Chapter 1

MODAL LOGICS OF SPACE

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1. Modal logics and spatial structures

1.1 What does modal logic have to do with space?

Despite historical links between the foundations of mathematics and development of axiomatic geometry, substantial logics for significant spatial structures have been scarce. Perhaps the best-known examples are both due to Tarski. The first is his still amazing work on the first-order theory of elementary Euclidean geometry, including the surprising proof of its decidability, and the resulting abstract theory of real-closed fields. This was the metamathematical finale to Hilbert's *Foundations of Geometry*, itself the culmination of Euclid's *Elements*. This strand is taken up by several chapters in this handbook (`Ch.~\ref{BG::c}`, `Ch.~\ref{PH::c}`), but it will be only mentioned in passing in this chapter. For our purposes here, the founding event is Tarski's topological interpretation of modal logic, culminating in his proof with McKinsey that the simple decidable modal logic **S4** is complete for interpreting modal $\Diamond$ as topological closure on the reals or any metric space like it. In what follows we concentrate on the latter *modal* direction in spatial logics, which is also represented in several other chapters of the handbook (`Ch.~\ref{BG::c}`, `Ch.~\ref{BD::c}`, `Ch.~\ref{KM::c}`).

It seems fair to say that there are mostly scattered results in this modal line, suggestive though they may be. To quickly survey several diverse directions in this line we recall Segerberg, 1973 on two-dimensional modal logics, Shehtman, 1983 on logics of physical structures (which was part of Dragalin's program of investigating modal logics of geometrical structures in physical spaces), Goldblatt, 1980 on the logic of Minkowski space-time, Chellas, 1980 on neighborhood semantics (originally proposed by Montague and Scott in the 1960s), the appendix of Benthem, 1983 on calculi for relative nearness, the work of the 'Georgian School' in modal logics of topology (partly surveyed in Esakia, 2004), Venema, 1999 on 'compass logic' in the two-dimensional plane, and Stebletsova, 2000; Stebletsova and Venema, 2001 on modal logics for projective geometry. So far, these ingredients have never added up to one coherent tradition of 'spatial logic', although some attempts have been made occasionally (cf. Anger et al., 1996). In contrast to this state of affair, *temporal logic* has been a thriving research program for many years (cf. Benthem, 1995 or Hodkinson and Reynolds, 2006). One of the goals of this handbook in general and our chapter in particular is to fill in this gap.

Our starting point is the topological interpretation of modal logic Tarski, 1938; McKinsey and Tarski, 1944, which we state in the modern truth-conditional format. The basic language $\mathcal{L}$ has a countable set P of proposition letters, boolean connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, and modal operators $\Box$, $\Diamond$. A *topological model* or simply a *topo-model* is a topological space $\langle X, \tau \rangle$ equipped with a valuation function $\nu : P \rightarrow \mathcal{P}(X)$.

Definition 1.1 (basic topological semantics) Truth of modal formulas is defined inductively at points x in a topo-model $M = \langle X, \tau, \nu \rangle$:

$$\begin{array}{lll}
M, x \models p & \text{iff} & x \in \nu(p) \text{ (with } p \in P) \\
M, x \models \neg\varphi & \text{iff} & \text{not } M, x \models \varphi \\
M, x \models \varphi \wedge \psi & \text{iff} & M, x \models \varphi \text{ and } M, x \models \psi \\
M, x \models \Box\varphi & \text{iff} & \exists U \in \tau \ (x \in U \wedge \forall y \in U \ M, y \models \varphi) \\
M, x \models \Diamond\varphi & \text{iff} & \forall U \in \tau \ (x \in U \rightarrow \exists y \in U : M, y \models \varphi)
\end{array}$$

As usual we can economize by defining, e.g., $\varphi \vee \psi$ as $\neg(\neg\varphi \wedge \neg\psi)$, and $\Diamond\varphi$ as $\neg\Box\neg\varphi$. We will do this whenever convenient.

This looks like the usual symbolic truth definition, and it is. But there is also an immediate spatial interpretation. Given any concrete model, each formula of the language denotes a region of the topological space being modelled. For instance, take the real plane $\mathbb{R}^2$ with the standard topology. Consider a valuation function having some spoon shaped region as the value of the proposition letter p, as depicted in Fig. 1.1.a. Then, the formula $\neg p$ denotes the region not occupied by the spoon, i.e., the background. The formula $\Box p$ denotes the interior of the spoon region p and so on, as explained in Fig. 1.1.

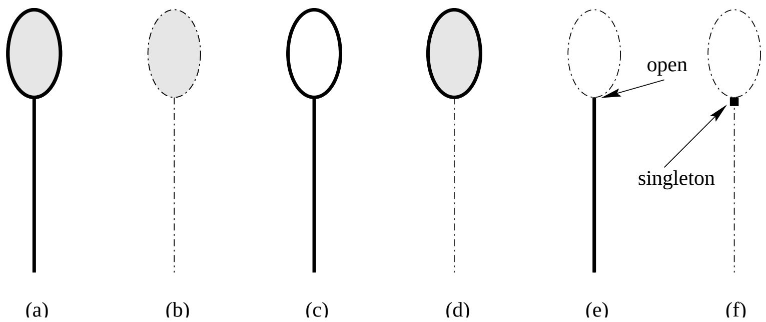


Figure 1.1. Each modal formula identifies a region in a topological space. (a) A spoon, p. (b) The container part of the spoon, $\Box p$. (c) The boundary of the spoon, $\Diamond p \wedge \Diamond\neg p$. (d) The container part of the spoon with its boundary, $\Diamond\Box p$. (e) The handle of the spoon, $p \wedge \neg\Diamond\Box p$. In this case the handle does not contain the junction handle-container point. (f) The junction handle-container point of the spoon, $\Diamond\Box p \wedge \Diamond(p \wedge \neg\Diamond\Box p)$: a singleton in the topological space.

Thus, a simple modal language can define regions in space in a perspicuous and appealing notation, and allow us to check assertions about them. Moreover, the same modal notation also facilitates spatial reasoning. For instance, the valid axiom $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$ says that two ways of computing a region—either as the interior of intersection of sets or as the intersection of interiors of those sets—always amount to the same thing. Thus, modal logic is also a small inference engine for basic spatial manipulations.

We will consider other modal languages and logics for spatial structures later on. For the moment, we merely point out that the preceding example contains two different perspectives on the encounter between modal logic and space. Some modal logicians see topological models as a means of providing *new semantics* for existing modal languages, mostly for logic-internal purposes. This can be motivated a bit more profoundly by thinking of topologies as models for *information*, making this interest close to central logical concerns. But someone primarily interested in Space as such will not worry about the semantics of modal languages. She will rather be interested in spatial structures by themselves, and *spatial logics* will be judged by how well they analyze old structures, discover new ones, and help in reasoning about them. Both perspectives will play in our presentation, with the mathematics largely the same, but the sort of issues suggested sometimes a bit different.

1.2 Relational semantics for modal logic

The standard models for modal logic are the well-known binary relational graphs, with necessity interpreted as truth in all accessible worlds, and possibility as truth in at least one accessible world:

$$M, s \models \Box\varphi \text{ iff } \forall t(sRt \to M, t \models \varphi)$$

$$M, s \models \Diamond\varphi \text{ iff } \exists t(sRt \ \wedge \ M, t \models \varphi)$$

In this chapter we presuppose a basic acquaintance with modal logic in this style. We refer to (Blackburn et al., 2001) and (Benthem and Blackburn, 2006) for a quick introduction in a modern spirit. In particular, here are some core themes that will occur below.

The natural measure of expressive power for the basic modal language over the class of arbitrary relational models is the invariance of all formulas for *bisimulations* between models M, w and N, v, which provides the right measure for structural equivalence as far as the language is concerned. This invariance analysis can be fine-tuned to play Ehrenfeucht-Fraisse-type model comparison games between models in which the Duplicator player has a winning strategy over a k-round game iff the two models M, w and N, v satisfy the same modal formulas up to modal operator depth k. As for axiomatics, the class of all standard models validates precisely the *minimal modal logic* **K**, whose most noteworthy principle is the above-mentioned distributivity of modal $\Box$ over conjunction. But deductive power goes up on special model classes. E.g., the modal logic **S4** with axioms $\Box p \to p$ and $\Box p \to \Box\Box p$ is complete for the class of all reflexive and transitive frames, and there is a host of other natural stronger logics. These *correspondences* between natural conditions on accessibility relations in graphs and modal axioms of certain shapes can also be studied per se, as a matter of semantic definability. There are even powerful methods for automatic analysis of modal axioms for their frame content. But in addition to deductive power and correspondence analysis of the basic language, there is also expressive power: the ability to say more about the same class of structures. Many modal languages in use *extend* the basic propositional formalism mentioned above by adding operators such as the 'universal modality' ("true in all worlds"), or temporal-style operators like 'Until' or 'Since'.

Finally, to complete this lightning summary, modal languages are designed with a certain *balance* in mind. E.g., the basic modal language is like the language of first-order logic in that it allows for quantification over objects. But this quantification is only 'local' or 'bounded', tied by accessibility to the current world. Trading in some first-order expressive power in this way comes with a bonus, however: validity and satisfiability in the basic modal language are *decidable*, indeed *PSPACE*-complete. Moreover, looking at other key tasks for a logical calculus, it may be noted that *model checking* for finite models is *PSPACE*-complete for the full first-order language, whereas it takes only polynomial time for modal logic. Likewise, testing two finite models for the existence of a bisimulation can be done in polynomial time, whereas the corresponding problem for the complete first-order language is the so-called Graph Isomorphism Problem, which is known to be in *NP*. More generally, extended modal languages try to boost expressive power on relevant structures, while skirting the cliffs of complexity. Well-known examples of such trade-offs much higher up are the 'Guarded Fragment' of first-order logic (Andreka et al., 1998) or the non-first-order modal 'μ-calculus' enriching the basic modal language with non-first-order operators for smallest and greatest fixed-points (Harel et al., 2000).

Even though these features of modal logic have not evolved for specific spatial reasons, they are often congenial with thinking about space. First of all, binary relational models themselves *are* a form of geometrization of modal semantics. Of course, they resemble abstract graphs and diagrams, rather than regions of Euclidean spaces, but still, geometrical intuitions play a role in understanding how it all works. Indeed, models of this sort can represent significant spatial structures. An example is the work of Shehtman, 1983 and Goldblatt, 1980 from the early 1980s (cf. also Andreka et al., 2006). Interestingly, in relativistic space-time, the crucial primitive notion is not the ternary 'Betweenness' of classical geometry, but the binary relation of

forward causal accessibility Cxy

which runs from a point x to all points y in the interior of its future light-cone, where causal signals can reach (see Fig. 1.2.a).

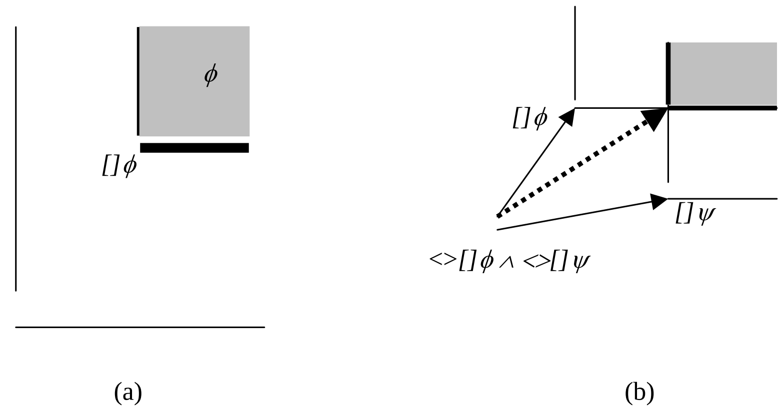


Figure 1.2. Forward modality in Minkowski space-time and validity of the **S4.2** Confluence Axiom.

Shehtman and Goldblatt independently proved that the complete modal logic of forward causal accessibility equals the modal logic **S4.2** which extends **S4** with the so-called 'Confluence Axiom' $\Diamond\Box p \rightarrow \Box\Diamond p$. The latter principle is illustrated in Fig. 1.2.b. It expresses the relativistic fact that any two different causal futures, as seen from the current point, even when not causally connected themselves, could potentially still lead to a common future history. Again we see how modal formulas express significant facts about space(–time).

All technical topics in our survey of relational semantics make spatial sense. Bisimulation-invariance analysis of expressive power is very close to thinking about geometrical *transformations and invariants* (Benthem, 2002), which goes back to the foundations of geometry in the 19th century. Also, modal logics can represent special styles of spatial reasoning, as we just saw. And issues of optimal language design have also emerged already. For instance, the above topological semantics for the basic modal language is still 'local', not in the sense of binary accessibility, but in being restricted to what is true in open neighborhoods of the current point. But many natural topological notions do not have this local character. E.g., a space is *connected* if it cannot be split into two non-empty clopen sets. This global property of topological spaces cannot be expressed in the basic modal language. But it can if one adds a universal modality. Finally, in all this, the issue of the 'Balance' returns. Modal systems are typically attempts at uncovering significant spatial structures, while providing low-complexity (decidable) calculi for reasoning with them.

1.3 Background: the many semantics of modal logic

In a sense, spatial interpretations of modal logic challenge the existing order. The now dominant relational semantics is really a product of the 1950s/1960s. Its historical predecessors include *algebraic semantics*, which has been used extensively in the technical literature, in the form of boolean algebras with operators. The chapter Venema, 2006 in the forthcoming *Handbook of Modal Logic* surveys the state of the art. Another earlier semantics of modality is Gödel's provability interpretation: a story which is told with many new historical details in the forthcoming chapter Artemov, 2006 in the same handbook. The latter paper is also an excellent broader source for mathematical uses of modal logic, including a brief, but useful account of spatial ones.

Clearly, our topological semantics is another 1930s challenger. This modelling was particularly vivid and attractive for the language of *intuitionistic logic*, where open sets may be viewed as information stages concerning some underlying point — an interpretation which returns in much greater sophistication in the topos semantics (see `Ch.~\ref{VI::c}`). The informational interpretation of topology will not be a major concern in this chapter, but we do mention topological semantics for *epistemic logic* briefly in Sec. 3.4 as it raises some interesting new issues that do not become visible in the standard binary relational modelling.

Also worth noting is that the topological semantics generalizes easily to the so-called *neighborhood models* for modal languages. Here one just assumes some binary relation RxY associating worlds x with sets of worlds Y (not necessarily open environments), with the same truth condition as above:

$\Box\varphi$ is true at world x iff there exists a set Y with RxY all of whose members y satisfy the formula φ

Neighborhood models are used, e.g., to express output relations for concurrent computation (Peleg, 1987), relations of 'support' or 'dependence' in logic programming (Benthem, 1992), or relations of 'power' for forcing a game to end in certain sets of outcomes in games, starting from some current node (Pauly, 2001). Neighborhood semantics is an interesting counterpoint to topological semantics, because it shows what happens further down the road. The minimal modal logic now loses Distributivity, retaining only *upward monotonicity* for the two modalities. There is still a notion of generalized bisimulation, however, whose topological version will return in Sec. 1.4. Finally, as to the Balance, the complexity of satisfiability in neighborhood semantics goes down from *PSPACE-complete* to *NP*. The latter is not true, however, for the topological interpretation, as it retains the Distribution Axiom, and its minimal modal logic **S4** is still *PSPACE-complete*.

All these different semantics are related. In particular, topological models are a special case of neighborhood models, and reflexive and transitive relational models are a special case of topological models, as will be explained below. Neighborhood models are also related to algebraic ones, but we will forego such details in this chapter. Even so, these technical connections have their uses. For instance, topological semantics still includes binary relational semantics as the special case of 'Alexandroff topologies' (cf. Sec. 2.4.1). Thus, its generalizations of standard modal notions, such as bisimulations, may be viewed as a significant extension of the latter's scope of applicability. Likewise, we will see in Sec. 3.2 how a topological perspective actually clarifies issues in binary relational model theory, viz. the axiomatization problem for classes of products of modal frames. And finally, topological viewpoints have suggested new modal languages and structures such as the 'Chu spaces' of Pratt, 1999 (cf. Benthem, 2000a on a first-order/modal style analysis of invariance and expressive power).

We conclude with one illustration going the other way. Despite its immediate spatial appeal, the topological semantics is also *more complex* than the binary relational semantics. Instead of matching up one modal operator $\Box$ with one quantifier, it matches it up with the $\exists\forall$ combination of *two* nested quantifiers: "there exists an open set such that for all its elements...". This makes things less perspicuous, and it may in fact be the reason why the topological interpretation, though historically first, was eventually supplanted by the simpler Kanger-Hintikka-Kripke graph-based version. But this is not all there is to be said. For, one can analyze the above $\exists\forall$ in terms of two consecutive modalities $\Diamond_{open}\Box_{element}$, where the first states the existence of an open set, while the second accesses its elements. From this point of view, the topological semantics lives inside a standard *bimodal* language over two-sorted binary relational models having both points and sets as objects. There are even mathematical reduction results showing precisely how far this reduction goes. This amounts to a richer many-sorted view of space, where both points and sets can be 'objects' on par. This style of thinking, too, has geometrical precedents, witness Hilbert's use of points, lines, and spaces as objects on par, rather than ascending stages in some abstract set-theoretic hierarchy. Benthem, 1999 presents a defense of many-sorted reformulations of complex modal semantics in temporal and spatial settings. The only framework that we know of where this 'unravelling' into separate modal stages is taken seriously in a spatial sense is the 'topological logic of knowledge' of Dabrowski et al., 1996 (cf. also `Ch.~\ref{MP::c}`). The bulk of existing work, however, is squarely within the standard topological framework, to which we now return.

1.4 Modal logic and topology. First steps

The topological interpretation explained above brings some interesting shifts in perspective. E.g., the crucial modal feature of *locality* in graph models now means that a formula is true at M, x iff it is true at x in any submodel whose domain is that of M restricted to some *open neighborhood* of x. Thus, *regions* are essential, and more generally, a modal approach provides a calculus of regions de-emphasizing constellations of points. As such, it is close to 'region versus points' theories of time and space (Allen, 1983; Allen and Hayes, 1985; Benthem, 1983; Randell et al., 1992).

There are also subtle differences with modal logic that lie just below the surface. E.g., binary relational semantics validates unlimited *Distributivity*: the modal box distributes over arbitrary infinite conjunctions of formulas. This is not so in topological semantics:

(-1, (-1/2, (-1/4, (....... 0, ..., +1/4), +1/2), +1)

Figure 1.3. Nested intervals refuting countable distributivity.

Let the proposition letters p_i be interpreted as the open intervals $(-i, +i)$. Then the point 0 satisfies the countable conjunction of all formulas $\Box p_i$. But 0 does not satisfy the related infinitary modal formula with the box over the conjunction, since that intersection is just the set $\{0\}$, whose topological interior is empty (see Fig. 1.3).

1.4.1 Expressive power: topo-bisimulation and topo-games.

To understand the expressive power of a modal language, a suitable notion of bisimulation is needed. The following definition reflects the semantic definition of the modal operators and can be seen as composed of two sub-moves: one in which points are linked, and one in which containing opens are matched.

Definition 1.2 (topo-bisimulation) A *topological bisimulation* or simply a *topo-bisimulation* between two topo-models $M = \langle X, \tau, \nu \rangle$ and $M' = \langle X', \tau', \nu' \rangle$ is a non-empty relation $T \subseteq X \times X'$ such that if xTx' then:

1 $x \in \nu(p) \Leftrightarrow x' \in \nu'(p)$ for each $p \in P$

2 (forth): $x \in U \in \tau \rightarrow \exists U' \in \tau' : x' \in U'$ and $\forall y' \in U' \; \exists y \in U : yTy'$

3 (back): $x' \in U' \in \tau' \rightarrow \exists U \in \tau : x \in U$ and $\forall y \in U \; \exists y' \in U' : yTy'$

A topo-bisimulation is *total* if its domain is X and its range is X'. If only the atomic clause (i) and the forth condition (ii) hold, we say that the second model *simulates* the first.

Topo-bisimulation captures the adequate notion of 'model equivalence' for the basic language $\mathcal{L}$ topologically interpreted. Evidence for this comes from the following two results (cf. Aiello and Benthem, 2002a).

Theorem 1.3 *Let $M = \langle X, \tau, \nu \rangle$ and $M' = \langle X', \tau', \nu' \rangle$ be two topo-models, and $x \in X$ and $x' \in X'$ be two topo-bisimilar points. Then for each modal formula φ we have $M, x \models \varphi$ iff $M', x' \models \varphi$. That is, modal formulas are invariant under topo-bisimulations.*

Theorem 1.4 *Let M, M' be two finite models, and $x \in X$, $x' \in X'$ be such that for each φ we have $M, x \models \varphi$ iff $M', x' \models \varphi$. Then there exists a topo-bisimulation between M and M' connecting x and x'. That is, finite modally equivalent models are topo-bisimilar.*

Topo-bisimulation is a standard model-theoretic tool for assessing expressivity of our language with respect to spatial patterns. Nevertheless, when comparing e.g. two image representations, it may still be too coarse. To refine the similarity matching, one can define a topological *model comparison game* $TG(M, M', n)$ between two topo-models M, M'. The idea of the game is that two players challenge each other picking

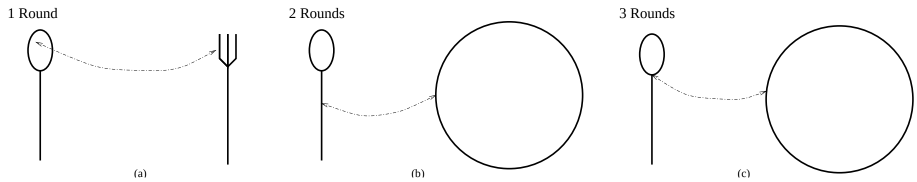


Figure 1.4. Game rounds needed for distinguishing shapes.

elements from the two models to compare. One player wins if he can show the models to be different, the other wins if he can show the models to be 'similar'. Winning strategies for the similarity player 'Duplicator' in infinite games, requiring never-ending continued responses, match up precisely with topo-bisimulations. Furthermore, for finite-length games, games and modal formulas are connected by the Adequacy Theorem:

THEOREM 1.5 *(Aiello and Benthem, 2002a) Duplicator has a winning strategy in the topo-game $TG(M, M', n, x, x')$ iff x and x' satisfy the same formulas of modal operator depth up to n in their respective models M, M'.*

Fig. 1.4 shows how many rounds Spoiler will need to distinguish positions on cutlery. The number of rounds corresponds to the depth of a modal 'difference formula' for the points under comparison. For instance, the single round in 2(a) corresponds to $\Box p$ versus $\neg\Box p$, while the three rounds in 2(c) correspond to the earlier-mentioned depth-three definition $\Diamond\Box p \wedge \Diamond(p \wedge \neg\Diamond\Box p)$ for the special point in the middle. The formal definition of a game, and an extensive discussion of plays and strategies are in Aiello and Benthem, 2002a, while the use of topo-games to compare models deriving from image descriptions is illustrated in Aiello, 2002b.

Excursion: Our examples show how logical games match topological notions very well. But there is a much earlier historical precedent. Van Dalen shows in the second volume of the *Brouwer* biography how Brouwer defined the crucial topological notion of *dimension* in terms of the following game:

Player 1 chooses two disjoint closed subsets A_1, B_1 of the space. Player 2 then chooses a closed separating set S_1. Player 1 now chooses two disjoint closed subsets of S_1. Player 2 then chooses a closed separating set S_2 inside S_1. Etcetera.

Player 2 wins if a separating set S_n is reached after n rounds which is totally disconnected. According to Brouwer, the dimension of a space is the lowest natural number n for which Player 2 has a winning strategy in the n-round game.

