EXTENSIONS OF THE AXIOM OF BLACKWELL DETERMINACY

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Abstract. We define extensions of the *Axiom of Blackwell Determinacy* in analogy to extensions of the *Axiom of Determinacy*. We prove that the *Axiom of Real Blackwell Determinacy* is strictly stronger than the *Axiom of Blackwell Determinacy*, and show that several strong variants of Blackwell determinacy are inconsistent (as is known for their perfect information analogues).

1. Introduction

Rather surprisingly, the theory of Blackwell determinacy (*i.e.*, determinacy properties of Blackwell's "games of slightly imperfect information" [$\mathrm{Bl}_0$69, $\mathrm{Bl}_0$97]) turned out to be parallel to the usual theory of perfect information (Gale-Stewart) determinacy: Martin, Neeman and Vervoort have proved in [MaNeVe03] that many axioms of Blackwell determinacy are equivalent to the corresponding axioms of perfect information determinacy. The pattern of these equivalences is so strikingly universal that Tony Martin made the following conjecture:

> For every boldface[1] pointclass $\mathbf{\Gamma}$ the following two statements are equivalent:
> (1) "Every $A \in \mathbf{\Gamma}$ is determined," and
> (2) "Every $A \in \mathbf{\Gamma}$ is Blackwell determined."

This conjecture is widely believed to be true but despite the fact that the conjecture has been proved in many cases, it is still open in its full generality, and even for the case $\mathbf{\Gamma} = \wp(\mathbb{R})$, *i.e.*, the claim that the Axiom of Determinacy AD and the Axiom of Blackwell Determinacy $\mathsf{Bl\text{-}AD}$ are equivalent. In this paper, we go beyond $\mathsf{Bl\text{-}AD}$ and start an investigation of its extensions in order to see whether the development of extensions of AD is mirrored in analogous results for extensions of $\mathsf{Bl\text{-}AD}$.

For AD, there are extensions of two types known: long games and games with uncountable sets of possible moves. By a theorem of Blass' from [$\mathrm{Bl}_1$75], these two types of extensions are connected: the Axiom of Determinacy for

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[1]A pointclass $\mathbf{\Gamma} \subseteq \wp(\mathbb{R})$ is called **boldface** if it is closed under continuous preimages, *i.e.*, for every continuous function f and every $A \in \mathbf{\Gamma}$, we have that $f^{-1}[A] \in \mathbf{\Gamma}$.

games of length ω^2 and the Axiom of Determinacy for real number moves are equivalent.[2]

For Blackwell determinacy, in this paper we shall only explore the extensions by allowing uncountable sets X of possible moves, and leave the very interesting topic of long Blackwell games for future research.

As it turns out, we have to be careful with the definitions here, as the mere definition of Blackwell determinacy presupposes the existence of a measure on the Borel σ-algebra of X^ω which in turn requires some fragment of the Axiom of Choice (*cf.* § 2). The central argument of this paper uses the fact that in the relevant cases choice games (games whose winning strategies induce choice functions) are essentially finite. This is discussed in detail in § 3.

The following sections, §§ 4 and 5 contain our main applications. In § 4, we discuss the Axiom of Real Blackwell Determinacy $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$. Using the choice games of § 3, we are able to prove that $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ is strictly stronger than $\mathsf{Bl\text{-}AD}$. Finally, in § 5, we use the techniques of § 3 to limit possible extensions of $\mathsf{Bl\text{-}AD}$. Similar to the well-known limitations to extending AD, we can prove that $\mathsf{Bl\text{-}AD}_{\wp(\mathbb{R})}$ and $\mathsf{Bl\text{-}AD}_{\omega_1{}^\omega}$ are inconsistent with ZF. It is currently still open whether $\mathsf{Bl\text{-}AD}_{\omega_1}$ is inconsistent.

2. Definitions

Throughout we shall work in the theory $\mathsf{ZF}+\mathsf{AC}_\omega(\mathbb{R})$. This small fragment of the axiom of choice is necessary for the definition of axioms of Blackwell determinacy. As customary in set theory, we write $\mathbb{R}$ for ω^ω and $[X]^\kappa$ for the set of subsets of X of cardinality κ.

We shall not go into details of the motivation and definition of the original Axiom of Blackwell Determinacy $\mathsf{Bl\text{-}AD}$ here and refer the reader to [Ma98], [Lö02a], [Lö02b], and [Lö04]. Our definition is equivalent to Vervoort's original definition in [Ve95]; *cf.* [Lö04, Theorem 2.5 (b)].

