Brouwer's incomplete objects

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Abstract. The theory of the idealized mathematician has been developed to formalize a method that is characteristic for Brouwer's papers after 1945. The method has been supposed to be radically new in his work. We replace the standard theory about this method by, we think, a more satisfactory one. We do not use an idealized mathematician. We claim that it is the systematic application of incomplete sequences, already introduced by Brouwer in 1918, that makes the method special. An investigation of earlier work by Brouwer (including an unpublished lecture in Geneva of 1934) in our opinion fully supports our position and shows that the method was not at all new for him.

Résumé. La théorie du mathématicien idéal a été développée pour formaliser une méthode caractéristique des travaux de Brouwer postérieurs à 1945. On a supposé que cette méthode représente une nouveauté importante. Nous en proposons une nouvelle théorie qui, croyons-nous, est plus adéquate que celle couramment acceptée. Nous n'y utilisons pas l'idée du mathématicien idéal, mais plutôt avanons que c'est l'application systématique des séquences incomplètes, déjà introduites par Brouwer en 1918, qui rend cette méthode particulire. Selon nous, un examen des travaux antérieurs de Brouwer (incluant les notes inédites d'un cours donné Genève en 1934) confirme notre thèse et montre que cette méthode n'était pas du tout nouvelle pour lui.

1 Introduction

In his papers after 1945 Brouwer applied a method which has been supposed to be for him a radically new approach. Characteristic is Heyting's comment when Kreisel presented his formalization of the method: "It is true that Brouwer in his lecture introduced an entirely new idea for which the subject is of essential importance"[1]. In this paper we want to show that the subject is no more essential in the method then in any other intuitionistic proof, and that it was not new at all.

We start in section 2 with a description of the standard theory for the reconstruction of the method, the theory of the idealized mathematician. In section 3 we discuss what we think are two of its shortcomings. First, the theory does not explain Brouwer's argument satisfactorily. Second, the notion of the idealized mathematician is very problematic, if not paradoxical. In section 4 we present our reconstruction. We do not use an idealized mathematician, only a principle of reasoning about the future that we think is obvious. We state that it is not the introduction of a subject, but the application of incomplete objects, also known as choice sequences, that makes the method special and interesting.

The above material has been discussed previously in our [Nie87]. Two sources opened up after this publication we think are of importance for the subject treated here. We shall discuss them after a brief introduction of incomplete sequences in section 5. The first, see section 6, is the text of Brouwer's 1927 Berlin lecture, [Bro91]. This work contains Brouwer's first examples with incomplete objects. We claim that these examples are the same ones as he used in this supposedly new method from after 1945. The second is [Bro], a manuscript from a lecture of 1934 that has remained unpublished, see section 7. Brouwer is very explicit about the special character of incomplete objects in [Bro], more than at any other place in his work. We think the text fully supports our position.

2 The Theory of the Creative Subject

After a break of more than fifteen years Brouwer started to publish again in 1948. In the first paper of this new period, which is [Bro48], he defines a real number for which he proves that it is different from 0, but not apart from 0. The definition runs as follows:

> Let α be a mathematical assertion that cannot be tested, i.e. for which no method is known to prove either its absurdity or the absurdity of its absurdity. Then the creating subject can, in connection with this assertion α, create an infinitely proceeding sequence $a_1, a_2, a_3, \ldots$ according to the following direction: As long as, in the course of choosing the a_n, the creating subject has

[1] See [Lak67], p. 173. The lecture Heyting refers to is Consciousness, Philosophy and Mathematics, [Bro75], pp. 480–495.

> experienced neither the truth, nor the absurdity of α, a_n is chosen equal to 0. However, as soon as between the choice of a_{r-1} and a_r the creating subject has obtained a proof of the truth of α, a_r as well as a_{r+v} for every natural number v is chosen equal to 2^{-r}. And as soon as between the choice of a_{s-1} and a_s the creating subject has experienced the absurdity of α, a_s, as well as a_{s+v} for every natural number v is chosen equal to $-(2)^{-s}$. This infinitely proceeding sequence $a_1, a_2, a_3, \ldots$ is positively convergent, so it defines a real number ρ ([Bro48]).

