DESCRIPTIONS AND CARDINALS BELOW $\boldsymbol{\delta}^1_5$

STEVE JACKSON AND FARID T. KHAFIZOV

1. Introduction

We work throughout in the theory $ZF+AD+DC$. In the mid 80's, Jackson computed the values of the projective ordinals $\boldsymbol{\delta}^1_n$. The upper bound in the general case appears in [J2], and the complete argument for $\boldsymbol{\delta}^1_5$ appears in [J1]. We refer the reader to [Mo] or [Ke] for the definitions and basic properties of the $\boldsymbol{\delta}^1_n$. A key part of the projective ordinal analysis is the concept of a description. Intuitively, a description is a finitary object "describing" how to build an equivalence class of a function $f : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ with respect to certain canonical measures W^m_3 which we define below. The proof of the upper bound for the $\boldsymbol{\delta}^1_{2n+3}$ proceeds by showing that every successor cardinal less than $\boldsymbol{\delta}^1_{2n+3}$ is represented by a description, and then counting the number of descriptions. The lower bound for $\boldsymbol{\delta}^1_{2n+3}$ was obtained by embedding enough ultrapowers of $\boldsymbol{\delta}^1_{2n+1}$ (by various measures on $\boldsymbol{\delta}^1_{2n+1}$) into $\boldsymbol{\delta}^1_{2n+3}$. A theorem of Martin gives that these ultrapowers are all cardinals, and the lower bound follows. A question left open, however, was whether every description actually represents a cardinal. The main result of this paper is to show, below $\boldsymbol{\delta}^1_5$, that this is the case. Thus, the descriptions below $\boldsymbol{\delta}^1_5$ exactly correspond to the cardinals below $\boldsymbol{\delta}^1_5$. Aside from rounding out the theory of descriptions, the results presented here also serve to simplify some of the ordinal computations of [J1]. In fact, implicit in our results is a simple (in principle) algorithm for determining the cardinal represented by a given description. This, in itself, could prove useful in addressing certain questions about the cardinals below the projective ordinals.

The results of this paper are self-contained, modulo basic AD facts about $\boldsymbol{\delta}^1_1, \boldsymbol{\delta}^1_3$ which can be found, for example, in [Ke]. In particular, $\boldsymbol{\delta}^1_1 = \omega_1$, $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$, $\boldsymbol{\delta}^1_1$ has the strong partition relation, and $\boldsymbol{\delta}^1_3$ has the weak partition relation (actually, the strong relation as well, but we do not need this here). $\omega, \omega_1, \omega_2$ are the regular cardinals below $\boldsymbol{\delta}^1_3$, and they, together with the c.u.b. filter, induce the three normal measures on $\boldsymbol{\delta}^1_3$.

Since we are not assuming familiarity with [J1], we present in the next section the definition of description and some related concepts. A few of our definitions are changed slightly from [J1]. We carry along through the paper some specific examples to help the reader through the somewhat technical

2000 *Mathematics Subject Classification.* 03E60.

Key words and phrases. Cardinals, Descriptions, Determinacy, Projective Ordinals.

definitions. In § 4 we give an application, and present a computational example.

2. Preliminaries

We define first the three families of canonical measures, W_1^m, S_1^m, W_3^m. If $f : \alpha \to ON$, we say f has the *correct type* if is strictly increasing, everywhere discontinuous, and of uniform cofinality ω; that is, there is a strictly increasing function $g : \omega \cdot \alpha \to ON$ such that $\forall \beta < \alpha \; f(\beta) = \sup_{\gamma < \omega \cdot (\beta+1)} f(\gamma)$. Recall κ has the strong partition property, $\kappa \to (\kappa)^\kappa_2$ if for all partitions $\mathcal{P} : (\kappa)^\kappa \to \{0,1\}$ of the increasing functions, there is an $A \subseteq \kappa$ of size κ and an $i \in \{0,1\}$ such that $\mathcal{P}(f) = i$ for all $f \in (A)^\kappa$. This is easily seen to be equivalent to the following variation: for every partition $\mathcal{P}$ of the functions from κ to κ of the correct type into two pieces, there is a c.u.b. $C \subseteq \kappa$ and an $i \in \{0,1\}$ such that for all $f : \kappa \to C$ of the correct type, $\mathcal{P}(f) = i$. In using this form of the partition relation, we usually have some well-order $\prec$ specified, and apply it to functions $f : \text{ dom }(\prec) \to \kappa$ of the correct type. Formally, we are just identifying $x \in \text{ dom }(\prec)$ with $|x|_\prec$.

For $r \in \omega$, let $<_r$ be the well-ordering of $(\omega_1)^r$ defined by: $(\alpha_1, \ldots, \alpha_r) <_r (\beta_1, \ldots, \beta_r)$ iff $(\alpha_r, \alpha_1, \ldots, \alpha_{r-1}) <^{lex} (\beta_r, \beta_1, \ldots, \beta_{r-1})$. If $h : <_r \to \omega_1$ is of the correct type, we define the *invariants* of f as follows: for $0 \leq j \leq r-2$, we define

$$h(j)(\alpha_1, \ldots, \alpha_{j+1}) \\ = \sup_{\alpha_j < \beta_{j+1} < \cdots < \beta_{r-1} < \alpha_{j+1}} h(\alpha_1, \ldots, \alpha_j, \beta_{j+1}, \ldots, \beta_{r-1}, \alpha_{j+1}).$$

We also define $h(r-1) = h$. Similarly, for $1 \leq j \leq r-1$ we define

$$\widetilde{h(j)}(\alpha_1, \ldots, \alpha_{j+1}) \\ = \sup_{\beta_j < \alpha_j, \beta_j < \beta_{j+1} < \cdots < \beta_{r-1} < \alpha_{j+1}} h(\alpha_1, \ldots, \alpha_{j-1}, \beta_j, \beta_{j+1}, \ldots, \beta_{r-1}, \alpha_{j+1}).$$

If $\alpha = [h]_{W_1^r}$, where $h : <_r \to \omega_1$ is of the correct type (where W_1^m is defined below), let $\alpha(j) = [h(j)]_{W_1^{j+1}}$ for $0 \leq j \leq r-1$. This is easily well-defined.

Definition 2.1 (Canonical Measures)**.**

1. W_1^m is the m-fold product of the normal measure on ω_1.
2. S_1^m is the measure on $\aleph_{m+1}$ defined as follows: $A \subseteq \aleph_{m+1}$ has measure one iff $\exists$ c.u.b. $C \subseteq \omega_1 \; \forall f : <_m \to C$ of the correct type, $[f]_{W_1^m} \in A$.
3. W_3^m is the measure on $\boldsymbol{\delta}^1_3$ defined as follows: $A \subseteq \boldsymbol{\delta}^1_3$ has measure one iff $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3 \; \forall f : \aleph_{m+1} \to C$ of the correct type, $[f]_{S_1^m} \in A$.

The strong, weak partition relations on $\boldsymbol{\delta}^1_1, \boldsymbol{\delta}^1_3$ respectively and our previous remarks easily show that these are measures (i.e., countably additive ultrafilters). These are the measures used in [J1]. For our purposes, it is convenient to introduce a variation of the family W_3^m. For each of the $(m-1)!$ permutations $\pi = (m, i_1, \ldots, i_{m-1})$ of m beginning with m, let $<^\pi$ be the corresponding well-ordering of $(\omega_1)^m$; that is, $(\alpha_1, \ldots, \alpha_m) <^\pi (\beta_1, \ldots, \beta_m)$

iff $(\alpha_m, \alpha_{i_1}, \dots, \alpha_{i_{m-1}}) <^{lex} (\beta_m, \beta_{i_1}, \dots, \beta_{i_{m-1}})$. Let S^π_1 denote the corresponding measure on $\aleph_{m+1}$ (as in the definition of S^m_1). W^m is the measure on $(m-1)!$ tuples $(\dots, \alpha_\pi, \dots)$ of ordinals $< \boldsymbol{\delta}^1_3$ defined by: A has measure one iff $\exists$ c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ $\forall f : \aleph_{m+1} \to C$ which are strictly increasing and continuous, $(\dots, [f]_{S^\pi_1}, \dots) \in A$. The weak partition relation on $\boldsymbol{\delta}^1_3$ easily shows that this is a measure.

We turn now to the definition of descriptions. A description is a finitary object, and has an index associated with it. An index is of the form (f_m) or $()$, and written as a superscript of the description. Descriptions indexed as $d^{(f_m)}$ will be called *type–0* descriptions, and those of the form $d^{()}$, *type–1* descriptions. Later we will suppress writing the index when it is understood or irrelevant. The descriptions defined directly will be also referred at as *basic* descriptions, and the ones defined in terms of the other descriptions will be called *non–basic*.

The following definitions are from [J1].

Fix $m, t \in \omega$, let $r(i) \in \omega$ and $K_i = S^{r(i)}_1$ or $W^{r(i)}_1$ for $i = 1, \dots, t$ be a sequence of canonical measures of length t. A set of descriptions, $\mathcal{D}_m = \mathcal{D}_m(K_1, \dots, K_t)$, is defined relative to this sequence of measures. Along with D_m is also defined a numerical function $k : \mathcal{D} \to \{1, \dots, t\} \cup \{\infty\}$.

Definition 2.2 (Descriptions)**.** $\mathcal{D}_m(K_1, \dots, K_t)$ and $k : \mathcal{D} \to \{1, \dots, t\} \cup \{\infty\}$ are defined by reverse induction on $k(d)$ through the following cases:

Basic Type-1:

$d^{()} := (k; p)^{()}$ where $1 \le k \le t$, $K_k = W^r_1$, and $1 \le p \le r$. $k(d) := k$.

Basic Type-0:

1. $d^{(f_m)} := (k; p)^{(f_m)}$ where $1 \le k \le t$, $K_k = W^r_1$, and $1 \le p \le r(k)$. $k(d) := k$.
2. $d^{(f_m)} := (p)^{(f_m)}$ where $1 \le p \le m$. $k(d) := \infty$.

Non–Basic Descriptions:

1. $d^{(f_m)} := (k; d^{(f_m)}_r, d^{(f_m)}_1, d^{(f_m)}_2, \dots, d^{(f_m)}_l)^{(f_m)}$ where $1 \le k \le t$, $K_k = S^r_1$, $l \le r-1$, and $k(d_1), \dots, k(d_l), k(d_r) > k$. $k(d) := k$.
2. $d^{(f_m)} := (k; d^{(f_m)}_r, d^{(f_m)}_1, d^{(f_m)}_2, \dots, d^{(f_m)}_l)^{s(f_m)}$ (Here s stands for "sup"), where $r \ge 2$, $1 \le k \le t$, $K_k = S^r_1$, $l \le r-1$, and $k(d_1), \dots, k(d_l), k(d_r) > k$.
   $k(d) := k$.
3. Same as 1. with $()$ replacing (f_m) everywhere.
4. Same as 2. with $()$ replacing (f_m) everywhere.

Now let $\mathcal{D}(K_1, \dots, K_t) := \cup_m \mathcal{D}_m(K_1, \dots, K_t)$ to be the set of descriptions relative to $K_1, \dots, K_t$. We will suppress the background sequence of measures simply writing $\mathcal{D}$ or $\mathcal{D}_m$. We call $\mathcal{D}_m$ the set of m–descriptions. Note that if $\bar{K}$ is a subsequence of $\bar{K}'$, then $\mathcal{D}_m(\bar{K}) \subseteq \mathcal{D}_m(\bar{K}')$.

Next we give the definition of the function h which interprets descriptions. Fix $d \in \mathcal{D}$, let $h_1, \dots, h_t$ be functions of type $K_1, \dots, K_t$, i.e., if $K_i =$

W_1^r, then $h_i : r \to \aleph_1$, and if $K_i = S_1^r$, then $h :<_r \to \aleph_1$ of correct type. We define the ordinal $h(d;\bar{h}) = h(d;h_1,\dots,h_t)$ through cases by reverse induction on $k(d)$. If $d = d^{()}$ then $h(d;h_1,\dots,h_t) < \aleph_1$ and if $d = d^{(f_m)}$ then $h(d;h_1,\dots,h_t) < \aleph_{m+1}$ and is represented with respect to W_1^m by a function which is also denoted by $h(d;h_1,\dots,h_t)(\alpha_1,\dots,\alpha_m)$.

Definition 2.3 (Interpretation of Descriptions)**.**

Basic Type-1: If $d^{()} = (k;p)$, then $h(d;\bar{h}) = h_k(p)$.

