusions

We have presented some results which propose a new topic for research in logic — definability of topological spaces in hybrid languages. Although we have obtained a Goldblatt-Thomason-like theorem, a lot of questions remain to be answered. It would be interesting to examine how definability is affected by enriching the hybrid languages with new operator like, for example, binding operator. Another quite fascinating topic would be to describe some interesting classes of topological spaces in hybrid languages (or, using the results obtained, proving it impossible) and thus tracing the boundaries of expressive power of hybrid languages for topological reasoning.

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