Signalling in IF games: a tricky business

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Abstract. The paper studies the phenomenon of signalling in logics for imperfect information, such as Hintikka's IF logic. It is shown that signalling is a phenomenon that at the one hand is essential for the semantics, but at the other hand is a source of tricky phenomena. Several properties which in the literature are claimed to hold for such logics, turn out to fail due to signalling: IF logic is not a conservative extension of predicate logic, renaming of variables is not always allowed, the prenex normal form theorem does not hold, and slashed connectives cannot easily be eliminated.

1. Introduction

IF logic is introduced by J. Hintikka, and advocated in a number of publications; the main ones are Hintikka (1996) and Hintikka and Sandu (1997). The difference with predicate logic concerns the interdependency of quantifiers. In predicate logic quantifiers may depend on the quantifiers in whose scope they occur, e.g. in the quantifier sequence $\forall x \exists y \psi$ the value chosen for y may depend on x. So scope indicates possible dependencies. It is not possible in predicate logic to express that a quantifier should be independent of another one. In IF logic it is possible to express that an existential quantifier is independent of a preceding universal quantifier (IF abbreviates 'Independence Friendly'); the issue of (in)dependence also applies to disjunctions.

In IF logic existential quantifiers (and disjunctions) are by definition independent of other existential quantifiers. A natural generalization of IF would be that these can at choice be dependent or independent of existential quantifiers. We indicate this generalization by IFG (IF Generalized). The generalizations that are investigated by Hodges (1997) and Caicedo and Krynicki (1999), go even further and allow for independency with universal quantifiers and conjunctions.

IF and IFG are logics with unexpected properties. Hodges (1997) has a section called *Deathtraps*, and says (p. 546): 'the idea "what a player is allowed to know", though it has strong intuitive content, can be very misleading'. Caicedo and Krynicki (1999) state (p. 22): 'One has to be careful about wrong extrapolations from classical semantics.' Janssen (2002) concludes (p. 375): 'The examples given in the previous sections show that strategies are on several points in conflict with intuitions on independence'.

The cause of the unexpected properties lies in a phenomenon which is called 'signalling', and which will be explained in the next sections. Although Caicedo and Krynicki (1999) have warned the reader, they fell in the trap of wrong extrapolations. Several of their theorems are incorrect, and this is, in our opinion, not due to some accidental oversight, but because the deathtraps of signalling are not well known.

In this paper we will present several tricky examples of signalling. The results of Caicedo and Krynicki (1999) and the fundamental claim by Hintikka that IF is a conservative extension of predicate logic all are incorrect due to signalling. Probably they can reformulated in a weaker sense (Caicedo et al. (200X)).

2. What is signalling?

In the context of game theoretical semantics for IF, 'signalling' is the phenomenon that the value of a variable one is supposed not to know, is available through the value of another variable. Below we present the earliest example of this phenomenon, but first we explain informally the game theoretical interpretation of IF and IFG. Formal definitions will be given in section 5.

The interpretation of a formula proceeds by a game between two players; ∀belard and ∃loise. ∀belard aims to refute the formula, ∃loise to confirm it. We suppose the formula to be in negation normal form. In that case ∀belard makes the choices for $\forall$ and $\wedge$, and ∃loise for $\exists$ and $\vee$. If a choice is to be made independent of the values of certain variables, that is indicated by mentioning those variables after a slash that is attached to the quantifier or connective. For instance, in $\exists x_{/y}$ the x has to be chosen independent of y, and in $\vee_{/x}$ a disjunct has to be chosen independent of x. Again, ∃loise has to make the choices in these cases, and ∀belard for the other ones. A formula is defined to be true if ∃loise has a winning strategy (and false if ∀belard has a winning strategy, but this will play no role in the paper).

Time for an example. We start with a classical formula:

(1) $\forall x \exists y[x = y]$

The game proceeds as follows. First ∀belard chooses a value for x, and next ∃loise chooses one for y. If she chooses the same value as ∀belard she wins this play. As a matter of fact, that method always is successful; it is a winning strategy, and therefore the formula is true.

An IFG formula that resembles (1) is:

(2) $\forall x\, \exists y_{/x}[x = y]$

Here ∃loise has to make her choice independently of x. Maybe she by accident, but she does not have a winning strategy for the game. Hence the formula is not true. It is not false either, because also ∀belard has no winning strategy.

Consider now

(3) $\forall x \exists z\, \exists y_{/x}[x = y]$

A vacuous quantifier is inserted, and in classical predicate logic that would make no difference; its truth value remains unchanged. But in IFG it makes a difference. The winning strategy for ∃loise is to choose z to be equal to x, which is allowed because no independence restrictions are put on z. Next she chooses y to be equal to z, which is allowed because she does not ask for the value of x. The result is that x equals y. So by proceeding in this way, ∃loise has a winning strategy. The strategy recognizes the value of z as a *signal* for the value of x, and uses that signal to choose the value of y. For original IF logic this example of signalling is not possible (because existential quantifiers are there by definition independent of each other), but we will meet several other examples concerning IF logic where signalling is used.

Example (1) was the first example of signalling that was discovered (Hodges (1997) p. 548). The example of a wrong extrapolation from classical logic given by Caicedo and Krynicki (1999), p.22, has a completely different appearance, but in fact it is a reformulation of (1), so it is based upon signalling. Janssen (2002) gives many other examples of signalling, and so does the present paper.

3. Signalling is needed

Before we consider cases where signalling causes a problem, it first will be illustrated that signalling is an essential component of the semantics of IF and IFG.

Consider (4), interpreted in a model with 2 elements: $\{0, 1\}$.

(4) $\forall x[x = 1 \vee x \neq 1]$

This formula is classically true, and it is so in IF and IFG semantics: ∃loise can make her choice based upon the value of x, and has as strategy: if $x = 1$ then L else R.

A related example is:

(5) $\exists u[u = 1 \vee u \neq 1]$

This formula is true in classical logic, and it is true in IF. But the winning strategy is *not*, as you might expect, to choose for $\vee$ the left disjunct in case $u = 1$ and the right disjunct otherwise. This is not allowed: the choices of ∃loise are in IF by definition independent of her own earlier choices. This independence is analogous to the formation of the Skolem form for classical predicate logic: the Skolem function for an existential quantifier has as arguments only variables that are universally quantified. For example: $\forall x \exists y \exists z[x < y < z]$ has Skolem form $\exists f \exists g \forall x[x < f(x) < g(x)]$ The fact that z $(= g(x))$ is greater than y $(= f(x))$, is accounted for by first recalculating internally in g what $f(x)$ is; one might say $g(x) = g'(x, f(x))$.

As explained, when playing (5) it is not allowed to recalculate the value of u for the decision on $\vee$. What can than be a winning strategy? The solution is to use constant strategies: for $\exists u$ the value 0, and for $\vee$ always choose R.

In IFG the same problem arises for (6) where it is made explicit that the value of u may not be used for making a choice for the disjunction. In (6) the same strategy is winning as for (5).