Basic Type-0:

1. If $d^{(f_m)} = (k;p)$, then $h(d;\bar{h})(\alpha_1,\dots,\alpha_m) = h_k(p)$.
2. If $d^{(f_m)} = (p), 1 \le p \le m$, then $h(d;\bar{h})(\alpha_1,\dots,\alpha_m) = \alpha_p$.

Non–Basic:

1. $d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \dots, d_l^{(f_m)})^{(f_m)}$ where $1 \le k \le t$, $K_k = S_1^r$, $l \le r-1$, and $k(d_1),\dots,k(d_l),k(d_r) > k$.
   a. If $l = r-1$, then $h(d;\bar{h})(\bar{\alpha}) := h_k(\ h(d_1;\bar{h})(\bar{\alpha}),\dots,h(d_r;\bar{h})(\bar{\alpha})\ )$
   b. If $l < r-1$, then
   $h(d;\bar{h})(\bar{\alpha}) := \sup_{\beta_{l+1}<\dots<\beta_{r-1}<h(d_r;\bar{h})(\bar{\alpha})} h_k(\ h(d_1;\bar{h})(\bar{\alpha}),\ \dots,$ $h(d_l;\bar{h})(\bar{\alpha}),\ \beta_{l+1},\dots,\beta_{r-1}, h(d_r;\bar{h})(\bar{\alpha})\ )$.
2. Let $d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \dots, d_l^{(f_m)}\ )^{s(f_m)}$ where $1 \le k \le t$, $K_k = S_1^r$, $l \le r-1$, and $k(d_1),\dots,k(d_l),k(d_r) > k$. Then $h(d;\bar{h})(\bar{\alpha}) := \sup_{\beta_l < h(d_l;\bar{h})(\bar{\alpha}),\ \beta_{l+1}<\dots<\beta_{r-1}<h(d_r;\bar{h})(\bar{\alpha})} h_k(\ h(d_1;\bar{h})(\bar{\alpha}),\dots,$ $h(d_{l-1};\bar{h})(\bar{\alpha}),\ \beta_l,\beta_{l+1},\dots,\beta_{r-1},\ h(d_r;\bar{h})(\bar{\alpha})\ )$
3. Same as 1., except now $h(d;\bar{h})$ is a single ordinal $< \aleph_1$.
4. Same as 2., except now $h(d;\bar{h})$ is a single ordinal $< \aleph_1$.

Next we put an ordering $<$ on $\mathcal{D}_m(K_1,\dots,K_t)$ as follows.

Definition 2.4 (Order $<$ on $\mathcal{D}(K_1,\dots,K_t)$)**.**
If $d_1, d_2 \in \mathcal{D}$ have the same index, then $d_1 < d_2$ iff for almost all $h_1,\dots,h_t$, $h(d_1,\bar{h}) < h(d_2,\bar{h})$.

This ordering can be easily checked to be a well-ordering on $\mathcal{D}_m(K_1,\dots,K_t)$.

The following definition give a condition which descriptions must satisfy in order to be well defined with repect to the equivalence classes of $h_1,\dots,h_t$, as made precise in lemma 2.1 below.

Definition 2.5 (Condition C)**.** Inductively, we say $d \in \mathcal{D}$ satisfies condition C if either d is basic or else d is non–basic, say of the form $d = (k; d_r, d_1, \dots, d_l)^s$, where s may or may not appear, and $d_1 < d_2 < \dots < d_l < d_r$, and $d_1,\dots,d_l,d_r$ satisfy condition C.

Lemma 2.1. *Suppose d satisfies C. Then for $\forall^* h_1$, if $h_1 = h_1'$ a.e., then $\forall^* h_2$, if $h_2 = h_2'$ a.e., …, $\forall^* h_t$, if $h_t = h_t'$, then $h(d;\bar{h}) = h(d;\bar{h'})$.*

The lemma is proved by a straightforward induction on the definition of description. We omit the details.

Having formally defined descriptions and their interpretations, we introduce now a simpler, less formal notation to represent them, which we refer to as the *functional representation* of the description. In the functonal representation, the notation more closely identifies the description with its interpretation. The functional representation of a description can be viewed as a term in the language with function symbols $h_i(j), \widetilde{h_i(j)}$, and variables $\alpha_{i,j}, \cdot_r$. A basic (type 0 or -1) description, of the form $(k;p)$ will be represented as $\alpha_{k,p}$. The basic type 0 description (p) will be represented as $\cdot_p$. A non-basic description of the form $d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \dots, d_l^{(f_m)})^{(f_m)}$ will then be represented as $h_k(l)(g_1, \dots, g_l, g_r)$, where $g_1, \dots, g_l, g_r$ are the representations of $d_1, \dots, d_l, d_r$. Similarly,

$$d^{(f_m)} := (k; d_r^{(f_m)}, d_1^{(f_m)}, d_2^{(f_m)}, \dots, d_l^{(f_m)})^{s(f_m)}$$

is represented as $\widetilde{h_k(l)}(g_1, \dots, g_l, g_r)$.

Thus, $\alpha_{i,j}$ is identified with the description whose interpretation relative to $h_1, \dots, h_t$ is the ordinal $\alpha_{i,j}$, where $h_i = (\alpha_{i,1}, \dots, \alpha_{i,j}, \dots)$. Also, $\cdot_p$ corresponds to the description whose interpretation is represented by the function $(\alpha_1, \dots, \alpha_m) \to \alpha_p$.

Examples . For the sequence of measures $K_1 = S^4_1$, $K_2 = S^4_1$, $K_3 = S^3_1$, $K_4 = W^4_1$, some descriptions (satisfying condition C) in $\mathcal{D}_4$ are: $d = h_1(2)(\alpha_{4,2}, h_2(1)(\alpha_{4,1}, \cdot_3), \cdot_4)$, $d = h_1(0)(h_2(1)(\alpha_{4,4}, h_3(0)(\cdot_4)))$. For the first of these, and for fixed $h_1, \dots, h_4 = (\alpha_{4,1}, \dots, \alpha_{4,4})$, the interpretation of d is the ordinal represented with repect to W^4_1 by the function $(\beta_1, \dots, \beta_4) \to h_1(2)(\alpha_{4,2}, h_2(1)(\alpha_{4,1}, \beta_3), \beta_4)$.

Definition 2.6 (Sup of a description)**.** If $q \in \mathcal{D}_m(K_1, \dots, K_t)$, and $1 \le n \le t$, then by $\sup_{K_n,\dots,K_t} q$ we mean a description $q' \ge q$, $q' \in \mathcal{D}_m(K_1, \dots, K_{n-1})$ such that $\forall^\star_{K_1} h_1 \forall^\star_{K_2} h_2 \dots \forall^\star_{K_{n-1}} h_{n-1}\ \forall \alpha < h(q'; \bar{h})\ \forall^\star_{K_n} h_n \dots \forall^\star_{K_t} h_t\ \alpha < h(q; \bar{h})$.

Formally, q' may be defined inductively through the following cases:

(1) If $q = \alpha_{i,j}$, then $q' = \alpha_{i,j}$ if $i < n$, and $q' = \cdot_1$ if $i \ge n$.

(2) If $q = \cdot_r$, $q' = q$. If $q = g(f_1, \dots, f_l, f_0)$, where $g = h_i(l)$ or $\widetilde{h_i(l)}$ and $i \ge n$, then $q' = \cdot_{r+1}$ if $f_0 = \cdot_r$, and otherwise $q' = f'_0$.

(3) If $q = h_i(l)(f_1, \dots, f_l, f_0)$ where $i < n$, then $q' = f'_0$ if $f'_0 > f_0$. Otherwise, let $k > 0$ be least such that $f'_k > f_k$. If $f'_k < f_0$, set $q' = h_i(k)(f_1, \dots, f_{k-1}, f'_k, f_0)$. If $f'_k = f_k$, set $q' = h_i(k-1)(f_1, \dots, f_{k-1}, f_0)$ if $k > 1$, and for $k = 1$, $q' = h_i(0)(f_0)$.

A straightforward induction on the definition of description shows that q' has the stated supremum property. Also, $q' = q$ iff $q \in \mathcal{D}_m(K_1, \dots, K_{m-1})$.

Example . If $K_1 = S^3_1$, $K_2 = S^3_1$, $K_3 = W^3_1$, $K_4 = S^3_1$, and

$$q = h_1(1)(\alpha_{3,1}, h_2(1)(h_4(0)(\cdot_2), \cdot_3)),$$

then $\sup_{K_3,K_4}(q) = h_2(0)(\cdot_3)$.

Definition 2.7 (Cofinality of d)**.** If $d \in \mathcal{D}_m(K_1, \dots, K_t)$ (and satisfies condition C), we say d has cofinality κ ($= \omega, \omega_1$, or ω_2) if $\forall^* h_1, \dots, h_t$ $\operatorname{cof} h(d; h_1, \dots, h_t) = \kappa$.

This may also be defined formally as follows.

(1) If $q = \alpha_{i,j}$, then $\kappa = \omega$.
(2) If $q = \cdot_r$, then $\kappa = \omega_1$ if $r = 1$, and $\kappa = \omega_2$ if $r > 1$.
(3) If $q = h_i(l)(f_1, \dots, f_l, f_0)$, and $K_i = S_1^r$, then $\kappa = \omega$ if $l = r-1$, and if $l < r-1$ then $\kappa = \operatorname{cof} f_0$.
(4) If $q = \widetilde{h_i(l)}(f_1, \dots, f_l, f_0)$, then $\kappa = \operatorname{cof} f_l$.

In [J1], the set of descriptions $\mathcal{D}$ was extended to a set $\overline{\mathcal{D}}$, and a property called "condition D" was introduced. Here, we have no need of $\overline{\mathcal{D}}$, and condition D simpifies to a fairly trivial condition. Nevertheless, to maintain consistency with [J1] we define:

Definition 2.8 (Condition D)**.** If $d = d^{f_m} \in \mathcal{D}_m(K_1, \dots, K_t)$ (and satisfies condition C), then we say d satisfies condition D if $d > \cdot_m$.

If d satisfies condition D, then $\forall^* h_1, \dots \forall^* h_t$ $h(d; h_1, \dots, h_t) > \aleph_m$, that is, $\forall^* h_1, \dots, h_t$ $\forall^* \alpha_1, \dots, \alpha_m$ $h(d; h_1, \dots, h_t)(\alpha_1, \dots, \alpha_m) > \alpha_m$. The significance of this is explained in remark 2.1 below.

Next, we show how to use descriptions to generate equivalence classes of functions from $\boldsymbol{\delta}^1_3$ to $\boldsymbol{\delta}^1_3$ with respect to the measures W^m (in [J1], the measures W^3_m were used).

Definition 2.9 (Ordinal represented by description)**.** Fix $m \in \omega$, and let $d = d^{f_m} \in \mathcal{D}_m(K_1, \dots, K_t)$ satisfy condition D. Let $g : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ be given.

- We define $(g; d; K_1, \dots, K_t)(W^m_3)$ to be the ordinal represented w.r.t. W^m by the function which assigns to $(\dots, [f]_{S^\pi_1}, \dots)$ the ordinal $(g; f; d; \bar{K})$, where $f : \aleph_{m+1} \to \boldsymbol{\delta}^1_3$ is continuous and represents $(\dots, [f]_{S^\pi_1}, \dots)$.
- $(g; f; d; \bar{K})$ is represented w.r.t. K_1 by the function which assigns to $[h_1]$ the ordinal $(g; d; h_1, K_2, \dots, K_t)$.
- In general, $(g; d; h_1, \dots, h_{i-1}, K_i, \dots, K_t)$ is represented w.r.t. K_i by the function which assigns to $[h_i]$ the ordinal

$$(g; d; h_1, \dots, h_{i-1}, h_i, K_{i+1}, \dots, K_t).$$

- Finally, $(g; d; h_1, \dots, h_t) = g(f(h(d; h_1, \dots, h_t)))$.