Topo-bisimulations are coarsenings of the basic structural equivalence in topology:

THEOREM 1.6 *Let* $M = \langle X, \tau, \nu \rangle$ *and* $M' = \langle X', \tau', \nu \rangle$ *be two topo-models. If* $\langle X, \tau \rangle$ *and* $\langle X', \tau' \rangle$ *are homeomorphic, then there exists a total topo-bisimulation between the topo-models* M *and* M'.

See (Aiello and Benthem, 2002a) for details as well as connections with other topological notions of structural similarity such as homotopy.

1.4.2 Deductive power: topo-logics. Now consider logical validity and hence the general calculus for spatial reasoning in this language. The logic **S4** is defined by the KT4 axioms and the rules of Modus Ponens and Necessitation (see Sec. 2.2 below). In the topological setting these principles translate into the following ones, with an informal explanation added:

$\Box\top$	(N)	the whole space is open
$(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$	(R)	open sets are closed under finite intersections
$\Box p \rightarrow \Box\Box p$	(4)	the interior operator is idempotent
$\Box p \rightarrow p$	(T)	the interior of any set is contained in the set

Then the universally valid formulas topologically interpreted are precisely the theorems of **S4**. But McKinsey and Tarski, 1944 proved a much more striking result.

THEOREM 1.7 **S4** *is complete for any dense-in-itself metric separable space.*

Thus, **S4** is also the logic of any Euclidean space $\mathbb{R}^n$ with the standard topology. Mints, 1998 proved completeness of **S4** for the Cantor space in a particularly elegant manner.

More restricted spatial structures generate stronger modal logics on top of **S4**. Take, for instance, the *serial* subsets of the real line, being the finite unions of convex intervals (Aiello et al., 2003). These have been used to model life-spans of 'events' in linguistics and computer science. Now consider the following additional axioms:

$$(\neg p \wedge \Diamond p) \rightarrow \Diamond\Box p \qquad (\mathbf{BD}_2)$$

$$\neg(p \wedge q \ \wedge \ \Diamond(p \wedge \neg q) \ \wedge \ \Diamond(\neg p \wedge q) \ \wedge \ \Diamond(\neg p \wedge \neg q)) \qquad (\mathbf{BW}_2)$$

These are complete for the serial sets. To give an impression of what is going on, look at Fig. 1.5, with a serial set denoted by p, and take the

p	..... (______] ..... x .... (_____)(_____] ....
	0 $\quad\sqrt{2}\quad$ 2 $\quad$ 3 $\quad$ 4 $\quad$ 5
$\neg p \wedge \Diamond p$	..... x .... x .... x ....
$\Diamond\Box p$	..... [______] .... [____________] ....

Figure 1.5. A serial set of $I\!R$ and the defined sub-formulas by the axiom $\mathbf{BD}_2$.

axiom $\mathbf{BD}_2$. In relational semantics, this axiom bounds the depth of the model to 2. In topological semantics, it states that the points that are both in the complement $\neg p$ of a region and in its closure $\Diamond p$, must be in the regular closed portion $\Diamond\Box p$ of the region itself.

Similarly, one can look at interesting 2-dimensional topological spaces. Here is a modal axiom

$$\Diamond(\Box p_3 \wedge \Diamond(\Box p_2 \wedge \Diamond\Box p_1 \wedge \neg p_1) \wedge \neg p_2) \rightarrow p_3 \qquad (\mathbf{BD}_3)$$

valid in the 'rectangular serial' sets of the plane $I\!R^2$. These special structures are investigated in Aiello et al., 2003 and Benthem et al., 2003. The latter provides an axiomatization for logics of this sort for Euclidean spaces of any dimension (see Sec. 2.6 below).

1.5 Modal logics of other spatial structures

Our account so far may have suggested that modal logic of space *must* be about topology. But this is not the case at all. Moving on from topology to more 'rigid' spatial structures, modal logic returns just as well, though in new guises. For instance, consider *affine geometry*, where the major notion is a ternary notion of *Betweenness* $\beta(xyz)$ between points. This says that point y lies in between x and z, allowing y to be one of these endpoints. Now define a binary betweenness modality $<B>$:

$$M, x \models <B>(\varphi, \psi) \quad \text{iff} \quad \exists y, z : \beta(yxz) \wedge M, y \models \varphi \wedge M, z \models \psi$$

Again, this leads to very concrete spatial pictures. This time, standard geometrical figures can be described by modal means. Consider Fig. 1.6. Let the proposition letter p denote the set of three points on the left forming the vertices of a triangle. The next two phases of the picture show how the formula $<B>(p, p)$ holds on the sides of the triangle, while the whole triangle, including its interior, is defined by the modal formula $<B>(<B>(p, p), p)$.

Clearly all of the earlier technical modal notions make sense once more. For instance, we can study modal bisimulation between geomet-

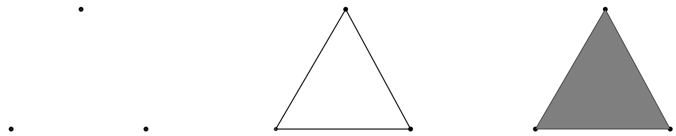


Figure 1.6.

ric figures, or modal deduction about triangles or convex figures. We will study such issues for geometrical modal logics later in Sec. 4. For the moment, we just note how this formalism can express significant geometric facts in surprising ways. Consider the following basic geometrical principle, known as *"Pasch's Axiom"* (see Fig. 1.7), written as follows in the first-order notation:

$$\forall txyzu(\beta(xtu) \wedge \beta(yuz) \to \exists v : \beta(xvy) \wedge \beta(vtz))$$

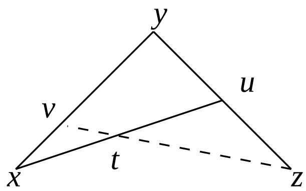


Figure 1.7. Pasch's property.

It says that any line drawn through a vertex of a triangle and continuing into its interior must cross the opposite side to that vertex at some point. This does not look modal at all, but in fact it is! Consider the following axiom of *associativity* for the Betweenness modality:

$$<B>(p, <B>(q, r)) \to <B>(<B>(p, q), r)$$

We will see in Sec. 4 that:

FACT 1.8 *Modal Associativity corresponds to Pasch's Axiom.*

At this stage a useful exercise for the reader would be to check how, unpacking the nested modalities $<B>$ in the antecedent, Pasch's Axiom is in fact precisely what is needed to see the validity of Associativity. In Sec. 4 and 5 we will develop these ideas further to also include metric geometry and eventually even linear algebra.

1.6 Logical analysis of space once more

Let us summarize the methodology of this chapter once more. Topologists or geometers do not worry about formal systems: they just state what they see in whatever formalism at hand. Logicians, however, propose a trade-off: specify a formal language restricting the notions one can talk about, and then see what complete logic comes out, perhaps even in the form of a decidable calculus. Tarski's elementary geometry is still a paradigm for this approach, and so are other logics discussed in `Ch.~\ref{PH::c}` of this handbook on first-order theories of polygons and of mereotopology. Incidentally, the spatial perspective also highlights quite different uses of a logical formalism. One is its *descriptive role* in defining spatial patterns, allowing us to describe these, check whether they hold in given situations, and compare different properties of spatial structures. Another is its *deductive role* as a calculus of reasoning about space, which is associated with other tasks, such as mathematical theorizing, information extraction from spatial databases, or reasoning by a robot trying to plan actions in a partially unknown environment.

Used in either mode, modal languages are fragments of first-order ones, restricting expressive power even further, but promising better complexity. The very multiplicity of modal languages is an advantage here, as we can work at different levels of structure, measured by different notions of invariance, whether topo-bisimulation, or some logical or geometrical strengthening thereof. This fits with the mathematical idea that Space can be studied legitimately at various levels of detail. Finally, consider the issue of the Modal Balance between expressive power and computational complexity. Indeed, there are low-complexity modal logics for some spatial structures. But there are also some phenomena showing that things can be complicated. For instance, in affine geometry, while the minimal modal logic of our binary Betweenness modality $<B>(\varphi, \psi)$ is decidable, this same language becomes undecidable over the special class of associative relations that we just associated with Pash's Axiom. The reason is that it can then straightforwardly encode the word problem for semigroups.

Moreover, our two guiding examples from Tarski's work do not point in the same direction in terms of the sources of their decidability. Modal Topology is indeed decidable for reasons of modal parsimony by abstract general methods having little to do with peculiarities of topological spaces. But Elementary Geometry is decidable not because its language has judiciously toned-down expressive power, but because its intended model of Euclidean Space is so rich that it happens to support a procedure of *quantifier elimination*, providing us with the decision algorithm, which is very special for this geometric setting. We refer to `Ch.~\ref{KK::c}` for more accumulated evidence on complexity of logics for spatial reasoning.

1.7 Contents of this chapter

This concludes our introduction to modal languages of spatial structures. The rest of this chapter is organized as follows. Sec. 2 contains a more extensive formal treatment of topological models, leading up to the general completeness theorem for topological models, and from there, to the landmark completeness theorem for the reals using modern techniques. Following that, we also discuss richer modal logics for more special topological structures, including restrictions on sets that can serve as values for propositions. Next, Sec. 3 surveys a number of more recent special topics in this area. These include (a) alternative interpretations of the modalities in terms of the topological derivative, (b) combining modal logics for describing products of topological spaces, (c) language extensions that can express further topological structure, and finally (d) topological models for epistemic logic, with an excursion into fixed-point extensions of the language that can define various notions of common knowledge. Sec. 4 is a discussion of modal languages for geometric structures, starting with affine cases, and then moving to modal languages for metric relations of relative nearness. We also provide a comparison with first-order languages for these structures. Finally, Sec. 5 looks at modal logics for mathematical morphology, which basically amounts to analyzing certain subsets of vector spaces. This topic also involves connections with modal 'arrow logics' for analyzing structures in relational algebra. Sec. 6 contains our conclusions.

2. Modal logic and topology. Basic results

2.1 Topological preliminaries

We start by surveying briefly the basic topological concepts that will be used throughout this chapter. They can be found in any textbook on general topology (see, e.g., Engelking, 1989; Kelley, 1975; Kuratowski, 1966).

DEFINITION 1.9 *A* topological space *is a pair $\mathcal{X} = \langle X, \tau \rangle$, where X is a nonempty set and τ is a collection of subsets of X satisfying the following three conditions:*

- $\emptyset, X \in \tau$*;*

- *If $U, V \in \tau$, then $U \cap V \in \tau$;*
- *If $\{U_i\}_{i\in I} \in \tau$, then $\bigcup_{i\in I} U_i \in \tau$.*

The elements of τ are called *open sets*. The complements of open sets are called *closed sets*. An open set containing $x \in X$ is called an *open neighborhood* of x.

A family $\mathcal{B} \subseteq \tau$ is called a *basis* for the topology if every open set can be represented as the union of elements of a subfamily of $\mathcal{B}$. It is well-known that a family $\mathcal{B}$ of subsets of X is a basis for some topology on X iff (i) for each $x \in X$ there exists $U \in \mathcal{B}$ such that $x \in U$, and (ii) for each $U, V \in \mathcal{B}$, if $x \in U \cap V$, then there exists $W \in \mathcal{B}$ such that $x \in W \subseteq U \cap V$.

For $A \subseteq X$, a point $x \in X$ is called an *interior point* of A if there is an open neighborhood U of x such that $U \subseteq A$. Let $\mathrm{Int}(A)$ denote the set of interior points of A. Then it is easy to see that $\mathrm{Int}(A)$ is the greatest open set contained in A, called the *interior* of A. A point $x \in X$ is called a *limit point* of $A \subseteq X$ if for each open neighborhood U of x, the set $A \cap (U - \{x\})$ is nonempty. The set of limit points of A is called the *derivative* of A and is denoted by $d(A)$. Let $\mathrm{Cl}(A) = A \cup d(A)$. Then it is easy to see that $x \in \mathrm{Cl}(A)$ iff $U \cap A$ is nonempty for each open neighborhood U of x, and that $\mathrm{Cl}(A)$ is the least closed set containing A, called the *closure* of A.

Let Int and Cl denote the interior and closure operators of $\mathcal{X}$, respectively. Then it is well known that the following are satisfied for each $A, B \subseteq X$:

$\mathrm{Int}(X) = X$	$\mathrm{Cl}(\emptyset) = \emptyset$
$\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$	$\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$
$\mathrm{Int}(A) \subseteq A$	$A \subseteq \mathrm{Cl}(A)$
$\mathrm{Int}(A) \subseteq \mathrm{Int}(\mathrm{Int}(A))$	$\mathrm{Cl}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(A)$

Moreover, there is a duality $\mathrm{Int}(A) = X - \mathrm{Cl}(X - A)$, and a topological space can also be defined in terms of an interior operator or a closure operator satisfying the above four conditions.

We also let $t(A)$ denote $X - d(X - A)$. Then $x \in t(A)$ iff there exists an open neighborhood U of x such that $U \subseteq A \cup \{x\}$. We call $t(A)$ the *co-derivative* of A. Let d and t denote the derivative and co-derivative operators of $\mathcal{X}$, respectively. Then it is well known that the following are satisfied for each $A, B \subseteq X$:

$d(A \cup B) = d(A) \cap d(B)$	$t(A \cap B) = t(A) \cap t(B)$
$A \subseteq A \cup d(A)$	$A \cap t(A) \subseteq A$

Definition 1.10 *Let $\mathcal{X}$ be a topological space and A be a subset of X.*

1 A is called clopen *if it is both closed and open.*

2 *A is called* dense *if* $Cl(A) = X$.

3 *A is called* nowhere dense *or* boundary *if* $Int(A) = \emptyset$.

4 *A is called* dense-in-itself *if* $A \subseteq d(A)$.

For a topological space $\mathcal{X}$, the family $\{U_i\}_{i \in I} \subseteq \tau$ is called an *open cover* of X if $\bigcup_{i \in I} U_i = X$.

Definition 1.11 *Let $\mathcal{X}$ be a topological space.*

1 *$\mathcal{X}$ is called* discrete *if every subset of X is open.*

2 *$\mathcal{X}$ is called* trivial *if $\emptyset$ and X are the only open subsets of X.*

3 *$\mathcal{X}$ is called* dense-in-itself *if $d(X) = X$.*

4 *$\mathcal{X}$ is called* separable *if there exists a countable dense subset of X.*

5 *$\mathcal{X}$ is called* compact *if every open cover of X has a finite subcover.*

6 *$\mathcal{X}$ is called* connected *if $\emptyset$ and X are the only clopen subsets of X.*

7 *$\mathcal{X}$ is called* 0-dimensional *if clopen subsets of X form a basis for the topology.*

8 *$\mathcal{X}$ is called* extremally disconnected *if the closure of each open subset of X is clopen.*

In the next definition we recall the separation axioms $T_0, T_{\frac{1}{2}}, T_1$, and T_2.

Definition 1.12 *Let $\mathcal{X}$ be a topological space.*

1 *$\mathcal{X}$ is called a* T_0-space *if for each pair of different points there exists an open set containing one and not containing the other.*

2 *$\mathcal{X}$ is called a* T_d-space *or a* $T_{\frac{1}{2}}$-space *if for each $x \in X$ there exists an open neighborhood U of x such that $\{x\}$ is closed in U. Equivalently, $\mathcal{X}$ is a T_d-space iff $dd(A) \subseteq d(A)$.*

3 *$\mathcal{X}$ is called a* T_1-space *if for each pair of different points there exists an open set containing exactly one of the points. Equivalently, $\mathcal{X}$ is a T_1-space iff each $\{x\}$ is closed in X.*

4 *$\mathcal{X}$ is called a* T_2-space *or a* Hausdorff space *if for each pair $x, y \in X$ of different points there exit disjoint open neighborhoods of x and y.*

It is well known that every T_2-space is a T_1-space, that every T_1-space is a T_d-space, and that every T_d-space is a T_0-space, but not vice versa.

Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. We call $\mathcal{Y}$ a *subspace* of $\mathcal{X}$ if $Y \subseteq X$ and U is an open subset of Y iff there exists an open subset V of X such that $U = V \cap Y$.

Definition 1.13 *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces and $f : X \to Y$ be a map.*

1 *f is called* continuous *if U open in Y implies that $f^{-1}(U)$ is open in X.*

2 *f is called* open *if U open in X implies that $f(U)$ is open in Y.*

3 *f is called* interior *if it is both continuous and open.*

We call $\mathcal{Y}$ a *continuous image* of $\mathcal{X}$ if there exists a continuous map from X onto Y. Open and interior images of $\mathcal{X}$ are defined analogously.

Let $\{\mathcal{X}_i\}_{i\in I}$ be a family of pairwise disjoint topological spaces. We define the *topological sum* of $\{\mathcal{X}_i\}_{i\in I}$ as the pair $\bigoplus_{i\in I} \mathcal{X}_i = \langle \bigcup_{i\in I} X_i, \tau \rangle$, where $U \in \tau$ iff $U \cap X_i \in \tau_i$. If the members of the family $\{\mathcal{X}_i\}_{i\in I}$ are not pairwise disjoint, then the topological sum is defined using disjoint union instead of set-theoretic union.

2.2 Relational semantics and some modal logics

2.2.1 The uni-modal case. We recall that a *frame* is a relational structure $\mathfrak{F} = \langle W, R \rangle$ such that W is a nonempty set and R is a binary relation on W. A *valuation* of the basic modal language $\mathcal{L}$ in $\mathfrak{F}$ is a function ν from the set P of propositional variables of $\mathcal{L}$ to the powerset of W. A pair $M = \langle \mathfrak{F}, \nu \rangle$ is called a *model* (based on $\mathfrak{F}$). Given a model M, we define when a formula φ is *true at a point* $w \in W$ by induction on the length of φ:

- $w \models p$ iff $w \in \nu(p)$;
- $w \models \neg\varphi$ iff not $w \models \varphi$;
- $w \models \varphi \wedge \psi$ iff $w \models \varphi$ and $w \models \psi$;
- $w \models \Box\varphi$ iff $(\forall v \in W)(wRv \to v \models \varphi)$;

and hence, also

- $w \models \Diamond\varphi$ iff $(\exists v \in W)(wRv \;\&\; v \models \varphi)$.

We say that φ is *true* in M if φ is true at every point in W, and that φ is *valid* in $\mathfrak{F}$ if φ is true in every model M based on $\mathfrak{F}$. Finally, we say that φ is *valid* in a class of frames if φ is valid in every member of the class.

Below we list several standard modal logics and their axiomatizations.

Definition 1.14

1 *The basic logic* **K** *of all frames is axiomatized by the axiom:*

$$\Box(p \to q) \to (\Box p \to \Box q) \quad \text{(K)}$$

with Modus Ponens and Necessitation as the only rules of inference:

$$\frac{\varphi \qquad \varphi \to \psi}{\psi} \quad \text{MP} \qquad\qquad \frac{\varphi}{\Box\varphi} \quad \text{N}$$

2 *The logic* **T** *of reflexive frames is axiomatized by adding to* **K** *the axiom:*

$$\Box p \to p \quad \text{(T)}$$

3 *The logic* **K4** *of transitive frames is axiomatized by adding to* **K** *the axiom:*

$$\Box p \to \Box\Box p \quad (4)$$

4 *The logic* **S4** *of reflexive and transitive frames is axiomatized by adding to* **K** *the axioms* (T) *and* (4).

5 *The logic* **S5** *of reflexive, transitive, and symmetric frames is axiomatized by adding to* **S4** *the axiom:*

$$p \to \Box\Diamond p \quad \text{(B)}$$

Each logic listed above is complete with respect to its relational semantics. In fact, each of these logics is complete with respect to its finite frames, and therefore has the *finite model property* (see, e.g., the textbook Blackburn et al., 2001).

2.2.2 Multi-modal cases. Multi-modal languages are conspicuous in modern applications of modal logic, which often call for combining operators. This happens in a spatial setting, e.g., when describing different topologies at the same time. Such combinations arise by performing certain operations on component logics. Here we recall several basic facts about ‘fusion’ and ‘product’ of uni-modal logics. Most of this material can be found in the textbook Gabbay et al., 2003.

The fusion: Let $\mathcal{L}_{\Box_1\Box_2}$ be a bimodal language with modal operators $\Box_1$ and $\Box_2$.

DEFINITION 1.15 *The* fusion *of* $\mathbf{K}$ *with itself, denoted by* $\mathbf{K} \oplus \mathbf{K}$*, is the least set of formulas of* $\mathcal{L}_{\Box_1\Box_2}$ *containing the axiom* (K) *for both* $\Box_1$ *and* $\Box_2$*, and closed under Modus Ponens,* $\Box_1$*-Necessitation, and* $\Box_2$*-Necessitation.*

The $\mathbf{K}\oplus\mathbf{K}$-frames are triples $\mathfrak{F} = \langle W, R_1, R_2\rangle$, where W is a nonempty set and R_1 and R_2 are binary relations on W. It is known that $\mathbf{K}\oplus\mathbf{K}$ is complete with respect to this semantics; in fact, it has the finite model property.

We will be interested in the fusion of $\mathbf{S4}$ with itself, which we denote by $\mathbf{S4} \oplus \mathbf{S4}$. It is defined similar to the fusion of $\mathbf{K}$ with itself. The $\mathbf{S4} \oplus \mathbf{S4}$-frames are triples $\mathfrak{F} = \langle W, R_1, R_2\rangle$, where W is a nonempty set and R_1 and R_2 are reflexive and transitive. We call such a frame *rooted* if there is a $w \in W$ such that for all $v \in W$ it holds that $w(R_1 \cup R_2)^* v$, where $(R_1 \cup R_2)^*$ is the reflexive and transitive closure of $R_1 \cup R_2$. It is known that $\mathbf{S4} \oplus \mathbf{S4}$ is complete with respect to this semantics; in fact, $\mathbf{S4} \oplus \mathbf{S4}$ is complete with respect to finite rooted $\mathbf{S4} \oplus \mathbf{S4}$-frames.

Let $\mathcal{T}_{2,2}$ denote the *full infinite quaternary tree* whose each node is R_1-related to two of its four immediate successors and R_2-related to the other two (see Fig. 1.8). We will make use of the next proposition in Sec. 3.2.2.

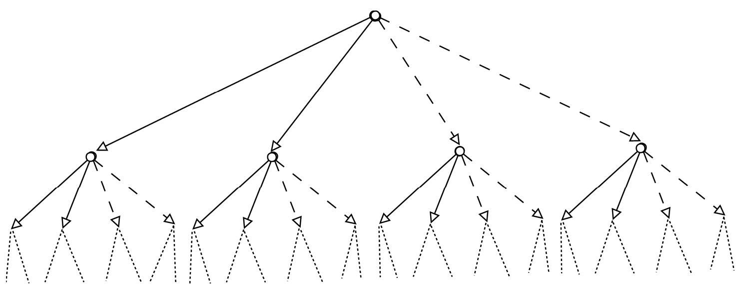


Figure 1.8. $\mathcal{T}_{2,2}$. The solid lines represent R_1 and the dashed lines represent R_2. The dotted lines at the final nodes indicate that the pattern repeats on infinitely.

PROPOSITION 1.16 *(Benthem et al., 2005)* $\mathbf{S4} \oplus \mathbf{S4}$ *is complete with respect to* $\mathcal{T}_{2,2}$.