Although Brouwer stated in the introduction of the paper that he had used this example in his lectures already from 1927 onwards, this way of defining has been supposed to be for him a radically new one. Brouwer seemingly introduced a subject into his mathematical practice, and he used his activity to define a sequence. In this manner it was interpreted in the reconstruction of Kreisel ([Kre67]), which was further elaborated by Myhill ([Myh68]), and especially by Troelstra ([Tro69]). In the resulting theory the expression *creating* subject was modified into *creative* subject and this creative subject was taken to be an idealized mathematician, for short IM, all of its mathematical activities supposedly covered by a discrete sequence of ω stages. The key notion in this theory of the creative subject, for short TCS, is: *the creative subject has evidence for φ at stage n*, formally expressed by

$$I_n\varphi.$$

Analyzing the properties of an idealized mathematician then leads to the acceptance of the following axioms (n and m are natural numbers, φ can be any mathematical assertion):

(1) $I_n\varphi \vee \neg I_n\varphi$,

(2) $I_n\varphi \rightarrow I_{n+m}\varphi$,

(3) $\varphi \rightarrow \exists n I_n\varphi$,

(4) $\exists n I_n\varphi \rightarrow \varphi$.

We will not discuss these axioms, but we shall look at its consequences. We just mention the following. They can be added to the intuitionistic logic with preservation of consistency, as appears from the construction of a model by van Dalen ([Dal78]). Further, the theory has become the standard in intuitionistic research; it has been applied by recent authors on the subject, e.g. Dummett ([Dum00]) and van Atten ([Att04]), for the reconstruction of Brouwer's supposedly new method.

With the TCS Brouwer's result can be obtained as follows. Let A be an undecided proposition, i.e. neither A nor $\neg A$ is known. We define an infinite sequence $a_1,\ a_2,\ a_3 \ldots,$ by

$$a_n = 0 \ \leftrightarrow\ (\neg I_n A \wedge \neg I_n \neg A),$$
$$a_n = 2^{-m} \ \leftrightarrow\ (m < n \wedge \neg I_m A \wedge I_{m+1} A),$$
$$a_n = -2^{-m} \ \leftrightarrow\ (m < n \wedge \neg I_m \neg A \wedge I_{m+1} \neg A).$$

The sequence $a_1, a_2, a_3, \ldots$ defines a real number, say ρ. If for this real number $\rho > 0$ were to hold, than A would hold as well because of

$\rho > 0 \rightarrow \exists n I_n A,$ by the definition of ρ, and
$\exists n I_n A \rightarrow A,$ by (4).

Since A is undecided, A does not hold, so $\rho > 0$ does not hold either. Analogously, $\rho < 0$ does not hold, because then $\neg A$ would follow. So ρ is not apart from 0. But since

$\rho = 0 \rightarrow \neg \rho > 0$
$\neg \rho > 0 \rightarrow \neg \exists n I_n A,$ by the definition of ρ, and
$\neg \exists n I_n A \rightarrow \neg A,$ by the contraposition of (3)

$\rho = 0 \rightarrow \neg A$ holds. Analogously we have $\rho = 0 \rightarrow \neg\neg A$, and consequently $\rho = 0 \rightarrow (\neg A \wedge \neg\neg A)$. So $\rho \neq 0$ does hold; apartness and equality are not equivalent, concluding the derivation of Brouwer's result.

For any mathematical assertion A we can define in a similar way as above $(a(n) = 0 \leftrightarrow \neg I_n A)$ and $(a(n) = 1 \leftrightarrow I_n A)$. This results in the axiom scheme which is known as Kripke's Scheme:

KS $\exists a(\forall n(a(n) = 0 \vee a(n) = 1) \ \wedge\ \exists n(a(n) \neq 0 \leftrightarrow A)).$

KS is often accepted as a reasonable principle. It is strong enough to derive most of Brouwer's counterexamples and for that reason it is sometimes added to the intuitionistic logic instead of the axioms of the TCS. Its advantage is that it does not explicitly refer to an idealized mathematician.

As stated in the introduction, we do not accept the standard reconstruction embodied in the TCS, and neither do we see any basis in Brouwer's work accepting the principle KS. We shall explain this in the next section.