We want to define an extension of the Axiom of Blackwell Determinacy for an arbitrary set X of possible moves. Let us denote by X^{Even} the set of finite sequences of elements of X of even length, by X^{Odd} the set of such sequences of odd length, and by $\mathrm{Prob}(X)$ the set of probability measures on X with countable support, *i.e.*, those measures μ on X such that there is a countable subset $Y \subseteq X$ such that $\mu(Y) = 1$. The restriction to measure with countable support is crucial here and is discussed briefly in [MaNeVe03, Remark 3.1].

We call a function $\sigma : X^{\mathrm{Even}} \to \mathrm{Prob}(X)$ a **(mixed) strategy for player I** and a function $\sigma : X^{\mathrm{Odd}} \to \mathrm{Prob}(X)$ a **(mixed) strategy for player II**.

We now understand X as a topological space with the discrete topology and take the product topology on X^ω, *i.e.*, the topology whose basic open

[2] *Cf.* [LöRo02] for an improvement of this equivalence to games of length $\omega \cdot 2$ based on a theorem of Woodin's.

sets are of the form

$$[s] := \{x \in X^\omega\,;\, s \subseteq x\}$$

for some $s \in X^{<\omega}$. Given mixed strategies σ and τ, let

$$\nu(\sigma,\tau)(s) := \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even, and} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then for any $s \in X^{<\omega}$, we can define

$$\mu_{\sigma,\tau}([s]) := \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s{\restriction}i)(\{s_i\}).$$

This defines a function on the basic open sets of our topological space X^ω. We would like to extend it to the Borel σ-algebra on X^ω, denoted by $\mathfrak{B}(X^\omega)$. However, we have to watch out that we don't use more choice than we have at our disposal. The distinction between *Borel sets* and *Borel codes* is at the heart of this subtle problem: while the definition of $\mu_{\sigma,\tau}$ extends naturally to Borel codes, extending it to Borel sets requires some choice.

A tree $T \subseteq \omega^{<\omega}$ is called **wellfounded** if it has no infinite branches, it is called **labelled** if there is a function mapping its leaves (terminal nodes) to elements of $X^{<\omega}$. The wellfounded labelled trees $\mathbf{T} = \langle T, \ell\rangle$ are called X^ω**-Borel codes**, and we can recursively read off the definition of a Borel set from a Borel code. Note that we can code such a $\mathbf{T}$ by an element of $\mathbb{R} \times X^\omega$. If $\mathbf{T}$ is an X^ω-Borel code, we let $\mathrm{B}_\mathbf{T}$ be the Borel subset of X^ω defined by $\mathbf{T}$ in the usual way. We write $\mathrm{BC}(X^\omega) := \{\mathrm{B}_\mathbf{T}\,;\, \mathbf{T} \text{ is an } X^\omega\text{-Borel code}\}$. Given the function $\mu_{\sigma,\tau}$ defined on the basic open sets, it naturally extends to $\mathrm{BC}(X^\omega)$.[3]

Proposition 1. Assume $\mathsf{AC}_\omega(\mathbb{R} \times X^\omega)$. Then $\mathrm{BC}(X^\omega) = \mathfrak{B}(X^\omega)$.

Proof. We only have to show that BC is closed under countable unions. Let $A_n \in \mathrm{BC}(X^\omega)$, so $C_n := \{\mathbf{T}\,;\, \mathrm{B}_\mathbf{T} = A_n\}$ is non-empty. By the above remark, this is essentially a countable family of subsets of $\mathbb{R}\times X^\omega$. Therefore, $\mathsf{AC}_\omega(\mathbb{R} \times X^\omega)$ allows us to simultaneously pick $\mathbf{T}_n \in C_n$ and then construct a Borel code for $\bigcup_{n\in\omega} A_n$. □

In the following, we let WO be the set of all reals coding a wellorder, and WO_α be the set of all reals coding a wellorder of length α. We let W be the set $\{\mathrm{WO}_\alpha\,;\, \alpha \in \omega_1\}$ (a set of sets of reals in bijection with the ordinal ω_1).

Proposition 2. The following fragments of the Axiom of Choice are provable in our base theory:

(1) $\mathsf{AC}_\omega(\mathbb{R} \times \mathbb{R}^\omega)$,

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[3] It is useful to keep in mind that the statement "Lebesgue-measure is σ-additive" is a fragment of the Axiom of Choice that can be false in some models of ZF. In the "Consequences of the Axiom of Choice" project [HoRu98], it is Form 37; *cf.* also [KeTa03]. For the case of $\mathsf{Bl\text{-}AD}_\mathbb{R}$, de Kloet [dK05] has a proof that $\mu_{\sigma,\tau}$ extends to the Borel σ-algebra that doesn't mention Borel codes, but follows more traditional measure-theoretic ideas.