The product: For two $\mathbf{K}$-frames $\mathfrak{F} = \langle W, S\rangle$ and $\mathfrak{G} = \langle V, T\rangle$, define the *product frame* $\mathfrak{F} \times \mathfrak{G}$ to be the frame $\langle W \times V, R_1, R_2\rangle$, where for $w, w' \in W$ and $v, v' \in V$:

$$(w,v)R_1(w',v') \text{ iff } wSw' \text{ and } v = v'$$
$$(w,v)R_2(w',v') \text{ iff } w = w' \text{ and } vTv'$$

The frame $\mathfrak{F} \times \mathfrak{G}$ can be viewed as a $\mathbf{K} \oplus \mathbf{K}$-frame by interpreting the modalities $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ as follows.

$$\begin{array}{lll} (w,v) \models \Box_1\varphi & \text{iff} & \forall(w',v') \text{ if } (w,v)R_1(w',v') \text{ then } (w',v') \models \varphi \\ (w,v) \models \Box_2\varphi & \text{iff} & \forall(w',v') \text{ if } (w,v)R_2(w',v') \text{ then } (w',v') \models \varphi \end{array}$$

Let $\mathbf{K} \times \mathbf{K}$ denote the logic of products of $\mathbf{K}$-frames. It is well known that $\mathbf{K} \times \mathbf{K}$ is axiomatized by adding the following two axioms to the fusion $\mathbf{K} \oplus \mathbf{K}$:

$$com = \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p$$

$$chr = \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$$

In a similar fashion we define the product of two **S4**-frames. Let $\mathbf{S4} \times \mathbf{S4}$ denote the logic of products of **S4**-frames. Similar to $\mathbf{K} \times \mathbf{K}$, the product logic $\mathbf{S4} \times \mathbf{S4}$ is axiomatized by adding *com* and *chr* to the fusion $\mathbf{S4} \oplus \mathbf{S4}$.

2.3 Interpreting $\Box$ as interior and $\Diamond$ as closure

Let $M = \langle \mathcal{X}, \nu \rangle$ be a topo-model, where $\mathcal{X} = \langle X, \tau \rangle$ is a topological space and $\nu : P \to \mathcal{P}(X)$ is a valuation. We gave the inductive definition of when a formula φ is true at a point x of the model M in Definition 1.1 of Sec. 1.1. The $\Box$ and $\Diamond$ clauses of Definition 1.1 imply that if φ is interpreted as a subset A of a topological space $\mathcal{X}$, then $\Box\varphi$ stands for $\mathrm{Int}(A)$ and $\Diamond\varphi$ for $\mathrm{Cl}(A)$. Either notion can be used as a primitive. In what follows we emphasize one or the other, depending on the ease of exposition.

Definition 1.17 *We say that φ is* true *in $M = \langle \mathcal{X}, \nu \rangle$ if φ is true at every $x \in X$. We say that φ is* valid *in $\mathcal{X}$ if φ is true in every model based on $\mathcal{X}$. Finally, we say that φ is* valid *in a class of topological spaces if φ is valid in every member of the class.*

Example 1.18 *Let* **Top** *denote the class of all topological spaces.*

1. *First we show that (T) is valid in* **Top**. *Let $\mathcal{X} \in$* **Top**, *$M = \langle \mathcal{X}, \nu \rangle$ be a topological model, and $x \models \Box p$ for $x \in X$. Then there exists an open neighborhood U of x such that $y \models p$ for each $y \in U$. In particular, since $x \in U$, we obtain that $x \models p$.*

2. *Next we show that (4) is valid in* **Top**. *Let $\mathcal{X} \in$* **Top**, *$M = \langle \mathcal{X}, \nu \rangle$ be a topological model, and $x \models \Box p$ for $x \in X$. Then there exists*

an open neighborhood U of x such that $y \models p$ for each $y \in U$. But then $y \models \Box p$ for each $y \in U$, implying that $x \models \Box\Box p$.

3 *Now we show that (K) is valid in* **Top**. *Let $\mathcal{X} \in$* **Top**, *$M = \langle \mathcal{X}, \nu \rangle$ be a topological model, and for $x \in X$ we have $x \models \Box(p \to q)$ and $x \models \Box p$. Then there exist open neighborhoods U and V of x such that $y \models p \to q$ for each $y \in U$ and $z \models p$ for each $z \in V$. Let $W = U \cap V$. Then W is an open neighborhood of x and for each $w \in W$ we have $w \models p \to q$ and $w \models p$. Therefore, $w \models q$ for each $w \in W$, implying that $x \models \Box q$.*

4 *Finally, we show that the necessitation rule preserves validity. If $\Box\varphi$ is not valid, then there exists a topological model $M = \langle \mathcal{X}, \nu \rangle$ and $x \in X$ such that $x \not\models \Box\varphi$. Therefore, there exists $y \in X$ such that $y \not\models \varphi$, implying that φ is not valid.*

Consequently, we obtain that the modal logic **S4** is sound with respect to interpreting $\Diamond$ as closure. In fact, as was shown by McKinsey and Tarski, 1944, **S4** is also complete with respect to this semantics. The details of the proof will be discussed below.

2.4 Basic topo-completeness of S4

As we already pointed out, **S4** is sound with respect to interpreting $\Diamond$ as the closure operator of a topological space. We are ready to show that **S4** is in fact complete with respect to this semantics. But first we discuss the well-known connection between relational and topological semantics of **S4** (see Aiello et al., 2003; Bezhanishvili and Gehrke, 2005).

2.4.1 Connection with relational semantics of S4.

Definition 1.19 *A topological space $\mathcal{X}$ is called an* Alexandroff *space if the intersection of any family of open subsets of X is again open.*

Equivalently, X is Alexandroff iff every $x \in X$ has a least open neighborhood. There is a close connection between Alexandroff spaces and **S4**-frames. Suppose $\mathfrak{F} = \langle X, R \rangle$ is an **S4**-frame. A subset A of X is called an *upset* of $\mathfrak{F}$ if $x \in A$ and xRy imply $y \in A$. Dually, A is called a *downset* if $x \in A$ and yRx imply $y \in A$.

For a given **S4**-frame $\mathfrak{F} = \langle X, R \rangle$ we define the topology τ_R on X by declaring the upsets of $\mathfrak{F}$ to be open. Then the downsets of $\mathfrak{F}$ turn out to be closed, and it is routine to verify that the obtained space is Alexandroff, that a least neighborhood of $x \in X$ is $R(x) = \{y \in X : xRy\}$, that the closure of a set $A \subseteq X$ is

$$R^{-1}(A) = \{x \in X : \exists y \in A \text{ with } xRy\},$$

and that the interior of A is

$$X - R^{-1}(X - A) = \{x \in X \,:\, (\forall y \in X)(xRy \to y \in A)\}.$$

Conversely, for a topological space $\mathcal{X}$ we define the *specialization order* on X by setting $xR_\tau y$ iff $x \in \mathrm{Cl}(y)$. Then it is routine to check that the specialization order is reflexive and transitive, and that it is a partial order iff $\mathcal{X}$ is T_0. Moreover, one can easily check that $R = R_{\tau_R}$, that $\tau \subseteq \tau_{R_\tau}$, and that $\tau = \tau_{R_\tau}$ iff X is Alexandroff.

These observations immediately imply that there is a 1-1 correspondence between Alexandroff spaces and **S4**-frames, and between Alexandroff T_0-spaces and partially ordered **S4**-frames. Since every finite topological space is an Alexandroff space, this immediately gives a 1-1 correspondence between finite topological spaces and finite **S4**-frames, and between finite T_0-spaces and finite partially ordered **S4**-frames. It is straightforward to see that this also implies a 1-1 correspondence between continuous maps and order-preserving maps, as well as between interior maps and p-morphisms. As an immediate consequence of all this, we obtain the following:

COROLLARY 1.20 *Every normal extension of* **S4** *that is complete with respect to relational semantics is also complete with respect to topological semantics.*

2.4.2 Canonical topo-model of S4. Corollary 1.20 says that standard modal models are a particular case of general topological semantics. Hence, the known completeness of **S4** plus the topological soundness of its axioms immediately give us general topological completeness. Even so, we now give a direct model-theoretic proof of this result, taken from Aiello et al., 2003. It is closely related to the standard modal Henkin construction and is much like completeness proofs for neighborhood semantics (Chellas, 1980), but with some nice topological twists.

DEFINITION 1.21

1. *Call a set* Γ *of formulas of* $\mathcal{L}$ *(*__S4__*–)*consistent *if for no finite set* $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ *we have that* $\mathbf{S4} \vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$.
2. *A consistent set of formulas* Γ *is called* maximally consistent *if there is no consistent set of formulas properly containing* Γ.

It is well known that Γ is maximally consistent iff, for each formula φ of $\mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both. Now we define a topological space out of maximally consistent sets of formulas.

Definition 1.22 (canonical topological space) *The* canonical topological space *is the pair* $\mathcal{X}^{\mathcal{L}} = \langle X^{\mathcal{L}}, \tau^{\mathcal{L}} \rangle$ *where:*

- $X^{\mathcal{L}}$ *is the set of all maximally consistent sets;*
- $\tau^{\mathcal{L}}$ *is the set generated by arbitrary unions of the following* basic sets $B^{\mathcal{L}} = \{\widehat{\Box\varphi} : \varphi$ *is any formula*$\}$, *where* $\widehat{\varphi} =_{def} \{x \in X^{\mathcal{L}} : \varphi \in x\}$. *In other words, basic sets are the families of the form:* $U_\varphi = \{x \in X^{\mathcal{L}} : \Box\varphi \in x\}$.

We first check that $\mathcal{X}^{\mathcal{L}}$ is indeed a topological space.

Lemma 1.23 $B^{\mathcal{L}}$ *forms a basis for the topology.*

Proof We only need to show the following two properties:

- For each $U_\varphi, U_\psi \in B^{\mathcal{L}}$ and each $x \in U_\varphi \cap U_\psi$, there is $U_\chi \in B^{\mathcal{L}}$ such that $x \in U_\chi \subseteq U_\varphi \cap U_\psi$;
- For each $x \in X^{\mathcal{L}}$, there is $U_\varphi \in B^{\mathcal{L}}$ such that $x \in U_\varphi$.

The necessitation rule implies that $\Box\top \in x$ for each x. Hence, $X^{\mathcal{L}} = \widehat{\Box\top}$, and so the second item is satisfied. As for the first item, thanks to the axiom (K), one can easily check that $\widehat{\Box(\varphi \wedge \psi)} = \widehat{\Box\varphi} \cap \widehat{\Box\psi}$. Hence, $U_\varphi \cap U_\psi \in B^{\mathcal{L}}$, and so $B^{\mathcal{L}}$ is closed under finite intersections, whence, the first item is satisfied. QED

Next we define the canonical topo-model.

Definition 1.24 (canonical topo-model) *The* canonical topo-model *is the pair* $M^{\mathcal{L}} = \langle \mathcal{X}^{\mathcal{L}}, \nu^{\mathcal{L}} \rangle$ *where:*

- $\mathcal{X}^{\mathcal{L}}$ *is the canonical topological space;*
- $\nu^{\mathcal{L}}(p) = \{x \in X^{\mathcal{L}} : p \in x\}$.

The valuation $\nu^{\mathcal{L}}$ equates truth of a propositional variable *at* a maximally consistent set with its membership *in* that set. We now show this harmony between the two viewpoints lifts to all formulas.

Lemma 1.25 (truth lemma) For all modal formulas φ,

$$M^{\mathcal{L}}, x \models_{\mathcal{L}} \varphi \text{ iff } x \in \widehat{\varphi}.$$

Proof Induction on the complexity of φ. The base case follows from the definition above. The case of the booleans follows from the following well-known identities for maximally consistent sets:

- $\widehat{\neg\varphi} = X^{\mathcal{L}} - \widehat{\varphi}$;
- $\widehat{\varphi \wedge \psi} = \widehat{\varphi} \cap \widehat{\psi}$.

The interesting case is that of the modal operator $\Box$. We do the two relevant implications separately, starting with the easy one.

$\Leftarrow$ 'From membership to truth.' Suppose $x \in \widehat{\Box\varphi}$. By definition, $\widehat{\Box\varphi}$ is a basic set, hence open. Moreover, thanks to the axiom (T), we have $\widehat{\Box\varphi} \subseteq \widehat{\varphi}$. Therefore, there exists an open neighborhood $U = \widehat{\Box\varphi}$ of x such that $y \in \widehat{\varphi}$, for any $y \in U$, and by the induction hypothesis, $M^{\mathcal{L}}, y \models_{\mathcal{L}} \varphi$. Thus $M^{\mathcal{L}}, x \models_{\mathcal{L}} \Box\varphi$.

$\Rightarrow$ 'From truth to membership.' Suppose $M^{\mathcal{L}}, x \models_{\mathcal{L}} \Box\varphi$. Then there exists a basic set $\widehat{\Box\psi} \in B^{\mathcal{L}}$ such that $x \in \widehat{\Box\psi}$ and $M^{\mathcal{L}}, y \models_{\mathcal{L}} \varphi$ for all $y \in \widehat{\Box\psi}$. By the induction hypothesis, $y \in \widehat{\varphi}$ for all $y \in \widehat{\Box\psi}$. Therefore, $\widehat{\Box\psi} \subseteq \widehat{\varphi}$. This implies that **S4** can prove the implication $\Box\psi \to \varphi$. But then **S4** can prove $\Box\Box\psi \to \Box\phi$, and hence, using the axiom (4), it can also prove $\Box\psi \to \Box\phi$. It follows that $\widehat{\Box\psi} \subseteq \widehat{\Box\phi}$, and so the world x belongs to $\widehat{\Box\phi}$. QED

Now we can clinch the proof of our main result.

THEOREM 1.26 (COMPLETENESS) *For any set of formulas* Γ*,*

$$\text{if} \quad \Gamma \models_{\mathcal{L}} \varphi \quad \text{then} \quad \Gamma \vdash_{\mathbf{S4}} \varphi.$$

Proof Suppose that $\Gamma \nvdash_{\mathbf{S4}} \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is consistent, and by the Lindenbaum lemma it can be extended to a maximally consistent set x. By the truth lemma, $M^{\mathcal{L}}, x \models_{\mathcal{L}} \neg\varphi$, whence $M^{\mathcal{L}}, x \not\models_{\mathcal{L}} \varphi$, and we have constructed the required counter-model. QED

COROLLARY 1.27 **S4** *is the logic of the class of all topological spaces.*

We note that the whole construction in the completeness proof above would also work if we restricted attention to the *finite* language consisting of the initial formula and all its subformulas. This means that we only get finitely many maximally consistent sets, and so non-provable formulas can be refuted on *finite models*, whose size is effectively computable from the formula itself.

COROLLARY 1.28

1 **S4** *is the logic of the class of all* finite *topological spaces.*

2 **S4** *has the* effective finite model property *with respect to the class of topological spaces.*

Incidentally, this also shows that validity in **S4** is *decidable*, but we forego such complexity issues in this chapter.

Comparing our construction with the standard modal Henkin model $\langle X^{\mathcal{L}}, R^{\mathcal{L}}, \models_{\mathcal{L}}\rangle$ for **S4**, the basic sets of our topology $\mathcal{X}^{\mathcal{L}}$ are $R^{\mathcal{L}}$-upward closed. Hence, every open of $\mathcal{X}^{\mathcal{L}}$ is $R^{\mathcal{L}}$-upward closed, and $\mathcal{X}^{\mathcal{L}}$ is weaker than the topology $\tau_{R^{\mathcal{L}}}$ corresponding to $R^{\mathcal{L}}$. In particular, our canonical topological space is *not* an Alexandroff space. For further aspects of the above construction consult (Aiello et al., 2003, Sec. 3).

2.5 Completeness in special spaces

We have already seen that **S4** is the logic of all topological spaces. But there are classical results with much more mathematical content such as McKinsey and Tarski's beautiful theorem that **S4** is also the logic of any dense-in-itself metric separable space. Here we concentrate on three spaces that play an important role in mathematics — the Cantor space $\mathbb{C}$, the rational line $\mathbb{Q}$, and the real line $\mathbb{R}$ — and sketch three proofs, taken respectively from Aiello et al., 2003; Benthem et al., 2005; Bezhanishvili and Gehrke, 2005, that **S4** is the logic of each of these spaces.

2.5.1 Completeness w.r.t. $\mathbb{C}$. We first show that **S4** is the logic of the Cantor space $\mathbb{C}$. Our exposition is rather sketchy. For full details we refer the reader to (Aiello et al., 2003, Sec. 4.1).

Suppose an **S4**-frame $\mathfrak{F} = \langle W, R\rangle$ is given. We recall that $\mathfrak{F}$ is *rooted* if there exists $r \in W$ — called a *root* of $\mathfrak{F}$ — such that rRw for each $w \in W$. We call $C \subseteq W$ a *cluster* if for all $w, v \in C$ we have wRv and vRw. A cluster C is called *simple* if it consists of a single point, and *proper* if it consists of more than one point. The next theorem will aid in proving that **S4** is the logic of $\mathbb{C}$.

THEOREM 1.29 *(Aiello et al., 2003)* **S4** *is complete with respect to finite rooted* **S4***-frames whose every cluster is proper.*

Now let a formula φ be not provable in **S4**. By Theorem 1.29, φ can be refuted in a finite rooted **S4**-model $M = \langle W, R, \nu\rangle$, with a root r, whose every cluster is proper. We transform the latter into a counterexample on the Cantor space $\mathbb{C}$. Our technique is *selective unravelling*, a refinement of the technique of *unravelling* in modal logic (see, e.g., Blackburn et al., 2001). We select those infinite paths of M that are in a 1-1 correspondence with infinite paths of the full infinite binary tree $\mathcal{T}_2$ (see Fig. 1.9).

We start with a root r and announce (r) as a selective path. Then if $(w_1, \ldots, w_k)$ is already a selective path, we introduce a *left* move by

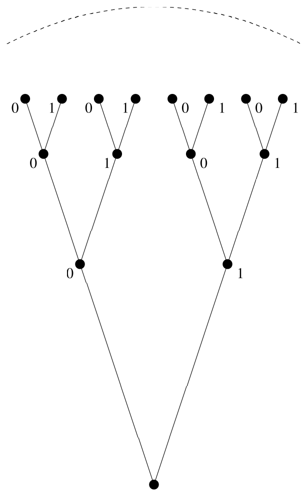


Figure 1.9. $\mathcal{T}_2$.

announcing $(w_1, \ldots, w_k, w_k)$ as a selective path; and we introduce a *right* move by announcing $(w_1, \ldots, w_k, w_{k+1})$ as a selective path if $w_k R w_{k+1}$ and $w_k \neq w_{k+1}$. (Since we assumed that every cluster of W is proper, such w_{k+1} exists for every w_k.) We call an infinite path σ of W *selective* if every initial segment of σ is a finite selective path of W. We denote by Σ the set of all infinite selective paths of W. For a finite selective path $(w_1, \ldots, w_k)$, let

$$B_{(w_1,\ldots,w_k)} = \{\sigma \in \Sigma : \ \sigma \text{ has an initial segment } (w_1, \ldots, w_k)\}.$$

Define a topology τ_Σ on Σ by introducing

$$\mathcal{B}_\Sigma = \{B_{(w_1,\ldots,w_k)} : (w_1, \ldots, w_k) \text{ is a finite selective path of } W\}$$

as a basis.

To see that $\mathcal{B}_\Sigma$ is a basis, observe that $B_{(r)} = \Sigma$, and that

$$B_{(w_1,\ldots,w_k)} \cap B_{(v_1,\ldots,v_m)} = \begin{cases} B_{(w_1,\ldots,w_k)} & \text{if } (v_1, \ldots, v_m) \text{ is an initial} \\ & \text{segment of } (w_1, \ldots, w_k), \\ B_{(v_1,\ldots,v_m)} & \text{if } (w_1, \ldots, w_k) \text{ is an initial} \\ & \text{segment of } (v_1, \ldots, v_m), \\ \emptyset & \text{otherwise.} \end{cases}$$

In order to define ν_Σ, note that every infinite selective path σ of W either gets stable or keeps cycling. In other words, either $\sigma =$

$(w_1, \ldots, w_k, w_k, \ldots)$ or $\sigma = (w_1, \ldots, w_n,\ w_{n+1},\ \ldots)$ where w_i belongs to some cluster $C \subseteq W$ for $i > n$. In the former case we say that w_k *stabilizes* σ, and in the latter — that σ *keeps cycling* in C. Now define ν_Σ on Σ by putting

$$\sigma \in \nu_\Sigma(p) \text{ iff } \begin{cases} w_k \models p & \text{if } w_k \text{ stabilizes } \sigma, \\ \\ \rho(C) \models p & \text{if } \sigma \text{ keeps cycling in } C \subseteq W, \text{ where } \rho(C) \text{ is} \\ & \text{some arbitrarily chosen representative of } C. \end{cases}$$

All we need to show is that $\langle \Sigma, \tau_\Sigma \rangle$ is homeomorphic to the Cantor space, and that $M_\Sigma = \langle \Sigma, \tau_\Sigma, \nu_\Sigma \rangle$ is topo-bisimilar to the initial M. In order to show the first claim, let us recall that the Cantor space is homeomorphic to the countable topological product of the two element set $\mathbf{2} = \{0, 1\}$ with the discrete topology. To picture the Cantor space, one can think of the full infinite binary tree $\mathcal{T}_2$; starting at the root, one associates 0 to every left-son of a node and 1 to every right-son. Then points of the Cantor space are infinite paths of $\mathcal{T}_2$. This together with the construction of Σ immediately gives us that $\langle \Sigma, \tau_\Sigma \rangle$ is homeomorphic to $\mathbb{C}$.

Finally, we show that M_Σ is topo-bisimilar to M. Define $F : \Sigma \to W$ by putting

$$F(\sigma) = \begin{cases} w_k & \text{if } w_k \text{ stabilizes } \sigma, \\ \rho(C) & \text{if } \sigma \text{ keeps cycling in } C. \end{cases}$$

Obviously F is well-defined, and is actually surjective. (For any $w_k \in W$, we have $F(\sigma_0,\ w_k, w_k, \ldots) = w_k$, where σ_0 is a selective path from w_1 to w_k.)

Proposition 1.30 *F is a total topo-bisimulation between $M_\Sigma = \langle \Sigma, \tau_\Sigma, \nu_\Sigma \rangle$ and $M = \langle W, R, \nu \rangle$.*

Proof (Sketch) With $\langle W, R \rangle$ we can associate a finite topological space $\langle W, \tau_R \rangle$. The set $\{R(v) : v \in W\}$ forms a basis for τ_R. Now the function $F : \langle \Sigma, \tau_\Sigma \rangle \to \langle W, \tau_R \rangle$ is continuous because

$$F^{-1}(R(v)) = \bigcup \{B_{(w_1, \ldots, w_k)} :\ vRw_k\}$$

for each $v \in W$, and F is open because

$$F(B_{(w_1, \ldots, w_k)}) = R(w_k)$$

for each basic open $B_{(w_1, \ldots, w_k)}$ of $\langle \Sigma, \tau_\Sigma \rangle$. Therefore, F is an interior map. Moreover, as follows from the definition of ν_Σ,

$$\sigma \in \nu_\Sigma(p) \text{ iff } F(\sigma) \in \nu(p).$$

Since every interior map satisfying this condition is a topo-bisimulation, so is our F. QED

THEOREM 1.31 **S4** *is the logic of* $\mathbb{C}$.

Proof Suppose $\mathbf{S4} \nvdash \varphi$. Then there is a finite rooted model M such that every cluster of M is proper and M refutes φ. Since $\mathbb{C}$ is homeomorphic to $\langle \Sigma, \tau_\Sigma \rangle$, by Proposition 1.30, there exists a valuation $\nu_{\mathbb{C}}$ on $\mathbb{C}$ such that $\langle \mathbb{C}, \nu_{\mathbb{C}} \rangle$ is topo-bisimilar to M. Hence, φ is refuted on $\mathbb{C}$. QED

2.5.2 Completeness w.r.t. $\mathbb{Q}$. Now we show that **S4** is also the logic of the rational line $\mathbb{Q}$. Our proof is taken from Benthem et al., 2005 and in it we rely on the following two well-known results.

THEOREM 1.32 *(van Benthem-Gabbay)* **S4** *is complete with respect to* $\mathcal{T}_2$.

Proof For a proof see, e.g., Goldblatt, 1980, Theorem 1 and the subsequent discussion. QED

THEOREM 1.33 *(Cantor) Every countable dense linear ordering without endpoints is isomorphic to* $\mathbb{Q}$.