(6) $\exists u[u = 1 \vee_{/u} u \neq 1]$

We return to IF, and make the example more complicated. Consider:

(7) $\forall x \exists u[u = x \wedge [u = 1 \vee u \neq 1]]$

This formula is classically true, but for IF semantics the situation seems difficult. The constant strategies from the previous example will not work. ∃loise cannot take a strategy that always yields the same value for u because it must satisfy $u = x$, and the strategy for $\vee$ must vary with the value of u. The solution is to use the value of x as signal for the value of u. ∃loise wins by choosing u equal to x, and making for $\vee$ the choice L if $x = 1$ and R otherwise.

Example (7) is an important example because it illustrates the need for signalling in IF. The language of IF logic is an extension of the language of predicate logic, and it is claimed to be a conservative extension (Hintikka (1996) p. 65). Without signalling ∃loise has no winning strategy in example (7). *Without* signalling, (7) would be a formula without slashes that is not true in IF, whereas classically it is true; it would be a counterexample to the claim of conservative extension. This shows: signalling is needed.

This story can directly be transferred to IFG. The IF example (7) is reformulated in IFG as (8), where of course the same strategy is winning.

(8) $\forall x \exists u[u = x \wedge [u = 1 \vee_{/u} u \neq 1]]$

When Hodges (1997) proposes to switch to IFG, he says (p. 22): 'Obviously this won't diminish the expressive power of the language.' But in order to allow a reconstruction of IF in IFG, we must have the same possibilities for signalling in IFG (for (8)) as in IF (for (7)).

So, signalling has effects that cannot easily be missed: giving it up would cause considerable changes in the semantics. An alternative semantics, in which signalling cannot occur, is given in Janssen (2002), but at the cost of loosing some of the expressive power. However, the argument given above for signalling is not the end of the story: in sections 4.4 and 7.4 we will show that even *with* signalling there are problems with the claim of conservative extension.

4. Counterexamples

In the literature one finds some theorems and claims which are incorrect due to the fact that the possibilities of signalling are not well known. In this section we present the counterexamples informally; in section 7 we will prove the results in a more technical way. For instance, in this section we do not quote the original versions of the theorems, and furthermore we will not give complete proofs here. Although we will prove that certain formulas are true, by providing winning strategies for ∃loise, we will not prove that certain formulas are *not* true. The reason is that negative proofs are more difficult to obtain: the collection of possible strategies is rather unwieldy. Instead we will (following Caicedo and Krynicki (1999)) first define (in section 5) an alternative for the game interpretation which they also use for formulating their theorems: an interpretation which uses sets of valuations.

4.1. Renaming of bound variables

Claim 1. (cf. Lemma 3.1a Caicedo and Krynicki (1999), p. 26)
Let Qx be $\forall x$ or $\exists x$ where the quantifiers may be with or without a slash. If ϕ is a sentence in which $Qx[\psi(x)]$ occurs as subformula, and z does not occur (free or bound) in ψ, then the subformula $Qx[\psi(x)]$ may be replaced by $Qz[\psi(z)]$ without changing the truth value.

We will show that this claim is *incorrect*. A counterexample is given by the following two formulas, where (10) is obtained from (9) by replacing s by y. They are played on a model with 2 elements: $\{0, 1\}$, and it does not matter whether you see them as IFG or IF formulas.

(9) $\forall x \forall y \forall z[x \neq y \vee \exists s\, \exists u_{/x}[u = x \wedge s = z]]$

(10) $\forall x \forall y \forall z[x \neq y \vee \exists y\, \exists u_{/x}[u = x \wedge y = z]]$

In (9) the winning strategy is as follows. We let $f_\vee \equiv$ *if* $x \neq y$ *then* L *else* R; $f_{\exists s} \equiv s := z$ and $f_{\exists u_{/x}} \equiv u := y$. This strategy is winning because y signals the value of x to $\exists u_{/x}$. In (10) the corresponding strategy is $f_{\exists y} \equiv y := z$. That strategy is not winning because when it comes to the choice of u, y is equal to z and not to x: the $\exists y$ blocks the signal from $\forall y$.

We expect that a general version of the renaming theorem is not possible. No matter how we restrict the choice of the new variable: there might always be a context in which this new variable blocks a signal from outside. So we always run the risk of changing the truth value in some context if we rename the variable. In order to obtain a kind of renaming theorem, we have to change the notion of equivalence. Instead of the absolute notion 'equivalent in all contexts', a stricter notion seems to be required.

4.2. Prenex normal form

In predicate logic the prenex normal form theorem describes how quantifiers can be shifted to the front of a formula, e.g. $\forall x[\phi] \vee \psi$ can be replaced by $\forall x[\phi \vee \psi]$, under the condition that x does not occur free in ψ. In IF and IFG these two formulas are not necessarily equivalent because quantifiers in ψ become dependent on $\forall x$. Therefore Caicedo and Krynicki (1999) present a rephrasing.

Claim 2. (cf. Lemma 3.1.c,d Caicedo and Krynicki (1999) p.26)
Let Qx *be* $\forall x$ *or* $\exists x$ *where the quantifiers may be with or without a slash and let* $\phi_{/x}$ *denote the result of adding to all quantifiers in* ϕ *the independence condition* $_{/x}$. *Then any subformula of the form* $[Qx[\psi] \vee \theta]$ *can be replaced by* $Qx[\psi \vee \theta_{/x}]$ *without changing the truth value.*

This quantifier extraction claim is *incorrect* because it may interfere with signalling. There are two ways in which this can happen.

The first counterexample will show that in $Qx[\psi \vee \theta_{/x}]$ the x can be used to send signals to $\theta_{/x}$. Indeed, the formulation of the claim given above, was intended to prevent this, but as we have seen in section 3, also decisions on disjunctions may depend on signals. The second counterexample is based upon the possibility that in $Qx[\psi \vee \theta_{/x}]$ the quantifier may block signals from outside to $\theta_{/x}$.

Counterexample 1

This first example is based upon a situation where a formula that is not true, becomes true by quantifier extraction. It is an IFG example; the situation for IF formulas still has to be investigated further. Consider:

(11) $\forall z[\forall x[x \neq z] \vee \exists u_{/z}[u = z \vee_{/z} u \neq z]]$

It will be clear that no z satisfies the left disjunct. In the right disjunct there is for $f_{\exists u_{/z}}$ no argument available, so it is a constant strategy. Furthermore $f_{\vee_{/z}}$ is not allowed to depend on z, but it may depend on u. Since $f_{\exists u_{/z}}$ is constant, this means that $f_{\vee_{/z}}$ is constant: always L or always R. For at least one value of z such a choice will not be winning, hence ∃loise has no winning strategy; so, formula (11) is not true.

The result of the quantifier extraction transformation in (11) is:

(12) $\forall z \forall x[x \neq z \vee \exists u_{/z,x}[u = z \vee_{/z} u \neq z]]$

This formula is true as the following strategy shows. Let $f_{\vee} \equiv$ *if* $x \neq z$ *then* L *else* R. So either the left disjunct is satisfied, or the right disjunct has to be satisfied in a situation where $x = z$. For u we take an arbitrary (but fixed) value. For $\vee_{/z}$ the value of x can be used as a signal: $f_{\vee_{/z}} \equiv$ *if* $u = x$ *then* L *else* R. So if the players arrive at the subformula $u = z$, this one is satisfied because $u = x$ and $x = z$, and if they arrive at the subformula $u \neq z$, that one is satisfied because $u \neq x$ and $x = z$. So these choice functions for the disjunctions form a winning strategy for ∃loise.