(2) $\mathsf{AC}_\omega(\mathbb{R} \times {\omega_1}^\omega)$,
(3) $\mathsf{AC}_\omega(\mathbb{R} \times (\mathrm{W} \cup \mathbb{R})^\omega)$,
(4) $\mathsf{AC}_\omega(\mathbb{R} \times (\omega_1 \cup \mathbb{R})^\omega)$, and
(5) $\mathsf{AC}_\omega(\mathbb{R} \times ({\omega_1}^\omega)^\omega)$.

Proof. Clearly, if there is a surjection from X to Y, then $\mathsf{AC}_\omega(X)$ implies $\mathsf{AC}_\omega(Y)$. Based on this, it is easy to see that all of our claims follow if we can construct a surjection from $\mathbb{R}$ onto ${\omega_1}^\omega$:

Fix a surjection $\pi : \mathbb{R} \to \omega_1$ and a bijection $\ulcorner\cdot,\cdot\urcorner : \omega \times \omega \to \omega$. We shall now define a surjection $\hat{\pi} : \mathbb{R} \to {\omega_1}^\omega$. As usual, for $n \in \omega$ and $x \in \mathbb{R}$, we define

$$(x)_n(m) := x(\ulcorner n, m\urcorner)$$

splitting up the real number x into countably many reals. We define $\hat{\pi}(x)(n) := \pi((x)_n)$.

Let us show that $\hat{\pi}$ is a surjection: if $f \in {\omega_1}^\omega$, let us use $\mathsf{AC}_\omega(\mathbb{R})$ in order to simultaneously get a family of $x_n \in \mathrm{WO}_{f(n)}$. Let $x(\ulcorner n, m\urcorner) := x_n(m)$. Then $\hat{\pi}(x) = f$. □

Since we ensured that we have enough choice available to extend $\mu_{\sigma,\tau}$ to all Borel subsets of X^ω, we can continue with our definition of Blackwell determinacy for $X \in \{\mathbb{R}, \omega_1, \mathrm{W} \cup \mathbb{R}, \omega_1 \cup \mathbb{R}, {\omega_1}^\omega\}$:

Given a Borel probability measure μ on $\mathfrak{B}(X^\omega)$, we denote outer and inner measure in the usual sense with μ^+ and μ^-, *i.e.*, $\mu^-(A) = 1$ if there is a Borel set $B \subseteq A$ such that $\mu(B) = 1$, and $\mu^+(A) = 0$ if there is a Borel set $B \supseteq A$ such that $\mu(B) = 0$.

Given a mixed strategy σ for player I, and a mixed strategy τ for player II, we say that σ is **optimal** for the payoff set $A \subseteq X^\omega$ if for all strategies τ_* for player II, $\mu^-_{\sigma,\tau_*}(A) = 1$, and similarly, we say that τ is **optimal** for the payoff set $A \subseteq X^\omega$ if for all strategies σ_* for player I, $\mu^+_{\sigma_*,\tau}(A) = 0$.

For $X \in \{\mathbb{R}, \omega_1, \mathrm{W} \cup \mathbb{R}, \omega_1 \cup \mathbb{R}, {\omega_1}^\omega\}$, we call a set $A \subseteq X^\omega$ **Blackwell determined** if either player I or player II has an optimal strategy. Now we can define $\mathsf{Bl\text{-}AD}_X$ to mean "Every $A \subseteq X^\omega$ is Blackwell determined".

3. Finite Blackwell Games

If $A \subseteq \omega^\omega$, σ is a pure strategy for player I and τ a winning strategy for player II in the game on $\omega^\omega \backslash A$, then there is a definable function

$$\hat{\tau} : \sigma \mapsto \sigma * \tau$$

such that $\mathrm{ran}(\hat{\tau}) \subseteq A$. It is discussed in [Lö∞] that the lack of a result like this is the difference between AD and $\mathsf{Bl\text{-}AD}$, and the non-existence of such a definable function is the main obstacle in proving Martin's conjecture.