3 The theory of the creative subject-II

A striking feature of the reconstruction of section 2 is that the untestedness in Brouwer's definition of ρ is not used and not needed; undecidability seems to be sufficient. Inspecting Brouwer's original proof shows that his argument for $\rho \neq 0$ is the same as in the reconstruction, but that his argument for $\rho > 0$ *does not hold* is different. It runs as follows:

> If for this real number ρ the relation $\rho > 0$ were to hold, then $\rho < 0$ would be impossible, so it would be certain α could never be proved to be absurd, so the absurdity of the absurdity of α would be known, so α would be tested, which it is not. Thus the relation $\rho > 0$ does not hold ([Bro75], pp. 478–479).

This reasoning to conclude testedness from $\rho > 0$ can be expressed in language of the TCS by

$$\begin{aligned}
&\rho > 0 \rightarrow \neg\rho < 0, \\
&\neg\rho < 0 \rightarrow \neg\exists n I_n \neg A, \quad \text{by the definition of } \rho \text{, and} \\
&\neg\exists n I_n \neg A \rightarrow \neg\neg A, \quad \text{by the contraposition of (3).}
\end{aligned}$$

As we may observe, Brouwer's reasoning is more complicated than the reconstruction of the TCS in section 2. The reason is that Brouwer does not use (4) here. Neither he does in his proof for $\rho < 0$, which is analogously, nor in his proof for $\rho \neq 0$, which is the same as in the TCS, see section 2. The use of (4) would simplify his argument, and he would not have to resort to an untested proposition, but could have used an undecided one. As it seems to us, he does not want to use (4).

But the reason that the TCS is widely considered to be controversial is another one. The question is whether the activity of the idealized mathematician ought to be applied in mathematics. To demonstrate the difficulties involved we show a paradox discovered by Troelstra ([Tro69], pp. 105-107).

Suppose that the CS proves his results one by one. By sufficiently narrowing the stages this amounts to one new result by the CS at each stage. Let $A_0, A_1, A_2, \ldots$ be the list of new results. We now define a predicate $L(\alpha)$ such that $L(\alpha)$ holds if and only if α is a lawlike sequence. We define a sequence β such that

$$\beta(n) = \alpha(n) + 1 \;\leftrightarrow\; A_n = L(\alpha) \text{ for some } \alpha$$
$$\beta(n) = 0 \text{ otherwise.}$$

Now Troelstra argues that intuitively β is lawlike, since it is determined by some fixed recipe, so $L(\beta)$ holds. Because of (4) we have $\exists n I_n L(\beta)$, so for some n_0, $I_{n_0} L(\beta)$ holds. But then $\beta(n_0) = \beta(n_0) + 1$, which is a contradiction.

Troelstra discusses two ways out. The first is to drop the condition of one new result at each stage; the second is to bring onto the stages a type structure of levels of selfreflection. He judges neither of them satisfactory, and he concludes that "the attempts to formalize the theory of the IM as envisaged by Brouwer cannot be said to be satisfactory examples of "informal rigour"". ([DT88], p. 846).

An argument for taking β to be lawlike may be that in the TCS the stages seem to have a definite description, expressed by (1). But in the intuitionistic interpretation, for a disjunction to hold we need a proof of one of the disjunctive parts. In the case of β this seems not evident to us.

Let us return to Brouwer's original use of *creating subject*. Let us interpret it as ourselves and let the stages cover our future. We can define β as above. Then its values depend on our future results. We have no way to determine these values, other than going in time to these stages, which are not specified at all. We think decidability is questionable, and we do not want to call this sequence lawlike.

Our conclusion of this section is that (1) and (4) are problematic. In the next section we shall look for principles for the reconstruction of Brouwer's argument, interpreted as we did above. We shall see that our analogues of these two, turn out to be not valid.[2]

[2] Just before submitting this paper we discovered to what consequences an intuitionistic interpretation of the TCS may lead. Suppose A is an undecided proposition. Because of (4) there can be no n for which $I_n A$ holds. Hence, because of (1), for each n we have $\neg I_n A$. Analogously for each n we have $\neg I_n \neg A$. So the values of Brouwer's sequence remain 0, conflicting his result that $\rho \neq 0$ holds. Note that (1) and (4) are the cause of the trouble.