Note that in variant (13) of (12) for $f_{\vee_{/z,x}}$ no information about x can be used. So (13) is not true, just as the sentence in which the quantifier extraction has not yet taken place, viz. (11).

(13) $\forall z \forall x[x \neq z \vee \exists u_{/z,x}[u = z \vee_{/z,x} u \neq z]]$

This counterexample suggests that in claim 2 not only the quantifiers, but also the connectives have to be slashed.

Counterexample 2

This example, again an IFG example, shows that by giving ∀ wide scope a signal can be blocked. The counterexample is obtained from:

(14) $\forall y \exists u[\forall u[u \neq u] \vee \exists x_{/y}[x = y]]$

For this formula the winning strategy is: $f_{\exists u} \equiv u := y$, $f_{\vee} \equiv R$, and let $f_{\exists x_{/y}} \equiv x := u$. Since $u = y$ it follows that $x = y$. Hence (14) is true in our model.

Quantifier extraction changes formula (14) into

(15) $\forall y \exists u \forall u[[u \neq u] \vee \exists x_{/y,u}[x = y]]$

The strategy $x := u$ given above for $\exists x_{/y}$ is not allowed for $\exists x_{/y,u}$, and the formula is not true. The quantifier $\forall u$ blocks the signal $u = y$ from outside to $\exists x_{/y,u}[x = y]$.

We think that a general version of the quantifier extraction lemma (as in claim 2) is not possible. One can always construct a context in which a moved quantifier blocks a signal. Therefore we suggest investigating the idea that the normal form theorem has to be restricted to situations where this cannot arise.

4.3. Slashed disjunction elimination

It is claimed that slashed disjunction can be eliminated. In Hintikka's work this follows from his translation procedure from IF logic to second order logic and back (cf. Hintikka (1996), p 52 and p. 62–63). Caicedo and Krynicki (1999) give the result within IFG.

Claim 3. (cf. Lemma 3.2, Caicedo and Krynicki (1999) p. 25)
$\phi \vee_{/Y} \psi \equiv_{\mathsf{G}} \exists u_{/Y}\, \exists s_{/Y,u}[[[u = s \wedge \phi] \vee [u \neq s \wedge \psi]] \vee \exists ! u[u = u \wedge [\phi \vee \psi]]]$

The last part of the formula in the claim deals with the situation that there is one element in the model. In our counterexample we assume that the model has at least three elements, so we may neglect that part.

Consider now:

(16) $\forall y \forall t[\exists x_{/t}[x = t] \vee_{/y} \exists x_{/t}[x = t]]$

The strategies for the two occurrences of $\exists x_{/t}$ may depend on y, but not on u. ∃loise may follow for the left occurrence of $\exists x_{/t}$ another strategy than for the right one. Because ∀belard can choose from at least 3 different values for t, there will always be one occurrence $\exists x_{/t}$ that ∃loise has to satisfy for two or more different values of t. This she cannot do, so the sentence is not true.

The proposed elimination rule changes (16) into (17), where we omitted the last part (since it is false in our model).

(17) $\forall y \forall t\, \exists u_{/y}\, \exists s_{/u,y}[[u = s \wedge \exists x_{/t}\, x = t] \vee [u \neq s \wedge \exists x_{/t}\, x = t]]$

∃loise has a winning strategy for this sentence: $f_{\exists u_{/y}} \equiv u := t$, $f_{\exists s_{/y,u}} \equiv s := t$, $f_{\vee} \equiv L$, and $f_{\exists x_{/t}} \equiv x := s$. This strategy never violates the independence restrictions, and it guarantees ∃loise to win. Note that the y plays no essential role in the strategies.

We conclude that also the rule for the elimination of slashed disjunctions has to formulated in some restricted way.

4.4. Conservative extension

Claim 4. (Hintikka (1996), p.65) *Technically speaking IF first-order logic is a conservative extension of ordinary first-order logic*

Also this claim is undermined by the tricky properties of signalling, as will be explained below.

A variant of example (7) from section 3 is the IFG example:

(18) $\forall x \exists u[u = x \wedge [u = 1 \vee_{/u} u \neq 1]]$

This example is true: the x could be used as a signal for the value of u at the disjunction. An extended version of (18) is:

(19) $\forall x \forall y \exists u\ [u = x \wedge \forall s[s = y \vee [u = 1 \vee_{/u} u \neq 1]]]$

This formula is again true: for the first $\vee$ choose R, and for the second $\vee$ use x as a signal for the value of u. Next we replace s by x.

(20) $\forall x \forall y \exists u\ [u = x \wedge \forall x[x = y \vee [u = 1 \vee_{/u} u \neq 1]]]$

Here the signal is blocked, and consequently the formula is not true.

Next we write (20) in the language of IF logic, what means that the independence of $\vee$ from $\exists u$ is not expressed. Then we get:

(21) $\forall x \forall y \exists u\ [u = x \wedge \forall x[x = y \vee [u = 1 \vee u \neq 1]]$

Above we have argued that $\exists$loise has no winning strategy for (20), hence not for (21); a proof will be given in Section 7.4. So, according to the IF interpretation, (21) is not true. At the same time it is a formula from classical predicate logic, and classically it is true. Hence IF-logic is *not* a conservative extension of predicate logic.

5. Definitions

5.1. The language

Definition 5. *The language of IFG-logic is defined as follows:*

1. The language has **variables**. Typical examples are x, y, z, u, s, and t. In general discussions variables range over a domain A of model $\mathfrak{A}$, but in the examples the domain is $\{0, 1\}$ or $\{0, 1, 2\}$.
2. The language has **constants**. In general discussions typical constants are a and b. In the examples 0 and 1 are used.
3. A **term** is a variable or a constant We choose our fragment not to contain any function symbols.
4. The **relation symbols** are $R_1, R_2, \ldots$; each with a fixed arity. In the examples the binary relation symbols $=, \neq, <$, and $\leq$ are used.
5. If $t_1, \ldots, t_n$ are terms, and n is the arity of R, then $R(t_1, \ldots, t_n)$ and $\neg R(t_1, \ldots, t_n)$ are **formulas**.

6. If ψ and θ are formulas, z is a variable, and Y a set of variables, then also the following expressions are **formulas**: $\psi \wedge \theta$, $\psi \vee \theta$, $\psi \vee_{/Y} \theta$, $\forall z\psi$, $\exists z\psi$. If $z \notin Y$ then $\exists z_{/Y}\, \psi$ is a formula.

After a slash we will omit brackets of set denotations, and write $\exists y_{/x}$ and $\exists y_{/x,u}$.

