Notions of Strong Compactness without the Axiom of Choice

Afstudeerscriptie

written by

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Chapter 1

Introduction

The study of large cardinal axioms is an active part of contemporary set theory. For a large cardinal notion there are often several definitions possible. For example, two common ways to define a large cardinal notion is as a critical point of an elementary embedding with certain properties, or in terms of ultrafilters. Many other types of definitions exist. With the axiom of choice these definitions are often equivalent. Without the axiom of choice, these definitions may not be equivalent anymore. Moreover the consistency strength of the large cardinal axiom may change with the ambient set theory, depending on which definition you choose.

In this thesis we study several different definitions related to the notion of a compact cardinal. We will be guided by two main questions: What is the structure of implications between different definitions? And: What is the relative consistency strength of these definitions? In both cases the answers may depend on the presence or absence of the axiom of choice.

We develop the preliminaries on infinitary languages, filters and ultrafilters, elementary embeddings and the ultraproduct construction in Chapter 2. In Chapter 3 we study the following four possible definitions of a compact cardinal. These notions are defined later in the text.

1. Alfred Tarski (1962) defined strongly compact and weakly compact cardinals by generalizing the compactness property of first-order logic to infinitary languages. This leads to the first definition: 'language-compactness' (Section 3.1).

2. Connected to the compactness property of first-order logic is the ultrafilter theorem: every filter can be extended to an ultrafilter. This leads to the second definition: the 'extension property' that every κ-complete filter can be extended to a κ-complete ultrafilter (Section 3.2).

3. The ultraproduct proof of the compactness theorem for first-order logic extends to the case of infinitary languages. An ingredient of this proof are fine ultrafilters. This leads to the third definition: the existence of 'fine measures' (Section 3.3).

4. These fine ultrafilters induce elementary embeddings of the universe with a certain covering property. A cardinal can be a critical point of an elementary embedding (not necessarily induced by an ultrafilter) with this covering property. This leads to the fourth definition: the 'embedding property' (Section 3.4).

We will investigate the structure of implications between these notions. In the proofs of some of the implications the axiom of choice is used. This raises the question whether this is really necessary. One method to answer this question is to produce a model of $\mathsf{ZF} + \neg\mathsf{AC}$ in which some cardinal has one property but not the other. In Chapter 4 we use the axiom of determinacy for this purpose. It implies that small cardinals (like ω_1 for example) have some properties normally ascribed to large cardinals.

A few questions which arise from the material in the text are collected in the appendix.

I happily recall the many hours Benedikt Löwe spent explaining set theory to me. In the past year he showed me what it is to be a mathematician. Thank you.

Stefan Bold not only wrote an inspiring thesis, but also carefully read a draft of my text. Thank you for your valuable comments.

Chapter 2

Preliminaries

In this chapter we develop the basic set-theoretic and model-theoretic preliminaries needed for this text.

We take Zermelo–Fraenkel set theory ZF without the axiom of choice (AC) as our basic set theory. The use of any additional axiom in a proof is always indicated. Two useful fragments of AC are defined in Section 2.1. As usual, ZFC abbreviates $\mathsf{ZF} + \mathsf{AC}$.

Section 2.2 contains a brief account of the syntax and semantics of infinitary languages. We derive some lemmas on ultrafilters in Section 2.3. Elementary embeddings are defined in Section 2.4 and some basic properties are derived without the axiom of choice.

The purpose of the last two sections is to develop reduced products in the context of infinitary languages. Section 2.5 introduces reduced products. Ultrafilters and elementary embeddings are connected in Section 2.6 on the ultrapower construction. Note that we can only use this construction in the context of ZFC.

We assume familiarity with the basic development of set theory and model theory. The reader may wish to consult for example [Dr74] or [Je78] on set theory, and [CK77] or [Ho97] on model theory. Our set-theoretic notation is mostly standard. Infinite cardinals are denoted by $\kappa, \lambda, \mu, \ldots$ and infinite ordinals by $\alpha, \beta, \gamma, \ldots$. Limit ordinals are often denoted by δ. We use $\wp(S)$ for the power set of S.