4 A new reconstruction

If we interpret the expression *creating subject* as ourselves, the sequence $a_1, a_2, a_3, \ldots$ from the definition in section 2 is a sequence that we *may* construct. We then start by choosing $a_1 = 0$ and we let $a_n = a_{n+1}$ with one exception. If we find a proof of A between the choice of a_{r-1} and a_r we choose $a_{r+v} = 2^{-r}$ for every v; and if we find a proof of A between the choice of a_{r-1} and a_r we choose $a_{r+v} = -2^{-r}$ for every v.

The values of the sequence now depend on our future mathematical results. We want intuitionisticaly valid principles to reason about them. For our basic term we shall use a G instead of an I. The G is used in tense logic to express "it is going to be the case that", and we shall use it similarly.

We suppose our future to be covered by a discrete sequence of ω stages, starting with the present stage as stage 0, and we define for a mathematical assertion φ

$$G_n\varphi$$

as: at the n-th stage from now we shall have a proof of φ. The introduction of this term enables us to refine the notion of proof.

In intuitionism stating φ means stating to have a proof of φ. We now demand of such a proof that it can be carried out *here and now*, i.e. all information for the proof is available at the present stage. If future information is involved we use $G_n\varphi$. A proof of $G_n\varphi$ may depend on information coming free before stage n. Of course $G_n\varphi$ may also hold because we have a proof for φ already now; we suppose a proof to remain valid. So we have (for any φ, n and m):

(5) $\varphi \rightarrow \exists n G_n\varphi$.

Of course we also have for all n and m

(6) $G_n\varphi \rightarrow G_{n+m}\varphi$.

But we can not accept the analogues of (1) and (4). That

(7) $G_n\varphi \vee \neg G_n\varphi$

is not valid for every φ and n follows immediately from letting the present stage be stage 0, which amounts to $\varphi \leftrightarrow G_0\varphi$, and from the undecidability of intuitionistic logic. That

(8) $\exists n G_n\varphi \to \varphi$

is not valid can we see as follows. Let B be an undecided proposition. We define a sequence $b_1, b_2, b_3 \ldots$ as follows. We choose $b_n = 0$ until we have found a proof for $B \vee \neg B$, after which we keep $b_n = 1$.

Take φ to be $(b_{n_0} = 0 \vee b_{n_0} = 1)$. We do not have $(b_{n_0} = 0 \vee b_{n_0} = 1)$ for any n_0, because this would mean that we we would have one of the disjuncts, and so we would already know now, whether we shall have a proof of $B \vee \neg B$ at the time of the choice of b_{n_0}. On the other hand, we do have $G_{n_0+1}(b_{n_0} = 0 \vee b_{n_0} = 1)$, i.e. we do have $G_{n_0+1}\varphi$, so (8) does not hold.

So, as extra principles above intuitionistic logic, we only have (5) and (6). But this is enough, because an inspection of Brouwer's proof, see above, shows that we only need the contraposition of (5). Thus, the untestedness of α becomes crucial.

Our basic term $G_n\varphi$ has a very natural interpretation in a certain kind of Kripke model, see [Nie87]. For these models (5) and (6) are valid formulas for every φ. But (7) is not valid, and (8) is only valid for negative formulas. In these models we have, for every n, $G_n\varphi \to \neg\neg\varphi$, but not vice versa. KS is no longer derivable: since we do not have decidability, we do not have $a(n) = 0 \vee a(n) = 1$ for the a in KS.

We interpreted the expression *creating subject* as "we", and anybody else can do the same. Brouwer's definition is a description of a construction, as any intuitionistic definition. But the construction is not completely determined. The values of the sequence under consideration depend on the mathematical experience of the maker of the sequence, the creating subject. But the activity of the creating subject is not used in the proof. The reasoning is done on the basis of the incomplete description only, before the construction has started.

In 1918 Brouwer introduced incomplete sequences, also known as choice sequences, to solve his foundational problems i.e. to reach the power of the continuum from the discrete. We claim that Brouwer is using an incomplete sequence in the example discussed above. It is not the introduction of an idealized mathematician that makes the creating subject arguments special, but the application of individual incomplete objects. We will now see in the work of Brouwer support for our position.