Definition 6. *$FV(\phi)$ is the set of* **free variables** *in ϕ. It consists of those variables in ϕ which do not occur in ϕ bound. $FV(\phi)$ includes variables in Y's occurring in $\vee_{/Y}$ and $\exists x_{/Y}$ as far as they are unbound in ϕ.*

A comparison of our definition of the logic with the literature gives rise to the following remarks:

1. We assume all formulas to be in negation normal form, as is done in almost all publications of Hintikka and in Väänänen (2002). In some other publications about variants of IF-logic negation may occur freely (e.g. Hodges (1997), Caicedo and Krynicki (1999)). We do not need it, however, for our discussion of signalling. A happy consequence is that there is no role switch between ∀belard and ∃loise, what makes the discussion easier to understand.
2. In Hintikka's version choices depend only on moves of the opponent, but can be made independent of them by slashing those moves away. We allow also (in)dependency between own choices. Therefore also existentially quantified variables may arise after a slash.
3. We do not consider strategies for ∀belard, as they are not necessary to make our points.

5.2. Playing

The main ingredient of a game is a formula from IFG logic. The aim of the game is to determine the truth of the formula in a model $\mathfrak{A}$. The two players have different aims: ∃loise tries to confirm the truth and ∀belard to refute it. We are, however, not so much interested whether ∃loise accidently wins (or looses), but whether she has a winning strategy for the game, because that is the criterium whether the formula is true or not. Therefore we will describe IFG on two levels: the level of actual playing the game, where the two players move, and win or lose, and the level of a set of sets of plays where the players may have a winning strategies.

We first describe how a game is played: which player has to move in a given position, what are his/her possible moves, and what is the effect of the move. In the course of the game the players will encounter subformulas like $\psi \vee_{/Y} \theta$ or $\exists x_{/Y}\, \psi$. The subscript indicates that the choice of the move has to be made independent of the variables in Y. This is a restriction on the motivation for the choice, but not on the choice itself. Therefore in the description of playing it makes no difference whether $_{/Y}$ occurs as subscript or not. Its role will be defined when we consider strategies in section 5.3.

A valuation describes at least the values of the free variables in ϕ; an alternative name would be finite assignment. For valuations we mainly use v and w.

Definition 7. *A* **valuation** *v for a formula ϕ in a model $\mathfrak{A}$ is a function $v \in \mathrm{dom}(\mathfrak{A})^X$ where $FV(\phi) \subseteq X$.*

Definition 8. *We use the following notations concerning valuations:*

$\{x{:}\, a\}$	the valuation that assigns a to x ($a \in \mathrm{dom}(\mathfrak{A})$)
$v * \{x{:}\, a\}$	the valuation obtained from v by changing the value assigned to x into a if v was defined for x, or by extending the domain of v such that it now assigns a to x.
ϵ	(*the empty valuation*) the valuation that is defined for no variable at all
$v \sim_Y w$	(v *is an* Y*-variant of* w) valuations v and w are defined for the same variables, the values they assign may differ for the variables in Y, but are the same outside Y

Definition 9. *A* **play** *is a triple $\langle \mathfrak{A}, \phi, v\rangle$ where ϕ is a formula from IFG-logic, $\mathfrak{A}$ a model and v a valuation where $FV(\phi) \subseteq \mathrm{dom}(v)$. A* **position** *is a pair $\langle \psi, w\rangle$, where ψ is a subformula of ϕ and w a valuation where $FV(\phi) \subseteq \mathrm{dom}(w)$. A* **move** *is a transition from a position to a position. The possible moves are determined by the form of ϕ. We distinguish the following cases:*

1. $\langle \phi, v\rangle \equiv \langle \psi \wedge \theta, v\rangle$
   ∀belard chooses L or R. If he chooses L, then the play is continued from position $\langle \psi, v\rangle$, otherwise from position $\langle \theta, v\rangle$.
2. $\langle \phi, v\rangle \equiv \langle \psi \vee \theta, v\rangle$ or $\langle \phi, v\rangle \equiv \langle \psi \vee_{/Y} \theta, v\rangle$
   ∃loise chooses L or R. If she chooses L, the play is continued from position $\langle \psi, v\rangle$ otherwise from position $\langle \theta, v\rangle$. For the role of $_{/Y}$ see section (5.3) below.
3. $\langle \phi, v\rangle \equiv \langle \forall x \psi, v\rangle$
   ∀belard chooses a value for x, say a, and the play proceeds from position $\langle \psi, v * \{x{:}\, a\}\rangle$.
4. $\langle \phi, v\rangle \equiv \langle \exists x \psi, v\rangle$ or $\langle \phi, v\rangle \equiv \langle \exists x_{/Y}\, \psi, v\rangle$
   ∃loise chooses a value for x, say b. Then the play is continued from position $\langle \psi, v * \{x{:}\, b\}\rangle$.

5. $\langle \phi, v \rangle \equiv \langle R(t_1, \ldots, t_n), v \rangle$
Here the play ends. Let $a_i = v(t_i)$ if t_i is a variable, en $a_i = t_i^{\mathfrak{A}}$ if t_i is a constant. If $(a_1, \ldots, a_n) \in R^{\mathfrak{A}}$ then ∃loise has won the play, otherwise she has lost.
6. $\langle \phi, v \rangle \equiv \langle \neg R(t_1, \ldots, t_n), v \rangle$
Here the pay ends. Let a_i be defined as in the previous clause. If $(a_1, \ldots, a_n) \notin R^{\mathfrak{A}}$ then ∃loise has won the instance of the game, otherwise she has lost.

5.3. The game

We now switch to the level where strategies can be defined. We consider a game as set of plays (Hodges (1997) calls this level a 'contest').

Definition 10. *A game is a triple $\langle \mathfrak{A}, \phi, V \rangle$, where $\mathfrak{A}$ is a model with domain A, ϕ a formula of IFG-logic and V a collection of valuations such that there is a set X of variables with $FV(\phi) \subseteq X$ and $V \subseteq A^X$.*

A choice function f_ϕ is a function that describes which choices ∃loise may make, depending on the values previously chosen for the variables. The requirement that a choice does not depend on the variables in a set Y is formalized by requiring that f_ϕ yields the same choice for different values assigned to the variables in Y.

Definition 11. *The possible* **choice functions** *f_ϕ for ∃loise at position $\langle \phi, v \rangle$ in a game $\langle \mathfrak{A}, \eta, V \rangle$ are defined by the following cases*

$\phi \equiv \psi \vee \theta$	$f_{\psi\vee\theta}: V \to \{L, R\}$
$\phi \equiv \psi \vee_{/Y} \theta$	$f_{\psi\vee_{/Y}\theta}: V \to \{L, R\}$ such that from $v_1 \sim_Y v_2$ follows that $f_{\psi\vee_{/Y}\theta}(v_1) = f_{\psi\vee_{/Y}\theta}(v_2)$.
$\phi \equiv \exists x \psi$	$f_{\exists x\psi}: V \to A$, where A is the domain of $\mathfrak{A}$.
$\phi \equiv \exists x_{/Y}\, \psi$	$f_{\exists x_{/Y}\,\psi}: V \to A$, where A is the domain of interpretation and $v_1 \sim_Y v_2$ implies $f_{\exists x_{/Y}\,\psi}(v_1) = f_{\exists x_{/Y}\,\psi}(v_2)$.