Proof For a proof see, e.g., Kuratowski and Mostowski, 1976, p. 217, Theorem 2. QED

REMARK 1.34 *We recall that if* $\langle X, < \rangle$ *is a linearly ordered set and* $x, y \in X$ *with* $x < y$, *then the* open interval (x, y) *is the set* $\{z \in X : x < z < y\}$. *If we view linearly ordered sets as topological spaces using the set of open intervals as a basis for the topology, then it follows from Cantor's theorem that every countable dense linear ordering without endpoints is (as a topological space) homeomorphic to* $\mathbb{Q}$.

We are now ready to proceed with the proof.

THEOREM 1.35 **S4** *is complete with respect to* $\mathbb{Q}$.

Proof Our strategy is as follows. We use completeness of **S4** with respect to $\mathcal{T}_2$, view $\mathcal{T}_2$ as an Alexandroff space, define a dense subset X of $\mathbb{Q}$ without endpoints, and establish a topo-bisimulation between X and $\mathcal{T}_2$. This will allow us to transfer counterexamples from $\mathcal{T}_2$ to X, which by Cantor's theorem is order-isomorphic, and hence homeomorphic to $\mathbb{Q}$.

Let $X = \bigcup_{n\in\omega} X_n$, where $X_0 = \{0\}$ and

$$X_{n+1} = X_n \cup \{x - \frac{1}{3^n}, x + \frac{1}{3^n} \ : \ x \in X_n\}$$

Claim 1.36 *For $n > 0$ and $x, y \in X_n$, $x \neq y$ implies $|x - y| \geq \frac{1}{3^{n-1}}$.*

Proof By induction on n. If $n = 1$, then $X_1 = \{0, 1, -1\}$, and so $x \neq y$ implies $|x - y| \geq 1$. That the claim holds for $n = k+1$ is also not hard to see. Note that if $u, v \in X_{n-1}$ with $u \neq v$, then, by induction hypothesis, $|u - v| \geq \frac{1}{3^{n-2}}$ and hence $|(u + \frac{1}{3^{n-1}}) - (v - \frac{1}{3^{n-1}})| \geq \frac{1}{3^{n-1}}$. QED

It follows from Claim 1.36 that $\langle X, <\rangle$ is a countable dense linear ordering without endpoints, thus order-isomorphic, and hence homeomorphic to $\mathbb{Q}$. It also follows that for each $x \in X$ with $x \neq 0$ there exists n_x with $x \in X_{n_x}$ and $x \notin X_{n_x - 1}$, and that there is a unique $y \in X_{n_x - 1}$ with $x = y - \frac{1}{3^{n_x - 1}}$ or $x = y + \frac{1}{3^{n_x - 1}}$. Therefore, the open X-intervals $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ form a basis for the order-topology on X.

Now we define f from X onto $\mathcal{T}_2$ by recursion (see Fig. 1.10): If $x = 0$ then we let $f(0)$ be the root r of $\mathcal{T}_2$; if $x \neq 0$ then $x \in X_{n_x} - X_{n_x - 1}$ and we let

$$f(x) = \begin{cases} \text{the left successor of } f(y) & \text{if } x = y - \frac{1}{3^{n_x - 1}} \\ \text{the right successor of } f(y) & \text{if } x = y + \frac{1}{3^{n_x - 1}} \end{cases}$$

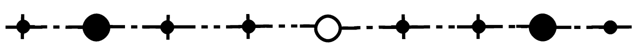


Figure 1.10. The first stages of the labelling in the completeness proof for **S4**.

Claim 1.37 *f is an interior map.*

Proof (Sketch) We recall that a basis for the Alexandroff topology on $\mathcal{T}_2$ is $\mathcal{B} = \{B_t\}_{t\in\mathcal{T}_2}$ where $B_t = \{s \in \mathcal{T}_2 \ : \ tRs\}$. Now f is open because for a basic open X-interval $(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$, we have $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)}$. Also f is continuous because for each $t \in \mathcal{T}_2$, the f-inverse image of B_t is open. Indeed, if $x \in f^{-1}(B_t)$, then $f(x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) = B_{f(x)} \subseteq B_t$, implying that there exists an open interval $I = (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}})$ of x such that $I \subseteq f^{-1}(B_t)$. Thus, f is interior. QED

To complete the proof, if $\mathbf{S4} \not\vdash \varphi$, then by Theorem 1.32, there is a valuation ν on $\mathcal{T}_2$ such that $\langle \mathcal{T}_2, \nu\rangle, r \not\models \varphi$. Define a valuation ξ on X

by $\xi(p) = f^{-1}(\nu(p))$. Since f is an interior map, and $f(0) = r$, we have that 0 and r are topo-bisimilar. Therefore, $\langle X, \xi\rangle, 0 \not\models \varphi$. Now since X is homeomorphic to $\mathbb{Q}$, we obtain that φ is also refutable on $\mathbb{Q}$. QED

2.5.3 Completeness w.r.t. $\mathbb{R}$. Finally, we show that **S4** is also the logic of the real line $\mathbb{R}$. There are at least three different proofs of this result. The original one is a particular case of a more general theorem (McKinsey and Tarski, 1944) that **S4** is the logic of any dense-in-itself metric separable space (see also Rasiowa and Sikorski, 1963). The other two can be found in (Aiello et al., 2003; Bezhanishvili and Gehrke, 2005). Here we sketch the proof given in Bezhanishvili and Gehrke, 2005, where the construction of the Cantor set on any bounded interval of $\mathbb{R}$ is used to show that every finite rooted **S4**-frame is an interior image of $\mathbb{R}$.

Suppose $a, b \in \mathbb{R}$, $a < b$, and $I = (a, b)$. We recall that the Cantor set $\mathbb{C}$ is constructed inside I by taking out open intervals from I infinitely many times. More precisely, in step 1 of the construction the open interval

$$I_1^1 = (a + \frac{b-a}{3}, a + \frac{2(b-a)}{3})$$

is taken out. We denote the remaining closed intervals by J_1^1 and J_2^1. In step 2 the open intervals

$$I_1^2 = (a + \frac{b-a}{3^2}, a + \frac{2(b-a)}{3^2}) \text{ and } I_2^2 = (a + \frac{7(b-a)}{3^2}, a + \frac{8(b-a)}{3^2})$$

are taken out. We denote the remaining closed intervals by J_1^2, J_2^2, J_3^2, and J_4^2. In general, in step m the open intervals $I_1^m, \ldots, I_{2^{m-1}}^m$ are taken out, and the closed intervals $J_1^m, \ldots, J_{2^m}^m$ remain.

Our immediate goal is to show that every finite tree is an interior image of I. We first show that the tree T of depth 2 and branching n shown in Fig. 1.11 is an interior image of I, and then extend this result to any finite tree by induction on the depth of the tree.

LEMMA 1.38 *T is an interior image of I.*

Proof Define $f_I^T : I \to T$ by putting

$$f_I^T(x) = \begin{cases} t_k, & \text{if } x \in \bigcup_{m \equiv k (mod\ n)} \bigcup_{p=1}^{2^{m-1}} I_p^m \\ r, & \text{otherwise} \end{cases}$$

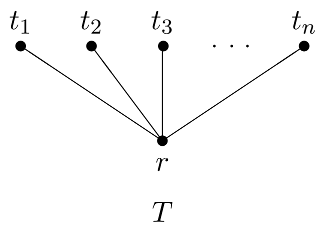


Figure 1.11. The tree of depth 2 and branching n.

Obviously, f_I^T is a well-defined onto map. Moreover,

$$(f_I^T)^{-1}(t_k) = \bigcup_{m \equiv k (mod\ n)} \bigcup_{p=1}^{2^{m-1}} I_p^m \quad \text{and} \quad (f_I^T)^{-1}(r) = \mathbb{C}$$

Since $\{\emptyset, \{t_1\}, \dots, \{t_n\}, T\}$ is a family of basic open subsets of T, it is obvious that f_I^T is continuous. Suppose U is an open interval of I. If $U \cap \mathbb{C} = \emptyset$, then $f_I^T(U) \subseteq \{t_1, \dots, t_n\}$, and so $f_I^T(U)$ is open. If $U \cap \mathbb{C} \neq \emptyset$, then there exists $c \in U \cap \mathbb{C}$. Since $c \in \mathbb{C}$ we have $f_I^T(c) = r$. From $c \in U$ it follows that there is $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U$. We pick m so that $\frac{b-a}{3^m} < \varepsilon$. As $c \in \mathbb{C}$, there is $k \in \{1, \dots, 2^m\}$ such that $c \in J_k^m$. Moreover, since the length of J_k^m is equal to $\frac{b-a}{3^m}$, we have that $J_k^m \subseteq U$. Therefore, U contains the points removed from J_k^m in the subsequent iterations in the construction of $\mathbb{C}$. Thus, $f_I^T(U) \supseteq \{t_1, \dots, t_n\}$ and $f_I^T(U) = T$. Hence, $f_I^T(U)$ is open for any open interval U of I. It follows that f_I^T is an onto interior map. QED

THEOREM 1.39 *Every finite tree of branching $n \geq 1$ is an interior image of I.*

Proof Suppose T is a finite tree of branching $n \geq 1$. Without loss of generality we may assume that the depth of T is $d + 1$, where $d \geq 2$. Then we can represent T as shown in Fig. 1.12, where $t_1, \dots, t_{n^d}$ are the elements of T of depth 2, and T_d is the subtree of T of all elements of T of depth ≥ 2. We note that for each $k \in \{1, \dots, n^d\}$ the upset $R(t_k)$ is isomorphic to the tree of depth 2 and branching n, and that T_d is the tree of depth d and branching n. So by the induction hypothesis, there is an onto interior map from I onto T_d. Also, by Lemma 1.38, there exists an onto interior map from I onto each $R(t_k)$. Now putting these maps together produces an onto interior map from I onto T. For the details we refer to Bezhanishvili and Gehrke, 2005, Theorem 8. QED

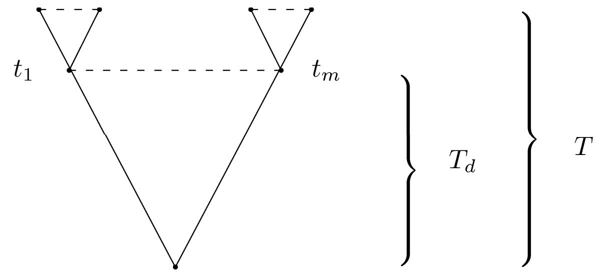


Figure 1.12. T and T_d.

The following useful assertion is now an easy consequence.

COROLLARY 1.40 *Every finite rooted partially ordered* **S4***-frame is an interior image of* $\mathbb{R}$*.*

Proof Since every finite rooted partially ordered **S4**-frame is a p-morphic image of some finite tree of branching $n \geq 1$ (Bezhanishvili and Gehrke, 2005, Lemma 4), it follows from Theorem 1.39 that every finite rooted partially ordered **S4**-frame is an interior image of any bounded open interval $I \subseteq \mathbb{R}$. Since I is homeomorphic to $\mathbb{R}$, the corollary follows.

QED

We now extend on Corollary 1.40 and show that all finite rooted **S4**-frames are interior images of $\mathbb{R}$. Suppose $\mathfrak{F}$ is an **S4**-frame. We define an equivalence relation $\sim$ on $\mathfrak{F}$ by $w \sim v$ iff w, v belong to the same cluster. Let $\mathfrak{F}/\sim$ denote the *skeleton* of $\mathfrak{F}$. That is, $\mathfrak{F}/\sim$ is the quotient of $\mathfrak{F}$ by $\sim$, and $R_\sim$ is defined on $W/\sim$ componentwise.

DEFINITION 1.41 *We call an* **S4***-frame* $\mathfrak{F}$ *a* quasi-tree *if* $\mathfrak{F}/\sim$ *is a tree.*

Suppose Q is a quasi-tree. We say that the *swelling* of Q is q if every cluster of Q consists of exactly q elements. Again, our immediate goal is to show that every finite quasi-tree is an interior image of I. This we show by first obtaining the quasi-tree Q of depth 2, branching n, and swelling q, shown in Fig. 1.13, as an interior image of I, and then extending this result to any finite quasi-tree by induction on the depth of the quasi-tree. For this we use the following lemma.

LEMMA 1.42 *(Bezhanishvili and Gehrke, 2005, Lemma 11) If* X *has a countable basis and every countable subset of* X *is nowhere dense, then*

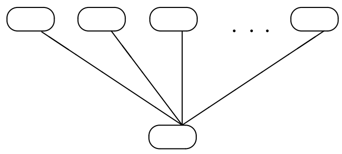


Figure 1.13. Quasi-tree of depth 2, branching n, and swelling q.

for each natural number n there exist disjoint dense and nowhere dense subsets $A_1, \ldots, A_n$ of X such that $X = \bigcup_{i=1}^{n} A_i$.

LEMMA 1.43 *Q is an interior image of I.*

Proof We denote the least cluster of Q by r and its elements by $r_1, \ldots, r_q$. Also for $1 \leq i \leq n$ we denote the ith maximal cluster of Q by t^i and its elements by $t^i_1, \ldots, t^i_q$. Since the Cantor set $\mathbb{C}$ satisfies the conditions of Lemma 1.42, it can be divided into q-many disjoint dense and nowhere dense subsets $\mathbb{C}_1, \ldots, \mathbb{C}_q$. Also each I^m_p ($1 \leq p \leq 2^{m-1}$, $m \in \omega$) satisfies the conditions of Lemma 1.42, and so each I^m_p can be divided into q-many disjoint dense and nowhere dense subsets $(I^m_p)^1, \ldots, (I^m_p)^q$. Suppose $1 \leq k \leq q$. We define $f^Q_I : I \to Q$ by putting

$$f^Q_I(x) = \begin{cases} t^i_k, & \text{if } x \in \bigcup_{m \equiv i (mod\ n)} \bigcup_{p=1}^{2^{m-1}} (I^m_p)^k \\ r_k, & \text{if } x \in \mathbb{C}_k \end{cases}$$

It is clear that f^Q_I is a well-defined onto map. Similar to Lemma 1.38 we have

$$(f^Q_I)^{-1}(t^i) = \bigcup_{m \equiv i (mod\ n)} \bigcup_{p=1}^{2^{m-1}} I^m_p \quad \text{and} \quad (f^Q_I)^{-1}(r) = \mathbb{C}$$

Hence, f^Q_I is continuous. To show that f^Q_I is open let U be an open interval in I. If $U \cap \mathbb{C} = \emptyset$, then $f^Q_I(U) \subseteq \bigcup_{i=1}^{n} t^i$. Moreover, since $(I^m_p)^1, \ldots, (I^m_p)^q$ partition I^m_p into q-many disjoint dense and nowhere dense subsets, $U \cap I^m_p \neq \emptyset$ implies $U \cap (I^m_p)^k \neq \emptyset$ for every $k \in \{1, \ldots, q\}$. Hence, if $f^Q_I(U)$ contains an element of a cluster t^i, it contains the whole cluster. Thus, $f^Q_I(U)$ is open. Now suppose $U \cap \mathbb{C} \neq \emptyset$. Since

$\mathbb{C}_1, \ldots, \mathbb{C}_q$ partition $\mathbb{C}$ into q-many disjoint dense and nowhere dense subsets, $U \cap \mathbb{C}_k \neq \emptyset$ for every $k \in \{1, \ldots, q\}$. Hence, $r \subseteq f_I^Q(U)$. Moreover, the same argument as in the proof of Lemma 1.38 guarantees that every point greater than points in r also belongs to $f_I^Q(U)$. Thus $f_I^Q(U) = Q$, implying that f_I^Q is an onto interior map. QED

THEOREM 1.44 *Every finite quasi-tree of branching n and swelling q is an interior image of I.*

Proof This follows along the same lines as the proof of Theorem 1.39 but is based on Lemma 1.43 instead of Lemma 1.38. QED

COROLLARY 1.45 *Every finite rooted* **S4***-frame is an interior image of $\mathbb{R}$.*

Proof This follows along the same lines as the proof of Corollary 1.40 but is based on the fact that every finite rooted **S4**-frame is a p-morphic image of some finite quasi-tree of branching n and swelling q (Bezhanishvili and Gehrke, 2005, Lemma 5) and Theorem 1.44. QED

THEOREM 1.46 **S4** *is complete with respect to $\mathbb{R}$.*

Proof If $\mathbf{S4} \not\vdash \varphi$, then there exists a finite rooted **S4**-model $M = \langle W, R, \nu \rangle$, with a root r, such that $M, r \not\models \varphi$. By Corollary 1.45 there exists an onto interior map $f : \mathbb{R} \to W$. Define a valuation ξ on $\mathbb{R}$ by putting $\xi(p) = f^{-1}(\nu(p))$. Then f is a total topo-bisimulation between $\langle \mathbb{R}, \xi \rangle$ and M. Thus, there exists a point $x \in \mathbb{R}$ such that $x \not\models \varphi$. QED

We recall that a subset A of $\mathbb{R}$ is *convex* if $x, y \in A$ and $x \leq z \leq y$ imply that $z \in A$.

COROLLARY 1.47 **S4** *is complete with respect to boolean combinations of countable unions of convex subsets of $\mathbb{R}$.*

Proof Let $f : \mathbb{R} \to W$ be the onto interior map from the proof of Theorem 1.46. Observe that for each $w \in W$ we have that $f^{-1}(w)$ is a boolean combination of countable unions of convex subsets of $\mathbb{R}$ (see Bezhanishvili and Gehrke, 2005, Theorem 15). The result follows. QED

2.6 The landscape of spatial logics over S4

As we have already seen, **S4** is the logic of all topological spaces when interpreting $\Diamond$ as closure. In addition, **S4** turned out to be the logic of the Cantor space $\mathbb{C}$, the rational line $\mathbb{Q}$, the real line $\mathbb{R}$, or more generally, any dense-in-itself metric separable space. These results, although with a lot of mathematical content, also indicate serious limitations of the basic modal language in expressing various topological properties. For example, the completeness of **S4** with respect to any dense-in-itself metric separable space already implies that such topological properties as being dense-in-itself, metric, or separable are not definable in the basic modal language. Here we address the topological definability issue, as well as review several normal extensions of **S4** that are complete with respect to interesting classes of topological spaces.

Topological definability and undefinability: Suppose a class K of topological spaces is given. We say that K is *topologically definable* or simply *topo-definable* if there exists a set of modal formulas Γ such that for each topological space $\mathcal{X}$ we have $\mathcal{X} \in K$ iff $\mathcal{X} \models \Gamma$. Topological completeness of **S4** tells us that the class **Top** of *all* topological spaces is topo-definable (by the formula $\top$ over **S4**). However, as we will see below, many important classes of topological spaces such as the classes of compact or connected spaces are *not* topo-definable. For this we will need the following theorem, first established in Gabelaia, 2001 and Benthem et al., 2003.

Theorem 1.48 *Suppose φ is an arbitrary modal formula.*

1 If $\mathcal{Y}$ is an interior image of $\mathcal{X}$, then $\mathcal{X} \models \varphi$ implies $\mathcal{Y} \models \varphi$.

2 If $\mathcal{Y}$ is an open subspace of $\mathcal{X}$, then $\mathcal{X} \models \varphi$ implies $\mathcal{Y} \models \varphi$.

3 If $\mathcal{X}$ is the topological sum of $\{\mathcal{X}_i\}_{i\in I}$, then $\mathcal{X} \models \varphi$ iff $\mathcal{X}_i \models \varphi$ for each $i \in I$.

Now we are ready to show that compactness, connectedness, and the separation axioms T_0, T_d, T_1, and T_2 are not topo-definable.

Proposition 1.49 *(Gabelaia, 2001)*

1 Neither compactness nor connectedness is topo-definable.

2 None of the separation axioms T_0, T_d, T_1, and T_2 is topo-definable.

Proof (1) Let $\mathcal{X} = \langle \{x\}, \tau \rangle$ be a singleton set with the discrete topology. Then obviously $\mathcal{X}$ is both compact and connected. On the other hand, any infinite topological sum of $\mathcal{X}$ is neither compact nor connected. Now apply Theorem 1.48(3).

(2) Let $\mathcal{X} = \langle\{x, y\}, \tau\rangle$ be a two point set with the trivial topology. Then obviously $\mathcal{X}$ does not satisfy any of the four separation axioms. Define $f : \mathbb{R} \to \{x, y\}$ by

$$f(r) = \begin{cases} x, & \text{if } r \in \mathbb{Q} \\ y, & \text{otherwise} \end{cases}$$

Then it is easy to see that f is an onto interior map. Now observe that $\mathbb{R}$ satisfies all the four separation axioms and apply Theorem 1.48(1).

QED

Remark 1.50 *$\mathbb{R}$ also satisfies stronger separation axioms such as T_3 (regularity), $T_{3\frac{1}{2}}$ (complete regularity), T_4 (normality), T_5, and T_6. Therefore, Proposition 1.49 also implies that none of $T_3, T_{3\frac{1}{2}}, T_4, T_5$, and T_6 is topo-definable.*

Proposition 1.49 indicates that the basic modal language $\mathcal{L}$ is not expressive enough for topological purposes. In Sec. 3 we will consider several enrichments of $\mathcal{L}$ and show that some of the topological properties not expressible in $\mathcal{L}$ can be expressed in its various enrichments. Nevertheless, it seems to be a natural question to characterize those classes of topological spaces that *can* be defined in $\mathcal{L}$. An answer to this question was given in Gabelaia, 2001, where a topological analogue of the Goldblatt-Thomason theorem was established.

Definition 1.51 *(Gabelaia, 2001) Let $\mathcal{X} = \langle X, \tau\rangle$ be a topological space. We let uf(X) denote the set of ultrafilters of the powerset $\mathcal{P}(X)$ and define R on uf(X) by*

$$wRu \text{ iff } A \in u \text{ implies } Cl(A) \in w$$

for each $A \subseteq X$. It is easy to verify that R is reflexive and transitive on uf(X). Let τ_R denote the Alexandroff topology on uf(X) generated by R. We call ae$(\mathcal{X}) = \langle$uf$(X), \tau_R\rangle$ the Alexandroff extension *of $\mathcal{X}$.*

We note that if the original topology on X is Alexandroff, then the Alexandroff extension of $\mathcal{X}$ can be obtained by first taking the ultrafilter extension of $\mathcal{X}$ and then taking the corresponding Alexandroff space. Alexandroff extensions of topological spaces turn out to be crucial for obtaining a topological version of the Goldblatt-Thomason theorem.

Definition 1.52 *We say that a class K of topological spaces* reflects *Alexandroff extensions if for each topological space $\mathcal{X}$ we have ae$(\mathcal{X}) \in K$ implies $\mathcal{X} \in K$.*

Theorem 1.53 *(Gabelaia, 2001) Let K be a class of topological spaces closed under formation of Alexandroff extensions. Then K is modally definable iff it is closed under taking open subspaces, interior images, topological sums, and it reflects Alexandroff extensions.*

Several refinements of Theorem 1.53 and extensions to richer languages can be found in Gabelaia and Sustretov, 2005. Below we present a number of topo-definable classes of spaces, as well as normal extensions of **S4** being complete with respect to topologically interesting classes of spaces.

The logic of discrete spaces: It is rather easy to see that $\mathcal{X} \models p \to \Box p$ iff every subset of X is open iff $\mathcal{X}$ is discrete. Therefore, $p \to \Box p$ (or equivalently $\Diamond p \to p$) topo-defines the class of discrete spaces.

S5 and trivial topologies: We observe that

$$\begin{aligned} \mathcal{X} \models p \to \Box\Diamond p \quad & \text{iff } A \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \text{ for each } A \subseteq X \\ & \text{iff } \mathrm{Cl}(A) \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \text{ for each } A \subseteq X \\ & \text{iff every closed subset of } X \text{ is open.} \end{aligned}$$

Therefore, $p \to \Box\Diamond p$ (or equivalently $\Diamond p \to \Box\Diamond p$) topo-defines the class of topological spaces in which every closed subset is open.