Remark 2.1. If d satisfies condition D, then $(g; d; \bar{K})(W^m)$ is well defined. To see this, let $f, f' : \aleph_{m+1} \to \boldsymbol{\delta}^1_3$ be strictly increasing, continuous, and $(\dots, [f]_{S^\pi_1}, \dots) = (\dots, [f']_{S^\pi_1}, \dots)$. Then there is a c.u.b. $C \subseteq \omega_1$ such that $\forall \pi = (m, i_1, \dots, i_m)$ $\forall h :<^\pi \to \omega_1$ of the correct type, $f([h]) = f'([h])$. Now, $\forall^* h_1, \dots, h_t$ $\forall^* \alpha_1, \dots, \alpha_m$ $h(d; h_1, \dots, h_t)(\alpha_1, \dots, \alpha_m) \in C$. Since $\forall^* h_1, \dots, h_t$ $h(d; h_1, \dots, h_t) > \aleph_m$, it follows there is a permutation $\overline{\pi}$ such that $\forall^* h_1, \dots, h_t$ $h(d; h_1, \dots, h_t)$ can be represented by a function h such that either $h :<^{\bar\pi} \to C$ is of the correct type, or $[h]$ is the supremum of ordinals represented by such functions. Since f, f' are continuous, in either case we have $f([h]) = f'([h])$.

Finally in this section we introduce the lowering operator $\mathcal{L}$ on $\mathcal{D}$. For every description $d \in \mathcal{D}_m(K_1, \ldots, K_t)$, $\mathcal{L}$ applied to d gives the largest description $\mathcal{L}(d) \in \mathcal{D}_m(K_1, \ldots, K_t)$ below d. First, given measures $K_1, \ldots, K_t$ and an integer k ($1 \le k \le t$ or $k = \infty$), an operator $\mathcal{L}^k$ is defined on those d satisfying $k(d) \ge k$, except for a unique $d = d(k)$ which is called the *minimal* description with respect to $\mathcal{L}^k$. Then $\mathcal{L} := \mathcal{L}^1$. $\mathcal{L}^k$ is defined by reverse induction on k as follows:

Definition 2.10 (Operator $\mathcal{L}^k$)**.**

I. $k = \infty$. So, d is basic type–0 with $d = d^{(f_m)} = \cdot_i$ for $1 \le i \le m$. If $i > 1$, then $\mathcal{L}^\infty := \cdot_{i-1}$. If $i = 1$, d is minimal with respect to $\mathcal{L}^\infty$.

II. $1 \le k \le t$.

1. $k = k(d)$
   - a. d is basic type–1, so $d = \alpha_{k,p}$. If $p > 1$, then $\mathcal{L}^k := \alpha_{k,p-1}$. If $p = 1$, d is minimal.
   - b. $d = d^{(f_m)} = h_k(l)(d_1, \ldots, d_l, d_0)$, with $l = r - 1$ and $K_k = S^r_1$. Then $\mathcal{L}^k(d) := \widetilde{h_k(l)}(d_1, \ldots, d_l)$ if $l \ge 1$, and if $l = 0$, that is, $d = h_k(0)(d_0)$, then $\mathcal{L}^k(d) := d_0$.
   - c. d as in (b), but $l < r - 1$. If $\mathcal{L}^{k+1}(d_0)$ is defined, and also $> d_l$ in case $l \ge 1$, then

     $$\mathcal{L}^k(d) := h_k(l+1)(d_1, \ldots, d_l, \mathcal{L}^{k+1}(d_l), d_0).$$

     If $\mathcal{L}^{k+1}(d_l)$ is not defined, or is $\le d_l$ (and $l \ge 1$), then we set $\mathcal{L}^k(d) := \widetilde{h_k(l)}(d_1, \ldots, d_l, d_0)$ if $l \ge 1$; otherwise $\mathcal{L}^k(d) := d_0$.
   - d. $d = \widetilde{h_k(l)}(d_1, \ldots, d_l, d_0)$. If $\mathcal{L}^{k+1}(d_l)$ is defined and $\mathcal{L}^{k+1}(d_l) > d_l$ if $l \ge 2$, set

     $$\mathcal{L}^k(d) := h_k(l)(d_1, \ldots, d_{l-1}, \mathcal{L}^{k+1}(d_l), d_0).$$

     Otherwise, set $\mathcal{L}^k(d) := \widetilde{h_k(l-1)}(d_1, \ldots, d_{l-1}, d_0)$ if $l \ge 2$, and for $l = 1$, $\mathcal{L}^k(d) := d_0$.
2. $k < k(d), K_k = W^{r(k)}_1$.
   - a. d is not minimal with respect to $\mathcal{L}^{k+1}$. Then $\mathcal{L}^k(d) := \mathcal{L}^{k+1}(d)$.
   - b. d is minimal with respect to $\mathcal{L}^{k+1}$. Then $\mathcal{L}^k(d) := \alpha_{k,r(k)}$.
3. $k < k(d), K_k = S^{r(k)}_1$
   - a. d is not minimal with respect to $\mathcal{L}^{k+1}$. Then $\mathcal{L}^k(d) := h_k(0)(\mathcal{L}^{k+1}(d))$.
   - b. d is minimal with respect to $\mathcal{L}^{k+1}$. Then d is minimal with respect to $\mathcal{L}^k$.

A straightforward induction on the definition of description shows that $\mathcal{L}(d)$, when defined, is the largest description strictly smaller than d (in $\mathcal{D}_m(K_1, \dots, K_t)$).

Example . For the sequence of measures $K_1 = S^4_1$, $K_2 = S^4_1$, $K_3 = S^3_1$, $K_4 = W^4_1$, and

$$d^{(f_4)} = h_1(2)(\alpha_{4,2}, h_2(1)(\alpha_{4,1}, \cdot_3), \cdot_4), \text{ and}$$
$$\mathcal{L}(d) = h_1(3)(\alpha_{4,2}, h_2(1)(\alpha_{4,1}, \cdot_3), h_2(0)(h_3(0)(\cdot_3)), \cdot_4).$$

3. Representation of cardinals below $\boldsymbol{\delta}^1_5$

We state our main result.

Theorem 3.1. *Let $m \in \omega$, $S_1, \dots, S_t \in \cup_i (W^i_1 \cup S^i_1)$ be a sequence of canonical measures. Let $d = d^{f_m} \in \mathcal{D}_m(S_1, \dots, S_t)$ be defined and satisfy condition D with respect $S_1, \dots, S_t$. Then,* $(\mathrm{id}; d; \overline{S})(W^m)$ *is a cardinal, where* $\mathrm{id} : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$ *is the identity function.*

Remark 3.1*.* As mentioned previously, the converse is also true [J1], that is, every cardinal below the predecessor of $\boldsymbol{\delta}^1_5$ is of this form. Also, if g is strictly greater than the identity function (almost everywhere with respect to the appropriate measure), then one can show that $(g; d; \overline{S})(W^m)$ is not a cardinal.

For the remainder of this paper, $\bar{d}$, etc., will denote a tuple $\bar{d} = (d; \overline{S})$, where $d \in \mathcal{D}_m(\bar{S})$.

The strategy of our proof is as follows. First we will define for each $\bar{d}$ a corresponding tree $T_{\bar{d}} = (T_{\bar{d}}, <)$. The tree $T_{\bar{d}}$ will have infinitely many nodes, which we will partition into finitely many blocks. For each such block we will assign an ordinal. Being added in a proper way these ordinals will give an ordinal $\xi_{\bar{d}}$. Then we will show that $(\mathrm{id}; d; \bar{S})(W^m) = \aleph_{\omega+\xi_{\bar{d}}}$.

Given $\bar{d}$, we define $<$ to be the transitive closure of $<'$, where $\bar{q} <' \bar{p} \iff [\bar{p} = (p; \bar{S})$, $\bar{q} = ((\mathcal{L}\bar{p}); \bar{S}, K)$, and $\bar{q}$ satisfies condition D]. Here $\mathcal{L}(\bar{p})$ denotes $\mathcal{L}(p)$ defined relative to the sequence $\bar{S}$. $\bar{d}$ is the root of $T_{\bar{d}}$. Intuitively, $T_{\bar{d}}$ is constructed by repeatedly applying the lowering operator $\mathcal{L}$ to $\bar{d}$, adding at most one new measure each time. In the definition of $<'$ above, the type of new measure K depends on $\mathrm{cof}(\mathcal{L}\bar{p})$: if it is ω_0, then no measure is added; if ω_1, then K must be of the form W^i_1; and if ω_2, then $K = S^i_1$. [We note that the restriction on K is a minor point, and could be dispensed with. For conceptual simplicity, we are restricting to only those K which are necessary.] As in [J1], we define the rank function on the nodes of the tree $T_{\bar{d}}$ by $|\bar{q}| := (\sup_{\bar{p} <' \bar{q}} |\bar{p}|) + 1$, and $|T_{\bar{d}}| = |\bar{d}|$.

Given $\bar{d} = (d; \bar{S})$, note that every node $\bar{q} \in T_{\bar{d}}$ is of the form $\bar{q} = (q; \bar{S}, \bar{M})$, for some sequence of measures $\bar{M}$. For such nodes in $T_{\bar{d}}$, we employ a notational convention when writing the functional representation of q. We will use the symbols $h_i(j), \widetilde{h_i(j)}$, $\alpha_{i,j}$ when refering to the measures in $\bar{S}$, and $k_i(j)$, $\widetilde{k_i(j)}$, $\gamma_{i,j}$ when to the measures in $\bar{M}$. For example, if $\bar{S} = (S^3_1, S^4_1, W^3_1)$, $\bar{M} = (S^4_1, W^4_1)$, then a functional representation for $q = q^{(f_4)}$ might look like $h_1(2)(\alpha_{3,1}, h_2(0)(k_4(1)(\gamma_{5,2}, \cdot_3)), \cdot_4)$.

For $\bar{d}$ as above and $\bar{q} \in T_{\bar{d}}$, we define a sequence, $\mathrm{oseq}_{\bar{d}}(\bar{q})$, which will be a sequence of terms of the form $\gamma_{i,j}$, $k_i(\cdot_r)$.

Definition 3.1 (The o–sequence of $\bar{q}$, $\mathrm{oseq}_{\bar{d}}(\bar{q})$)**.**
Given $\bar{d} = (d; \bar{S})$ and $\bar{q} = (q; \bar{S}, \bar{M})$, let $g(d_1, d_2, \ldots, d_l, d_0)$ be the functional representation of $\bar{q}$. Here g stands for an invariant of some h or some k function. We have numbered the arguments of g according to their significance in determining size of $h(q, \bar{S})$. (Each d_i is a subdescription defined relative to the same sequence of measures $\bar{S}, \bar{M}$.) We define recursively *the o–sequence of* $\bar{q}$ as follows.

$$\mathrm{oseq}_{\bar{d}}(\bar{q}) := \begin{cases} [\mathrm{oseq}_{\bar{d}}(d_0)^\frown \mathrm{oseq}_{\bar{d}}(d_1)^\frown \ldots {}^\frown \mathrm{oseq}_{\bar{d}}(d_l)]' & \\ & \text{if } g = h_i(j) \text{ or } \widetilde{h_i(j)} \\ \mathrm{oseq}_{\bar{d}}(d_0) & \text{if } (g = k_i(j) \text{ or } \widetilde{k_i(j)}) \text{ and } d_0 \neq \cdot_r \\ k_i(\cdot_r) & \text{if } g = k_i(j) \text{ and } d_0 = \cdot_r \\ k_i(\cdot_r) & \text{if } g = \widetilde{k_i(j)} (\text{ with } j \geq 1) \text{ and } d_0 = \cdot_r \\ \gamma_{i,j} & \text{if } q = \gamma_{i,j} \\ \emptyset & \text{if } q = \cdot_r \text{ or } \alpha_{i,j} \end{cases}$$

Here $'$ denotes the operation which eliminates repetition of ordinals and functions: we concatenate all $\mathrm{oseq}_d(d_i)$, and then if a symbol $\gamma_{i,j}$, or $k_i(\cdot_r)$ appears in the resulting sequence more than once, we keep it only in the first position where it appears.

We define also a variation of $\mathrm{oseq}_{\bar{d}}(\bar{q})$ which we denote $\mathrm{oseq}^*_{\bar{d}}(\bar{q})$. This is defined exactly as $\mathrm{oseq}_{\bar{d}}(\bar{q})$, except that in the first case we do not apply the deletion operation $'$ to the concatenated sequence. Now, each term $t = \gamma_{i,j}$ or $t = k_i(\cdot_r)$ may appear several times in the sequence. For each such term, say $k_i(\cdot_r)$, we will attach superscripts to the occurences of this term in $\mathrm{oseq}^*_{\bar{d}}(\bar{q})$. The occurences of this term will thus be of the form $k^1_i(\cdot_r), \ldots, k^a_i(\cdot_r)$. The attachment of the superscripts is defined (inductively) as follows. If t^a, t^b both correspond to subdescriptions of $p = g(p_1, \ldots, p_l, p_0)$ (where p is a subdescription of q) then $a < b$ if t^a corresponds to a subdescription of p_i which appears to the left of the subdescription p_j corresponding to t^b. If t^a, t^b both correspond to subdescriptions of p_i, the ordering of a, b is given by induction.