Now, matters change if we are looking at Blackwell games with a finite length: By the *von Neumann Minimax Theorem*, finite Blackwell games can be solved by backwards induction like perfect information games and give rise to a definable function. We exploit this idea in the following:

Lemma 3. If μ is a probability measure on X with countable support then there is a unique $\mathbf{m}_\mu > 0$ such that

$$\{x\,;\,\mu(\{x\}) = \mathbf{m}_\mu\} \text{ is nonempty and finite, and}$$
$$\{x\,;\,\mu(\{x\}) > \mathbf{m}_\mu\} = \varnothing.$$

Proof. For $\varepsilon > 0$, let $X_\varepsilon := \{x\,;\,\mu(\{x\}) \geq \varepsilon\}$. By σ-additivity, it is impossible that any set X_ε is infinite. Now pick $z \in \mathrm{supp}(\mu)$ and let $\varepsilon := \mu(\{z\})$, so that X_ε is finite and nonempty. Define

$$\mathbf{m}_\mu := \max\{\mu(\{x\})\,;\,x \in X_\varepsilon\}.$$

□

Theorem 4. Let X be linearly ordered by $\preceq$, $n \in \mathbb{N}$, $A \subseteq X^n$ and τ be a optimal strategy for player II in the game on $X^n \backslash A$. Then there is a definable function $\hat{\tau}$ defined on the mixed strategies such that $\mathrm{ran}(\hat{\tau}) \subseteq A$.

Proof. Given a mixed strategy σ, we define recursively

$$x_{2i} := \min_{\preceq}\{x\,;\,\sigma(\langle x_0,\ldots,x_{2i-1}\rangle)(\{x\}) = \mathbf{m}_{\sigma(\langle x_0,\ldots,x_{2i-1}\rangle)}\}, \text{ and}$$
$$x_{2i+1} := \min_{\preceq}\{x\,;\,\tau(\langle x_0,\ldots,x_{2i}\rangle)(\{x\}) = \mathbf{m}_{\tau(\langle x_0,\ldots,x_{2i}\rangle)}\}.$$

By Lemma 3, the minimum is taken over a nonempty finite linearly ordered set, so this sequence is definable in ZF. We let $\hat{\tau}(\sigma)$ be the sequence $\langle x_0,\ldots,x_{n-1}\rangle$ and claim that for all σ, $\hat{\tau}(\sigma) \in A$:

Clearly, $\mathbf{m} := \mu_{\sigma,\tau}(\{\langle x_0,\ldots,x_{n-1}\rangle\})$ is the finite product of the positive numbers $\mathbf{m}_{\sigma(\langle x_0,\ldots,x_{2i-1}\rangle)}$ and $\mathbf{m}_{\tau(\langle x_0,\ldots,x_{2i}\rangle)}$, and therefore strictly positive. If $\langle x_0,\ldots,x_{n-1}\rangle \notin A$, then $\mu_{\sigma,\tau}(X^n\backslash A) \geq \mathbf{m}$, contradicting the optimality of τ. □

For sets Y and Z, we define the class of Y-Z-choice games $\mathrm{CG}_{Y,Z}(A)$ as follows: If $A : Y \to \wp(Z)$ is a family of nonempty subsets of Z indexed by elements of Y, then the game $\mathrm{CG}_{Y,Z}(A)$ is the two-round game in which player I plays an element $y \in Y$, player II follows up with playing an element $z \in Z$, and player II wins if $z \in A(y)$.

Theorem 5. If $X := Y \cup Z$ is linearly ordered and $\mathsf{Bl\text{-}AD}_X$ (is defined and) holds, then $\mathsf{AC}_Y(Z)$ holds.[4]

Proof. Let $A : Y \to \wp(Z)$ be a family of nonempty sets. Because the sets are nonempty, player I cannot have an optimal strategy in $\mathrm{GC}_{Y,Z}(A)$. Let τ be an optimal strategy for player II. Let σ_y be defined by $\sigma_y(\varnothing)(\{y\}) := 1$. Then

$$f : y \mapsto \hat{\tau}(\sigma_y)$$

is a definable choice function by Theorem 4. □

[4]The requirement that $\mathsf{Bl\text{-}AD}_X$ be defined is just a reminder that the definition of $\mathsf{Bl\text{-}AD}_X$ needs a fragment of the axiom of choice, as we want to extend $\mu_{\sigma,\tau}$ to the Borel σ-algebra on X^ω. This will become relevant in the discussion before Corollary 11.