S4.2 and extremally disconnected spaces: We recall that

$$\mathbf{S4.2} = \mathbf{S4} + (\Diamond\Box p \to \Box\Diamond p)$$

Now observe that

$$\begin{aligned} \mathcal{X} \models \Diamond\Box p \to \Box\Diamond p \quad & \text{iff } \mathrm{Cl}(\mathrm{Int}(A)) \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \text{ for each } A \subseteq X \\ & \text{iff } \mathrm{Cl}(\mathrm{Int}(A)) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \text{ for each } A \subseteq X \\ & \text{iff the closure of every open subset of } X \text{ is open} \\ & \text{iff } \mathcal{X} \text{ is extremally disconnected.} \end{aligned}$$

Therefore, $\Diamond\Box p \to \Box\Diamond p$ topo-defines the class of extremally disconnected spaces.

S4.1 and filters of dense sets: We recall that

$$\mathbf{S4.1} = \mathbf{S4} + (\Box\Diamond p \to \Diamond\Box p)$$

For a topological space $\mathcal{X}$ let $\mathcal{D}(X)$ be the set of all dense subsets of X. Now $\mathcal{X} \models \Box\Diamond p \to \Diamond\Box p$ iff $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(\mathrm{Int}(A))$ for each $A \subseteq X$. As shown in (Bezhanishvili et al., 2003, p. 293, Proposition 2.1), the last condition is equivalent to $\mathcal{D}(X)$ being a filter. So, $\Box\Diamond p \to \Diamond\Box p$ topo-defines the class of topological spaces in which $\mathcal{D}(X)$ is a filter.

S4.Grz and hereditarily irresolvable spaces: We recall that

$$\mathbf{S4.Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to \Box p$$

We also recall that a space $\mathcal{X}$ is *resolvable* if it can be represented as the union of two disjoint dense subsets, that it is *irresolvable* if it is not resolvable, that it is *hereditarily irresolvable* if every subspace of $\mathcal{X}$ is irresolvable, and that it is *scattered* if every subspace of $\mathcal{X}$ has an isolated point.

For each $A \subseteq X$ let $\rho(A) = A \cap \mathrm{Cl}(\mathrm{Cl}(A) - A)$. We observe that $\mathcal{X} \models \Box(\Box(p \to \Box p) \to p) \to \Box p$ iff $A \subseteq \mathrm{Cl}(A - \rho(A))$ for each $A \subseteq X$. As is shown in (Bezhanishvili et al., 2003, p. 295, Theorem 2.4), the last condition is equivalent to $\mathcal{X}$ being hereditarily irresolvable. Therefore, $\Box(\Box(p \to \Box p) \to p) \to \Box p$ topo-defines the class of hereditarily irresolvable spaces. Now since **S4.Grz** is complete with respect to its relational semantics and since for an Alexandroff space $\mathcal{X}_{\mathfrak{F}} = \langle X, \tau_R \rangle$ the notions of hereditarily irresolvable and scattered coincide with each other and with the notion of $\mathfrak{F}$ having no infinite ascending chains (Gabelaia, 1999), we obtain that **S4.Grz** is the logic of hereditarily irresolvable spaces, and also the logic of scattered spaces. As scattered spaces are a proper subclass of hereditarily irresolvable spaces (Bezhanishvili et al., 2003), they are *not* topo-definable. Moreover, since **S4.Grz** is also the logic of ordinals (Abashidze and Esakia, 1987), ordinals are *not* topo-definable either.

Euclidean hierarchy: Corollary 1.47 shows that the logic of boolean combinations of countable unions of convex subsets of $I\!R$ is already **S4**. The logic becomes much stronger, however, if we restrict our attention to finite unions of convex subsets of $I\!R$.

We call a subset of $I\!R$ *serial* if it is a finite union of convex subsets of $I\!R$. Let $S(I\!R)$ denote the family of serial subsets of $I\!R$. Unlike the countable unions of convex subsets of $I\!R$, $S(I\!R)$ does form a boolean algebra. We call a valuation ν on $I\!R$ *serial* if $\nu(p) \in S(I\!R)$ for each propositional variable p. We call a formula φ *s-true* if it is true in $I\!R$ under a serial valuation, and we call φ *s-valid* if φ is s-true for each serial valuation. Let $L(S) = \{\varphi : \varphi \text{ is s-valid}\}$. It is easy to see that $L(S)$ is a normal extension of **S4**, we refer to as *the logic of serial subsets of* $I\!R$. The following theorem was first established in Aiello et al., 2003 (see also Benthem et al., 2003):

THEOREM 1.54 *$L(S)$ is the logic of the 2-fork frame $\mathfrak{F}$ shown in Fig. 1.14.*

For $n \geq 2$, we call $X \subseteq I\!R^n$ *hyper-rectangular convex* if $X = X_1 \times \cdots \times X_n$, where all the X_i's are convex subsets of $I\!R$. We also call $X \subseteq I\!R^n$ *n-chequered* if it is a finite union of hyper-rectangular convex subsets of $I\!R^n$. Let $CH(I\!R^n)$ denote the set of all n-chequered subsets of $I\!R^n$. Similar to $S(I\!R)$ we have that $CH(I\!R^n)$ forms a boolean algebra.

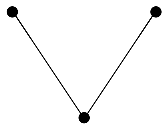


Figure 1.14. The 2-fork frame.

We call a valuation ν on $\mathbb{R}^n$ *n-chequered* if $\nu(p) \in CH(\mathbb{R}^n)$ for each propositional variable p. We call a formula φ *n-true* if it is true in $\mathbb{R}^n$ under an n-chequered valuation, and we call φ *n-valid* if φ is n-true for each n-chequered valuation. Let $L_n = \{\varphi : \varphi \text{ is n-valid}\}$. Similar to $L(S)$ we have that L_n is a normal extension of **S4**, we refer to as *the logic of n-chequered subsets of* $\mathbb{R}^n$. Moreover, the logics form a decreasing chain:

$$L(S) = L_1 \supset L_2 \supset L_3 \supset \cdots \supset L_n \supset \ldots$$

Let $\mathfrak{F}^n$ denote the Cartesian product of the 2-fork frame $\mathfrak{F}$ on itself n-times. The following theorem was proved in (Benthem et al., 2003):

THEOREM 1.55 *For $n \geq 2$ we have that L_n is the logic of $\mathfrak{F}^n$.*

In particular, the logic of chequered subsets of the real plane coincides with the logic of $\mathfrak{F}^2$. An illustration of $\mathfrak{F}^2$ is given in Fig. 1.15.

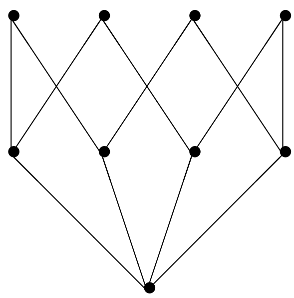


Figure 1.15. $\mathfrak{F}^2$.

We call $X \subseteq \mathbb{R}^\infty$ *∞-rectangular convex* if $X = \prod_{i=1}^{\infty} X_i$, where each X_i is a convex subset of $\mathbb{R}$, and all but finitely many of X_i's are equal to either $\mathbb{R}$ or $\emptyset$. We call $X \subseteq \mathbb{R}^\infty$ *∞-chequered* if it is a finite union of ∞-rectangular convex subsets of $\mathbb{R}^\infty$. Let $CH(\mathbb{R}^\infty)$ denote the set of ∞-chequered subsets of $\mathbb{R}^\infty$. Similar to each $CH(\mathbb{R}^n)$ we have that $CH(\mathbb{R}^\infty)$ forms a boolean algebra. We call a valuation ν on $\mathbb{R}^\infty$ *∞-chequered* if $\nu(p) \in CH(\mathbb{R}^\infty)$ for each propositional variable p. We call a formula φ ∞-*true* if it is true in $\mathbb{R}^\infty$ under a ∞-chequered valuation, and we call φ ∞-*valid* if φ is ∞-true for each ∞-chequered valuation. Let $L_\infty = \{\varphi : \varphi \text{ is } \infty\text{-valid}\}$. It is easy to see that L_∞ is a normal extension of **S4**, we refer to as *the logic of* ∞-*chequered subsets of* $\mathbb{R}^\infty$. The following theorem can be found in (Benthem et al., 2003):

THEOREM 1.56 $L_\infty = \bigcap L_n$.

To summarize, we obtained the logic $L(S)$ of serial subsets of the real line $\mathbb{R}$, as well as its natural generalizations — the logics L_n of sufficiently well-behaved n-chequered subsets of n-dimensional Euclidean spaces $\mathbb{R}^n$. Unlike the full modal logic of each Euclidean space $\mathbb{R}^n$, which coincides with **S4**, all logics L_n are different, forming a decreasing chain converging to the logic L_∞ of ∞-chequered subsets of $\mathbb{R}^\infty$. This provides us with a sort of *Euclidean hierarchy* in modal logic. It has been suggested by Litak, 2004 that L_∞ may be closely related to the quite differently motivated 'Logic of Problems' first defined by Medvedev.

3. Modal logic and topology. Further directions

In Sec. 2 we were chiefly concerned with the interpretation of $\Diamond$ as closure, the resulting logic **S4**, and the landscape of spatial logics over **S4**. We noticed that many important topological properties are not expressible in the basic modal language $\mathcal{L}$. In this section we will discuss several ways of increasing the expressive power of $\mathcal{L}$.

3.1 $\Diamond$ as derivative

There are at least two natural ways to increase the expressive power of $\mathcal{L}$. One is to add new modal operators to $\mathcal{L}$, and the other is to interpret the modal $\Diamond$ as a topological operator that is more expressive than the closure operator. We will consider adding new modal operators to $\mathcal{L}$ in Sec. 3.2 and 3.3. Here we outline some of the consequences of interpreting $\Diamond$ as derivative (first also suggested by McKinsey and Tarski, 1944). Since $\mathrm{Cl}(A) = A \cup d(A)$ for each $A \subseteq X$, the derivative operator is more expressive than the closure operator. We recall that $x \in d(A)$ iff $A \cap (U - \{x\}) \neq \emptyset$ for each open neighborhood U of x, that the co-derivative of A is $t(A) = X - d(X - A)$, and that $x \in t(A)$ iff there exists an open neighborhood U of x such that $U \subseteq A \cup \{x\}$.

Let $M = \langle \mathcal{X}, \nu \rangle$ be a topo-model. We define when a formula φ is *d-true at a point* $x \in X$ by induction on the length of φ:

- $x \models_d p$ iff $x \in \nu(p)$;
- $x \models_d \neg\varphi$ iff not $x \models_d \varphi$;

- $x \models_d \varphi \wedge \psi$ iff $x \models_d \varphi$ and $x \models_d \psi$;
- $x \models_d \Box\varphi$ iff $\exists U \in \tau(x \in U \;\&\; \forall y \in U - \{x\} \;\; y \models_d \varphi)$;

and hence, also

- $x \models_d \Diamond\varphi$ iff $\forall U \in \tau(x \in U \to \exists y \in U - \{x\} : y \models_d \varphi)$.

We say that φ is *d-true* in $M = \langle \mathcal{X}, \nu \rangle$ if φ is d-true at every $x \in X$. We say that φ is *d-valid* in $\mathcal{X}$ if φ is d-true in every model based on $\mathcal{X}$. Finally, we say that φ is *d-valid* in a class of topological spaces if φ is d-valid in every member of the class.

EXAMPLE 1.57

1 *We show that* $(p \wedge \Box p) \to \Box\Box p$ *is d-valid in* **Top**. *Let* $\mathcal{X} \in$ **Top**, $M = \langle \mathcal{X}, \nu \rangle$ *be a topo-model, and* $x \models_d p \wedge \Box p$ *for* $x \in X$. *Then* $x \models_d p$ *and there exists an open neighborhood* U *of* x *such that* $y \models_d p$ *for each* $y \in U - \{x\}$. *Therefore,* $y \models_d p$ *for each* $y \in U$. *But then* $y \models_d \Box p$ *for each* $y \in U$, *implying that* $x \models_d \Box\Box p$.

2 *That* $\Box(p \to q) \to (\Box p \to \Box q)$ *is d-valid in* **Top** *and that the necessitation rule preserves d-validity can be proved as in Example 1.18.*

DEFINITION 1.58 *Let* **wK4** *denote the modal logic* $\mathbf{K} + (p \wedge \Box p) \to \Box\Box p$. *Obviously* **wK4** *is weaker than* **K4**, *and we call* **wK4** weak **K4**.

It follows that the modal logic **wK4** is sound with respect to d-semantics. In fact, as was shown in Esakia, 2001, **wK4** is also complete with respect to d-semantics. First we discuss the connection between relational semantics of **wK4** and d-semantics; then we show that **wK4** is the d-logic of all topological spaces; after that we determine the connection between **S4** and **wK4**; finally, we discuss stronger spatial logics over **wK4**. Most results in this section are taken from (Esakia, 2001; Esakia, 2004; Bezhanishvili et al., 2005; Shehtman, 1990; Shehtman, 2006).

3.1.1 Weak K4.

DEFINITION 1.59 *Let* $\mathfrak{F}$ *be a frame. We call* $\mathfrak{F}$ weakly transitive *if* $\forall w, v, u \in W \;\; (wRv \;\&\; vRu \;\&\; w \neq u \to wRu)$.

It is known that **wK4** is sound and complete with respect to the class of all weakly transitive frames. In fact, **wK4** has the finite model property (Esakia, 2001). Because of this, we will sometimes refer to weakly transitive frames as **wK4**-frames. We call a weakly transitive frame $\mathfrak{F} = \langle W, R \rangle$ *rooted* if there exists $r \in W$ — called a *root* of $\mathfrak{F}$ — such that rRw for every $w \neq r$.

THEOREM 1.60 *(Esakia, 2001)* **wK4** *is complete with respect to finite rooted irreflexive* **wK4***-frames.*

DEFINITION 1.61 *Let $\mathfrak{F} = \langle X, R \rangle$ be a* **wK4***-frame. We denote by $\overline{\mathfrak{F}} = \langle X, \overline{R} \rangle$ the* reflexive closure *of $\mathfrak{F}$ (that is, $\overline{R}$ is obtained from R by adding* all *reflexive arrows), and by $\underline{\mathfrak{F}} = \langle X, \underline{R} \rangle$ the* irreflexive fragment *of $\mathfrak{F}$ (that is, $\underline{R}$ is obtained from R by deleting* all *reflexive arrows.*

For a **wK4**-frame $\mathfrak{F}$, it is obvious that $\overline{\mathfrak{F}}$ is an **S4**-frame, and that $\underline{\mathfrak{F}}$ is an irreflexive **wK4**-frame. Moreover, every **wK4**-frame is obtained either from an **S4**-frame by deleting some reflexive arrows or from an irreflexive **wK4**-frame by adding some reflexive arrows.

Given a **wK4**-frame $\mathfrak{F}$, we view $\overline{\mathfrak{F}}$ as an Alexandroff space. For $A \subseteq X$ we denote the derivative of A in $\overline{\mathfrak{F}}$ by $d_R(A)$. The next series of results is taken from (Esakia, 2001).

LEMMA 1.62 *Let $\mathfrak{F}$ be a* **wK4***-frame and $A \subseteq X$. In $\overline{\mathfrak{F}}$ we have $d_R(A) = \underline{R}^{-1}(A)$.*

Proof We observe that

$$\begin{array}{lll} x \in d_R(A) & \text{iff} & \text{for each open neighborhood } U \text{ of } x \text{ we have } U \cap (A - \{x\}) \neq \emptyset \\ & \text{iff} & \overline{R}(x) \cap (A - \{x\}) \neq \emptyset \\ & \text{iff} & \underline{R}(x) \cap A \neq \emptyset \\ & \text{iff} & x \in \underline{R}^{-1}(A) \end{array}$$

The result follows. QED

Now suppose $\mathcal{X}$ is a topological space. We define R_d on X by setting xR_dy iff $x \in d(y)$.

LEMMA 1.63 *$\langle X, R_d \rangle$ is an irreflexive* **wK4***-frame.*

Proof That R_d is irreflexive follows from $x \notin d(x)$. To see that R_d is weakly transitive suppose xR_dy, yR_dz, and $x \neq z$. Then $x \in d(y)$, $y \in d(z)$, and $x \neq z$. From $x \in d(y)$ it follows that for each open neighborhood U of x we have $y \in U - \{x\}$; from $y \in d(z)$ it follows that for each open neighborhood V of y we have $z \in V - \{y\}$; and from $x \neq z$ it follows that for each open neighborhood U of x we have $z \in U - \{x, y\} \subseteq U - \{x\}$. Thus, $x \in d(z)$, and so xR_dz. QED

LEMMA 1.64

1 If $\mathfrak{F}$ is a **wK4***-frame, then $R_{d_R} \subseteq R$.*

2 If $\mathfrak{F}$ is an irreflexive **wK4***-frame, then* $R_{d_R} = R$.

3 If $\mathcal{X}$ is a topological space, then $R_d^{-1}(A) \subseteq d(A)$.

4 If $\mathcal{X}$ is an Alexandroff space, then $R_d^{-1}(A) = d(A)$.

Proof (1) In a **wK4**-frame $\mathfrak{F}$, $xR_{d_R}y \to x \in d_R(y) \to x \in R^{-1}(y) \to xRy$.

(2) Suppose $\mathfrak{F}$ is an irreflexive **wK4**-frame. Then $xR_{d_R}y \leftrightarrow x \in d_R(y) \leftrightarrow x \in \underline{R}^{-1}(y) \leftrightarrow x \in R^{-1}(y) \leftrightarrow xRy$.

(3) Suppose $\mathcal{X}$ is a topological space. Then $x \in R_d^{-1}(A) \to (\exists y)(xR_dy$ & $y \in A) \to (\exists y)(x \in d(y)$ & $d(y) \subseteq d(A)) \to x \in d(A)$.

(4) Suppose $\mathcal{X}$ is an Alexandroff space. Then $x \in d(A) \leftrightarrow \overline{R_d}(x) \cap (A - \{x\}) \neq \emptyset \leftrightarrow R_d(x) \cap A \neq \emptyset \leftrightarrow x \in R_d^{-1}(A)$. QED

Corollary 1.65 *For a nonempty set X, there is a 1-1 correspondence between:*

(i) Alexandroff topologies on X;

(ii) Reflexive and transitive relations on X;

(iii) Irreflexive and weakly transitive relations on X.

It follows that there is a 1-1 correspondence between Alexandroff spaces, **S4**-frames, and irreflexive **wK4**-frames. Now we are in a position to show that **wK4** is the d-logic of all topological spaces.

Theorem 1.66

1 **wK4** *is the d-logic of all topological spaces.*

2 **wK4** *is the d-logic of all* finite *topological spaces.*

3 **wK4** *has the* effective finite model *property with respect to the class of topological spaces.*

Proof Obviously both (1) and (3) follow from (2). To see (2), Theorem 1.60 implies that **wK4** is complete with respect to finite irreflexive **wK4**-frames. By Corollary 1.65, irreflexive **wK4**-frames correspond to topological spaces. The result follows. QED

3.1.2 Connections between S4 and wK4. There is a close connection between **S4** and **wK4**. For the set $\mathbb{F}$ of formulas of $\mathcal{L}$, we define a *translation* $tr : \mathbb{F} \to \mathbb{F}$ by induction (Boolos, 1993):

- $tr(p) = p$;
- $tr(\varphi \wedge \psi) = tr(\varphi) \wedge tr(\psi)$;
- $tr(\neg\varphi) = \neg tr(\varphi)$;
- $tr(\Box\varphi) = tr(\varphi) \wedge \Box tr(\varphi)$.

DEFINITION 1.67

1 *Let L be a normal extension of* **wK4** *and S be a normal extension of* **S4**. *We say that L and S are* companions *if $S \vdash \varphi$ iff $L \vdash tr(\varphi)$.*

2 *For a normal extension L of* **wK4**, *let $T(L) = \{\varphi \ : \ L \vdash tr(\varphi)\}$.*

LEMMA 1.68

1 *$T(L)$ is a normal extension of* **S4**.

2 *$T(L)$ is a unique companion of L.*

Proof (1) It is easy to verify that

$$\mathbf{wK4} \vdash tr(\Box p \to p), tr(\Box p \to \Box\Box p), tr(\Box(p \to q) \to (\Box p \to \Box q))$$

and that $T(L)$ is closed under MP and N. So, $T(L)$ is a normal extension of **S4**.

(2) Suppose S is a companion of L. Then $S \vdash \varphi \leftrightarrow L \vdash tr(\varphi) \leftrightarrow T(L) \vdash \varphi$. Therefore, $S = T(L)$. QED

On the other hand, a given normal extension S of **S4** may have many different companions. For example, both **wK4** and **K4** (and all the normal logics in between) are companions of **S4**.

Speaking in terms of relational semantics, if a normal extension L of **wK4** is characterized by a class K of frames, then $T(L)$ is characterized by the class $\overline{K} = \{\overline{\mathfrak{F}} \ : \ \mathfrak{F} \in K\}$, where $\overline{\mathfrak{F}}$ denotes the reflexive closure of $\mathfrak{F}$.

Now we turn to the topological significance of tr. For a class K of topological spaces, let $L_d(K)$ denote the set of formulas of $\mathcal{L}$ that are d-valid in K. Since **wK4** is sound with respect to d-semantics, it is obvious that $L_d(K)$ is a normal extension of **wK4**. We call $L_d(K)$ the *d-logic* of the class K. The next two facts are taken from (Bezhanishvili et al., 2005, Lemma 2.1 and Theorem 2.2).

LEMMA 1.69 *Let K be a class of topological spaces and $\mathcal{X} \in K$.*

1 $\mathcal{X} \models \varphi$ iff $\mathcal{X} \models_d tr(\varphi)$.

2 $K \models \varphi$ iff $K \models_d tr(\varphi)$.

THEOREM 1.70 *If L is a d-logic, then $T(L)$ is topologically complete.*

We point out that $T(L)$ may be topologically complete without L being a *d*-logic. Thus, the converse of Theorem 1.70 is not in general true. In Sec. 3.1.3 we indicate several examples of topologically complete normal extensions of **S4** and **wK4** that are each others companions.

3.1.3 The landscape of spatial logics over wK4. The landscape of spatial logics over **wK4** is investigated less vigorously than that of spatial logics over **S4**. Nevertheless, there are several interesting results in this direction that we list below.

As we have already seen, **wK4** is the logic of all topological spaces when interpreting $\Diamond$ as derivative. In addition, we will see that **K4** is the logic of all T_d-spaces, that **KD4** is the logic of the Cantor space $\mathbb{C}$, the rational line $\mathbb{Q}$, or more generally, any 0-dimensional dense-in-itself metric separable space, that the *d*-logic of $\mathbb{R}$ is $\mathbf{KD4G}_2$, and that the *d*-logic of each $\mathbb{R}^n$, for $n \geq 2$, is $\mathbf{KD4G}_1$.

We say that a class K of topological spaces is *d-definable* if there exists a set of modal formulas Γ such that for each topological space $\mathcal{X}$ we have $\mathcal{X} \in K$ iff $\mathcal{X} \models_d \Gamma$. Since the derivative operator of a topological space is more expressible than the closure operator, topo-definability results will automatically transfer into *d*-definability results. However, there are *d*-definable topological properties that are not topo-definable. For example, the class of all T_d-spaces is *not* topo-definable. On the other hand, $\Box p \to \Box\Box p$ *d*-defines it. Also, the class of dense-in-itself spaces is *not* topo-definable, but it is *d*-definable by $\Diamond\top$ (or equivalently by $\Box p \to \Diamond p$). Below we present several results in this direction.