Example . For $q =$

$h_1(2)(h_2(2)(k_2(\cdot_2), k_3(\cdot_2), \cdot_3), h_2(2)(k_2(\cdot_2), k_4(\cdot_3), \cdot_4), h_2(2)(k_2(\cdot_2), k_4(\cdot_3), \cdot_5))$,

$\mathrm{oseq}_{\bar{d}}(q) = (k_2(\cdot_2), k_4(\cdot_3), k_3(\cdot_2))$, and $\mathrm{oseq}^*_{\bar{d}}(q) = (k^3_2(\cdot_2),\ k^2_4(\cdot_3),\ k^1_2(\cdot_2),\ k^1_3(\cdot_2),\ k^2_2(\cdot_2),\ k^1_4(\cdot_3))$.

Note that $\text{oseq}_{\bar{d}}(\bar{q})$, $\text{oseq}^*_{\bar{d}}(\bar{q})$ are uniquely determined by the functional representation of $\bar{q}$ (with our notational conventions). In particular, $\text{oseq}_{\bar{d}}(\bar{q})$, $\text{oseq}^*_{\bar{d}}(\bar{q})$ depend only on $\bar{d}, q$, and we may write $\text{oseq}_{\bar{d}}(q)$, $\text{oseq}^*_{\bar{d}}(q)$. While the measures $\bar{S}$ are fixed in considering $T_{\bar{d}}$, the other measures, $\bar{M}$, vary as we range over all possible nodes. The fact that the k–functions and γ–ordinals from $\text{oseq}_{\bar{d}}(\bar{q})$ are in some sense arbitrary is important in our computation.

For $\bar{d} = (d; \bar{S})$, $\bar{q} = (q; \bar{S}, \bar{M})$, we define $\sup_{\bar{d}} q := \sup_{\bar{M}} q$.

Proposition 3.2. *Let $p \in \mathcal{D}(\bar{S})$, and consider $\bar{p} = (p; \bar{S}, K)$ where $K = S^n_1$ if cof(p) $= \omega_2$, and $K = W^n_1$ if cof(p) $= \omega_1$. If* $\operatorname{cof} p = \omega_2$, *then k, which represents the function corresponding to K, occurs in the functional representation of $\mathcal{L}(\bar{p})$. If* $\operatorname{cof} p = \omega_1$, *then γ_n, which represents the largest ordinal corresponding to K, occurs in the functional representation of $\mathcal{L}(\bar{p})$.*

Proof. By induction on the definition of p. We suppose $\operatorname{cof} p = \omega_2$, the other case being similar. Then $K = S^n_1$ for some $n \geq 1$. We consider the following cases.

case 1.) $p = \cdot_r$. Then $r > 1$, and $k(0)(\cdot_{r-1})$ is a subdescription of $\mathcal{L}(\bar{p})$.

case 2.) $p = \widetilde{h_i(l)}(\dots, q, s)$. Since $\operatorname{cof} p = \omega_2$, we have $\operatorname{cof} q = \omega_2$. By induction, k appears in the functional representation of $\mathcal{L}(\bar{q})$. Since q is greater than all descriptions to its left, $\mathcal{L}(\bar{q})$ is $\geq$ all descriptions to the left of q. Since $\mathcal{L}(\bar{q})$ has k in its functional representation, and the others do not, $\mathcal{L}(\bar{q})$ is greater than these descriptions. Thus, $\mathcal{L}(\bar{p}) = h_i(l)(\dots, \mathcal{L}(\bar{q}), s)$.

case 3.) $p = h_i(l)(\dots, q, s)$. Then $h_i(l)$ is a proper invariant of h_i (as otherwise $\operatorname{cof}(p) = \omega$). Also, $\operatorname{cof}(s) = \omega_2$, and so k appears in the functional representation of $\mathcal{L}(\bar{s})$. Arguing as in the previous case, we have $\mathcal{L}(\bar{p}) = h_i(l+1)(\dots, q, \mathcal{L}(\bar{s}), s)$, and we are done.

□

Proposition 3.3. *If $\bar{d} = (d; \bar{S})$, $q \leq \mathcal{L}(\bar{d})$, and $q \in \mathcal{D}(\bar{S})$, then there is a node $\bar{q}$ in $T_{\bar{d}}$ with description q.*

Proof. By induction on $|T_{\bar{d}}|$. Let $p = \mathcal{L}(\bar{d})$. Consider $\bar{p} = (\mathcal{L}(\bar{d}); \bar{S}, K) \in T_{\bar{d}}$. If $q = p$, we are done, and if $q < p$, then since $q \in \mathcal{D}(\bar{S}) \subseteq \mathcal{D}(\bar{S}, K)$, there is by induction a node $\bar{q}$ in $T_{\bar{p}}$ with the description q. However $T_{\bar{p}} \subset T_{\bar{d}}$, hence we are done. □

Proposition 3.4. *Let $\bar{d} = (d; \bar{S})$, and $\bar{q}$ in $T_{\bar{d}}$. Then $p := \sup_{\bar{d}} q$ appears in some node $\bar{p} \in T_{\bar{d}}$.*

Proof. We easily have $p \leq d$. If $p = d$, then we may take $\bar{p} = \bar{d} \in T_{\bar{d}}$. Otherwise, $p \leq \mathcal{L}(\bar{d})$, and hence p in a node in $T_{\bar{d}}$ by proposition 3.3. □

Definition 3.2 (Level of $\bar{q}$ with respect to $\bar{d}$)**.** Let u be a sequence of terms of the form $\gamma_{i,j}$ or $k_i(\cdot_r)$. We define a linear order $<_u$ on the elements of the sequence u as follows:

1. $\gamma_{i,j} <_u \gamma_{k,l}$ iff $(i,j) <^{lex} (k,l)$

2. $\gamma_{i,j} <_u k_l(\cdot_r)$ for all i, j, l, r
3. $k_i(\cdot_r) <_u k_j(\cdot_s) \iff (r, i) <^{lex} (s, j)$

Next define a subsequence w of u as follows: $w(0) = u(0)$. Assume that $w(i)$ has been defined for all $i = 0, \ldots, l$, and $w(l) = u(r)$. If there is $r' > r$ such that $u(r) <_u u(r')$, then let r'' be the least such, and we put $w(l+1) = u(r'')$. If there is no such r', we stop. Let $\#k_i(\cdot_n) := n$ and let $\#\gamma_{i,j} := 0$, for all i, j, n. Then we set

$$lev(u) := \Sigma^0_{i=|w|-1} \omega^{\#w(i)}.$$

Suppose now $\bar{d} = (d; \bar{S})$, and $\bar{q} \in T_{\bar{d}}$. Let $u_{\bar{q},\bar{d}} = \mathrm{oseq}_{\bar{d}}(\bar{q})$. Then define $lev_{\bar{d}}(\bar{q}) = lev(u_{\bar{q},\bar{d}})$. If $\mathrm{oseq}_{\bar{d}}(\bar{q}) = \emptyset$, set $lev_{\bar{d}}(\bar{q}) = 0$.

Note that the ordering $<_u$ is just the ordering on descriptions translated to their functional representations.

Example . If $q = h_0(h_1(\gamma_{1,1}, \cdot_1), h_1(\gamma_{1,2}, k_2(\cdot_1)), \cdot_2)$, then $u = \mathrm{oseq}_{\bar{d}}(\bar{q}) = \langle \gamma_{1,1}, k_2(\cdot_1), \gamma_{1,2} \rangle$, and $w = \langle \gamma_{1,1}, k_2(\cdot_1) \rangle$. So, $lev_{\bar{d}}(\bar{q}) = \omega^{\#k_2(\cdot_1)} + \omega^{\#\gamma_{1,1}} = \omega + 1$.

Lemma 3.5. *Fix $\bar{d} = (d; \bar{S})$. Then $\{lev_{\bar{d}}(\bar{q}) \mid \bar{q} \in T_{\bar{d}}\}$ is finite.*

Proof. Consider a node $\bar{q} \in T_{\bar{d}}$ with functional representation $g(f_1, \ldots, f_l, f_0)$. Let us temporarily call the description $g(f_1, \ldots, f_l, f_0)$ of rank one. We refer to each subdescription f_i as having rank two, to subdescriptions of f_i, of rank three, and so on. Without loss of generality assume $g = h_i(j)$. Because the $\bar{S}$ measures are fixed (hence there are only finitely many $h_i(j)$, $\alpha_{i,j}$) there is $v < \omega$, such that all of the subdescriptions of $\bar{q}$ that do not start with $k_i(j)$, for some i, j, have rank less than v. This gives a bound on the length of $\mathrm{oseq}_{\bar{d}}(q)$. Also, for terms of $\mathrm{oseq}_{\bar{d}}(q)$ of the form $k_i(\cdot_r)$, we must have $r \leq m$. The result now follows. ☐

We now group the nodes of $T_{\bar{d}}$ into blocks.

Definition 3.3 (Block $\mathrm{B}_{\bar{d}}(q)$, Depth of a block $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q))$)**.** Fix $\bar{d} = (d; \bar{S})$, $d \in \mathcal{D}_m(\bar{S})$. For $q \in \mathcal{D}_m(\bar{S})$, $q \leq d$, we define the *block*, $\mathrm{B}_{\bar{d}}(q)$, as the set of all nodes $\bar{p} \in T_{\bar{d}}$ with $\sup_{\bar{d}} p = q$. We also define the depth of a block by $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q)) := \max\{lev_{\bar{d}}(\bar{p}) \mid \bar{p} \in \mathrm{B}_{\bar{d}}(q)\}$.

Observe that the number of blocks is determined by the number of descriptions $q \in \mathcal{D}_m(\bar{S})$, which is clearly finite. Let us enumerate them in decreasing order: $d = q_1 > q_2 > \cdots > q_n$. Therefore the number of blocks is also finite and equal to n.

Note that every node $\bar{q} \in T_{\bar{d}}$ is in one of these blocks. Now we define the ordinal

$$\xi_{\bar{d}} := \omega^{\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_n))} + \cdots + \omega^{\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_2))} + \omega^{\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_1))}$$

which as we shall see determines the cardinality of $(\mathrm{id}; d; \bar{S})(W^m)$.

Remark 3.2. The last summand in the definition of $\xi_{\bar{d}}$ is always 1. That is because $\mathcal{L}(\bar{d})$ is defined relative to $\bar{S}$, and therefore $\mathrm{B}_{\bar{d}}(q_1) = \mathrm{B}_{\bar{d}}(d) = \{\bar{d}\}$. Consequently, $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_1)) = 0$ and $\omega^{\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_1))} = 1$.

Proposition 3.6. *Fix $\bar{d} = (d; \bar{S})$ and $\bar{p} = (p; \bar{S}, S^*) \in T_{\bar{d}}$, with $p = \mathcal{L}(\bar{d})$. Suppose $\bar{q} \in T_{\bar{p}} \subseteq T_{\bar{d}}$. Then $lev_{\bar{p}}(\bar{q}) \leq lev_{\bar{d}}(\bar{q})$. Moreover, if $\mathrm{oseq}_{\bar{d}}(\bar{q})$ starts with the function induced by the S^* measure, then strict inequality holds, and if otherwise, then $\sup_{\bar{d}} q = \sup_{\bar{p}} q$.*

Proof. Assume $\bar{q} = (q; \bar{S}, S^*, \bar{M}) \in T_{\bar{p}} \subset T_{\bar{d}}$ for some sequence of measures $\bar{M}$. We consider the case $S^* = S^i_1$, the other case being easier. Extending our notational convention slightly, we use terms $h_i(j), \alpha_{i,j}$ corresponding to the $\bar{S}$ measures, k^* corresponding to S^*, and $k_i(j), \gamma_{i,j}$ corresponding to the $\bar{M}$ measures.