4. The Axiom of Real Blackwell Determinacy

In the perfect information context, the most famous extension of AD is the Axiom of Real Determinacy $\mathsf{AD}_{\mathbb{R}}$. The metatheoretical investigation of $\mathsf{AD}_{\mathbb{R}}$ was commenced in the short but seminal [So78] where Solovay proved (among many other things) that $\mathsf{AD}_{\mathbb{R}}$ has strictly higher consistency strength than AD [So78, Theorem 5.10, 1.].

In this section, we shall be proving the analogues of the most basic results from Solovay's paper for $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$:

Theorem 6. $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ proves $\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$.

Proof. In Theorem 5, let $Y := Z := \mathbb{R}$. □

Now this can be used to show that $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ is strictly stronger than $\mathsf{Bl\text{-}AD}$ (if they are consistent):

Corollary 7. If $\mathsf{Bl\text{-}AD}$ is consistent, then $\mathsf{Bl\text{-}AD} \not\vdash \mathsf{Bl\text{-}AD}_{\mathbb{R}}$.

Proof. This proof follows closely Solovay's proof of the analogous theorem for perfect information determinacy:

By an easy diagonalization argument, we see that the family $U : \mathbb{R} \to \wp(\mathbb{R})$ defined by $U(x) := \{y\,;\, y \text{ is not ordinal definable from } x\}$ cannot have choice function which is ordinal definable from a real. If any of the sets in the family U is empty, then all of the reals are ordinal definable from some real, and hence the reals would be wellorderable. Therefore, under $\mathsf{Bl\text{-}AD}$, U is a family of nonempty sets. But by the above, $\mathbf{L}(\mathbb{R})$ cannot have a choice function for U, so $\mathbf{L}(\mathbb{R}) \models \mathsf{Bl\text{-}AD} + \neg\mathsf{AC}_{\mathbb{R}}(\mathbb{R})$. Consequently, by Theorem 6, $\mathbf{L}(\mathbb{R}) \models \mathsf{Bl\text{-}AD} + \neg\mathsf{Bl\text{-}AD}_{\mathbb{R}}$. □

Corollary 7 establishes that $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ is not a consequence of $\mathsf{Bl\text{-}AD}$, but doesn't say anything about their consistency strengths. The most promising route to prove that the consistency strength of $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$ is higher than that of $\mathsf{Bl\text{-}AD}$ is to follow Solovay's footsteps. There are partial results along this line: Blackwell determinacy yields Solovay objects (*i.e.*, ultrafilters on $\wp_{\omega_1}(\mathbb{R})$; [dKKiLö∞]) that then witness that $\aleph_1$ is κ-strongly compact (under $\mathsf{Bl\text{-}AD}$) or κ-supercompact (under $\mathsf{Bl\text{-}AD}_{\mathbb{R}}$) for all $\kappa < \Theta$. These can then be used to complete the analogues of Solovay's arguments from [So78].

5. Inconsistent Extensions

As mentioned, the most well-known demarcations for the theory of perfect information determinacy are the inconsistency results for AD_{ω_1} and $\mathsf{AD}_{\wp(\mathbb{R})}$ [My63].

We shall now use the techniques of § 3 to give similar demarcation lines for axioms of Blackwell determinacy. Note that the standard arguments for the inconsistency of AD_{ω_1} and $\mathsf{AD}_{\wp(\mathbb{R})}$ go through the perfect set property of all sets of reals which in turn implies that there are no uncountable wellordered subsets of the reals. Since it is still unknown whether $\mathsf{Bl\text{-}AD}$ (or any of its extensions) proves that all sets of reals have the perfect set property, we have to use a different argument to make sure that there are no uncountable wellordered sets of reals:

Lemma 8. Bl-AD implies that there is no injection from ω_1 into the reals.

Proof. Under the assumption of Bl-AD, the cardinal ω_1 has the strong partition property [Lö04, Theorem 4.15], *i.e.*, every colouring of $[\omega_1]^{\omega_1}$ with two colours has a homogenous subset.

Suppose for a contradiction that $X \subseteq \mathbb{R}$ has cardinality ω_1, in particular, let $\pi : X \to \omega_1$ be a bijection. We shall define a colouring of $[X]^{\omega_1}$ that has no homogeneous subset which finishes the proof. For $A \subseteq X$, let $\pi_\alpha(A)$ be the αth element of A in the order induced by π and the natural order on ω_1.