We say that a choice function f is **independent of** *Y on V if for all $v, w \in V$ from $v \sim_Y w$ follows $f(v) = f(w)$.*

Definition 12. *A* **strategy** *F_ϕ for a game $\langle \mathfrak{A}, \phi, V \rangle$ is a collection choice functions $\{f_\psi\}_{\psi \in Sub(\phi)}$ which for each subformula ψ where ∃loise has to make a choice, provides a choice function f_ψ. Different occurrences of ψ in ϕ have their own choice function.*

Definition 13. *A strategy F_ϕ is called a* **winning strategy** *in game $\langle \mathfrak{A}, \phi, V \rangle$ if playing in accordance with that strategy guarantees ∃loise to win in all possible plays $\langle \mathfrak{A}, \phi, v \rangle$, $v \in V$, of the game. Notation: $\mathfrak{A} \models_{\mathsf{G}} \phi[V, F_\phi]$.*

Definition 14. *Sentence ϕ is called* **game-true**, *shortly 'true', if there exists a winning strategy F_ϕ such that $\mathfrak{A} \models_{\mathsf{G}} \phi[\{\epsilon\}, F_\phi]$*

6. Valuations

In this section we will present an alternative definition for IFG that resembles the classical definition of satisfiability using valuations. One of the reasons is that several theorems from Caicedo and Krynicki (1999) are formulated with such a definition. There is a difference however between the use of valuations for IF (and IFG), and in the traditional approach to predicate logic. Whereas classically a formula is interpreted with respect to a single valuation, for IF and IFG this will be done with respect to a set of valuations. Therefore several notions from definition 8 are lifted to the level of sets.

Definitions 15. *The following notations concern sets of valuations:*

$\{xy\colon aa, bb\}$	(*is an example of the explicit notation we use for a set of valuations*) the set of valuations that assign to x and y identical values from $\{a, b\}$
$V * \{x\colon a\}$	$\{v * \{x : a\} \mid v \in V\}$
$V * \{x\colon A\}$	$\{v * \{x : a\} \mid v \in V, a \in A\}$
$V \sim_Y W$	(V *is an Y-variant of W*) for each $v \in V$ there is a $w \in W$ such that $v \sim_Y w$ and for each $w \in W$ there is a $v \in V$ such that $w \sim_Y v$

In order to express a counterpart of independence in the realm of valuations, we borrow the notion of Y-saturatedness from Caicedo and Krynicki (1999). That a choice function is independent of Y will correspond with having a certain partition of its domain into Y-saturated sets (see definition 20).

Definition 16. *A subset W of V is called* **Y-saturated in** V *if for all $w, v \in V$ from $w \sim_Y v$ and $w \in W$ follows that also $v \in W$.*

Lemma 17. *Let V_1 and V_2 be Y-saturated subsets of V. Then $V_1 \cup V_2$, $V_1 \cap V_2$, and $V_1 \setminus V_2$ are Y-saturated subsets in V.*

Lemma 18. *The equivalence classes in V of the relation $\sim_Y$ are Y-saturated.*

Definition 19. *The partition $V/_{\sim_Y}$ of V into its $\sim_Y$ equivalence classes is called the* **Y-saturated partition** *of V.*

We are now prepared for the definition of the interpretation in a model $\mathfrak{A}$ of a formula ϕ with respect to set a of valuations V; notated as $\mathfrak{A} \models_{\mathsf{V}} \phi[V]$. Note that the subscript $_{\mathsf{V}}$ is fixed part of the notation, whereas for V any denotation for a set of valuations can be used.

Definition 20. *Let $\mathfrak{A}$ be a model with domain A, ϕ a formula, X a set of variables for which $FV(\phi) \subseteq X$, and $V \subseteq A^X$. Then ϕ* **is true under V in $\mathfrak{A}$**, *notated $\mathfrak{A} \models_{\mathsf{V}} \phi[V]$, iff:*

1. For atomic ϕ
   $\mathfrak{A} \models_{\mathsf{V}} R(t_1, \dots, t_n)[V]$ iff for all $v \in V$ we have $(a_1, \dots, a_n) \in R^{\mathfrak{A}}$, where $a_i = v(t_i)$ if t_i is a variable, and $a_i = t_i^{\mathfrak{A}}$ if t_i is a constant.
   $\mathfrak{A} \models_{\mathsf{V}} \neg R(t_1, \dots, t_n)[V]$ iff for no $v \in V$ we have $(a_1, \dots, a_n) \in R^{\mathfrak{A}}$, where a_i as just defined.
2. $\mathfrak{A} \models_{\mathsf{V}} [\psi \wedge \theta][V]$ iff $\mathfrak{A} \models_{\mathsf{V}} \psi[V]$ and $\mathfrak{A} \models_{\mathsf{V}} \theta[V]$.
3. $\mathfrak{A} \models_{\mathsf{V}} [\psi \vee \theta][V]$ iff $V = V_1 \cup V_2$ for some V_1 and V_2, such that $\mathfrak{A} \models_{\mathsf{V}} \psi[V_1]$ and $\mathfrak{A} \models_{\mathsf{V}} \theta[V_2]$.
4. $\mathfrak{A} \models_{\mathsf{V}} [\psi \vee_{/Y} \theta][V]$ iff $V = V_1 \cup V_2$ for some V_1 and V_2, such that V_1 and V_2 are Y-saturated in V and $\mathfrak{A} \models_{\mathsf{V}} \psi[V_1]$ and $\mathfrak{A} \models_{\mathsf{V}} \theta[V_2]$.
5. $\mathfrak{A} \models_{\mathsf{V}} \forall x \phi[V]$ iff $\mathfrak{A} \models_{\mathsf{V}} \phi[V * \{x : A\}]$.
6. $\mathfrak{A} \models_{\mathsf{V}} \exists x \psi[V]$ iff there is a $W \sim_x (V * \{x : A\})$ such that $\mathfrak{A} \models_{\mathsf{V}} \psi[W]$
7. $\mathfrak{A} \models_{\mathsf{V}} \exists x_{/Y}\, \psi[V]$ iff $V = \cup_i V_i$, where each V_i is Y-saturated in V and for each i there is an a_i such that $\mathfrak{A} \models_{\mathsf{V}} \psi[\cup_i (V_i * \{x : a_i\})]$

It might be helpful to compare the clause 7 with clause 4: the slashed existentional quantifier is seen as a slashed disjunction for the different values x may take. Clause 6 could be expressed in an analogous way.

Note that if we let $V = \emptyset$, this inductive definition of satisfaction yields, for any IFG-formula, $\mathfrak{A} \models_{\mathsf{V}} \phi[\emptyset]$. This might look anomalous, but it is actually necessary for the situation with disjunction, where the empty sets of valuations can occur if V is split into V and $\emptyset$. Be aware that this is different from saying that formulas are always satisfied by the singleton set $A^{\emptyset} = \{\epsilon\}$: this is not the case. In fact, satisfaction with respect to $A^{\emptyset}$ is only *defined* for formulas with no free variables, i.e. sentences.