There are many ways to define a recursive bijection $\ulcorner \cdot, \cdot \urcorner \colon \omega \times \omega \to \omega$. We will use the diagonal enumeration:

$$\ulcorner m, n \urcorner := \frac{1}{2}(m^2 + m + 2mn + 3n + n^2).$$

The real numbers $\mathbb{R}$ are identified with the Baire space ${}^\omega\omega$, and consequently a countable sequence of natural numbers is sometimes referred to as a real. Using any bijection $\ulcorner \cdot, \cdot \urcorner \colon \omega \times \omega \to \omega$, every real $x \in \mathbb{R}$ codes a countable sequence of reals $\langle (x)_m \colon m \in \omega \rangle$ defined by

$$(x)_m(n) := x(\ulcorner m, n \urcorner).$$

Using the bijection $\ulcorner \cdot, \cdot \urcorner$ it is easy to see that real numbers also code binary relations on ω. This can be used to obtain a surjection from $\mathbb{R} \to \omega_1$; see for example [Ro01, p. 15].

2.1 Fragments of the Axiom of Choice

Let $\{X_i \colon i \in I\}$ be a family of nonempty subsets of some set X. A *choice function* for this family is a function $f \colon I \to \bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ for every $i \in I$. Let $\mathsf{AC}_I(X)$ denote the following axiom:

Every family $\{X_i \colon i \in I\}$ of nonempty subsets of X has a choice function.

Lemma 2.1. *If there is a surjection $X \to Y$, then $\mathsf{AC}_I(X) \Rightarrow \mathsf{AC}_I(Y)$.*

Proof. Let $f \colon X \to Y$ be a surjection and assume $\mathsf{AC}_I(X)$. Let $\{Y_i \colon i \in I\}$ be a family of nonempty subsets of Y. We have to find a choice function for this family. Since $f \colon X \to Y$ is surjective,

$$X_i := f^{-1}[Y_i] = \{x \in X \colon f(x) \in Y_i\}$$

is a nonempty subset of X for every $i \in I$. By $\mathsf{AC}_I(X)$, there is a choice function $g \colon I \to \bigcup_{i \in I} X_i$ for this family. Since $g(i) \in X_i = f^{-1}[Y_i]$ for every $i \in I$,

$$(f \circ g)(i) = f(g(i)) \in f[f^{-1}[Y_i]] = Y_i.$$

Therefore, the composition $f \circ g$ is a choice function for the family $\{Y_i \colon i \in I\}$, as required. $\square$

The axiom of choice is equivalent to $\forall I \, \forall X \, \mathsf{AC}_I(X)$. Without the axiom of choice, it cannot be proven that every successor cardinal is regular. We will derive the regularity of ω_1 from $\mathsf{AC}_\omega(\mathbb{R})$.

Lemma 2.2 ($\mathsf{AC}_\omega(\mathbb{R})$)**.** *There is a surjection $\mathbb{R} \to {}^\omega\omega_1$.*

Proof. Let $f \colon \mathbb{R} \to {}^\omega\mathbb{R}$ and $g \colon \mathbb{R} \to \omega_1$ be surjections. We can define a function $h \colon \mathbb{R} \to {}^\omega\omega_1$ by

$$h(x) := \langle g(f(x)(i)) \colon i \in \omega \rangle .$$

We have to show that h is surjective. Let $\langle \alpha_i \colon i \in \omega \rangle$ be a countable sequence in ω_1. Since $g \colon \mathbb{R} \to \omega_1$ is surjective, we can use $\mathsf{AC}_\omega(\mathbb{R})$ to choose for every $i \in \omega$ an $x_i \in \mathbb{R}$ such that $g(x_i) = \alpha_i$. Then $\langle x_i \colon i \in \omega \rangle$ is a countable sequence in $\mathbb{R}$.

Since $f\colon \mathbb{R} \to {}^{\omega}\mathbb{R}$ is surjective, there is an $x \in \mathbb{R}$ such that $f(x) = \langle x_i \colon i \in \omega\rangle$. We verify that this is a suitable real:

$$h(x) = \langle g(f(x)(i)) \colon i \in \omega\rangle = \langle g(x_i) \colon i \in \omega\rangle = \langle \alpha_i \colon i \in \omega\rangle .$$

□

Since there is a surjection $\mathbb{R} \to {}^{\omega}\omega_1$ when $\mathsf{AC}_\omega(\mathbb{R})$ holds, $\mathsf{AC}_\omega(\mathbb{R})$ implies $\mathsf{AC}_\omega({}^{\omega}\omega_1)$ by Lemma 2.1.