5 The definition of a spread

In intuitionism mathematics is a creation of the human individual. It consists of mental constructions by the individual only. Prime material for these constructions is the sequence of the natural numbers N, which has its origin in "our perception of a move of time" ([Bro81], p. 4). From the natural numbers the integers Z and rational numbers Q can be constructed. Brouwer was this far already in his thesis in 1907 ([Bro75], pp. 11–98). But at that time N and Q were actual infinite sets in Brouwer's conception, and he did not have a satisfactory method to introduce the real numbers. By 1918 he had solved his foundational problems and he had drawn the full consequences of his constructivist point of view. The sequence of natural numbers was given by its first element and a law to construct every next one. So N, and thereby Q and Z, were potentially infinite sets. And the real numbers were also introduced by a method to construct them: the notion of a *spread*. It is this notion that made the intuitionistic reconstruction of mathematics possible. We give here a slightly modified version.

The definition of a spread is founded on a countable sequence A of mathematical objects already constructed. A may for example consist of natural numbers, rational numbers or intervals of rational numbers. A spread is a *law* that regulates the construction of infinitely proceeding sequences, their terms chosen more or less freely from A. The law says, in constructing a sequence, whether a choice from A is admissible as first element, and whether it is admissible as next in an already constructed initial segment. After each admitted choice there is at least one admissible successor. A sequence constructed according to a spread law is called an element of the spread. *Such an element is generally not completely representable.*

In his originally German text Brouwer used for *spread* the word *Menge*, which is German for *set*. This may be misleading, because a spread is not defined by its elements, but provides a way to construct them. So quantifying over elements of a spread is quantifying over sequences one *can* construct. This construction does not have to be fixed beforehand completely. The terms of an element are chosen one by one, and within the limitations of the spread law, these choices are free. But at any moment of the construction this freedom can be limited further.

If a sequence is completely determined from the first term onward, we call the sequence *lawlike*. Brouwer names them *sharp*, or *fertig*, which is German for completed. If a sequence is not completely determined we call it, following Brouwer, *incomplete*. In previous papers ([Nie87] and

[Nie02]) we used for incomplete sequences the term *choice sequences*, in the tradition of Heyting and Troelstra. In this meaning it is also used below. But in the literature *choice sequence* sometimes is used for an arbitrary element of a spread, lawlike or not.

The fact that an arbitrary element of a spread is constructed term by term, with no other restriction than the spread law, leads to the *continuity principle*: if a property holds for an arbitrary element of a spread, this must be evident on the basis of a finite initial segment, and then it holds as a matter of fact for every element with the same initial segment. Brouwer applies this principle to prove the existence of uncountable powers ([Bro75], p. 160).

Let C be the spread with as founding sequence the natural numbers N, and each choice is admissible. Its elements are sequences of natural numbers. If to each element of this spread a natural number is assigned by a function f, the assignment must be done on the basis of an initial segment of each element, and any element of C sharing the same initial segment, will be assigned the same number. So f cannot be 1-1. Conversely, a 1-1 function from N to C is easily indicated. Conclusion: C has a larger power than N.

The continuity principle is not valid classically: the function f assigning a 0 to sequences of natural numbers with all the terms even, and 1 otherwise, is classically a perfect definition. In Intuitionism it is not.

These general principles of a continuum with choice sequences were the main interest in the research on choice sequences. The standard text on the subject is Troelstra's monograph [Tro77]. It contains a huge number of technical results on formal systems of certain classes of choice sequences. For these formal systems Troelstra proves elimination theorems: a sentence with quantification over choice sequences can be translated into a, in that system equivalent, sentence without parameters for choice sequences.

There are no instances of individual choice sequences in [Tro77] or in [Tro82], a study of the origin and development of choice sequences in the work of Brouwer. From his view point there is no need for them, because from his technical results he draws the conclusion that choice sequences are eliminable. They have no mathematical relevance, he states in his recent lectures, their interest is philosophical ([Tro01] p. 227).

But individual incomplete sequences do occur in the work of Brouwer. We shall show them in the next section.