K4 and T_d-spaces:

PROPOSITION 1.71 **K4** *is the d-logic of all T_d-spaces.*

Proof Since $\mathcal{X}$ is a T_d-space iff $dd(A) \subseteq d(A)$ for each $A \subseteq X$, we obtain that **K4** is sound with respect to the class of T_d-spaces. To see completeness, recall that **K4** is complete with respect to the class of all (not necessarily finite) irreflexive **K4**-frames (see, e.g., Chagrov and Zakharyaschev, 1997, p. 102, Exercise 3.11). Since each one of these corresponds to a T_d-space, the result follows. QED

Another way to obtain d-completeness of **K4** is to construct a canonical topological model for **K4** similar to the one constructed in Sec. 2.4.2 (Steinsvold, 2005).

The d-logics of $\mathbb{C}$ and $\mathbb{Q}$: Let

$$\mathbf{KD4} = \mathbf{K4} + \Diamond\top$$

For a topological space $\mathcal{X}$ we have $\mathcal{X} \models_d \Diamond\top$ iff $d(X) = X$ iff $\mathcal{X}$ is dense-in-itself. Consequently, since both $\mathbb{C}$ and $\mathbb{Q}$ are dense-in-itself T_d-spaces, we have that $\mathbb{C}, \mathbb{Q} \models_d \mathbf{KD4}$. Moreover, as was shown in Shehtman, 1990, Theorem 29, **KD4** is the d-logic of any 0-dimensional dense-in-itself metric separable space. As an immediate consequence we obtain that $\mathbf{KD4} = L_d(\mathbb{C}) = L_d(\mathbb{Q})$.

The d-logics of Euclidean spaces: The situation here is different from that of **S4**. Indeed, we have that the d-logic of $\mathbb{C}$ and $\mathbb{Q}$ is different from the d-logic of $\mathbb{R}$, and that the d-logic of $\mathbb{R}$ is different from the d-logic of $\mathbb{R}^n$ for each $n \geq 2$. Let G_1 denote the formula

$$(\Diamond p \wedge \Diamond\neg p) \to \Diamond((p \wedge \Diamond\neg p) \vee (\neg p \wedge \Diamond p))$$

and $\mathbf{KD4G}_1$ denote the logic obtained from **KD4** by postulating the formula G_1. Also let $Q_1 = p_1 \wedge \neg p_2 \wedge \neg p_3$, $Q_2 = \neg p_1 \wedge p_2 \wedge \neg p_3$, $Q_3 = \neg p_1 \wedge \neg p_2 \wedge p_3$, G_2 denote the formula

$$\Box((Q_1 \wedge \Box Q_1) \vee (Q_2 \wedge \Box Q_2) \vee (Q_3 \wedge \Box Q_3)) \to (\Box\neg Q_1 \vee \Box\neg Q_2 \vee \Box\neg Q_3)$$

and $\mathbf{KD4G}_2$ denote the logic obtained from **KD4** by postulating the formula G_2. Then it follows from (Shehtman, 1990; Shehtman, 2006) that $\mathbf{KD4G}_1 = L_d(\mathbb{R}^n)$ for each $n \geq 2$, and that $\mathbf{KD4G}_2 = L_d(\mathbb{R})$.

GL and scattered spaces: Recall that

$$\mathbf{GL} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$$

As we already saw in Sec. 2.6, the class of scattered spaces is not topo-definable. On the other hand, as was shown in (Esakia, 1981), $\mathcal{X} \models_d \Box(\Box p \to p) \to \Box p$ iff $\mathcal{X}$ is scattered. Therefore, $\Box(\Box p \to p) \to \Box p$ d-defines the class of scattered spaces. Now as **GL** is the d-logic of ordinal spaces (Abashidze, ; Blass, 1990), we obtain that **GL** is the d-logic of both scattered spaces and ordinal spaces. Since the class of ordinal spaces is a proper subclass of the class of scattered spaces, it follows that the class of ordinal spaces is neither d-definable nor topo-definable.

K4.Grz and hereditarily irresolvable spaces: As we already saw in Sec. 2.6, $\Box(\Box(p \to \Box p) \to p) \to \Box p$ topo-defines the class of hereditarily irresolvable spaces. Consequently, the class of hereditarily irresolvable spaces is also d-definable. Interestingly enough, the same axiom d-defines the class of hereditarily irresolvable spaces (Esakia, 2002). Moreover, **K4.Grz** is the d-logic of hereditarily irresolvable spaces (see Gabelaia, 2004). Both **GL** and **K4.Grz** are companions of **S4.Grz**, but as was shown in (Esakia, 2002), **K4.Grz** is the least companion of **S4.Grz**.

Further results on d-definability and d-completeness can be found in Bezhanishvili et al., 2005.

3.2 Product logics

Products of relational models have been studied extensively in Gabbay and Shehtman, 1998 (see also Gabbay et al., 2003) for their uses in combining information) and the behavior of matching modal logics is well-known. This section is about products of topological spaces as a generalization of this methodology. In particular, we study two horizontal and vertical topologies along with the standard product topology. For each of the three topologies on the product we introduce a modal box in our language and give axiomatizations of the resulting logics. The material presented here is taken from (Benthem et al., 2005).

3.2.1 Products of topologies. Let $\mathcal{X} = \langle X, \eta \rangle$ and $\mathcal{Y} = \langle Y, \theta \rangle$ be two topological spaces. Recall that the *standard product topology* τ on $X \times Y$ is defined by letting the sets $U \times V$ form a basis for τ, where U is open in $\mathcal{X}$ and V is open in $\mathcal{Y}$. We define two additional one-dimensional topologies on $X \times Y$ by 'lifting' the topologies of the components.

Definition 1.72 *Suppose $A \subseteq X \times Y$. We say that A is* horizontally open (H-open) *if for any $(x, y) \in A$ there exists $U \in \eta$ such that $x \in U$ and $U \times \{y\} \subseteq A$. Similarly, we say that A is* vertically open (V-open) *if for any $(x, y) \in A$ there exists $V \in \theta$ such that $y \in V$ and $\{x\} \times V \subseteq A$. If A is both H- and V-open, then we call it* HV-open.

The H-closed, V-closed and HV-closed sets are defined similarly. Let τ_1 denote the set of all H-open subsets of $X \times Y$ and τ_2 denote the set of all V-open subsets of $X \times Y$. It is easy to verify that both τ_1 and τ_2 form topologies on $X \times Y$.

Definition 1.73 *We call τ_1 the* horizontal topology *and τ_2 the* vertical topology.

Remark 1.74 *It is obvious that a set open in the standard product topology is both horizontally and vertically open. That is $\tau \subseteq \tau_1$ and $\tau \subseteq \tau_2$. However, the converse inclusions do not hold in general.*

The interpretation of the modal operators $\Box_1$ and $\Box_2$ of $\mathcal{L}_{\Box_1\Box_2}$ in $\langle X \times Y, \tau_1, \tau_2\rangle$ is as expected:

$$(x,y) \models \Box_1\varphi \text{ iff } (\exists U \in \tau_1)((x,y) \in U \text{ and } \forall (x',y') \in U\ (x',y') \models \varphi)$$

$$(x,y) \models \Box_2\varphi \text{ iff } (\exists V \in \tau_2)((x,y) \in V \text{ and } \forall (x',y') \in V\ (x',y') \models \varphi)$$

The modalities $\Diamond_1$ and $\Diamond_2$ are defined dually. Furthermore, all the usual notions such as satisfiability and validity generalize naturally to this new language.

There are some similarities and differences between products of frames and of topological spaces. To see the similarities, let $\mathfrak{F} = \langle W, R\rangle$ and $\mathfrak{F}' = \langle W', R'\rangle$ be **S4**-frames, and let $\mathfrak{F} \times \mathfrak{F}' = \langle W \times W', R_1, R_2\rangle$ be their product. Then τ_{R_1} and τ_{R_2} are precisely the horizontal and vertical topologies on the product space $W \times W'$. This shows that our topological product construction is a faithful generalization of the usual product construction for frames. Thus, whenever topological spaces $\mathcal{X}$ and $\mathcal{Y}$ are representable as **S4**-frames (are Alexandroff), then the horizontal and vertical topologies on their product $X \times Y$ can be defined from the horizontal and vertical relations on the product of these frames. In other words, our topological setting generalizes the case for products of frames. To see the differences, we point out that both *com* and *chr*, while valid on products of frames, can be refuted on topological products. Below we exhibit their failure on $\mathbb{R} \times \mathbb{R}$.

(a) Failure of *com*: Let

$$\nu(p) = (\bigcup_{x \in (-1,0)} \{x\} \times (x, -x)) \cup (\{0\} \times (-1,1)) \cup (\bigcup_{x \in (0,1)} \{x\} \times (-x, x))$$

(see Fig. 1.16a). Then there is a basic horizontal open $(-1,1) \times \{0\}$ such that $(0,0)$ is in it and every point in $(-1,1) \times \{0\}$ sits in a vertically open subset of $\nu(p)$. Thus, $\Box_1\Box_2 p$ is true at $(0,0)$. On the other hand, there is no vertical open containing $(0,0)$ in which every point sits inside a horizontally open subset of $\nu(p)$, implying that $\Box_2\Box_1 p$ is false at $(0,0)$.

(b) Failure of *chr*: Let $\nu(p) = \bigcup\{\{\frac{1}{n}\} \times (-\frac{1}{n}, \frac{1}{n}) : n \geq 1\}$ (see Fig. 1.16b). Then in any basic horizontal open around $(0,0)$ there is a point that sits in a basic vertical open in which p is true everywhere. Thus, $\Diamond_1\Box_2 p$ is true at $(0,0)$. On the other hand, since the horizontal closure of $\nu(p)$ is $\nu(p) \cup \{(0,0)\}$ and since the vertical interior of $\nu(p) \cup \{(0,0)\}$ is $\nu(p)$, we have that $\Box_2\Diamond_1 p$ is false at $(0,0)$.

These counterexamples on $\mathbb{R} \times \mathbb{R}$ are not accidental. (Benthem et al., 2005, Sec. 4) shows when they can be reproduced in products of arbitrary topological spaces.

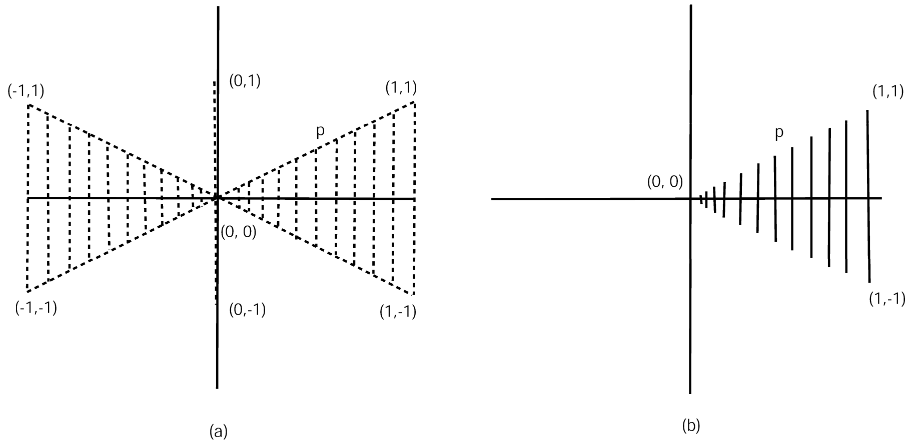


Figure 1.16. Counterexamples of *com* and *chr* on $\mathbb{R} \times \mathbb{R}$.

3.2.2 Completeness for products. Our main goal here is to show that the logic of all products of topological spaces is $\mathbf{S4} \oplus \mathbf{S4}$. In fact, we show that $\mathbf{S4} \oplus \mathbf{S4}$ is the logic of $\mathbb{Q} \times \mathbb{Q}$.

THEOREM 1.75 $\mathbf{S4} \oplus \mathbf{S4}$ *is the logic of* $\mathbb{Q} \times \mathbb{Q}$.

Proof As follows from Proposition 1.16, $\mathbf{S4} \oplus \mathbf{S4}$ is complete with respect to the infinite quaternary tree $\mathcal{T}_{2,2} = \langle W, R_1, R_2 \rangle$. We view $\mathcal{T}_{2,2}$ as equipped with two Alexandroff topologies defined from R_1 and R_2. To prove completeness of $\mathbf{S4} \oplus \mathbf{S4}$ with respect to $\mathbb{Q} \times \mathbb{Q}$ we take the X constructed in the proof of Theorem 1.35, define recursively an HV-open subspace Y of $X \times X$ and an interior map g from Y onto $\mathcal{T}_{2,2}$ with respect to both topologies: this will allow us to transfer counterexamples from $\mathcal{T}_{2,2}$ to Y, then from Y to $X \times X$, and finally from $X \times X$ to $\mathbb{Q} \times \mathbb{Q}$.

Let $Y = \bigcup_{n \in \omega} Y_n$ where $Y_0 = \{(0,0)\}$ and

$$Y_{n+1} = Y_n \cup \{(x - \frac{1}{3^n}, y), (x + \frac{1}{3^n}, y), (x, y - \frac{1}{3^n}), (x, y + \frac{1}{3^n}) : (x,y) \in Y_n\}$$

CLAIM 1.76 *Y is an HV-open subspace of $X \times X$.*

Proof Let $(x,y) \in Y$. Then $x \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) \subseteq X$. Therefore, $(x,y) \in (x - \frac{1}{3^{n_x}}, x + \frac{1}{3^{n_x}}) \times \{y\} \subseteq Y$. Thus, Y is an H-open subspace of $X \times X$. That Y is a V-open subspace of $X \times X$ is proved symmetrically.

QED

A similar argument as in the proof of Theorem 1.35 shows that for each $(x, y) \in Y$ such that $(x, y) \neq (0, 0)$ there exists $n_{(x,y)}$ with $(x, y) \in Y_{n_{(x,y)}}$ and $(x, y) \notin Y_{n_{(x,y)}-1}$, and that there is a unique $(u, v) \in Y_{n_{(x,y)}-1}$ such that $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v)$ or $(x, y) = (u, v \pm \frac{1}{3^{n_{(x,y)}-1}})$.

We define g from Y onto $\mathcal{T}_{2,2}$ by recursion (cf. Fig. 1.17): If $(x, y) = (0, 0)$ then we let $g(0, 0)$ be the root r of $\mathcal{T}_{2,2}$; if $(x, y) \neq (0, 0)$ then $(x, y) = (u \pm \frac{1}{3^{n_{(x,y)}-1}}, v)$ or $(x, y) = (u, v \pm \frac{1}{3^{n_{(x,y)}-1}})$ for a unique $(u, v) \in Y_{n_{(x,y)}-1}$, and we let

$$g(x,y) = \begin{cases} \text{the left } R_1\text{-successor of } g(u,v) & \text{if } (x,y) = (u - \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{the right } R_1\text{-successor of } g(u,v) & \text{if } (x,y) = (u + \frac{1}{3^{n_{(x,y)}-1}}, v) \\ \text{the left } R_2\text{-successor of } g(u,v) & \text{if } (x,y) = (u, v - \frac{1}{3^{n_{(x,y)}-1}}) \\ \text{the right } R_2\text{-successor of } g(u,v) & \text{if } (x,y) = (u, v + \frac{1}{3^{n_{(x,y)}-1}}) \end{cases}$$

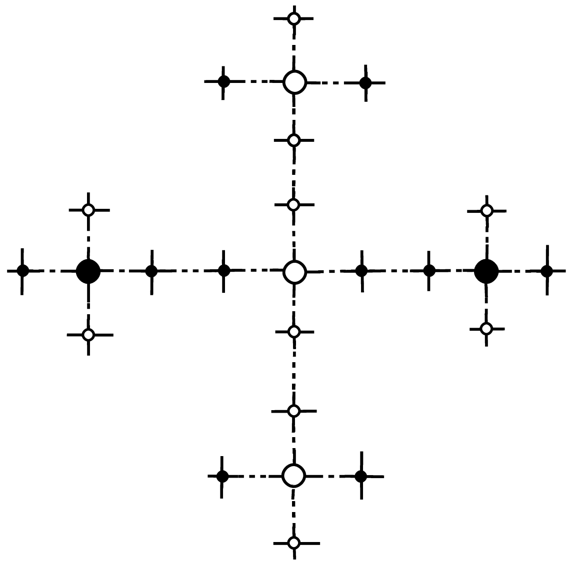


Figure 1.17. The first stages of the labelling in the completeness proof for **S4** $\oplus$ **S4**.

Claim 1.77 *g is an interior map with respect to both topologies.*

Proof Let τ_1 and τ_2 denote the restrictions of the horizontal and vertical topologies of $X \times X$ to Y, respectively. We prove that g is an interior map with respect to τ_1. That g is interior with respect to τ_2 is proved symmetrically. We observe that

$$\{(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\} \; : \; (x,y) \in Y\}$$

forms a basis for τ_1. We also recall that a basis for the Alexandroff topology on $T_{2,2}$ defined from R_1 is $\mathcal{B}_1 = \{B^1_t\}_{t \in T_{2,2}}$ where $B^1_t = \{s \in T_{2,2} : tR_1 s\}$.

To see that g is open, let $(x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}$ be a basic open for τ_1. Then the same argument as in Claim 1.37 guarantees that $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)}$. Thus g is open. To see that g is continuous it suffices to show that for each $t \in T_{2,2}$, the g-inverse image of B^1_t belongs to τ_1. Let $(x,y) \in g^{-1}(B^1_t)$. Then $tR_1 g(x,y)$. So $g((x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}) = B^1_{g(x,y)} \subseteq B^1_t$. Thus there exists an open neighborhood $U = (x - \frac{1}{3^{n(x,y)}}, x + \frac{1}{3^{n(x,y)}}) \times \{y\}$ of (x,y) such that $U \subseteq g^{-1}(B^1_t)$, implying that g is continuous. QED

To complete the proof, if $\mathbf{S4} \oplus \mathbf{S4} \not\vdash \varphi$, then there is a valuation ν on $\mathcal{T}_{2,2}$ such that $\langle \mathcal{T}_{2,2}, \nu\rangle, r \not\models \varphi$. Define a valuation ξ on Y by $\xi(p) = g^{-1}(\nu(p))$. Since g is an interior map with respect to both topologies and $g(0,0) = r$, we have that $\langle Y, \xi\rangle, (0,0) \not\models \varphi$. Now since Y is an HV-open subset of $X \times X$, we obtain that φ is refutable on $X \times X$. Finally, Theorem 1.35 implies that X is homeomorphic to $\mathbb{Q}$. Therefore, $X \times X$ is both horizontally and vertically homeomorphic to $\mathbb{Q} \times \mathbb{Q}$, and hence φ is also refutable on $\mathbb{Q} \times \mathbb{Q}$. QED

COROLLARY 1.78 **S4**$\oplus$**S4** *is the logic of products of arbitrary topologies.*

It follows that the logic of products of arbitrary topologies is decidable and has a $PSPACE$-complete satisfiability problem (Spaan, 1993). This stands in contrast to the satisfiability problem for $\mathbf{S4} \times \mathbf{S4}$, which turned out to be undecidable (Gabelaia et al., 2005).

3.2.3 Adding standard product interior. So far we only focused on the horizontal and vertical topologies on the product space, by analogy to products of relational structures. However, unlike products of relational structures, the standard product topology is not definable in terms of horizontal and vertical topologies (see Benthem et al., 2005, Sec. 3). Therefore, it is only natural to add an extra modal operator $\Box$ to the language $\mathcal{L}_{\Box_1 \Box_2}$ with the intended interpretation as the interior operator of the standard product topology.

For two topological spaces $\mathcal{X} = \langle X, \eta\rangle$ and $\mathcal{Y} = \langle Y, \theta\rangle$, we consider the product $\langle X \times Y, \tau, \tau_1, \tau_2\rangle$ with three topologies: the standard product topology τ, the horizontal topology τ_1, and the vertical topology τ_2. Then $\Box$ is interpreted as:

$$(x, y) \models \Box\varphi \quad \text{iff} \quad \exists U \in \eta \text{ and } \exists V \in \theta : U \times V \models \varphi$$

Since $\tau \subseteq \tau_1 \cap \tau_2$, the modal principle

$$\Box p \to \Box_1 p \wedge \Box_2 p$$

is valid in product spaces. Let **TPL** denote the logic in the new language $\mathcal{L}_{\Box,\Box_1,\Box_2}$ that contains all the axioms of **S4** $\oplus$ **S4** $\oplus$ **S4** plus the axiom $\Box p \to \Box_1 p \wedge \Box_2 p$. We call **TPL** the *topological product logic*. The main significance of **TPL** is that in the language $\mathcal{L}_{\Box,\Box_1,\Box_2}$ it is the logic of products of arbitrary topologies. This can be proved by generalizing the completeness of **S4** $\oplus$ **S4** with respect to $\mathbb{Q} \times \mathbb{Q}$ for this new case. As a result, we obtain that **TPL** is complete with respect to $\mathbb{Q} \times \mathbb{Q}$, hence is the logic of products of arbitrary topologies (with horizontal, vertical, and standard product topologies). For the details of the proof, see (Benthem et al., 2005, Sec. 6).

3.3 Extended modal languages

An extremely useful technique in modal logic is to gain expressive power by adding modal operators in such a way that decidability is retained. For instance, to express equality of states in relational models, one adds a difference operator $D\varphi$ which reads "there is a state different from the current one that satisfies φ." The same move makes sense for space. Topological relations not captured by the basic modal language can be safely expressed by adding appropriate new modal operators. We have entered the realm of extended modal languages, see (de Rijke, 1993; Benthem, 1991b).

3.3.1 Universal modalities and global properties. The basic language $\mathcal{L}$ interpreted on topological spaces has a 'local' view of the world. A global perspective comes from the addition of the universal modality that expresses accessibility to any point (Goranko and Passy, 1992). Universal modalities were brought to the spatial reasoning community in (Bennett, 1995). For this purpose, one adds:

$$\begin{aligned} M, x \models E\varphi \quad &\text{iff} \quad \exists y \in X :\ M, y \models \varphi \\ M, x \models U\varphi \quad &\text{iff} \quad \forall y \in X\ M, y \models \varphi \end{aligned}$$

More systematically the relevant new valid principles are those of **S5**:

$$\begin{array}{ll} Ep \leftrightarrow \neg U \neg p & \text{(Dual)} \\ U(p \to q) \to (Up \to Uq) & \text{(K)} \\ Up \to p & \text{(T)} \\ Up \to UUp & \text{(4)} \\ p \to UEp & \text{(B)} \end{array}$$

In addition, the following 'connecting' principle is part of the axioms:

$$\Diamond p \to Ep$$

Using these principles, the new enriched language $\mathcal{L}_u$ allows a normal form:

PROPOSITION 1.79 *(Aiello and Benthem, 2002a) Every formula of $\mathcal{L}_u$ is equivalent to one without nested occurrences of E, U.*

The definition of topo-bisimulation extends straightforwardly. It merely demands that topo-bisimulations be *total* relations.

THEOREM 1.80 *(Aiello and Benthem, 2002a)*

- *Extended modal formulas in $\mathcal{L}_u$ are invariant under total topo-bisimulations.*
- *Finite $\mathcal{L}_u$-modally equivalent models are totally topo-bisimilar.*

In the topological setting, fragments of this language can also be relevant. E.g., a continuous map has only one of the zig-zag clauses of topo-bisimulation. Now, consider 'existential' modal formulas constructed using only atomic formulas and their negations, $\wedge, \vee, \Box, E$, and U.