Definition 21. *For ϕ a sentence, we say that ϕ is* **valuation true** *iff $\mathfrak{A} \models_{\mathsf{V}} \phi[\{\epsilon\}]$. Notation: $\mathfrak{A} \models_{\mathsf{V}} \phi$*

Definition 20 differs for clause $\exists x_{/Y}$ essentially from the definition in Caicedo and Krynicki (1999). Their definition has a typo that turns $_{/Y}$ into a vacuous addition, and, after the obvious correction, gives rise to a difference between the game interpretation and the valuation interpretation in certain cases. We will not go into details.

Next we establish the equivalence of the game interpretation and the valuation interpretation.

Theorem 22. *A sentence ϕ is game true iff ϕ is valuation true.*

Proof
We will show that for any formula ϕ and for all $V \subseteq A^X$ with $FV(\phi) \subseteq X$ the statements (i) and (ii) are equivalent.
(i) $\mathfrak{A} \models_{\mathsf{V}} \phi[V]$
(ii) there is a winning strategy F_ϕ such that $\mathfrak{A} \models_{\mathsf{G}} \phi[V, F_\phi]$
In particular this shows for sentences ϕ: $\mathfrak{A} \models_{\mathsf{V}} \phi[\{\epsilon\}]$ *iff* there is a winning strategy F_ϕ such that $\mathfrak{A} \models_{\mathsf{G}} \phi[\{\epsilon\}, F_\phi]$, which proves the theorem.
Proof (⇒)
We only consider the clauses where a choice function for ∃loise has to be designed.

3. $\mathfrak{A} \models_{\mathsf{V}} \psi \vee \theta[V]$
   By definition 20 there are V_1 and V_2 such that $\mathfrak{A} \models_{\mathsf{V}} \psi[V_1]$ and $\mathfrak{A} \models_{\mathsf{V}} \theta[V_2]$. Then, by induction hypothesis, there are winning strategies F_ψ and F_θ such that $\mathfrak{A} \models_{\mathsf{G}} \psi[V_1, F_\psi]$ and $\mathfrak{A} \models_{\mathsf{G}} \theta[V_2, F_\theta]$. Define $f_{\psi\vee\theta}(v) = L$ if $v \in V_1$, and R otherwise. Let $F_\phi = \{f_{\psi\vee\theta}\} \cup F_\psi \cup F_\theta$. Then $\mathfrak{A} \models_{\mathsf{G}} \phi[V, F_\phi]$.
4. $\mathfrak{A} \models_{\mathsf{V}} \psi \vee_{/Y} \theta[V]$
   Follow the construction from the previous case. Since V_1 and V_2 are Y-saturated in V, so is $V_1 \setminus V_2$, hence $f_{\psi\vee_{/Y}\theta}$ indeed is independent of Y on V.
6. $\mathfrak{A} \models_{\mathsf{V}} \exists x\psi[V]$
   According to clause 6 of definition 20 there is a $W \sim_x (V * \{x\colon A\})$ such that $\mathfrak{A} \models_{\mathsf{G}} \psi[W]$. By induction hypothesis there is a winning strategy F_ψ such that $\mathfrak{A} \models_{\mathsf{G}} [W, F_\psi]$. For each $v \in (V * \{x\colon A\})$ choose a $w_v \in W$ such that $w_v \sim_x v$. Define $f_{\exists x\psi}(v) = w_v(x)$. Then $\mathfrak{A} \models_{\mathsf{G}} \exists x\psi[V, \{f_{\exists x\psi}\} \cup F_\psi]$
7. $\mathfrak{A} \models_{\mathsf{V}} \exists x_{/Y}\,\psi[V]$
   Let $\{W_j\}$ be the Y-saturated partition of V (see def. 19). Let V_i and a_i be as in clause 7 of definition 20. For each W_j choose a V_i such that $W_j \subseteq V_i$ and define $b_j = a_i$. Since $\mathfrak{A} \models_{\mathsf{V}} \psi[V_i * \{x\colon a_i\}]$ we have $\mathfrak{A} \models_{\mathsf{V}} \psi[W_j * \{x\colon b_j\}]$, and since the W_j are pairwise disjunct $\mathfrak{A} \models_{\mathsf{V}} \psi[\cup_j(W_j * \{x\colon b_j\})]$. By induction hypothesis there is a winning strategy F_ψ such that $\mathfrak{A} \models_{\mathsf{G}} \psi[\cup_j(W_j * \{x\colon b_j\}), F_\psi]$. Define $f_{\exists x_{/Y}\,\psi}(v) = b_j$ if $v \in W_j$. This defines a function because the sets W_j are pairwise disjunct, and this function is independent of Y

because the sets W_j are Y-saturated. Hence $\mathfrak{A} \models_{\mathsf{G}} \psi[V, \{f_{\exists x_{/Y}\,\psi}\} \cup F_\psi]$.

Proof (⇐).
We consider only the cases where ∃loise applies her strategy.

3. $\mathfrak{A} \models_{\mathsf{G}} (\psi \vee \theta)\,[V, F_{\psi\vee\theta}]$.
   Let $V_1 = f^{-1}_{\psi\vee\theta}(L)$ and $V_2 = f^{-1}_{\psi\vee\theta}(R)$, so $V = V_1 \cup V_2$. Then for the substrategy from F_ϕ for ψ, viz. F_ψ, holds $\mathfrak{A} \models_{\mathsf{G}} \psi[V_1, F_\psi]$. Analogously $\mathfrak{A} \models_{\mathsf{G}} \theta[V_2, F_\theta]$. By induction hypothesis $\mathfrak{A} \models_{\mathsf{V}} \psi[V_1]$ and $\mathfrak{A} \models_{\mathsf{V}} \theta[V_2]$. Then V satisfies the conditions of clause 3 in definition 20 , hence $\mathfrak{A} \models_{\mathsf{V}} (\psi \vee \theta)[V]$.

4. $\mathfrak{A} \models_{\mathsf{G}} (\psi \vee_{/Y} \theta)\,[V, F_{\psi\vee_{/Y}\theta}]$.
   Define V_1 and V_2 as in clause 3 above. Since $f_{\psi\vee_{/Y}\theta}$ is independent of Y in V, sets V_1 and V_2 are Y-saturated in V. So V_1 and V_2 satisfy the conditions of clause 4 in definition 20, hence $\mathfrak{A} \models_{\mathsf{V}} (\psi \vee_{/Y} \theta)[V]$.

6. $\mathfrak{A} \models_{\mathsf{G}} \exists x\psi[V, F_{\exists x\psi}]$
   Let B be the range of $f_{\exists x\psi}$ and define $V_b = f^{-1}_{\exists x\psi}(b)$ for each $b \in B$. Then $\mathfrak{A} \models_{\mathsf{G}} \psi[\cup_b(V_b * \{x{:}\,b\}), F_\psi]$. By induction hypothesis we know that $\mathfrak{A} \models_{\mathsf{V}} \psi[\cup_b(V_b * \{x{:}\,b\})]$. So $W = \cup(V_b * \{x{:}\,b\})$ satisfies clause 6 from definition 20. Hence $\mathfrak{A} \models_{\mathsf{V}} \exists x\psi[V]$.