6 Incomplete objects

The real numbers can be introduced by a spread, with as founding sequence an enumeration of the rational numbers. Any rational number q is admissible as first choice. A rational number q is admissible as next in an admitted initial segment $q_1, q_2, \ldots, q_n$ if $| q - q_n |< 2^{-n}$. So the elements of this spread are convergent sequences of rational numbers. Two elements (a_n) and (b_n) are *coincident* if their termwise difference converges to 0, i.e. if $\forall k \exists n \forall m > n(| a_{m+n} - b_{n+m} |< 2^{-k})$. Real numbers are introduced as equivalence classes of this relation.

All handling of real numbers is done via their generating sequences. For example, for the real numbers a and b, generated by (a_n) and (b_n), $a < b$ holds if $\exists n \exists k \forall m((b_{n+m} - a_{n+m}) > 2^{-k})$ holds.

By no means is this the only way. Brouwer had a preference to introduce the real numbers by a spread with as founding sequence an enumeration of the $\lambda^{(n)}$-intervals; a $\lambda^{(n)}$-interval is an interval of rational numbers of the form $[a \cdot 2^{-n}, (a+2) \cdot 2^{-n}]$, with a an integer. An element of this spread is a sequence with as n-th term a $\lambda^{(n)}$- interval which is contained in its predecessor. Two sequences of $\lambda^{(n)}$-intervals, Brouwer calls them *points*, are now coincident, when each interval of one of the sequences, has an interval in common with every interval of the other sequence. Real numbers, *point cores*, are again the equivalence classes of the coincidence relation. For real numbers a and b we now define $a < b$ if an interval of the generating sequence of a is lying to the left of an interval of the generating sequence of b.

There is no essential difference between these methods of introducing real numbers. They all result in the same continuum.

For real numbers Brouwer made the following distinction. The real numbers generated by lawlike sequences form the *reduced continuum*; all real numbers together, generated by lawlike or incomplete elements, form the *(full) continuum*. He mentions this distinction already in 1919 ([Bro75], p. 235) but he started to work with it from 1927. In that year Brouwer gave lectures in Berlin; the text of these lectures is in [Bro91].

Relevant for our purpose in this text is Brouwer's study of the notion of order. He shows that $<$ is not a complete order on the reduced continuum, i.e. he shows that $a = 0 \vee a < 0 \vee a > 0$ does not hold generally for the real numbers generated by lawlike sequences. In doing this he uses a technique already applied by him in 1908 ([Bro75],p. 108).[3]

[3] The citations in this section are translations by the author from the German original.

> Further, we denote with K_1 the smallest natural number n with the property that the n-th up to the $(n+9)$-th digit in the decimal expansion of π form the sequence 0123456789, and we define as follows a point r of the reduced continuum: the n-th λ-interval λ_n is a $\lambda^{(n-1)}$-interval centered around 0, as long as $n < K_1$; however, for $n \geq K_1$ λ_n is a $\lambda^{(n-1)}$-interval centered around $(-2)^{-K_1}$. The point core of the reduced continuum generated by r is neither $=0$, nor < 0 nor > 0, as long as the existence of K_1 neither has been proved nor has been proved to be absurd. So until one of these discoveries has taken place the reduced continuum is not completely ordered ([Bro91], pp. 31–32).

Note the role of time in this argument. Neither $r < 0$ nor $r > 0$ did hold for Brouwer *then and there*, because he could not give a specific natural number n_0 such that $K_1 = n_0$. Recently it has been discovered that K_1 exists and that it is even, so $r > 0$ holds *now* ([Bor98]). For the full continuum he proves that the relation $<$ is not an order at all, i.e. $a \neq 0 \rightarrow (a > 0 \vee a < 0)$ does not hold for every a. This proof is new:

> Therefore we consider a mathematical entity or species S, a property E, and we define as follows the point s of the continuum: the n-th λ-interval λ_n is a $\lambda^{(n-1)}$-interval centered around 0, as long as neither the validity nor the absurdity of E for S is known, but it is a $\lambda^{(n)}$-interval centered around 2^{-m} (-2^{-m}), if $n \geq m$ and between the choice of the $(m-1)$-th and the m-th interval a proof of the validity (absurdity) of E for S has been found. The point core belonging to s is $\neq 0$, but as long as neither the absurdity, nor the absurdity of the absurdity of E for S is known, neither > 0 nor < 0. Until one of these discoveries has taken place, the continuum can not be ordered ([Bro91], pp. 31–32).