We may consider the o–sequences of $\bar{q}$ defined relative to $\bar{p}$ and $\bar{d}$. Let us fix them: $u_p := \mathrm{oseq}_{\bar{p}}(\bar{q})$ and $u_d := \mathrm{oseq}_{\bar{d}}(\bar{q})$. We want to analyze the relationship between these two sequences. Recall the definition of the o–sequence. In that definition we concatenated recursively o–sequences of the corresponding subdescriptions. We can repeat the same constructions with the only difference that we stop when the subdescription is $k^*(j)(\dots)$, for some j. Suppose that happens t times. Then

$$u_d = [u_1 {}^\frown \mathrm{oseq}_{\bar{d}}(k^*(j_1)(\dots)) {}^\frown \dots {}^\frown u_2 {}^\frown \mathrm{oseq}_{\bar{d}}(k^*(j_t)(\dots)) {}^\frown u_{t+1}]'$$

$$u_p = [u_1 {}^\frown \mathrm{oseq}_{\bar{p}}(k^*(j_1)(\dots)) {}^\frown \dots {}^\frown u_2 {}^\frown \mathrm{oseq}_{\bar{p}}(k^*(j_t)(\dots)) {}^\frown u_{t+1}]'$$

In other words, the difference between u_d and u_p is determined only by the o–sequences of the subdescriptions starting with an invariant of k^*. Let us fix such a subdescription $s_m = k^*(j_m)(f_1, \dots, f_l, f_0)$, for some $1 \leq m \leq t$. Note that every f_i either starts with an invariant of some k–function (different from k^*), is an ordinal $\gamma_{i,j}$, or it is $\cdot_r$, for some r. We first argue that $lev_{\bar{p}}(s_m) \leq lev_{\bar{d}}(s_m)$.

Suppose $f_0 = \cdot_r$. Then $\mathrm{oseq}_{\bar{d}}(s_m) = k^*(\cdot_r)$, hence $lev_{\bar{d}}(s_m) = \omega^r$, and $\mathrm{oseq}_{\bar{p}}(s_m) = [\mathrm{oseq}_{\bar{p}}(f_1) {}^\frown \dots {}^\frown \mathrm{oseq}_{\bar{p}}(f_l)]'$. Because for each $1 \leq i \leq l$, $f_i < \cdot_r$, f_i can not have k–functions with dot variables $\geq \cdot_r$. Thus $lev_{\bar{p}}(f_i) < \omega^r$, and hence $lev_{\bar{p}}(s_m) < lev_{\bar{d}}(s_m)$.

Suppose now f_0 begins with some k–function and has the highest dot variable $\cdot_r$, for some r. Then $\mathrm{oseq}_{\bar{d}}(s_m) = k_i(\cdot_r)$ for some i, and $\mathrm{oseq}_{\bar{p}}(s_m) = k_i(\cdot_r) {}^\frown \mathrm{oseq}_{\bar{p}}(f_1) \dots {}^\frown \mathrm{oseq}_{\bar{p}}(f_l)$. Note that for all $1 \leq i \leq l$, f_i can not have a k–function with a dot variable higher than $\cdot_r$. If $\mathrm{oseq}_{\bar{p}}(f_i)$ contains some $k_j(\cdot_r)$, then $j \leq i$, because $f_i < f_0$. Thus $k_j(\cdot_r)$ will be canceled when we compute $lev_{\bar{p}}(s_m)$. Therefore, $lev_{\bar{p}}(s_m) = \omega^r = lev_{\bar{d}}(s_m)$. Similarly $lev_{\bar{p}}(s_m) = lev_{\bar{d}}(s_m)$, when $f_0 = \gamma_{i,j}$.

From the results of the last two paragraphs, an easy argument shows that $lev_{\bar{p}}(\bar{q}) \leq lev_{\bar{d}}(\bar{q})$.

Finally, suppose $\mathrm{oseq}_{\bar{d}}(\bar{q})$ begins with the term b, which is of the form $k^*(\cdot_r)$, $k_i(\cdot_r)$, or $\gamma_{i,j}$. If $b = k^*$, we must have $s_1 = k^*(j_1)(\dots, \cdot_r)$. Then, as we argued above, $lev_{\bar{p}}(s_1) < lev_{\bar{d}}(s_1)$, and an easy argument then shows $lev_{\bar{p}}(\bar{q}) < lev_{\bar{d}}(\bar{q})$.

If $b = k_i(\cdot_r)$ or $\gamma_{i,j}$, then both $\mathrm{oseq}_{\bar{d}}(\bar{q})$ and $\mathrm{oseq}_{\bar{d}}(\bar{q})$ begin with b, which corresponds to the most important subdescription in determining the rank of $\bar{q}$. Let $f(g_1, \dots, g_l, g_0)$ be the functional representation of $\bar{q}$. Let i be the least integer so that a subdescription with term b appears in g_i. Then $\mathrm{oseq}_{\bar{p}}(g_i)$ and $\mathrm{oseq}_{\bar{d}}(g_i)$ both begin with b. By induction we may assume $\sup_{\bar{p}} g_i = \sup_{\bar{d}} g_i$, which implies $\sup_{\bar{p}} q = \sup_{\bar{d}} q$. □

Lemma 3.7. *Let $\bar{d} = (d, \bar{S})$, and $\bar{p}$ be a node in $T_{\bar{d}}$ below $\bar{d}$. Then $\xi_{\bar{p}} \le \xi_{\bar{d}} - 1$.*

Proof. By induction on the rank of $\bar{d}$, we may assume that $\bar{p}$ has description $p = \mathcal{L}(\bar{d})$. If $\operatorname{cof} \bar{p} = \omega$, i.e., the tree $T_{\bar{d}}$ does not split at the root $\bar{d}$, then the proof is trivial. Suppose $\operatorname{cof} \bar{p} = \omega_2$. Thus, $\bar{p} = (p; \bar{S}, S^*)$, where $S^* = S^i_1$ for some i. Keeping with the previous conventions, we denote the function corresponding to the measure S^* by k^*. A node $\bar{s}$ whose o-sequence, $\mathrm{oseq}_{\bar{d}}(\bar{s})$, begins with a term of the form $k^*(\cdot_i)$ will be called a *star* node. Otherwise $\bar{s}$ is called a *nonstar* node.

Let $\mathrm{B}_{\bar{d}}(q_1), \dots, \mathrm{B}_{\bar{d}}(q_n)$ be all the blocks of $T_{\bar{d}}$ where $q_1 = d > q_2 = p > q_3 > \cdots > q_n$ and $q_i \in \mathcal{D}_m(\bar{S})$. Note that all the q_i with $i > 1$ are in $T_{\bar{p}}$ as well.

It is a trivial observation that $\mathrm{oseq}_{\bar{d}}(\bar{s}) = \emptyset \Rightarrow \mathrm{oseq}_{\bar{p}}(\bar{s}) = \emptyset$. The converse, however, is not true: there could be a node $\bar{s}$ with $\mathrm{oseq}_{\bar{p}}(\bar{s}) = \emptyset$ while $\mathrm{oseq}_{\bar{d}}(\bar{s}) \ne \emptyset$. If we fix a d–block, $\mathrm{B}_{\bar{d}}(q_i)$ with $i > 1$, then some of the nodes $\bar{q} \in \mathrm{B}_{\bar{d}}(q_i)$ may be such that $\mathrm{oseq}_{\bar{p}}(q) = \emptyset$, whence a $\bar{d}$–block may split into several $\bar{p}$–blocks. The idea of the proof then is to show that $\omega^{\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))}$ is no less than the sum of the ordinals assigned to the corresponding $\bar{p}$–blocks.

Let us fix for the moment some q_i, for $2 \le i \le n$. Let $s_{i_1} = q_i > s_{i_2} > \dots s_{i_t}$ enumerate the $s \in \mathcal{D}_m(\bar{S}, S^*)$ such that $\sup_{S^*} s = q_i$. Thus, the $\bar{d}$ block corresponding to q_i splits into $\bar{p}$ blocks determined by the s_{i_j}.

Part (1) of the following claim is true in general, while part (2) uses our assumption that $\operatorname{cof} \bar{p} > \omega$.

Claim . With the notation as above:

(1) $\Sigma^1_{j=t} \omega^{\mathrm{depth}(\mathrm{B}_{\bar{p}}(s_{i_j}))} \le \omega^{\mathrm{depth}(\mathrm{B}_{\bar{d}}(s_1))}$.
(2) If $i = 2$ (that is, $s_{i_1} = p$), then $\Sigma^1_{j=t} \omega^{\mathrm{depth}(\mathrm{B}_p(s_{i_j}))} < \omega^{\mathrm{depth}(\mathrm{B}_d(s_1))}$.

Proof. From proposition 3.6, $lev_{\bar{p}}(\bar{s}) \le lev_{\bar{d}}(\bar{s})$ for all $\bar{s} \in T_{\bar{p}}$, and in particular for all $\bar{s} \in \mathrm{B}_{\bar{p}}(s_{i_1})$. Since $\mathrm{B}_{\bar{p}}(s_{i_1}) \subseteq \mathrm{B}_{\bar{d}}(q_i)$, it follows that $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s_{i_1})) \le \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$.

Now let $2 \le j \le t$, and consider $\bar{s} \in \mathrm{B}_{\bar{p}}(s_{i_j})$. Then s must be a star node, because otherwise $s_{i_1} = q_i = \sup_{\bar{d}} s = \sup_{\bar{p}} s$, by proposition 3.6, and hence $\bar{s} \in \mathrm{B}_{\bar{p}}(s_{i_1})$, a contradiction. So, for every $s \in \mathrm{B}_{\bar{p}}(s_{i_j})$, $lev_{\bar{p}}(s) < lev_{\bar{d}}(s)$. Consequently, $\mathrm{depth}(\mathrm{B}_{\bar{p}}(s_{i_j})) < \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$, for all $j = 2, \dots, t$. The first part of the claim now follows.

Suppose now $i = 2$, so $s_{i_1} = p$. By proposition 3.2, k^* appears in a term in the functional representaton of $\mathcal{L}(\bar{p})$. Since $\sup_{S^*}(\mathcal{L}(\bar{p})) = p$, it follows that $\text{depth}(\mathrm{B}_{\bar{d}}(p)) > 0$. However, $\mathrm{B}_{\bar{p}}(p) = \{\bar{p}\}$. So, $0 = \text{depth}(\mathrm{B}_{\bar{p}}(p)) < \text{depth}(\mathrm{B}_{\bar{d}}(p))$, and the second part of the claim follows from proposition 3.6. □

Lemma 3.7 is an immediate consequence of the last claim:

$$\xi_{\bar{p}} = \Sigma^2_{i=n}[\Sigma^1_{j=t_i}\omega^{\text{depth}(\mathrm{B}_p(s_{i_j}))}] \leq \Sigma^2_{i=n}\omega^{\text{depth}(\mathrm{B}_d(q_i))} = \xi_{\bar{d}} - 1.$$

The proof of the case when $\operatorname{cof} p = \omega_1$ is entirely similar. □

Corollary 3.8. *Let $d \in \mathcal{D}_m(\bar{K})$, and satisfy condition D. Then*

$$(\text{id}; d; \bar{K})(W^m) \leq \aleph_{\omega+\xi_{\bar{d}}}.$$

Proof. Lemma 3.7 and a trivial induction show that $|T_{\bar{d}}| \leq \xi_{\bar{d}}$. By the results of [J1], $(\text{id}; d; \bar{S})(W^m) \leq \aleph_{\omega+|T_{\bar{d}}|}$. So $(\text{id}; d; \bar{S})(W^m) \leq \aleph_{\omega+\xi_{\bar{d}}}$. □

To show that the lower bound for $(\text{id}; d; \bar{s})(W^m)$ is also $\aleph_{\omega+\xi_{\bar{d}}}$, we recall the following fact.

Theorem 3.9 (Martin)**.** *Assume $\kappa \to \kappa^\kappa$. Then for any measure ν on κ, the ultrapower $j_\kappa(\kappa)$ is a cardinal.*

Proof. See [J1]. □

Our stategy for the rest of the proof is to embed the ultrapower of $\boldsymbol{\delta}^1_3$ by the measure corresponding to $\xi_{\bar{d}}$ (made precise below) into $(\text{id}; \bar{d}; \bar{S})(W^m)$. We require first some embedding lemmas.

Definition 3.4 (Strong embedding)**.** Let $(D_i, <_{D_i}), (E_i, <_{E_i})$, $1 \leq i \leq n$ be well-orderings of length $< \boldsymbol{\delta}^1_3$, and M_i, N_i measures on D_i, E_i. Let $D = D_1 \oplus \cdots \oplus D_l$, $E = E_1 \oplus \cdots \oplus E_l$, the sum of the order types. We say $(D, \{M_i\})$ strongly embeds into $(E, \{N_i\})$ if there is a measure μ on $\kappa < \boldsymbol{\delta}^1_3$, and a function H with the following properties:

(1) $\forall^*_\mu \theta \;\; H(\theta) = ([\phi_1]_{M_1}, \ldots, [\phi_l]_{M_l})$, where $\phi_i : D_i \to E_i$ is order-preserving.
(2) For all $A_i \subseteq E_i$, $1 \leq i \leq n$, of N_i measure 1, $\forall^*_\mu \theta \; \forall i \; \forall^*_{M_i} \alpha \in D_i$ $\phi_i(\alpha) \in A_i$.