COROLLARY 1.81 *(Aiello and Benthem, 2002a) Let the simulation $\rightharpoondown$ run from M to M' with $x \rightharpoondown x'$. Then, for any existential modal formula φ, $M, x \models \varphi$ only if $M', x' \models \varphi$. In words, existential modal formulas are preserved under simulations.*

The language $\mathcal{L}_u$ is more expressive than the basic modal language $\mathcal{L}$. Indeed, as we already saw in Sec. 2.6, connectedness of a topological space is not expressible in $\mathcal{L}$. However, as it was shown in (Shehtman, 1999; Aiello and Benthem, 2002a), it *is* expressible in $\mathcal{L}_u$ by the formula:

$$U(\Diamond p \to \Box p) \to Up \vee U\neg p \tag{1.1}$$

In fact, it was shown in Shehtman, 1999 that (1.1) axiomatizes any connected dense-in-itself metric separable space, which is a generalization of the McKinsey-Tarski theorem. Another generalization was obtained in (Bezhanishvili and Gehrke, 2005), where it was shown that

RCC	$\mathcal{L}_u$	Description
$\mathtt{DC}(A, B)$	$\neg E(A \wedge B)$	A is disconnected from B
$\mathtt{EC}(A, B)$	$E(\Diamond A \wedge \Diamond B) \wedge \neg E(\Box A \wedge \Box B)$	A and B are externally connected
$\mathtt{P}(A, B)$	$U(A \rightarrow B)$	A is part of B
$\mathtt{EQ}(A, B)$	$U(A \leftrightarrow B)$	A and B are equal

Figure 1.18. Expressing RCC relations via $\mathcal{L}_u$.

(1.1) axiomatizes the boolean combinations of countable unions of convex subsets of the real line.

By encoding a fragment of the Region Connection Calculus (RCC) (Randell et al., 1992) in the language $\mathcal{L}_u$, Bennett showed the power of the language in expressing spatial arrangement of regions. The relevant elementary relations between regions that one can express are those of parthood and connectedness. The encoding is reported in Fig. 1.18, which is the basis for the appropriate calculus in computer science and AI.

3.3.2 Temporal operators and boundaries. Another extension of the modal language of topology comes from temporal formalisms. Consider the 'Until' logic of Kamp, 1968. Abstracting from linear temporal behavior gives a natural notion of spatial 'Until', describing truth in a neighborhood up to some 'fence' in topological models:

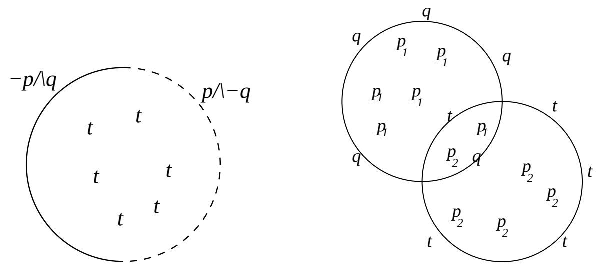
$M, x \models \varphi \mathcal{U} \psi$ iff there exists an open neighborhood U of x such that $\forall y \in U$ we have $\varphi(y)$ and $\forall z$ on the boundary of U we have $\psi(z)$

Here the boundary is definable in the earlier modal language:

$$\text{boundary}(U) = \Diamond U \wedge \Diamond \neg U$$

Fig. 1.19 has graphical representations of spatial 'Until'. As in temporal logic, this operator can define various further notions of interest. This richer language still has topo-bisimulations in line with the proposals in Kurtonina and de Rijke, 1997 for dealing with the $\exists\forall$-complexity of 'Until'.

Borrowing from temporal logic is an interesting phenomenon per se. Many temporal principles valid in $\mathbb{R}$ survive the move to the spatial interpretation. For instance, two key equivalences for obtaining temporal



(a) (b)

Figure 1.19. Examples of Until models.

normal forms are

$$t\mathcal{U}(p \vee q) \leftrightarrow (t\mathcal{U}p) \vee (t\mathcal{U}q)$$
$$(p \wedge q)\mathcal{U}t \leftrightarrow (p\mathcal{U}t) \wedge (q\mathcal{U}t)$$

In a two-dimensional spatial setting, the first equivalence fails: Fig. 1.19a refutes the implication $\rightarrow$. But the other remains a valid principle of monotonicity. The direction $\rightarrow$ of the second equivalence is a general monotonicity principle again. Conversely, we even have a stronger valid law:

$$p_1\mathcal{U}q \wedge p_2\mathcal{U}t \rightarrow (p_1 \wedge p_2)\mathcal{U}(q \vee t)$$

We mention that (Aiello, 2002a) contains a more sustained analysis of the spatial content of the $I\!R$ complete Until logic of (Burgess, 1984).

3.3.3 Extended spatio-temporal formalisms. Another use for the preceding ideas is in combined logics of *space-time*, treated extensively in `Ch.~\ref{KK::c}`. In particular, Shehtman, 1993 axiomatized the complete logic of the rationals in this language, while Gerhardt, 2004 added "Since"/"Until" using methods of de Jongh and Veltman, 1985 that go back to Burgess, 1979. But axiomatizing the complete logic of the reals remains open. We refer to Bezhanishvili and Kupke, 2005 for some interesting progress in tackling this problem using products of modal logics, while also introducing the "Since"/"Until" operators for the temporal component.

3.4 Topological semantics for epistemic logic

Spatial models seem to serve only geometrical purposes. But as we noted in Sec. 1.3, the earliest topological semantics were actually proposed for modelling *intuitionistic logic* based on evidence and knowledge.

Nowadays, however, standard relational semantics holds sway in modelling intuitionistic logic, or explicit knowledge-based *epistemic logic* in the tradition of Hintikka in philosophy (Hintikka, 1962), Aumann in economics (Binmore, 1994), or Halpern and Parikh in computer science (Fagin et al., 1995; Wooldridge, 2002). Nevertheless, Benthem and Sarenac, 2004 have shown recently how even the more technical results obtained in the spatial tradition are illuminating for knowledge once we switch to a topological interpretation. This section is a brief survey of the main ideas.

The main relational models have reflexive and transitive accessibility relations, and the key semantic clause about an agent's knowledge of a proposition says that $K_i\varphi$ holds at a world x iff φ is true in all worlds y accessible for i from x. For an illustration of how this works cf. Fig. 1.20.

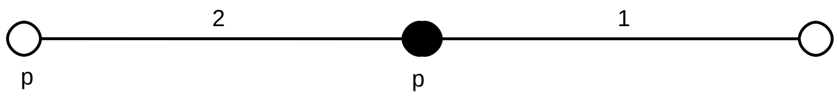


Figure 1.20. In the black central world, 1 does not know if p, while 2 knows that p. Therefore, in the world to the left, 1 knows that 2 knows that p, but 2 does not know if 1 knows p.

Thus, the epistemic knowledge modality is a modal box $\Box_i\varphi$, and the basic logic is that of the spatial interpretation, viz. **S4**. In an epistemic setting, the spatial modal axioms get a special flavor. E.g., the iteration axiom $\Box_1 p \to \Box_1\Box_1 p$ now expresses 'positive introspection': agents who know something, know that they know it. More precisely, we have **S4**-axioms for each separate agent, but no further 'mixing axioms' for iterated knowledge of agents such as $\Box_1\Box_2 p \to \Box_2\Box_1 p$. Indeed, the latter implication fails in the above example because in the world on the left, 1 knows that 2 knows that p, but 2 does not know if 1 knows p. Another way of describing the set of valid principles is as the *fusion* of separate logics **S4** for each agent. In what follows, we shall mostly work with the two-agent groups $G = \{1, 2\}$.

3.4.1 Group knowledge: agents as relations. A striking discovery in an interactive epistemic setting has been various notions of what may be called *group knowledge*. Two well known examples are the following (Fagin et al., 1995):

1 $E_G\varphi$: every agent in group G knows that φ,

2 $C_G\varphi$: φ is *common knowledge* in the group G.

The latter notion has been proposed in the philosophical, economic, and linguistic literature as a necessary precondition for coordinated behavior between agents (cf. Lewis, 1969). The usual semantic definition of common knowledge runs as follows:

$$M, x \models C_{1,2}\varphi \text{ iff for all } y \text{ with } x\ (R_1 \cup R_2)^* y \text{ we have } M, y \models \varphi$$

where $x(R_1 \cup R_2)^* y$ if x is connected to y by a finite sequence of successive steps from either of the two accessibility relations. This is the familiar reflexive and transitive closure of the union of the relations for both agents. The key valid principles for common knowledge are as follows:

Equilibrium Axiom:	$C_{1,2}\varphi \leftrightarrow (\varphi \wedge (\Box_1 C_{1,2}\varphi \wedge \Box_2 C_{1,2}\varphi))$
Induction Rule:	$\frac{\vdash \varphi \rightarrow (\Box_1(\psi \wedge \varphi) \wedge \Box_2(\psi \wedge \varphi))}{\vdash \varphi \rightarrow C_{1,2}\psi}$

This logic is known in the literature as $\mathbf{S4}_2^C$. It has been shown to be complete and decidable (Fagin et al., 1995).

A further interesting notion of knowledge for a group of agents is the so-called *implicit knowledge* $D_G\varphi$, which roughly describes what a group would know if its members decided to merge their information:

$$M, x \models D_{1,2}\varphi \text{ iff for all } y \text{ with } x(R_1 \cap R_2)y \text{ we have } M, y \models \varphi$$

where $R_1 \cap R_2$ is the intersection of the accessibility relations for the separate agents. Unlike universal and common knowledge, this notion is not invariant under modal *bisimulations*. It also involves a new phenomenon of merging information possessed by different agents. The latter topic will return below.

New notions of group knowledge introduce *new agents*. E.g., C_G defines a new kind of **S4**-agent since $(R_1 \cup R_2)^*$ is again reflexive and transitive. Note that $R_1 \cup R_2$ is not necessarily transitive, so the new 'agent' corresponding to the fact that 'everybody knows' would have different epistemic properties. In particular, it would lack positive introspection as to what it knows. In contrast, the relation $R_1 \cap R_2$ for D_G is again an **S4**-agent since Horn conditions like reflexivity and transitivity are preserved under intersections of relations. Thus, with two **S4**-agents $1, 2$, two additional agents supervene, one weaker and one stronger:

$$\begin{array}{ccccc} & & R_1 \cap R_2 & & \\ & \nearrow & & \nwarrow & \\ R_1 & & & & R_2 \\ & \nwarrow & & \nearrow & \\ & & (R_1 \cup R_2)^* & & \end{array}$$

3.4.2 Alternative views of common knowledge. Despite the success of the standard epistemic logic framework, there are still doubts about its expressive power and sensitivity. Notably, in his well-known critical paper (Barwise, 1988), Barwise claimed that a proper analysis of common knowledge must distinguish three different approaches:

1 countably *infinite iteration* of individual knowledge modalities,

2 the *fixed-point view* of common knowledge as 'equilibrium',

3 agents' having a *shared epistemic situation*.

Barwise's distinctions are hard to implement in standard relational semantics. But they make sense in topological semantics! Here is some technical groundwork.

The Equilibrium Axiom for the operator $C_G\varphi$ describes it as a fixed-point of an epistemic operator $\lambda X.\varphi \wedge \Box_1 X \wedge \Box_2 X$. In conjunction with the Induction Rule, it may even be seen to be the *greatest fixed-point* definable in the standard modal *μ-calculus* as:

$$C_G\varphi := \nu p.\varphi \wedge \Box_1 p \wedge \Box_2 p$$

The greatest fixed-point is computed as the first stabilization stage of a descending approximation sequence for a *monotonic* set function through the ordinals. We write $[|\varphi|]$ for the truth set of φ in the relevant model where evaluation takes place:

$C^0_{1,2}\varphi := [|\varphi|]$,
$C^{\kappa+1}_{1,2}\varphi := [|\varphi \wedge \Box_1(C^\kappa_{1,2}\varphi) \wedge \Box_2(C^\kappa_{1,2}\varphi)|]$,
$C^\lambda_{1,2}\varphi := [| \bigwedge_{\kappa<\lambda} C^\kappa_{1,2}\varphi|]$, for λ a limit ordinal.

Finally, we let $C_{1,2}\varphi := C^\kappa_{1,2}\varphi$ where κ is the least ordinal for which the approximation procedure halts; that is, $C^{\kappa+1}_{1,2}\varphi = C^\kappa_{1,2}\varphi$. In general, reaching this stopping point may take any number of ordinal stages. E.g., the least-fixed-point formula $\mu p.\Box p$ computing the 'well-founded part' of the binary accessibility relation may stabilize only at the cardinality of the model. But in certain cases we can do much better, as the following well-known fact shows:

FACT 1.82 *In every relational epistemic model, the approximation procedure for the common knowledge modality stabilizes at $\kappa \leq \omega$.*

This result shows that Barwise's fixed-point and countable-iteration views of common knowledge coincide in relational models. More precisely, $\nu p.\varphi \wedge \Box_1 p \wedge \Box_2 p$ is equivalent to

$$K_{1,2}\varphi := \varphi \wedge \Box_1\varphi \wedge \Box_2\varphi \wedge \Box_1\Box_2\varphi \wedge \ldots$$

The simple stabilization behavior at ω is most easily understood by observing that the knowledge modalities $\Box_i$ distribute over any infinite conjunction. Therefore, $\Box_i(\bigwedge_{n<\omega} C^n_{1,2}\varphi)$ is simply $\bigwedge_{n<\omega} \Box_i C^n_{1,2}\varphi$ which is equivalent to $\bigwedge_{n<\omega} C^n_{1,2}\varphi$. More generally, stabilization for the formula $\nu p.\varphi(p)$ is guaranteed by stage ω in any model in case the syntax defining the monotone approximation operator is constrained to a 'universal-conjunctive' format (Benthem, 1996).

3.4.3 Topological models for epistemic logic with fixed-points. The language of epistemic logic can be interpreted just as well in topological models, although the presence of many agents calls for an indexed *family of topologies* on the base set of worlds. All the notions such as bisimulation, axiomatic systems, and the product constructions of Sec. 3.2 also make sense epistemically. But these now acquire a special flavor—putting together topological models into one product space amounts to *merging information spaces* for different agents. The earlier horizontal and vertical topologies on the products encode the agents' original individual spaces. Our earlier result that the modal logic of the product construction is the fusion **S4** $\oplus$ **S4** then says epistemically that we have really defined a good 'conservative merge' without side-effects.

Further topologies on the product space encode further emergent group-oriented information structures. The earlier definitions of common knowledge still make sense in topological models. As before, the countably infinite iteration of all finite sequences of alternating knowledge modalities for the individual agents $1, 2$ is

$$K_{1,2}\varphi := \bigwedge_{n<\omega} K^n_{1,2}\varphi$$

with $K^n_{1,2}\varphi$ defined inductively as follows:

$K^0_{1,2}\varphi := \varphi$
$K^{n+1}_{1,2}\varphi := \Box_1(K^n_{1,2}\varphi) \wedge \Box_2(K^n_{1,2}\varphi)$

The same is true for the fixed-point definition

$$C_{1,2}\varphi := \nu p.\varphi \wedge \Box_1 p \wedge \Box_2 p$$

However, the definitions of common knowledge by fixed-points and by countably infinite iteration will now diverge, because one can show that given an interpretation of p, the interpretation of $K_{1,2}p$ does not always define a horizontally and vertically open set in the product model. Since the fixed-point version of $C_{1,2}p$ is always horizontally and vertically open, it follows that the two are not the same. We refer to (Benthem and Sarenac, 2004) for the details.

We can also view product spaces as introducing new 'collective agents' via new topologies. In particular, common knowledge as a greatest fixed-point now corresponds to the *intersection* $\tau_1 \cap \tau_2$ of the horizontal and vertical topologies on the product space. On the other hand, the topological meaning of the implicit group knowledge D_G is the *join* $\tau_1 \vee \tau_2$ of the horizontal and vertical topologies. Its basis is the pairwise intersection of horizontal and vertical opens. The latter topology need not always be of great interest. For instance, $\tau_1 \vee \tau_2$ is discrete on $\mathbb{Q} \times \mathbb{Q}$. From an informational perspective, this means that merging the information that we get about points in the horizontal and vertical directions fixes their position uniquely. The result of all this is again an inclusion diagram:

$$\begin{tikzcd} & \tau_1 \vee \tau_2 & \\ \tau_1 \arrow[ur] & & \tau_2 \arrow[ul] \\ & \tau_1 \cap \tau_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Returning to the three distinctions made in (Barwise, 1988), what about the third view of having a 'shared situation'? One good candidate for it would be the standard product topology τ. The agent corresponding to this new group concept τ only accepts very strong collective evidence for any proposition. And we know the complete logic of adding this agent from the joint axiomatization of horizontal, vertical, and standard product topologies from Sec. 3.2.3.

4. Modal logic and geometry

Many mathematical theories of space exist beyond topology such as affine and metric geometry, linear algebra, or newer theories like mathematical morphology, covered in `Ch.~\ref{BH::c}`. Modal structures emerge in all of them.

We start by recalling that affine geometry is given by the following three axioms involving points, lines, and an incidence relation (Blumenthal, 1961; Goldblatt, 1987, and `Ch.~\ref{BG::c}`):

A1 Any two distinct points lie on exactly one line.

A2 There exist at least three non-collinear points.

A3 Given a point a and a line L, there is exactly one line M that passes through a and is parallel to L.

Affine spaces have a strong modal flavor (see Balbiani et al., 1997; Balbiani, 1998; Venema, 1999; Stebletsova, 2000). Approaches include two-sorted versions with matching bimodal operators, and merging points and lines into one sort of pairs $\langle$point, line$\rangle$ equipped with two incidence relations. By contrast, the classical approach to affine structure is (Tarski, 1959), which contains a complete first-order axiomatization of elementary geometry in terms of a ternary betweenness predicate $\beta(xyz)$, as well as quaternary equidistance $\delta(xyzu)$, interpreted as x is as distant from y as z is from u. Yet, Tarski's beautiful decidable axiomatization still leaves things to be desired. First, the system has high complexity, viz. exponential space (Ben-Or et al., 1986). And from an expressive viewpoint, the axioms mix betweenness and equidistance, whereas one would like to understand affine and metric structure separately. A complete axiomatization of pure affine first-order geometry was given in Szczerba and Tarski, 1965. We now turn to the modal view.

4.1 Affine geometry in modal logic

4.1.1 Basic modal language and affine transformations.

Define a binary betweenness modality $<B>$:

$$M, x \models <B>(\varphi, \psi) \quad \text{iff} \quad \exists y, z : \beta(yxz) \wedge M, y \models \varphi \wedge M, z \models \psi$$

Thus, our language is a propositional language enriched with the betweenness modal operator $<B>$. Models for this language are triples $\langle X, \beta, \nu \rangle$, where X is a nonempty set, β is a ternary betweenness relation on X, and ν is a valuation function. Now we define *affine bisimulations* — modal counterparts of affine transformations — which are mappings relating points verifying the same proposition letters, and maintaining the betweenness relation:

DEFINITION 1.83 (AFFINE BISIMULATION) Given two affine models $\langle X, \beta, \nu\rangle$ and $\langle X', \beta', \nu'\rangle$, with x, y, z ranging over X and x', y', z' over X', an *affine bisimulation* is a nonempty relation $B \subseteq X \times X'$ such that if xBx' then:

1 x and x' satisfy the same proposition letters

2 (forth condition): $\beta(yxz) \Rightarrow \exists y'z' : \beta'(y'x'z')$ and yBy' and zBz'

3 (back condition): $\beta'(y'x'z') \Rightarrow \exists yz : \beta(yxz)$ and yBy' and zBz'.

In (Goldblatt, 1987) isomorphisms are considered the only interesting maps across affine models. But in fact, just as with topological bisimulations versus homeomorphisms (Theorem 1.6), affine bisimulations are interesting coarser ways of comparing spatial situations. In the true modal spirit they only consider behavior of points inside local line environments. Fig. 1.21 shows a case of non-isomorphic yet bisimilar triangles with atomic properties indicated. This affine bisimulation can be regarded as a sort of 'modal contraction' to the smallest model with the same structure.

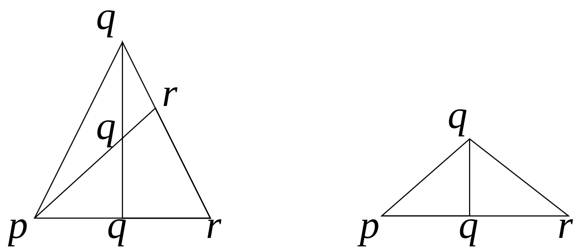


Figure 1.21. Affine bisimilar models.

By contrast, the models in Fig. 1.22 are not bisimilar. Affine bisimulations preserve truth of modal formulas in an obvious way, but $q \wedge$ $<B>(r, r)$ holds at the q point of the left model and nowhere on the right.

Incidentally, there *is* a smaller affine bisimulation contraction for the left-hand triangle in Fig. 1.22. But the resulting model is not 'planar': it cannot be represented in two-dimensional Euclidean space. Now consider a new valuation shown in Fig. 1.23. In this case there does *not* exist a bisimilar contraction: every point of the triangle is distinguishable by a formula which is not true on any other point, see Fig. 1.24.

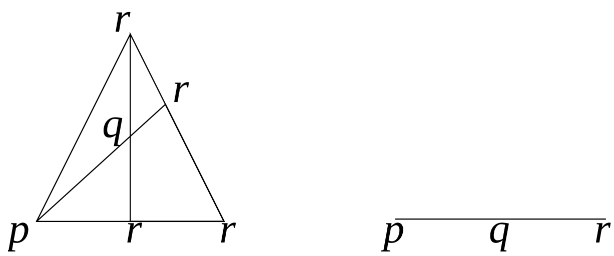


Figure 1.22. Affine bisimilar reduction.

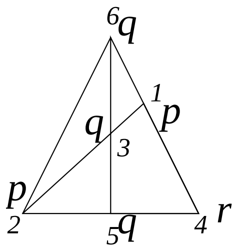


Figure 1.23. An irreducible affine model.

Point	Formula	
1	$\varphi_1 =$	$p \wedge {<}B{>}(q, r)$
2	$\varphi_2 =$	$p \wedge \neg\varphi_1$
3	$\varphi_3 =$	$q \wedge {<}B{>}(\varphi_1, \varphi_2)$
4	$\varphi_4 =$	r
5	$\varphi_5 =$	$q \wedge {<}B{>}(\varphi_2, \varphi_4)$
6	$\varphi_6 =$	$q \wedge \neg\varphi_3 \wedge \neg\varphi_5$

Figure 1.24. Formulas true at points of the model in Fig. 1.23.

4.1.2 Modal logics of betweenness. The preceding language has a minimal logic as usual, which does not yet have much geometric content. Its key axioms are two distribution laws:

$$
{<}B{>}(p \vee q, r) \leftrightarrow {<}B{>}(p, r) \vee {<}B{>}(q, r)
$$
$$
{<}B{>}(p, q \vee r) \leftrightarrow {<}B{>}(p, q) \vee {<}B{>}(p, r)
$$

This minimal logic has all the usual modal properties, including decidability. Further axioms would express basic universal frame conditions such as betweenness being symmetric at end-points and all points lying 'in between themselves':

$$<B>(p, q) \to <B>(q, p)$$
$$p \to <B>(p, p)$$

These are simple modal *frame correspondences*. A more interesting example was already mentioned in Sec. 1.5, involving an existential affine axiom. Consider *associativity* of the betweenness modality:

$$<B>(p, <B>(q, r)) \to <B>(<B>(p, q), r)$$

FACT 1.84 *Modal Associativity corresponds to Pasch's Axiom.*

Proof We spell out the simple correspondence argument to show how easy matches can be between modal axioms and geometric laws. Consider Pasch's Axiom (Sec. 1.5). Suppose that

$$\forall txyzu(\beta(xtu) \land \beta(yuz) \to \exists v : \beta(xvy) \land \beta(vtz))$$

holds in a frame. Assume that a point t satisfies $<B>(p, <B>(q, r))$. Then there exist points x, u with $\beta(xtu)$ such that $x \models p$ and $u \models <B>(q, r)$. Therefore, there also exist points y, z with $\beta(yuz)$ such that $y \models q$ and $z \models r$. Now by Pasch's Axiom, there must be a point v with $\beta(xvy)$ and $\beta(vtz)$. Thus, $v \models <B>(p, q)$ and hence $t \models <B>(<B>(p, q), r)$.