7. $\mathfrak{A} \models_{\mathsf{G}} \exists x_{/Y}\,\psi[V, F_{\exists x_{/Y}\,\psi}]$.
   Let V_b be as in clause 6 above. Since $f_{\exists x_{/Y}\,\psi}$ is independent of Y in V, the sets V_b are Y-saturated in V. So $V = \cup_b V_b$ satisfies the conditions from clause 7 in definition 20. Hence $\mathfrak{A} \models_{\mathsf{V}} \exists x_{/Y}\,\psi[V]$

7. Proofs

7.1. Renaming bound variables

Caicedo and Krynicki (1999), p.26, present a result that bound variables under standard conditions can be renamed. They formulate a general version for formulas with free variables, and therefore a general notion of equivalence is needed. Their definition requires that the two expressions for all assignments agree on truth ($\models^+_{\mathsf{G}}$ what is our $\models_{\mathsf{G}}$) and falsehood ($\models^-_{\mathsf{G}}$). For completeness of information we also quote the falsehood part, but in our discussion only truth plays a role.

Quote 23. (Caicedo and Krynicki (1999), p.24) *Formulas ϕ and ψ are* G-equivalent, *notated as $\phi \equiv_{\mathsf{G}} \psi$, if and only if for any set V of valuations on a fixed domain including $FV(\phi) \cup FV(\psi)$ in a structure $\mathcal{A}$, we have $\mathfrak{A} \models^+_{\mathsf{G}} \phi\,[V] \iff \mathfrak{A} \models^+_{\mathsf{G}} \psi\,[V]$, and $\mathfrak{A} \models^-_{\mathsf{G}} \phi\,[V] \iff \mathfrak{A} \models^-_{\mathsf{G}} \psi\,[V]$.*

The renaming theorem allows renaming by a fresh variable:

Quote 24. (Lemma 3.1a Caicedo and Krynicki (1999), p.26) *Let Q be $\exists$ or $\forall$. Then: $Qx_{/Y}\ \phi(x) \equiv_{\mathsf{G}} Qz_{/Y}\ \phi(z)$, if z does not occur in $Qx_{/Y}\ \phi(x)$.*

Our counterexample is obtained from the two sentences which were used in the discussion in section 4, viz (9) and (10). We consider here the situation after ∀belard has made his choices for the universal quantifiers, and ∃loise has chosen the right disjunct.

Lemma 25. *Let $V = \{xyz\colon 110, 111, 000{,}001\}$. Then:*

(22) $\mathfrak{A} \models_{\mathsf{V}} \exists s\, \exists u_{/x}\ [u = x \wedge s = z]\ [V]$*, whereas*

(23) $\mathfrak{A} \not\models_{\mathsf{V}} \exists y\, \exists u_{/x}\ [u = x \wedge y = z]\ [V]$

So renaming the bound variables may change the truth value of the formula.

Proof
The winning strategy for (22) is $f_{\exists s} \equiv s := z$ and $f_{\exists u_{/x}} \equiv u := y$. The negative result will be proved using the interpretation with valuations.

Assume that

(24) $\mathfrak{A} \models_{\mathsf{V}} \exists y\, \exists u_{/x}\ [u = x \wedge y = z]\ [\{xyz\colon 110, 111, 000{,}001\}]$

Then there must be a set $W \sim_y (V * \{y : A\})$ such that

(25) $\mathfrak{A} \models_{\mathsf{V}} \exists u_{/x}\ [u = x \wedge y = z]\ [W]$

The values of y and z will not change any further in the recursion to subformulas, so we have to restrict here our choice to valuations for which $y = z$ holds. So W consists of the y-variants for which $y = z$ holds: $W = \{xyz\colon 100, 111, 000, 011\}$. Since (25) holds, there must be a collection W_i of in W x-saturated subsets such that $W = \cup_i W_i$. The x-saturated subsets of W are W, $W_1 = \{xyz\colon 100, 000\}$ and $W_2 = \{xyz\colon 111, 011\}$ (and the empty set). Now we have to find for each element of the collection a value for u such that the left conjunct $u = x$ becomes satisfied (the right conjunct is already satisfied by our choice for W). For none of the three candidates for the collection such a value for u can be found. So there is no collection W_i that satisfies the requirements for $\exists u_{/x}$. So our initial assumption (24) is incorrect, which proves (23).

7.2. Prenex normal form

Caicedo and Krynicki present the following rephrase of the quantifier extraction part of the prenex normal form theorem.

Quote 26. (Lemma 3.1.c,d Caicedo and Krynicki (1999) p.26.)
Let Q be $\exists$ or $\forall$. Let $\psi_{/x}$ denote the result of adding to all quantifiers in ψ the independence condition $_{/x}$. Then:

1. $[Qx_{/Y}\ \phi \vee \psi] \equiv_{\mathsf{G}} Qx_{/Y}\,[\phi \vee \psi_{/x}]$
2. $[Qx_{/Y}\ \phi \wedge \psi] \equiv_{\mathsf{G}} Qx_{/Y}\,[\phi \wedge \psi_{/x}]$

In section 4 we have presented two counterexamples. The first one showed that new signalling possibilities emerge by quantifier extraction. We consider that example after the first choice by ∀belard.

Lemma 27. *Let $V = \{z\colon 1,0\}$. Then:*

(26) $\mathfrak{A} \not\models_{\mathsf{V}} \forall x[x \neq z] \vee \exists u_{/z}[u = z \vee_{/z} u \neq z]\ [V]$

(27) $\mathfrak{A} \models_{\mathsf{V}} \forall x[x \neq z \vee \exists u_{/z,x}[u = z \vee_{/z} u \neq z]]\ [V]$

So quantifier extraction as quoted in (26), may change the interpretation of a formula.

Proof The winning strategy for (27) is $f_{\vee} \equiv$ *if* $x \neq z$ *then* L *else* R, $f_{\exists u_{/z,x}} \equiv 0$ and $f_{\vee z} \equiv$ *if* $x = 0$ *then* L *else* R.

Next we prove the negative result using valuations. Assume that

(28) $\mathfrak{A} \models_{\mathsf{V}} \forall x[x \neq z] \vee \exists u_{/z}[u = z \vee_{/z} u \neq z]\ [\{z\colon 1,0\}]$

It will be clear that $\forall x[x \neq z]$ will not be satisfied for any nonempty subset of $\{z\colon 1,0\}$. Therefore

(29) $\mathfrak{A} \models_{\mathsf{V}} \exists u_{/z}[u = z \vee_{/z} u \neq z][\{z\colon 1,0\}]$

must hold. Then we have consider the z-saturated subsets of $\{z\colon 1,0\}$. That is only the set itself. So there must be a value a such that

(30) $\mathfrak{A} \models_{\mathsf{V}} [u = z \vee_{/z} u \neq z][\{zu\colon 1a, 0a\}]$

Again, there is only one way to divide the valuations into z-saturated subsets, the set itself and the empty set. But no matter what the value of a would be, neither $\mathfrak{A} \models_{\mathsf{V}} u = z\ [\{zu\colon 1a, 0a\}]$ nor $\mathfrak{A} \models_{\mathsf{V}} u \neq z\ [\{zu\colon 1a, 0a\}]$. Hence (28) cannot be true.