If (D_i, M_i) strongly embeds into (E_i, N_i) for all $1 \leq i \leq n$, then $D = \oplus D_i$ strongly embeds into $E = \oplus E_i$.

Given the ordering $D = D_1 \oplus \cdots \oplus D_l$ and measures M_i, let ν_D denote the measure on l-tuples from $\boldsymbol{\delta}^1_3$ induced by the weak partition relation on $\boldsymbol{\delta}^1_3$, functions $f : D \to \boldsymbol{\delta}^1_3$ of the correct type, and the M_i.

Proposition 3.10. *If $(D, \{M_i\})$, $1 \leq i \leq n$, strongly embeds into $(E, \{N_i\})$, then $j_{\nu_D}(\boldsymbol{\delta}^1_3) \leq j_{\nu_E}(\boldsymbol{\delta}^1_3)$.*

Proof. Let μ, H witness the strong embeddability. We define an embedding π from $j_{\nu_D}(\boldsymbol{\delta}^1_3)$ to $j_{\nu_E}(\boldsymbol{\delta}^1_3)$. Define $\pi([F]_{\nu_D}) = [G]_{\nu_E}$, where for $g = (g_1 \oplus \cdots \oplus g_l) : E \to \boldsymbol{\delta}^1_3$ of the correct type, $G([g_1]_{E_1}, \dots, [g_l]_{E_l}) = [\theta \to F([g_1 \circ \phi_1]_{M_1}, \dots, [g_l \circ \phi_l]_{M_l})]_\mu$, where $H(\theta) = ([\phi_1]_{M_1}, \dots, [\phi_l]_{M_l})$. Using the properties of H, this is easily well-defined and an embedding. □

Proposition 3.11. *Let $\mathcal{O}$ be an order type of length $< \boldsymbol{\delta}^1_3$, and ν a measure on $\mathcal{O}$. Let $0 \leq k < l$, $m > 0$. Let D be lexicographic order on $(\alpha_1, \dots, \alpha_m, \gamma)$ where $\alpha_i < \aleph_{k+1}$, $\gamma \in \mathcal{O}$, and let M be the product measure $M = S^k_1 \times \cdots \times S^k_1 \times \nu$, or $= W^1_1 \times \cdots \times W^1_1 \times \nu$ if $k = 0$. Let E be lexicographic order on (β, γ), where $\beta < \aleph_{l+1}$ and $\gamma \in \mathcal{O}$, and N the product measure $S^l_1 \times \nu$ on E. Then (D, M) strongly embeds into (E, N). Similarly if D is the sum of m copies of $\mathcal{O}$, and $l = 0$ (with measure $W^1_1 \times \nu$).*

Proof. We prove the result for $k \geq 1$, the other cases being similar. Let $\mu = S^{l+m}_1$. Define $H([h]_{W^{l+m}_1}) = [\phi]_M$, where $\phi : D \to E$ is defined as follows. $\phi([f_1]_{W^k_1}, \dots, [f_m]_{W^k_1}, \gamma) = ([g]_{W^l_1}, \gamma)$, where

$$\begin{aligned} &g(\delta_1, \dots, \delta_l) = \\ &h(\delta_1, \dots, \delta_k, f_1(\delta_1, \dots, \delta_k), \dots, f_m(\delta_1, \dots, \delta_k), \delta_{k+1}, \dots, \delta_l). \end{aligned}$$

This is easily well-defined, and gives a strong embedding. □

By a *basic order type*, we mean $D = D_1 \oplus \cdots \oplus D_l$, where for all $1 \leq i \leq l$, either $D_i = 1$ (i.e., the order type of a single point), or $D_i = \aleph_{k^i_m+1} \otimes \aleph_{k^i_{m-1}+1} \cdots \otimes \aleph_{k^i_1+1}$ (i.e., lexicographic ordering on tuples $(\alpha_1, \dots, \alpha_m)$ where $\alpha_j < \aleph_{k^i_j+1}$, and m depends on i). Let M_i be the product measure $M_i = S^{k^i_1}_1 \times \cdots \times S^{k^i_m}_1$. We refer to such a D_i as a *sub-basic* order type. To each such D, we associate an ordinal $c(D)$ as follows. If $D_i = 1$, $c(D_i) = 1$. If $D_i = \aleph_{k_m+1} \otimes \cdots \otimes \aleph_{k_1+1}$, then $c(D_i) = \omega^{\omega^{k_m}} \cdots \omega^{\omega^{k_2}} \cdot \omega^{\omega^{k_1}}$. Finally, $c(D) = c(D_1) + \cdots + c(D_l)$.

Lemma 3.12. *For D a basic order type with corresponding measure ν_D, $j_{\nu_D}(\boldsymbol{\delta}^1_3) \geq \aleph_{\omega + c(D) + 1}$.*

Proof. An easy induction on the length of D, $|D|$, using proposition 3.11. For example, the inductive step at $D = \aleph_3$ would be: $j_{\nu_{\aleph_3}}(\boldsymbol{\delta}^1_3) \geq \sup_n j_{\nu_{(\aleph_2)^n}}(\boldsymbol{\delta}^1_3) \geq \sup_n \aleph_{\omega + \omega^{\omega \cdot n} + 1} = \aleph_{\omega^{\omega^2}}$. Since $\operatorname{cof} j_\nu(\boldsymbol{\delta}^1_3) > \omega$ for any measure ν, we then have $j_{\nu_{\aleph_3}}(\boldsymbol{\delta}^1_3) \geq \aleph_{\omega^{\omega^2}+1} = \aleph_{\omega + \omega^{\omega^2} + 1}$. □

Suppose now $M = M_1 \times \cdots \times M_k = M^0_1 \times \cdots \times M^0_{a_0} \times \cdots \times M^n_1 \times \cdots \times M^n_{a_n}$ is a product measure, where $M^i_j = W^1_1$ if $i = 0$, and $M^i_j = S^i_j$ for $i > 0$. Let $\pi = (p_1, \dots, p_k)$ be a permutation of k. Let D be the M measure one set of $(\alpha_1, \dots, \alpha_k) = (\alpha^0_1, \dots, \alpha^0_{a_0}, \dots, \alpha^n_1, \dots, \alpha^n_{a_n})$ such that $\alpha^0_1 < \cdots < \alpha^0_{a_0}$, $\alpha^i_j > \aleph_i$, and $\alpha_i(0) < \alpha_j(0)$ for $i < j$ and $\alpha_i > \aleph_1$. Let $<_D$ be the ordering of D defined by: $(\alpha_1, \dots, \alpha_k) <_D (\beta_1, \dots, \beta_k)$ iff $(\alpha_{p_1}, \dots, \alpha_{p_k}) <^{lex} (\beta_{p_1}, \dots, \beta_{p_k})$.

We define the *canonical subsequence* π^* of π as follows. $\pi^* = (q_1, \dots, q_l) = (p_{s_1}, \dots, p_{s_l})$, where $s_1 = 1$, and $s_{i+1} > s_i$ is least such that $p_{s_{i+1}} > p_{s_i}$. Note that $q_l = k$. To fix notation, let $M_i = M^{r(i)}_{u(i)}$ for $1 \leq i \leq k$. Define N to be the product measure $N = M_{q_1} \times \cdots \times M_{q_l}$, and let E be lexicographic ordering on tuples $(\beta_1, \dots, \beta_l)$ with $\beta_i < \aleph_{r(q_i)+1}$.

Notice that $(E, <_E)$ is a basic order type.

Lemma 3.13. *With $(D, <_D)$, $(E, <_E)$ as above, $(E, <_E)$ strongly embeds into $(D, <_D)$.*

Proof. Let $\mu = M_1 \times \cdots \times M_{q_1-1} \times \prod_{j=q_1}^k M_j^+$, where $(W_1^1)^+ = S_1^1$, and $(S_1^r)^+ = S_1^{r+1}$. Fix $\theta = (\theta_1, \dots, \theta_k)$, and let $h_i :<_{r(i)+1} \to \aleph_1$ represent θ_i if $r(i) > 0$ and $i \geq q_1$. Set $H(\theta) = [\phi]_N$, where $\phi(\alpha_1, \dots, \alpha_l) = (\beta_1, \dots, \beta_k)$ is defined as follows. First, $\beta_1, \dots, \beta_{q_1-1} = \theta_1, \dots, \theta_{q_1-1}$. Next, suppose $q_i \leq j < q_{i+1}$. If $r(j) = 0$, set $\beta_j = h_j(\alpha_{q_i})$. If $r(j) > 0$ and $r(q_i) = 0$, set $\beta_j = [g_j]$, where $g_j(\gamma_1, \dots, \gamma_{r(j)}) = h_j(\alpha_{q_i}, \gamma_1, \dots, \gamma_{r(j)})$. If $r(q_i) > 0$, set $\beta_j = [g_j]$, where

$$g_j(\gamma_1, \dots, \gamma_{r(j)}) = \\ h_j(\gamma_1, \dots, \gamma_{r(q_i)}, f_i(\gamma_1, \dots, \gamma_{r(q_i)}), \gamma_{r(q_i)+1}, \dots, f_i(0)(\gamma_{r(j)})),$$

where $[f_i] = \alpha_{q_i}$, and the argument $\gamma_{r(i)}$ of h_j is omitted if $r(q_i) = r(j)$ (this is just to give the correct number of arguments). This is easily checked to be well-defined and a strong embedding. $\square$

Remark 3.3. The proof of lemma 3.13 also shows if π' is any subsequence of the canonical sequence π^* of π, and E', N' the corresponding order and product measure, then (E', N') strongly embeds into (D, M).

Proposition 3.14. *For every block $\mathrm{B}_{\bar{d}}(q_i), 1 \leq i \leq n$ with $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i)) > 0$, there is a node $\bar{p}_i$, with description p_i such that $\bar{p}_i \in \mathrm{B}_{\bar{d}}(q_i)$, $lev_d(\bar{p}_i) = \mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i))$, and p_i has functional representation $p_i = h_k(r)(f_1, \dots, f_r, f_0)$ where $S_k = S_1^{r+1}$ (that is, p_i has maximal possible length).*

Proof. Suppose q_i has functional representation $q_i = h_k(l)(f_1, \dots, f_l, f_0)$, and $S_k = S_1^{r+1}$. Let $\bar{q}_i = (q_i; \bar{S}) \in T_{\bar{d}}$ with description q_i. We must have $l < r$, as otherwise $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i)) = 0$. Likewise, we must have $\mathrm{cof}\, \bar{q}_i = \mathrm{cof}\, \bar{f}_0 > \omega$, as otherwise $\mathcal{L}(\bar{f}_0) \in \mathcal{D}(\bar{S})$ and hence $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i)) = 0$. Let $\bar{p} = (p; \bar{S}, \bar{K}) \in \mathrm{B}_{\bar{d}}(q_i)$ have maximum possible level. Easily, p is of the form $p = f(f_1, \dots, f_l, \dots, f_{l'}, f_0)$, where $f = h_k(l')$ or $\widetilde{h_k(l')}$. Since replacing $\widetilde{h_k(l')}$ by $h_k(l')$ does not change the level or the block, we may assume $p = h_k(l')(f_1, \dots, f_l, \dots, f_{l'}, f_0)$. If $l' < r$, then $\mathrm{cof}\, \bar{p} = \mathrm{cof}\, \bar{f}_0 > \omega$, and the last $\bar{K}$ measure, say K_t, does not appear in p. By proposition 3.2, $\mathcal{L}(\bar{f}_0)$ will have a term corresponding to the measure K_t in its functional representation, and hence $\mathcal{L}(\bar{f}_0) > f_{l'}$. Thus, $\mathcal{L}(\bar{p}) = h_k(l)(f_1, \dots, f_l, \dots, f_{l'}, \mathcal{L}(\bar{f}_0), f_0)$. Repeating the argument, we finish.