Conversely, assume that $\beta(xtu)$ and $\beta(yuz)$. Define a valuation on the space by setting $\nu(p) = \{x\}$, $\nu(q) = \{y\}$, and $\nu(r) = \{z\}$. Thus, $u \models <B>(q, r)$ and

$$t \models <B>(p, <B>(q, r)).$$

Then by the validity of Modal Associativity,

$$t \models <B>(<B>(p, q), r)$$

So there must be points v, w with $\beta(vtw)$ such that $v \models <B>(p, q)$ and $w \models r$. By the definition of ν, the latter implies $w = z$, and the former that $\beta(xvy)$. So indeed, v is the required point. QED

All these correspondences may even be *computed automatically* as they have 'Sahlqvist form' (cf. Blackburn et al., 2001 for more general theory).

Complete affine modal logics of special models may also be axiomatized, though only few examples have been dealt with so far. At least for the real line $I\!R$, the task is easy as one can take advantage of the binary ordering $\leq$, defining

$$M, x \models \text{<}B\text{>}(\varphi, \psi) \quad \text{iff} \quad \exists y, z : M, y \models \varphi \wedge M, z \models \psi \wedge y \leq x \leq z$$

Using this, we can define temporal operators Future and Past (both including the present). Conversely, these two unary operators define $\text{<}B\text{>}$ on $I\!R$:

$$\text{<}B\text{>}(\varphi, \psi) \leftrightarrow (P\varphi \wedge F\psi)$$

Thus, a complete and decidable axiomatization for our $\text{<}B\text{>}$-language can be found using the well-known tense logic of future and past on $I\!R$ (Segerberg, 1970).

4.1.3 Logics of convexity. An interesting and rich notion is that of the *convex closure* of a set, consisting of all points lying on a segment whose end-points are in the set. Convexity is important in many fields from computational geometry (Preparata and Shamos, 1985) to cognitive science (Gärdenfors, 2000). We can capture convexity modally by frames of points with the betweenness relation:

$$M, x \models C\varphi \text{ iff } \exists y, z : M, y \models \varphi \wedge M, z \models \varphi \wedge x \text{ lies in between } y \text{ and } z \tag{1.2}$$

This is a *one-step convexity* operator whose countable iteration yields the standard convex closure, cf. Fig. 1.6. A corresponding binary modality $C\varphi$ is defined as follows:

$$\exists yz : \beta(yxz) \wedge \varphi(y) \wedge \varphi(z)$$

Basic axioms are different here. In particular, distributivity fails. The one-step convex closure of a set of two distinct points is their whole interval, while the union of their separate one-step closures is just these points themselves. Thus, only monotonicity remains as a valid reasoning principle.

Another principle which is invalid in general is Idempotence of the convexity modality:

$$CC\varphi \leftrightarrow C\varphi$$

Iterating $C\varphi$ can lead to new sets, witness Fig. 1.6. Even so, the non-idempotence is of interest, as it helps distinguish dimensions. For instance, $CC\varphi \leftrightarrow C\varphi$ holds in $I\!R$, but not in $I\!R^2$.

One may now think that the stages $C^{n+1}\varphi \leftrightarrow C^n\varphi$ determine the dimensionality of the spaces $I\!R^n$ for all n. But here is a surprise.

THEOREM 1.85 *(Aiello, 2002a) The principle $CCC\varphi \leftrightarrow CC\varphi$ holds in $I\!R^3$.*

But convexity does provide dimension principles after all. Here is an old result from (Helly, 1923):

THEOREM 1.86 (HELLY) If $K_1, K_2, \ldots, K_m$ are convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$, in which $m > n + 1$, and if for every choice of $n + 1$ of the sets K_i there exists a point that belongs to all the chosen sets, then there exists a point that belongs to all the sets $K_1, K_2, \ldots, K_m$.

Our modal language formalizes this theorem as follows:

$$\bigwedge_{f:\{1,\ldots,n+1\}\to\{1,\ldots,m\}} E(\bigwedge_{i=1}^{n+1}(C^n\varphi_{f(i)}) \to E(\bigwedge_{i=1}^{m} C^n\varphi_i)$$

where E is the existential modality defined in terms of betweenness:

$$E\varphi \;\text{ iff }\; \langle B\rangle(\varphi, \top)$$

Thus again, modal languages capture significant geometrical facts.

4.1.4 First-order affine geometry. As usual, the above modal language is a fragment of a first-order language under the standard translation. The relevant first-order language is not quite that of Tarski's elementary geometry, however, as we also get unary predicate letters denoting regions. As in our discussion of topology, the affine first-order language of regions is a natural limit toward which affine modal languages can strive via various logical extensions. From a geometrical viewpoint, one might also hope that 'layering' the usual language in this modal way will bring to light interesting new geometrical facts.

Another major feature of standard geometry is the *equal status of points and lines.* This would suggest a reorganization of the modal logic to a *two-sorted* one (cf. Marx and Venema, 1997) stating properties of both points and segments viewed as independent semantic objects. One can think of this as a way of lowering the second-order complexity as the relevant subsets have now become first-order objects in their own right (cf. Benthem, 1999). Other analogies are with modal *Arrow Logic* (Benthem, 1996; Venema, 1996), where transitions between points become semantic objects in their own right. The two-sorted move seems very geometrical in spirit, and it would also reflect duality principles of the sort that led from affine to projective geometry.

4.2 Metric geometry in modal logic

There is more structure to geometry than just affine point-line patterns. We now bring out additional *metric* information using a notion of comparative distance.

4.2.1 Structures for relative nearness. Relative nearness was introduced in (Benthem, 1983, see Fig. 1.25):

$$N(x, y, z) \text{ iff } y \text{ is closer to } x \text{ than } z \text{ is, i.e., } d(x, y) < d(x, z)$$
$$\text{where } d(x, y) \text{ is the distance function.}$$

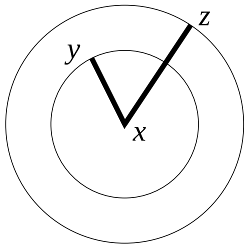


Figure 1.25. From point x, point y is closer than point z.

The function d can be a spatial metric, cognitive visual closeness, or even a utility function. (Randell et al., 2001) developed the logic of comparative nearness for the purpose of robot navigation, extending the calculus of regions RCC.

Relative nearness defines equidistance:

$$Eqd(x, y, z) : \neg N(x, y, z) \wedge \neg N(x, z, y)$$

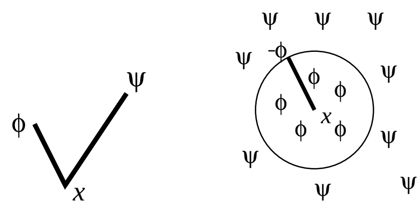
Affine betweenness is also definable in terms of N, at least in $\mathbb{R}^n$: cf. Sec. 4.2.2. Finally, even identity of points $x = y$ is expressible:

$$x = y \text{ iff } \neg N(x, x, y)$$

The further analysis of this structure can proceed as in the affine case. As it happens, though, the *universal first-order theory* of relative nearness for Euclidean spaces is still not axiomatized.

4.2.2 Modal logic of nearness. Now consider the obvious modal operator accessing ternary nearness N:

$$M, x \models \langle N\rangle(\varphi, \psi) \text{ iff } \exists y, z : M, y \models \varphi \wedge M, z \models \psi \wedge N(x, y, z)$$



(a) (b)

Figure 1.26. Interpreting the modal operator of nearness and its dual.

The universal dual of $<N>$ is also interesting in its spatial behavior:

$$M, x \models [N](\varphi, \psi) \text{ iff } \forall y, z\ (N(x, y, z) \wedge M, y \models \neg\varphi \rightarrow M, z \models \psi)$$

Dropping the negation, one gets the following appealing notion:

If any point y around the current point x satisfies φ, then all points z further out must satisfy ψ.

The basic modal logic of nearness again has distribution laws:

$$<N>(p \vee q, r) \leftrightarrow <N>(p, r) \vee <N>(q, r)$$
$$<N>(p, q \vee r) \leftrightarrow <N>(p, q) \vee <N>(p, r)$$

Universal frame constraints return as special axioms. Here are two examples:

$$<N>(p, q) \wedge \neg<N>(p, p) \wedge \neg<N>(q, q) \wedge <N>(q, r) \rightarrow <N>(p, r) \quad \text{(transitivity)}$$

$$<N>(p, q) \wedge \neg<N>(p, p) \wedge \neg<N>(q, q) \wedge Er \rightarrow <N>(p, r) \vee <N>(r, q) \quad \text{(connectedness)}$$

Modal logics of nearness on special structures may include further constraints computable by correspondence techniques. Here is a general technique covering many cases. Our language can define that φ holds in a unique point:

$$E!\,\varphi \text{ iff } E(\varphi \wedge \neg<N>(\varphi, \varphi))$$

Now a straightforward proof, known from extended modal logics with a difference modality (de Rijke, 1993), establishes the following:

Proposition 1.87 *Every universal first-order property of N is modally definable.*

4.2.3 First-order theory of nearness. As for the complete first-order theory of relative nearness, we merely list some illustrations, taken from Aiello and Benthem, 2002b, for background to the modal analysis.

FACT 1.88 *The single primitive of comparative nearness defines the two primitives of Tarski's Elementary Geometry in first order logic.*

Proof The following defines betweenness (see Fig. 1.27):

$$\beta(yxz) \text{ iff } \neg\exists x' : N(y, x', x) \wedge N(z, x', x)$$

This allows us to define parallel segments in the usual way as having

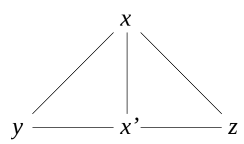


Figure 1.27. Defining betweenness via nearness.

no intersection points on their generated lines.

$$\begin{aligned} xx'||yy' \leftrightarrow \neg\exists c : \beta(xx'c) \wedge \beta(yy'c) \wedge \\ \neg\exists c' : \beta(c'xx') \wedge \beta(cyy') \wedge \\ \neg\exists c'' : \beta(xcx') \wedge \beta(ycy') \end{aligned}$$

Then one defines equal segment length by

$$\delta(x, y, z, u) \text{ iff } \exists y' : \ xu||yy' \wedge xy||uy' \wedge \neg N(u, z, y') \wedge \neg N(u, y'z)$$

QED

There are many other systems of first-order geometry with similar richness. For instance, see the axiomatization of constructive geometry in Plato, 1995.

5. Modal logic and linear algebra

Connections between *linear algebra* and spatial representation are well-known from a major qualitative visual theory, viz. *mathematical morphology* (Matheron, 1967; Serra, 1982). Our brief treatment follows the lines of (Aiello and Benthem, 2002a; Benthem, 2000b), to which we refer for details. A different connection between mathematical morphology and modal logic is found in (Bloch, 2000). The flavor of this spatial logic is different from what we had before, but similar modal themes emerge all the same.

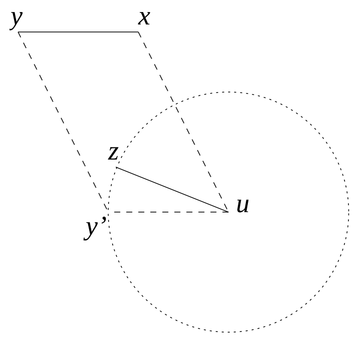


Figure 1.28. Equidistance in terms of nearness.

5.1 Mathematical morphology

In line with our spatial emphasis of this paper, we will stick with concrete vector spaces $\mathbb{R}^n$ in what follows. Images are regions consisting of sets of vectors. Mathematical morphology provides four basic ways of combining or simplifying images, viz. *dilation*, *erosion*, *opening* and *closing*. These are illustrated in Fig. 1.29.

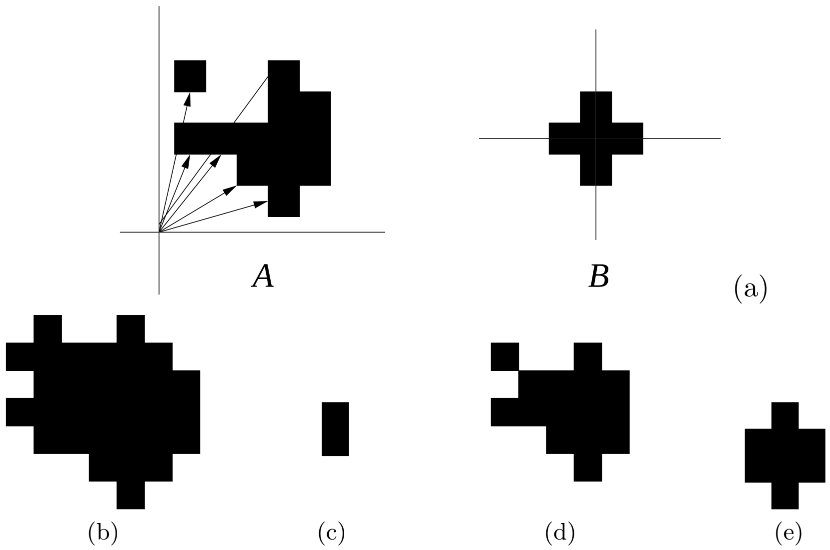


Figure 1.29. (a) Regions A and B of the vector space $\mathbb{R}^2$; (b) dilating A by B; (c) eroding A by B; (d) closing A by B; (e) opening A by B.

Intuitively, dilation adds regions together while, e.g., erosion is a way of removing 'measuring idiosyncrasies' from a region A by using region B as a kind of boundary smoothener. (If B is a circle, one can think of it as rolling tightly along the inside of A's boundary, leaving only a smoother interior version of A.) More formally, dilation or *Minkowski addition* $\oplus$ is the sum:

$$A \oplus B = \{a + b \ : \ a \in A \text{ and } b \in B\} \quad \text{dilation}$$

This is naturally accompanied by

$$A \ominus B = \{a \ : \ a + b \in A \text{ for all } b \in B\} \quad \text{erosion}$$

Openings and closings are combinations of dilations and erosions:

the structural *opening* of A by B	$(A \ominus B) \oplus B$
the structural *closing* of A by B	$(A \oplus B) \ominus B$

In addition, mathematical morphology also employs the usual boolean operations on regions: intersection, union, and complement. This is our third mathematization of real numbers $\mathbb{R}^n$ in various dimensions, this time focusing on their vector structure. Evidently, the above operations are only a small sub-calculus, chosen for its computational utility and expressive perspicuity.

5.2 Links with linear logic

The Minkowski operations behave a bit like the operations of *propositional logic*. Dilation is like a logical conjunction $\oplus$, and erosion like an implication $\longrightarrow$, as seems clear from their definitions ('combining an A and a B', and 'if you give me a B, I will give you an A'). The two are related by the following *residuation law*:

$$A \bullet B \subseteq C \text{ iff } A \subseteq B \longrightarrow C$$

which is also typical for conjunction and implication. Nevertheless, there are also some differences. For instance, $A \oplus A$ is not in general equal to A.

A logical calculus for these operations is known as *multiplicative linear logic* in computer science and as the *Lambek calculus with permutation* in linguistics (Troelstra, 1992; Kurtonina, 1995). The calculus derives 'sequents' of the form $A_1, \ldots, A_k \Rightarrow B$ where each expression A, B in the current setting stands for a region, and the intended interpretation—in our case—says that

The sum of the A's is included in the region denoted by B.

Here are the derivation rules, starting from basic axioms $A \Rightarrow A$:

$$\frac{X \Rightarrow A \qquad Y \Rightarrow B}{X, Y \Rightarrow A \bullet B} \qquad\qquad \frac{X, A, B \Rightarrow C}{X, A \bullet B \Rightarrow C} \qquad \text{(product rules)}$$

$$\frac{A, X \Rightarrow B}{X \Rightarrow A \longrightarrow B} \qquad\qquad \frac{X \Rightarrow A \qquad B, Y \Rightarrow C}{X, A \longrightarrow B, Y \Rightarrow C} \qquad \text{(arrow rules)}$$

$$\frac{X \Rightarrow A}{\pi[X] \Rightarrow A}\text{permutation} \qquad \frac{X \Rightarrow A \qquad A, Y \Rightarrow B}{X, Y \Rightarrow B}\text{cut} \qquad \text{(structural rules)}$$

Derivable sequents typically include:

$$A, A \longrightarrow B \Rightarrow B \qquad \text{(‘function application’)}$$
$$A \longrightarrow B, B \longrightarrow C \Rightarrow A \longrightarrow C \qquad \text{(‘function composition’)}$$

Another key example is the two ‘Currying’ laws, whose proof uses the $\bullet$ rules:

$$(A \bullet B) \longrightarrow C \Rightarrow (A \longrightarrow (B \longrightarrow C))$$
$$(A \longrightarrow (B \longrightarrow C)) \Rightarrow (A \bullet B) \longrightarrow C$$

The major combinatorial properties of this calculus **LL** are known, including proof-theoretic cut elimination theorems, and *decidability* of derivability in *NP* time. Moreover, there are several formal semantics underpinning this calculus (algebraic, game-theoretic, category-theoretic, and possible worlds-style, Benthem, 1991a). Still, no totally satisfying modelling has emerged so far. But mathematical morphology provides a new model for linear logic!

FACT 1.89 *(Aiello and Benthem, 2002b) Every space $\mathbb{R}^n$ with the Minkowski operations is a model for all* **LL***-provable sequents.*

This soundness theorem shows that every sequent derivable in **LL** must be a valid principle of mathematical morphology. The converse seems an open question of independent interest:

> *Is multiplicative linear logic complete w.r.t the class of all $\mathbb{R}^n$'s? Or even w.r.t. two-dimensional Euclidean space?*

Further, mathematical morphology laws ‘mix’ pure Minkowski operations $\oplus, \longrightarrow$ with standard boolean ones. E.g. they include the fact that $(A \cup B) \longrightarrow C$ is the same as $(A \longrightarrow C) \cap (B \longrightarrow C)$. This requires adding boolean operations to **LL**:

$$\frac{X, A \Rightarrow B}{X, A \cap C \Rightarrow B} \qquad \frac{X, A \Rightarrow B}{X, C \cap A \Rightarrow B} \qquad \frac{X \Rightarrow A \qquad X \Rightarrow B}{X \Rightarrow A \cap B}$$

$$\frac{X \Rightarrow A}{X \Rightarrow A \cup B} \qquad \frac{X \Rightarrow A}{X \Rightarrow B \cup A} \qquad \frac{X, A \Rightarrow B \qquad X, C \Rightarrow B}{X, A \cup C \Rightarrow B}$$

The boolean operations look like the 'additives' of linear logic, but they also resemble ordinary modal logic.

5.3 Arrow logic and hybrid modalities

The basic players in an algebra of regions in a vector space are the vectors themselves. For instance, Fig. 1.29.a represents the region A as a set of 13 vectors departing from the origin. Vectors come with some natural operations such as binary addition or unary inverse—witness the usual definition of a vector space. A vector v in our particular spaces may be viewed as an ordered pair of points (o, e), with o the origin and e the end point. Pictorially, this is an arrow from o to e. Now this provides a point of entry into one more area of modal logic.

In modal arrow logic objects are transitions or arrows structured by various relations. In particular, there is a binary modality for *composition* of arrows and a unary one for *converse*. The motivation comes from dynamic logics, treating transitions as objects in their own right, and from relational algebra, making pairs of points into separate objects. This allows for greater expressive power than the usual systems, while also lowering complexity of the core logics (Blackburn et al., 2001; Benthem, 1996). For instance, the pair-interpretation has arrows as pairs of points (a_o, a_e), and then defines these semantic relations:

composition $C(a_o, a_e)(b_o, b_e)(c_o, c_e)$ iff $a_o = c_o$, $a_e = b_o$, and $b_e = c_e$,

inverse $R(a_o, a_e)(b_o, b_e)$ iff $a_o = b_e$ and $a_e = b_o$,

identity $I(a_o, a_e)$ iff $a_o = a_e$.

An abstract model is a set of arrows as primitive objects, with three relations as above, and a valuation function as usual:

Definition 1.90 (arrow model) An *arrow model* is a tuple $M = \langle W, C, R, I, \nu \rangle$ such that $C \subseteq W \times W \times W$, $R \subseteq W \times W$, $I \subseteq W$, and $\nu : P \rightarrow \mathcal{P}(W)$.

Such models have a wide variety of interpretations, from linguistic syntax to category theory (Venema, 1996), but important here is the obvious connection with vector spaces. Think of $Cxyz$ as $x = y + z$, Rxy as $x = -y$ and Ix as $x = 0$. Now use a modal arrow language with proposition letters, the identity element 0, monadic operators $\neg, -$, and a dyadic operator $\oplus$. The truth definition then has the following key clauses:

$$\begin{array}{lll}
M, x \models p & \text{iff} & x \in \nu(p) \\
M, x \models 0 & \text{iff} & Ix \\
M, x \models -\varphi & \text{iff} & \exists y : Rxy \text{ and } M, y \models \varphi \\
M, x \models \neg\varphi & \text{iff} & \text{not } M, x \models \varphi \\
M, x \models \varphi \vee \psi & \text{iff} & M, x \models \varphi \text{ or } M, x \models \psi \\
M, x \models A \oplus B & \text{iff} & \exists y \exists z : Cxyz \wedge M, y \models A \wedge M, z \models B \\
M, x \models A \ominus B & \text{iff} & \forall yz\ (Cyxz \wedge M, z \models A \rightarrow M, y \models B)
\end{array}$$

Most modal topics make immediate sense in linear algebra or mathematical morphology. E.g., the above models support a natural notion of *bisimulation*, which will now compare vector spaces in coarser ways than their usual linear transformations.

Next, there is valid reasoning. Here is the basic system of arrow logic:

$$(p \vee q) \oplus r \leftrightarrow (p \oplus r) \vee (q \oplus r) \tag{1.3}$$

$$p \oplus (q \vee r) \leftrightarrow (p \oplus q) \vee (p \oplus r) \tag{1.4}$$

$$-(p \vee q) \leftrightarrow (-p \wedge -q) \tag{1.5}$$

$$p \wedge (q \oplus r) \rightarrow q \oplus (r \wedge (-q \oplus p)) \tag{1.6}$$

These principles either represent or imply obvious vector laws. To see the validity of (1.6), note that if a vector a is the composition of b and c, then c can also be written as the composition $-b \oplus a$.

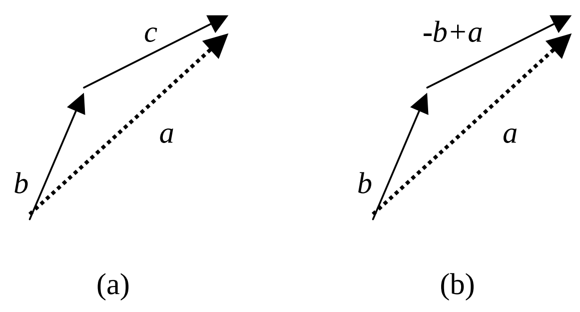


Figure 1.30. Triangle axiom for arrow composition.

On top of this, special arrow logics have been axiomatized with a number of additional frame conditions. In particular, the vector space interpretation makes composition *commutative* and *associative*, which leads to further axioms:

$p \oplus q \leftrightarrow q \oplus p$	commutativity
$p \oplus (q \oplus r) \leftrightarrow (p \oplus q) \oplus r$	associativity

The key fact about composition is now the vector law

$$a = b + c \quad \text{iff} \quad c = a - b$$

which derives the triangle inequality (see Fig. 1.30).

Again the soundness of arrow logic is clear, and we can freely derive old and new laws of vector algebra. But the central open question about our connection between arrow logic and mathematical morphology is again the converse:

> *What is the complete* axiomatization *of arrow logic over the standard vector spaces* $\mathbb{R}^n$*?*

6. Conclusions

The accumulated work surveyed in this chapter suggests that modal logic is a natural medium for analyzing spatial reasoning. The contact has interesting repercussions on both sides: modal logic acquires new models and new questions, while topology and geometry acquire new modally inspired notions. Moreover, we have shown how the modal connection also spreads new ideas into spatial reasoning from other application areas such as dynamic or epistemic logic.

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