We define

$$\chi(A) = 0 \iff \pi_0(A) < \pi_1(A)$$
$$\chi(A) = 1 \iff \pi_0(A) > \pi_1(A)$$

where $<$ is the ordinary ordering on the reals. Now we claim that χ cannot have an uncountable homogeneous set. Suppose that $H \subseteq X$ is uncountable (hence of cardinality ω_1), and without loss of generality assume that $\pi_0(H) < \pi_1(H)$, so $\chi(H) = 0$. Since ω_1 cannot be order-preservingly embedded into the reals (and hence not into H), there must be $h, h^* \in H$ such that $\pi(h) < \pi(h^*)$ but $h > h^*$. Let $H^* := \{h\} \cup \{x \in H\,;\, \pi(x) \geq \pi(h^*)\}$. Then $H^* \in [H]^{\omega_1}$, but $\chi(H^*) = 1$, so H was not homogeneous for χ. □

We can now exploit Theorem 4 to get inconsistency results:

Theorem 9. $\mathsf{Bl\text{-}AD}_{\omega_1 \cup \mathbb{R}}$ and $\mathsf{Bl\text{-}AD}_{\mathrm{W} \cup \mathbb{R}}$ are inconsistent.

Proof. Since $\omega_1 \cup \mathbb{R}$ and $\mathrm{W} \cup \mathbb{R}$ are in bijection, the two theories are obviously equivalent. Note that $\omega_1 \cup \mathbb{R}$ can be linearly ordered in ZF, so that we can apply Theorem 4.

Now, by Theorem 5, we get $\mathsf{AC}_{\omega_1}(\mathbb{R})$, and have a choice function for the set W which is an injection from ω_1 into the reals, contradicting Lemma 8. □

Corollary 10. $\mathsf{Bl\text{-}AD}_{{\omega_1}^\omega}$ is inconsistent.

Proof. As $\omega_1 \cup \mathbb{R}$ canonically embeds into ${\omega_1}^\omega$ as a subset, this follows immediately from Theorem 9. □

We would like to go on and apply the reasoning of the proof of Corollary 10 to the axiom $\mathsf{Bl\text{-}AD}_{\wp(\mathbb{R})}$, but we have to be cautious here. Notice that we didn't define $\mathsf{Bl\text{-}AD}_{\wp(\mathbb{R})}$ in § 2: in order to show that the definition of $\mu_{\sigma,\tau}$ extends to the Borel σ-algebra on X^ω, we needed the appropriate amount of choice which we got from a surjection from $\mathbb{R}$ onto $\mathbb{R} \times X^\omega$ (Proposition 2). Clearly, if $X = \wp(\mathbb{R})$, such a surjection doesn't exist.

There are two ways to remedy this problem, both of them axiomatic:

1. Since we are only interested in one particular finite game that will yield the contradiction (player I will be playing elements of W and player II will be playing sets $\{x\}$ for $x \in \mathbb{R}$), we could just be more careful with our definitions of what we mean by "optimal" and replace the rôle of the Borel σ-algebra by the set $\mathrm{BC}(X^\omega)$ of decoded Borel codes. This is not necessarily a σ-algebra, and therefore $\mu_{\sigma,\tau}$ might not extend to a measure, but rather to a premeasure on the ring BC. Then we would define optimality of a strategy with respect to BC instead of the Borel σ-algebra. For the game that we shall use to derive the contradiction, this difference is irrelevant.

2. We can just assume that $\mathsf{AC}_\omega(\wp(\mathbb{R}))$ is part of the definition of $\mathsf{Bl\text{-}AD}_{\wp(\mathbb{R})}$. This way, we ensure that BC and the Borel σ-algebra on $(\wp(\mathbb{R}))^\omega$ coincide, and that $\mu_{\sigma,\tau}$ extends to a measure.

In the following, the reader can choose whether he or she wants to read $\mathsf{Bl\text{-}AD}_{\wp(\mathbb{R})}$ according to **1.** or **2.**:

Corollary 11. $\mathsf{Bl\text{-}AD}_{\wp(\mathbb{R})}$ is inconsistent.

Proof. As in the proof of Corollary 10, we just observe that $\mathrm{W}\cup\mathbb{R}$ canonically embeds into $\wp(\mathbb{R})$ as a subset (identifying $\mathbb{R}$ as the subset of the singletons on $\wp(\mathbb{R})$), and then apply Theorem 9. □

Our methods do not allow to prove the inconsistency of $\mathsf{Bl\text{-}AD}_{\omega_1}$, as the game that would give a set of reals of cardinality ω_1 (and thus yield the contradiction) is an infinite Blackwell game, and thus Theorem 4 is not applicable. We close with the following open problem and conjecture:

Conjecture 12. $\mathsf{Bl\text{-}AD}_{\omega_1}$ is inconsistent.

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