Suppose now $q_i = \widetilde{h_k(l)}(f_1, \dots, f_l, f_0)$. As above, let $\bar{p} = (p; \bar{S}, \bar{K}) \in \mathrm{B}_{\bar{d}}(q_i)$ have maximum possible level. Easily, $p = h_k(l')(f_1, \dots, f_{l-1}, g_l, \dots, g_{l'}, f_0)$ for some $l \leq l' \leq r$. As $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i)) > 0$, $\mathrm{cof}\, f_l > \omega$. Let $u \geq l$ be largest such that g_u involves one of the $\bar{K}$ measures. We may assume p is chosen to maximize u (subject to having maximum level). If $u = r$, we are done, so assume $u < r$. Let $g' = \sup_{\bar{K}} g_u$. Thus, $\mathrm{cof}\, g' > \omega$. If $g' = f_0$, then $\mathrm{cof}\, f_0 > \omega$, and we finish as before. Otherwise, consider $p' = \widetilde{h_k(u+1)}(f_1, \dots, f_{l-1}, g_l, \dots, g_u, g', f_0)$. By considering a path in $T_{\bar{d}}$ from $\bar{d}$ to $\bar{p}$, one easily sees that

$$(h_k(u)(f_1, \dots, f_{l-1}, g_l, \dots, g_u, f_0); \bar{S}, \bar{L}) \in T_{\bar{d}},$$

for some subsequence $\bar{L}$ of $\bar{K}$. By proposition 3.3, for some sequence $\bar{M}$, $(p'; \bar{S}, \bar{L}, \bar{M}) \in T_{\bar{d}}$. Since $\mathrm{cof}\, p' = \mathrm{cof}\, g' > \omega$, $M \neq \emptyset$. By proposition 3.2, $\mathcal{L}(g'; \bar{S}, \bar{L}, \bar{M}) > g_u$, as it involves a measure from $\bar{M}$. Thus, $\mathcal{L}(p'; \bar{S}, \bar{L}, \bar{M}) = h_k(u+1)(f_1, \dots, f_{l-1}, g_l, \dots, g_u, \mathcal{L}(g'; \bar{S}, \bar{L}, \bar{M}), f_0)$. This, however, gives a node in $\mathrm{B}_{\bar{d}}(q_i)$ of maximum level which violates the maximality of u. $\square$

We now prove our main lemma.

Lemma 3.15. *Fix $\bar{d} = (d; \bar{S})$ where $d \in \mathcal{D}_m(\bar{S})$, and satisfies condition D. Then* $(\mathrm{id}; d; \bar{S})(W^m) \geq \aleph_{\omega+\xi_{\bar{d}}}$.

Let $d = q_1 > q_2 > \cdots > q_n$ enumerate the $q \in \mathcal{D}_m(\bar{S})$, so the number of $\bar{d}$–blocks is also n. Recall that $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_1)) = 0$.

For $2 \leq i \leq n$ such that $\mathrm{depth}(\mathrm{B}_{\bar{d}}(q_i)) > 0$, let $\bar{p}_i$ be as in proposition 3.14. We refer to these blocks as the *non-trivial* blocks. For the trivial blocks, let $p_i = q_i$. For non-trivial block i, let $\bar{p}_i = (p_i; \bar{S}, \bar{K}(i))$, where $\bar{K}(i) = (K_1(i), \dots, K_{t_i}(i))$.

Recall that for non-trivial blocks, the ordinal $lev_{\bar{d}}(p_i)$ was derived from w_i, the subsequence of $\mathrm{oseq}_{\bar{d}}(p_i)$ (see definition 3.2). Let $t^*_i = \mathrm{oseq}^*_{\bar{d}}(p_i)$, and $l_i = \mathrm{lh}\, w_i - 1$, $l^*_i = \mathrm{lh}\, t^*_i - 1$. Define two order types, D_i, E_i as follows.

For non-trivial blocks, set $E_i := \aleph_{1+\#w_i(l_i)} \cdot \cdots \cdot \aleph_{1+\#w_i(0)}$, that is lexicographic ordering on tuples $(\beta_0, \dots, \beta_{l_i})$, where $\beta_j < \aleph_{1+\#w_i(j)}$, and where $\beta_0 < \cdots < \beta_{l_i}$. Let N_i be the product measure $N_i = N(0) \times \cdots \times N(l_i)$, where $N(j) = W^1_1$ if $\#w_i(j) = 0$, and $N(j) = S^{\#w_i(j)}_1$ if $\#w_i(j) > 0$.

To define D_i, let $(t(0), \dots, t(l^*_i))$ be the sequence of terms from $\mathrm{oseq}^*_{\bar{d}}(p_i)$ written in increasing order (in the ordering of terms). Let M_i be the product measure $M_i = M(0) \times \cdots \times M(l^*_i)$, where $M(j) = W^1_1$ if $t(j) = \gamma^a_{b,c}$, and $M(j) = S^r_1$ if $t(j) = k^a_b(\cdot_r)$. Let π_i be the permutation of l^*_i defined by: $t^*_i(j) = t(\pi_i(j))$. Let D_i be the corresponding order type.

For trivial blocks, let $D_i = E_i = 1$. Let $E = E_n \oplus \cdots \oplus E_1$, $D = D_n \oplus \cdots \oplus D_1$. Let ν_E, ν_D be the corresponding measures on $(\boldsymbol{\delta}^1_3)^n$.

Notice that for all non-trivial blocks i, (E_i, N_i) is the order type and measure corresponding to a subsequence of the canonical sequence of π_i.

Thus, by lemmas 3.12, 3.13 we have:

$$j_{\nu_D}(\boldsymbol{\delta}^1_3) \;\geq\; j_{\nu_E}(\boldsymbol{\delta}^1_3) \;\geq\; \aleph_{\omega+\xi_{\bar{d}}} \geq (\mathrm{id}; d; \bar{S})(W^m)$$

We show now that $j_{\nu_D}(\boldsymbol{\delta}^1_3) \leq (\mathrm{id}; d; \bar{S})(W^m)$, which shows that equality holds in the previous inequalities, and completes the proof of lemma 3.15.

We define an embedding $\phi : j_{\nu_D}(\boldsymbol{\delta}^1_3) \to (\mathrm{id}; d; \bar{S})(W^m)$. Fix $[G]_{\nu_D}$, $G : \boldsymbol{\delta}^1_3 \to \boldsymbol{\delta}^1_3$. $\phi([G]_{\nu_D})$ will be represented with respect to $W^m, S_1, \dots, S_s$ (as in the definition of $(\mathrm{id}; d; \bar{S})(W^m)$) by $\phi([G]_{\nu_D})(f, h_1, \dots, h_s)$. We set $\phi([G]_{\nu_D})(f, h_1, \dots, h_s) = G([g])$, where $g : D \to \boldsymbol{\delta}^1_3$ is defined as follows. It suffices to define $g_i = g \restriction D_i$. If i is a trivial block, that is, $D_i = 1$, then set $g_i(0) = (\mathrm{id}; f; p_i; h_1, \dots, h_s)$. Fix a non-trivial block i, let $t^* = (t(0), \dots, t(l^*)) = \mathrm{oseq}^*_{\bar{d}}(p_i)$, and write $K_1, \dots, K_t$ for $K_1(i), \dots, K_{t_i}(i)$. Recall each term $t(l)$ of $\mathrm{oseq}^*_{\bar{d}}(p_i)$ is of the form $t_l = \gamma^{a_l}_{i_l, j_l}$ or $t(l) = k^{a_l}_{i_l}(\cdot_{r_l})$.

We must define $g_i(\beta_0, \dots, \beta_{l^*})$ where $\bar{\beta}$ is as in the definition of D_i. Fix such $\beta_0, \dots, \beta_{l^*}$, and for $\beta_l > \aleph_1$, let $\beta_l = [\tau_l]_{W^{r_l}_1}$, where $\tau_l : <_{r_l} \to \aleph_1$ is of the correct type.

Finally, define $g_i(\beta_0, \dots, \beta_{l^*}) = (\mathrm{id}; f; p_i; h_1, \dots, h_s; \beta_0, \dots, \beta_{l^*})^*$. Roughly speaking, this is defined as $(\mathrm{id}; p_i; h_1, \dots, h_s; k_1, \dots, k_t)$, except that for subdescriptions q corresponding to terms $t(l)$ of $\mathrm{oseq}^*_{\bar{d}}(p_i)$, the interpretation of the description, $h(q; \bar{h}, \bar{k})$, is replaced by β_l if $t(l) = \gamma^a_{i,j}$, and by $h(\alpha_1, \dots, \alpha_m) = \tau_l(\alpha_1, \dots, \alpha_{r_l})$ if $t(l) = k^a_i(\cdot_{r_l})$.

More formally, define $(\mathrm{id}; f; p_i, \bar{h}; \bar{\beta})^* = f((p_i; \bar{h}; \bar{\beta})^*)$, where $(q; \bar{h}; \bar{\beta})^* < \aleph_{m+1}$ is represented with respect to W^m_1 by the function $(q; \bar{h}; \bar{\beta})^*(\alpha_1, \dots, \alpha_m)$ defined inductively as follows:

(1) If $q = h_a(b)(q_1, \dots, q_l, q_0)$, $S_a = S^r_1$, and $l = r-1$, then $(q; \dots)^*(\bar{\alpha}) = h_a((q_1; \dots)^*(\bar{\alpha}), \dots, (q_l; \dots)^*(\bar{\alpha}), (q_0; \dots)^*(\bar{\alpha}))$.
(2) If $q = h_a(b)(q_1, \dots, q_l, q_0)$, $S_a = S^r_1$, and $l < r-1$, then $(q; \dots)^*(\bar{\alpha}) = \sup_{\gamma_{l+1} < \cdots < \gamma_{r-1} < (q_0; \dots)^*(\bar{\alpha})} h_a((q_1; \dots)^*(\bar{\alpha}), \dots, (q_l; \dots)^*(\bar{\alpha}), \gamma_{l+1}, \dots, \gamma_{r-1}, (q_0; \dots)^*(\bar{\alpha}))$.
(3) If $q = \widetilde{h_a(b)}(q_1, \dots, q_l, q_0)$, $S_a = S^r_1$, $1 \leq l \leq r-1$, then $(q; \dots)^*(\bar{\alpha}) = \sup_{\gamma_l < (q_l; \dots)^*(\bar{\alpha}), \gamma_{l+1} < \cdots < \gamma_{r-1} < (q_0; \dots)^*(\bar{\alpha})} h_a((q_1; \dots)^*(\bar{\alpha}), \dots, (q_{l-1}; \dots)^*(\bar{\alpha}), \gamma_l, \gamma_{l+1}, \dots, \gamma_{r-1}, (q_0; \dots)^*(\bar{\alpha}))$.
(4) If $q = \gamma_{i,j}$, and corresponds to $t(e) = \gamma^a_{i,j}$, then $(q; \dots)^*(\bar{\alpha}) = \beta_e < \aleph_1$.
(5) If $q = k_a(b)(q_1, \dots, q_l, q_0)$ or $= \widetilde{k_a(b)}(q_1, \dots, q_l, q_0)$, note that $\mathrm{oseq}^*_{\bar{d}}(q)$ consists of a single term in $\mathrm{oseq}^*_{\bar{d}}(p_i)$, and corresponds to a term, say $t(e) = k^b_i(\cdot_r)$ of $\mathrm{oseq}^*_{\bar{d}}(p_i)$. Then $(q; \dots)^*(\bar{\alpha}) = \tau_e(\alpha_1, \dots, \alpha_r)$.

First note that for fixed $G, f, h_1, \dots, h_u$ each $g_i(\beta_0, \dots, \beta_{l^*_i})$, and hence g is well-defined. Next, we claim that for fixed G, that $\forall^* f$, if $[f] = [f']$ then $\forall^*[h_1]$, if $[h_1] = [h'_1]$ $\dots$, $\forall^*[h_u]$ if $[h_u] = [h'_u]$ then $\forall 1 \leq i \leq n$ $\forall^* \beta_0, \dots, \beta_{l^*_i}$ $g_i(\bar{\beta}) = f((p_i; \bar{h}, \bar{\beta})^*) = f'((p_i; \bar{h}', \bar{\beta})^*) = g'_i(\bar{\beta})$. To see this, note that

$\forall^*\beta_0,\dots,\beta_{l_i^*}$ the functions τ_e (or β_e if $t(e)=\gamma^a_{i,j}$) have range (almost everywhere) in a c.u.b. set $C\subseteq\omega_1$ on which $h_j=h'_j$ and is closed under the $h_i(0)$, and without loss of generality, the h_j,h'_j have range in a c.u.b. set defining a S_1^m measure one set on which $f=f'$. Note also, and this is a key point, that in computing $(p_i;\bar{h};\bar{\beta})^*$, compositions of the form $h_k\circ h_l$ or $h_k\circ\tau_l$ may be used, but none of the form $\tau_k\circ\tau_l$. Also, for a subdescription $q=g(q_1,\dots,q_l,q_0)$ of p_i, it is straightforward to check that $(q_1;\bar{h};\bar{\beta})^*<\cdots<(q_l;\bar{h},\bar{\beta})^*<(q_0;\bar{h},\bar{\beta})^*$; it is here we use the definition of the ordering of the β_j in D_i. From these observations the claim is immediate.