Brouwer proofs here a stronger statement for the full continuum than he does for the reduced continuum: if a relation on a space is not an order, it cannot be a complete order. Therefore, the sequence generating s in the citation above must be an incomplete sequence.

As we remarked in section 2, Brouwer mentioned in the introduction of his 1948 example that he had used it in his lectures from 1927 onwards. We think it cannot be anything else than that he referred to this example above; there is no other candidate in [Bro91], nor is there in the other texts of lectures we shall discuss below. Note that Brouwer uses *we* in his definition, conform our interpretation of the *creating subject*.

This example has not played a role in the discussion about the creating subject arguments: it was not published until 1991. But a year later

Brouwer gave similar examples in a lecture in Vienna, and this text was published in 1930, see [Bro30].

Between Berllin and Vienna he had generalized the technique based on the expansion of π as used in [Bro91], by introducing the notion of *a fleeing property for natural numbers.* It satisfies the following conditions: for each natural number it is decidable whether it possesses f or not, no natural number possessing f is known, and the assumption of the existence of a number possessing f is not known to be contradictory. The critical number λ_f of a fleeing property f is the smallest natural number possessing f.

Brouwer's standard example of a fleeing property is being the smallest k, the k-th up to the $k + 9$-th digit in π's expansion of which form the sequence $0, 1, 2, \ldots, 9$, used in the first cited Berlin example above. The defined real number over there is an example of *a dual pendular number* (our translation of the German *duale Pendelzahl*). As we mentioned in the previous section, for Brouwer's standard example the critical number has become known, so for us this property is not fleeing anymore.

In [Bro30] Brouwer examines the continuum on seven properties, all valid classically, but not intuitionisticaly. Whenever it is possible, he uses a lawlike sequence, as in the following example. With 0 instead of 1/2 it is the same as our first cited Berlin example.

> That the continuum (and also the reduced continuum) is *not discrete* follows from e.g. the fact that the number $1/2+p_f$, where p_f is the dual pendular number of the fleeing property f, is neither equal to 1/2, nor apart from 1/2. ([Bro75], p. 435).

But if necessary he uses an incomplete sequence

> That $<$ is not *an order* on the continuum is demonstrated by the real number p, generated by the sequence (c_n), its terms chosen such that $c_1 = 0$ and $c_v = c_{v+1}$ with only the following exception. Whenever I find the critical number of some particular fleeing property f, I choose the next c_v equal to -2^{-v}, and when I find a proof this critical number does not exist, I choose c_v equal to 2^{-v}. This number p is unequal to 0, but nevertheless it is not apart from zero ([Bro75], p. 435–436).

Notwithstanding the superficial similarity, this sequence (c_n) is completely different from the sequence (b_n) used in the Berlin example to show that the reduced continuum was not completely ordered. There is no 1-1 connection between the values of (c_n) and the digits of the expansion of π as in the case of (b_n); (c_n) is used for a property of the full continuum, so it is an incomplete sequence. In the next section we shall cite Brouwer about the difference between two such sequences.

The resemblance with the 1948 example is less then in the case of the Berlin example, because Brouwer does not mention *untestedness* here. Wether it is the same depends on the question wether a fleeing property can be tested or not, which is not obvious from the definition Brouwer gives here. We think that clearly Brouwer had the intention to give the same example, and that he supposes untestedness, for the following reason. If the non-existence of a number possessing f is known to be contradictory, one of the possibilities in the definition above would be excluded beforehand. Furthermore, Brouwer gives in [Bro48] his standard example of a fleeing property also as an example of an untested proposition.

Let us return to our starting point of our historical review, which was finding evidence for our reconstruction of the 1948 creating subject argument. As we have seen in this section, the method of the CS was not new for Brouwer in 1948. We have also concluded that Brouwer applies in this method incomplete elements of a spread.

The citations in the next section, which are from our richest source, confirm these conclusions. But above all, these citations support our conception of an incomplete sequence.