The second counterexample showed that signals from outside can be blocked by quantifier extraction. We consider that example after the initial choices by ∀belard en ∃loise.

Lemma 28. *Let $V = \{yt\colon 00, 11\}$. Then*

(31) $\mathfrak{A} \models_{\mathsf{V}} \forall t\ [t \neq t] \vee \exists x_{/y}[x = y]\ [V]]$

(32) $\mathfrak{A} \not\models_{\mathsf{V}} \forall t\ [t \neq t \vee \exists x_{/y,t}[x = y]]\ [V]$

So quantifier extraction as described in claim 26, may change the interpretation of a formula.

Proof The winning strategy for (31) is given by $f_{\vee} \equiv R$ and $f_{\exists x_{/y}} \equiv x := t$. We show (32) using the interpretation with valuations. So, suppose:

(33) $\mathfrak{A} \models_{\mathsf{V}} \forall t\ [t \neq t \vee \exists x_{/y,t}[x = y]]\ [\{yt\colon 00, 11\}]$.

Due to the meaning of $\forall t$ (34) follows, so (35) holds:

(34) $\mathfrak{A} \models_{\mathsf{V}}\ [t \neq t \vee \exists x_{/y,t}[x = y]]\ [\{yt\colon 00, 01, 11, 10\}]$.

(35) $\mathfrak{A} \models_{\mathsf{V}}\ \exists x_{/y,t}[x = y]\ [\{yt\colon 00, 01, 11, 10\}]$

The set of valuations in (35) has only itself and the empty set as y-saturated subset. So there must be a value a such that:

(36) $\mathfrak{A} \models_{\mathsf{V}}\ x = y\ [\{xyt\colon a00,\ a01, a11, a10\}]$

There is, however, no value which does so for all valuations in the set. Hence (32) is proven.

7.3. Slashed disjunction elimination

Quote 29. (Lemma 3.2, Caicedo and Krynicki (1999) p.25)
$\phi \vee_{/Y} \psi \equiv_{\mathsf{G}} \exists u_{/Y}\, \exists s_{/Y,u}[[[u = s \wedge \phi] \vee [u \neq s \wedge \psi]] \vee \exists! u[u = u \wedge [\phi \vee \psi]]]$

Since the counterexample is in a domain with three elements, we omit the part after $\exists!$(that disjunct is than false).

Lemma 30. *Let* $\mathrm{dom}(\mathfrak{A}) = \{0, 1, 2\}$*. Then*

(37) $\mathfrak{A} \not\models_{\mathsf{V}} \forall y \forall t[\exists x_{/t}[x = t] \vee_{/y} \exists x_{/t}[x = t]]$

Proof Let $V = \{0, 1, 2\}^{\{y,t\}}$, and suppose that the formula mentioned in the lemma was true in $\mathfrak{A}$. Then (38) must hold.

(38) $\mathfrak{A} \models_{\mathsf{V}} [\exists x_{/t}[x = t] \vee_{/y} \exists x_{/t}[x = t][V]$

Then there must be sets V_1 and V_2, both y-saturated in V, such that each satisfies one of the conjuncts. The three elementary candidates for these y-saturated sets are $V_i = \{i\}^t \times \{0, 1, 2\}^y$, where $i \in \{0, 1, 2\}$. The other candidates are the union of two or three of these. Assume that V_1 consists of one of the elementary sets, then V_2 must consist of the union of the other two. In that union two values for t occur, hence there is no value for x which for all valuations satisfies $x = t$. Analogously for the other combinations of y-saturated subsets.

Since we have already shown (section 4) that the rule for slashed disjunction elimination transforms (37) into a true formula, the proposed rule cannot be correct.

7.4. Conservative Extension

Quote 31. (Hintikka (1996), p. 65) *Technically speaking IF first-order logic is a conservative extension of ordinary first-order logic.*

Lemma 32. *The IFG sentence (39) is not true.*

(39) $\forall x \forall y \exists u\ [u = x \wedge \forall x[x = y \vee [u = 1 \vee_{/u} u \neq 1]]$

Proof
Let $\mathfrak{A}$ be a model with two elements $\{0, 1\}$. Assume

(40) $\mathfrak{A} \models_{\mathsf{V}} \forall x \forall y \exists u\ [u = x \wedge \forall x[x = y \vee [u = 1 \vee_{/u} u \neq 1]]][\{\epsilon\}]$

By definition 20 the following must hold:

(41) $\mathfrak{A} \models_{\mathsf{V}} \exists u[u = x \wedge \forall x[x = y \vee [u = 1 \vee_{/u} u \neq 1]]]\ [\{xy\colon 11, 10, 01, 00\}]$

Since u must be equal to x, we take for the u-variant only those valuations where $x = u$.

(42) $\mathfrak{A} \models_{\mathsf{V}} [u = x \wedge \forall x[x = y \vee [u = 1 \vee_{/u} u \neq 1]]]\ [\{xyu\colon 111, 101, 010, 000\}]$

The right conjunct has to be satisfied by the same set of valuations. The $\forall x$ adds all x-variants, so it must be the case that:

(43) $\mathfrak{A} \models_{\mathsf{V}} [x = y \vee [u = 1 \vee_{/u} u \neq 1]]\ [\{xyu\colon 111, 011, 101, 001, 010, 110, 000, 100\}]$

Let V_1 consist of all valuations for which $x = y$, then V_1 satisfies the left disjunct. Let V_2 consist of the other valuations, so $V_2 = \{xyu\colon 011, 101, 010, 100\}$. Then it should be the case that:

(44) $\mathfrak{A} \models_{\mathsf{V}} [u = 1 \vee_{/u} u \neq 1]\ [V_2]$

If we would have made another division, the V_2 would have been larger, but (44) still should hold. The u-saturated subsets of V_2 are $\{xyu\colon 011, 010\}$ and $\{xyu\colon 101, 100\}$. None of these subsets satisfies $u = 1$, and none satisfies $u \neq 1$. So (44) cannot be the case. Hence (40) is not true, so there is no winning strategy for (39).

Lemma 33. *IF logic is* not *a conservative extension of ordinary predicate logic.*

Proof Consider the IF formulation of sentence (39):

(45) $\forall x \forall y \exists u\ [u = x \wedge \forall x[x = y \vee [u = 1 \vee u \neq 1]]]$

The previous lemma proves that there is no winning strategy for (39), hence not for (45), whereas classically (45) is valid.

8. Conclusions

Signalling is a tricky business. It disturbs several extrapolations from classical logic (change of bound variables, prenex normal form), and the interaction of signalling and implicit independence causes that Hintikka's IF is *not* a conservative extension of predicate logic.

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