The proofs that ϕ depends only on $[G]_{\nu_D}$, and that ϕ is one-to-one are similar. So, suppose $[G_1]=[G_2]$. Let $C\subseteq\boldsymbol{\delta}^1_3$ be c.u.b. such that if $g:<_D\to C$ is of the correct type, then $G_1([g])=G_2([g])$. Let $C'=\{\alpha\in C:\alpha$ is the α^{th} element of $C\}$. Consider $f,h_1,\dots,h_t$ such that f has range in C', the h_i are of the correct type, and h_{i+1} has range in a c.u.b. subset of ω_1 closed under $h_i(0)$. Let $g:<_D\to\boldsymbol{\delta}^1_3$ be the function defined in the definition of ϕ. Since f has range in C', so does g. Also, g is easily order-preserving restricted to a measure one set, since the terms of each $\text{oseq}^*_{\bar{d}}(p_i)$ were enumerated in order of their significance in determining the size of $(p_i;\bar{S};\bar{K})$. If i is a non-trivial block, then from proposition 3.14 p_i has the form $p_i=h_j(l)(q_1,\dots,q_l,q_0)$ where $S_j=S_1^{l+1}$. Then, g_i has uniform cofinality ω, since h_j does, and f is continuous. If i is a trivial block, then $g_i(0)=f((q_i;\bar{S}))$, and has cofinality ω since $\bar{q}_i$ does in this case. An easy argument now shows that there is a g' such that $[g']=[g]$, and g' is of the correct type with range in C.

This completes the proof of lemma 3.15, and of theorem 3.1. As we remarked in the proof of lemma 3.15, we have actually shown the following.

Theorem 3.16. *Let $d\in\mathcal{D}_m(K_1,\dots,K_t)$ satisfy condition D. Then*

$$(\mathrm{id};d;\bar{K})(W^m)=\aleph_{\omega+\xi_{\bar{d}}}$$

(where $\xi_{\bar{d}}$ is defined after definition 3.3).

Corollary 3.17. *The successor cardinals κ, $\boldsymbol{\delta}^1_3\le\kappa<\boldsymbol{\delta}^1_5$, are exactly the ordinals of the form $(\mathrm{id};d;\bar{K})(W^m)$ for some $d\in\mathcal{D}_m(K_1,\dots,K_t)$ satisfying condition D.*

Proof. Use the theorem 3.16 and [J1]. □

Remark 3.4. As mentioned previously, our definitions are slightly different from those of [J1]. However, a minor variation of our embedding argument shows that the ordinals $(\mathrm{id};d;\bar{K})(W_1^m)$ as defined in [J1] are also cardinals (essentially, one adds extra trivial blocks corresponding to $(d)\in\bar{\mathcal{D}}$, that is, without the symbol s).

4. Applications

Recall from § 3 the definitions of a basic order type, D, the ordinal $c(D)$, and the associated measure ν_D. Recall also lemma 3.12, which says $j_{\nu_D}(\boldsymbol{\delta}^1_3) \geq \aleph_{\omega+c(D)+1}$.

We show now that equality holds here, thereby providing another representation for the successor cardinals $\boldsymbol{\delta}^1_3 < \kappa < \boldsymbol{\delta}^1_5$.

Theorem 4.1. *For D a basic order type, and associated measure ν_D, we have $j_{\nu_D}(\boldsymbol{\delta}^1_3) = \aleph_{\omega+c(D)+1}$.*

Proof. Let $\kappa = \aleph_{\omega+c(D)+1}$. From Martin's theorem (theorem 3.9), $j_{\nu_D}(\boldsymbol{\delta}^1_3)$ is a cardinal, and since $\mathrm{cof}(j_{\nu_D}(\boldsymbol{\delta}^1_3)) > \omega$, it is a successor cardinal. From [J1], every successor $\boldsymbol{\delta}^1_3 < \kappa < \boldsymbol{\delta}^1_5$ is of the form $(\mathrm{id}; d; \bar{S})(W^m)$ for some $d \in \mathcal{D}_m(\bar{S})$. From the equality proved in lemma 3.15, $\kappa = (\mathrm{id}; d; \bar{S})(W^m) = j_{\nu_E}(\boldsymbol{\delta}^1_3) = \aleph_{\omega+c(E)+1}$ for some basic order type E.

To finish, it is enough to observe that if D, E are basic order types with $c(D) = c(E)$, then $j_{\nu_D}(\boldsymbol{\delta}^1_3) = j_{\nu_E}(\boldsymbol{\delta}^1_3)$. For this, it is enough to show that if A, B are sub-basic order types, with $c(A) < c(B)$, then $A \oplus B$ strongly embeds into B. This, however, follows from a trivial variation of proposition 3.11 (replacing $(\aleph_{k+1})^m$ with $\aleph_{p_m+1} \cdot \dots \cdot \aleph_{p_1+1}$, where $p_1, \dots, p_m \leq k$). □

We thus have two ways of representing the successor cardinals below $\boldsymbol{\delta}^1_5$, and the results of this paper give an algorithm for converting from one representation to the other. Questions about the cardinals below $\boldsymbol{\delta}^1_5$ may thus be approached in either manner. To illustrate this, we compute the cofinality of a successor cardinal below $\boldsymbol{\delta}^1_5$.

Theorem 4.2. *Suppose $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1} < \aleph_{\alpha+1} < \aleph_{\omega^{\omega^\omega}+1} = \boldsymbol{\delta}^1_5$. Let $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, where $\omega^\omega > \beta_1 \geq \cdots \geq \beta_n$ be the normal form for α. Then:*

- *If $\beta_n = 0$, then $\mathrm{cof}(\kappa) = \boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$.*
- *If $\beta_n > 0$, and is a successor ordinal, then $\mathrm{cof}(\kappa) = \aleph_{\omega\cdot 2+1}$.*
- *If $\beta_n > 0$ and is a limit ordinal, then $\mathrm{cof}(\kappa) = \aleph_{\omega^\omega+1}$.*

We note that $\aleph_{\omega+2}, \aleph_{\omega\cdot 2+1}$, and $\aleph_{\omega^\omega+1}$ are the three regular cardinals strictly between $\boldsymbol{\delta}^1_3$ and $\boldsymbol{\delta}^1_5$, and are the ultrapowers of $\boldsymbol{\delta}^1_3$ by the three normal measures on $\boldsymbol{\delta}^1_3$ (generated by the c.u.b. filter and the possible cofinalities $\omega, \omega_1, \omega_2$). This is proved in [J1].

sketch. The proof in all cases is similar, so suppose $\beta_n > 0$ and is a limit. Thus, $\beta_n = \omega^{m_l} + \omega^{m_{l-1}} + \cdots + \omega^{m_1}$, where $m_l \geq m_{l-1} \geq \cdots \geq m_1 > 0$. For $1 \leq i \leq n$, let D_i be the sub-basic order type corresponding to β_i, that is, $c(D_i) = \omega^{\beta_i}$.

Let $D = D_1 \oplus \cdots \oplus D_n$. Thus, $D_n = \aleph_{m_l+1} \cdot \dots \cdot \aleph_{m_1+1}$. Also, $\kappa := \aleph_{\alpha+1} = j_{\nu_D}(\boldsymbol{\delta}^1_3)$ from theorem 4.1. Let ν_2 be the ω_2–cofinal normal measure on $\boldsymbol{\delta}^1_3$. We embed $j_{\nu_2}(\boldsymbol{\delta}^1_3)$ cofinally into κ. Given $[F]_{\nu_2}$, let $\pi([F]) = [G]_{\nu_D}$, where for $g = (g_l, \dots, g_1) :<_D\to \boldsymbol{\delta}^1_3$ of the correct type, $G([g_l], \dots, [g_1]) = F(\sup g_1)$.

π is easily well-defined and strictly increasing. An easy partition argument using the weak partition relation on $\boldsymbol{\delta}^1_3$ shows that π is also cofinal. □

Finally, we close by considering an example which illustrates the arguments of this paper. Let $\bar{S} = (S^3_1, S^2_1)$, $m = 2$, and $d \in \mathcal{D}_m(\bar{S})$ with functional representation $d = h_0(0)(\cdot_2)$. Let $\kappa = (\mathrm{id}; d; \bar{S})(W^2)$. The following table lists the descriptions $q_1, \dots, q_7$ determining the blocks $B_1, \dots, B_7$, the p_i giving the depth of each block, and the rank $r_i := \omega^{\mathrm{depth}(B_i)}$ of eack block.

$$q_1 = h_0(0)(\cdot_2)$$
$$r_1 = 1$$
$$q_2 = h_0(1)(h_1(0)(\cdot_1), \cdot_2)$$
$$p_2 = h_0(2)(h_1(0)(\cdot_1), k_3(0)(\cdot_1), \cdot_2)$$
$$r_2 = \omega^\omega$$
$$q_3 = \widetilde{h_0(1)}(h_1(0)(\cdot_1), \cdot_2)$$
$$p_3 = h_0(2)(h_1(1)(\gamma_{4,1}, \cdot_1), k_5(0)(\cdot_1), \cdot_2)$$
$$r_3 = \omega^\omega \cdot \omega = \omega^{\omega+1}$$
$$q_4 = h_0(1)(\cdot_1, \cdot_2)$$
$$p_4 = h_0(2)(\cdot_1, k_6(0)(\cdot_1), \cdot_2)$$
$$r_4 = \omega^\omega$$
$$q_5 = h_0(2)(\cdot_1, h_1(0)(\cdot_1), \cdot_2)$$
$$p_5 = h_0(2)(\cdot_1, h_1(1)(\gamma_{7,1}, \cdot_1), \cdot_2)$$
$$r_5 = \omega$$
$$q_6 = \widetilde{h_0(2)}(\cdot_1, h_1(0)(\cdot_1), \cdot_2)$$
$$p_6 = h_0(2)(\cdot_1, h_1(1)(\gamma_{8,1}, \cdot_2)$$
$$r_6 = \omega$$

$$q_7 = \widetilde{h_0(1)}(\cdot_1, \cdot_2)$$
$$p_7 = h_0(2)(\gamma_{9,1}, k_{10}(0)(\cdot_1), \cdot_2)$$
$$r_7 = \omega \cdot \omega^\omega = \omega^{\omega+1}$$

Thus, $\kappa = \aleph_{\omega^{\omega+1}+\omega+\omega+\omega^\omega+\omega^{\omega+1}+\omega^\omega+1} = \aleph_{\omega^{\omega+1}\cdot 2+\omega^\omega+1}$. From theorem 4.2, $\mathrm{cof}(\kappa) = \aleph_{\omega^\omega+1}$.

References

[J1] Jackson, S., *Computation of $\boldsymbol{\delta}^1_5$*, to appear.

[J2] Jackson, S., *AD and the Projective Ordinals*, Cabal Seminar 81-85, Lecture Notes in Mathematics, vol. 1333, Springer–Verlag, 1988, p. 117–224.

[Ke] Kechris, A.S., *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer–Verlag, 1994.

[Ke1] Kechris, A.S., *AD and Projective Ordinals*, Cabal Seminar 76-77, Lecture Notes in Mathematics, vol. 689, Springer–Verlag, 1978, p.91–132.

[Ke2] Kechris, A.S., *Homogeneous Trees and Projective Ordinals*, Cabal Seminar 77-79, Lecture Notes in Mathematics, vol. 839, Springer–Verlag, 1978, p.33–73.

[Mo] Moschovakis, Y.N., *Descriptive Set Theory*, Studies in Logic and the Foundations of Mathematics, vol. 100, North–Holland, 1980.

Department of Mathematics, University of North Texas, Denton, Texas 76203-5116

E-mail address: jackson@unt.edu

Department of Mathematics, University of North Texas, Denton, Texas 76203-5116

E-mail address: farid@unt.edu