7 The Geneva Lectures

Brouwer stopped publishing after 1930 but he did not stop lecturing. [Bro] is the text of his Geneva lectures of 1934. In no way this manuscript has been made fit for publication. It is full of crossing outs and improvements. But there are no other places in the work of Brouwer were he spends so many words on the special character of incomplete objects. Furthermore, this text is the link between Brouwer's creating subject arguments after 1948 and the first choice sequences of the late twenties we cited above.

The text contains no new material. After the introduction of the real numbers with λ-intervals and the definition of order, he wants to show that the natural $<$ is not an order on the continuum. Therefore he defines a real number by giving a description of a construction:[4]

[4] [Bro] consists of six parts, probably corresponding with six lectures. All citations of this section are from the second part, pp. 22-26, translated by the author from the French original.

> The n-th interval $\lambda^{(n-1)}$ is of length $2/(n-1)$ and centered around 0. This is how one starts. But at the same time one works on a difficult problem, to know whether the property E for a species S is true, for example Fermat's problem. If for that problem a solution has been found between the $(n-1)$-th and the n-th choice, the choice of the intervals will be different.

As we pointed out in the example of [Bro30],if E could be tested, one of the possibilities in the definition would be excluded. We remark that Brouwer gives in [Bro48] also Fermat's problem as an example of a proposition that can not be tested.

> If the property is true for the species S, then the v-th interval will be for $v \geq n$ the interval $\lambda^{(v)}$ centered around 2^{-n}. The next interval will be placed according to this law, within its predecessor with the same center. If, on the other side, one finds that the property E is absurd for the species S, than the intervals will be centered around -2^{-n}.

Brouwer proves, just as he will do again in 1948, that the defined number cannot be $= 0$:

> This point s is defined completely correctly. The point is different from 0, because if it was equal to 0, then the possibility to continue the sequence around 2^{-n} would be excluded. So the supposition that one day a proof of E for S would be found, would be absurd and the supposition that one day one finds a proof of the absurdity for S would be absurd too. The truth and the absurdity of that property would be absurd both and that is impossible.

Next he argues that the point cannot have a positive distance from 0:

> We have a point which is different from 0, but it is neither positive nor negative, because if it was the one or the other, the problem in question would be solved.

But the following text has been struck out:

> We have a point that is different from 0, but at the same time neither the relation, (if we define that point by s), neither the relation $s < 0$, nor the relation $>$ than 0 holds, because, if the relation $s < 0$ would hold, one would have to exclude the first variant, that is to say, that one would have solved the problem positively, which is not the case, and if the other relation would hold, one should have to exclude the second variant, that is to say that one would have solved the problem negatively.

We do not know the reason why he scrapped the proof. But that he did not trust the proof seems unthinkable to us, because it is the same one he used later in 1948. Further, there can be no misunderstanding about the fact that he is using an incomplete sequence here:

> We show the same for the reduced continuum. The point above is not a sharp point, because the construction is not completely determined, but depends on the intelligence of the constructor relative to the posed problem.

Constructor is our translation of the French *constructeur*. As it seems, Brouwer would opt later for *creating subject*. That $>$ is not an order on the reduced continuum (actually Brouwer only shows that it not a complete order) is done in the familiar way. Let K_1 be defined as in the Berlin example. One starts with choosing as λ_n a $\lambda^{(n)}$-interval centered around 0 for $n < K_1$, and we choose a $\lambda^{(n)}$-interval centered around $(-2)^{-K_1}$ for $n \geq K_1$. About the difference between this point r and s, the incomplete sequence above, Brouwer remarks:

> When one hundred different persons are constructing the number r, one is always certain that any interval chosen by one of these persons is always covered, at least partly, by every interval chosen by one of the others. That is different for s. If I would give the definition of s to one hundred different persons, who are all going to work in a different room, it is possible that one of these one hundred persons once will choose an interval not covered by an interval chosen by one of the others.

As we may observe here, there is no idealized mathematician involved in these incomplete sequences. They are given by a description of a construction, their terms made to depend on the mathematical experience of the one who constructs them, which can be any one. The activity of this subject does not play a role, the reasoning about an incomplete sequence is done before the construction has started, on the basis of the incomplete description only.

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