Proposition 2.3 ($\mathsf{AC}_\omega(\mathbb{R})$)**.** *The uncountable cardinal ω_1 is regular.*

Proof. Suppose ω_1 is singular. Then there is family $\{X_i \colon i \in \omega\}$ of subsets of ω_1 such that $|X_i| \leq \omega$ for every $i \in \omega$ and $\bigcup_{i\in\omega} X_i = \omega_1$. We can regard every X_i ($i \in \omega$) as a subset of ${}^{\omega}\omega_1$. Since $\mathsf{AC}_\omega(\mathbb{R})$ implies $\mathsf{AC}_\omega({}^{\omega}\omega_1)$, we can choose for every $i \in \omega$ a surjection $f_i \colon \omega \to X_i$. Then $g\colon \omega \times \omega \to \bigcup_{i\in\omega} X_i$ defined by $g(m, n) := f_m(n)$ is surjective. The composition of g with a surjection $\omega \to \omega \times \omega$ gives a surjection $\omega \to \bigcup_{i\in\omega} X_i = \omega_1$. But then ω_1 is countable, which is a contradiction. □

Let R be a binary relation on a nonempty set X. For $S \subseteq X$, an element $x \in S$ is *R-minimal* if $x \not R\, y$ for every $y \in S$. If every nonempty subset of X has an R-minimal element, then R is said to be *well-founded*.

Lemma 2.4. *If R is well-founded, then there is no countable sequence $\langle x_i \colon i \in \omega\rangle$ in X such that $x_{i+1}\ R\ x_i$ for every $i \in \omega$.*

Proof. Suppose $\langle x_i \colon i \in \omega\rangle$ is a countable sequence in X such that $x_{i+1}\ R\ x_i$ for every $i \in \omega$. Clearly, $\{x_i \colon i \in \omega\}$ is a nonempty subset of X which has no R-minimal element and therefore R is not well-founded. □

We would like a relation to be well-founded if and only if there are no infinite descending chains, and we need some choice for that. Let $\mathsf{DC}(X)$ denote the following axiom:

For every binary irreflexive relation R on a nonempty set X such that for every $x \in X$ there is a $y \in X$ with $y\ R\ x$, there is a countable sequence $\langle x_i \colon i \in \omega\rangle$ in X such that $x_{i+1}\ R\ x_i$ for every $i \in \omega$.

The axiom of dependent choice (DC) is the axiom $\forall X\, \mathsf{DC}(X)$.

Lemma 2.5 ($\mathsf{DC}(X)$)**.** *If there is no countable sequence $\langle x_i \colon i \in \omega\rangle$ in X such that $x_{i+1}\ R\ x_i$ for every $i \in \omega$, then R is well-founded.*

Proof. Suppose R is not well-founded. Then there is a nonempty $S \subseteq X$ such that S has no R-minimal element. In other words, for every $x \in S$ there is a $y \in S$ such that $y\ R\ x$, because otherwise $x \in S$ would be an R-minimal element of S. Using $\mathsf{DC}(X)$, we get a countable sequence $\langle x_i \colon i \in \omega\rangle$ in X such that $x_{i+1}\ R\ x_i$ for every $i \in \omega$. □

Zermelo used the axiom of choice in his famous proof of the well-ordering theorem: in ZFC every set can be well-ordered. The well-ordering theorem is equivalent to the axiom of choice. In fact even the statement 'every powerset can be well-ordered' implies AC.

Without the axiom of choice we have to be somewhat careful about which definition of *cardinality* we adopt. We reserve the term *cardinal* for initial ordinals. Define

$$|X| := \{Y : \text{there is a bijection } X \to Y \text{ and } \operatorname{rank}(Y) \text{ is minimal}\}.$$

Furthermore we let $|X| \leq |Y|$ if and only if there is an *injection* $f\colon X \to Y$. This ensures that the Cantor-Bernstein theorem

$$\text{If } \ |X| \leq |Y| \ \text{ and } \ |Y| \leq |X|\,, \text{then } \ |X| = |Y|$$

is still valid. Without the axiom of choice this ordering need not be a total order, however.

2.2 Infinitary Languages and Structures

Let κ be an infinite cardinal. We shall introduce the infinitary language $\mathscr{L}_{\kappa\kappa}$ and its finite quantifier fragment $\mathscr{L}_{\kappa\omega} \subseteq \mathscr{L}_{\kappa\kappa}$. The language $\mathscr{L}_{\kappa\omega}$ allows infinitary conjunction and disjunction, $\mathscr{L}_{\kappa\kappa}$ further allows infinitary quantification. Since the reader is assumed to be familiar with the basics of (first-order) model theory, we give only a brief treatment of those aspects that are specific for the infinitary languages. A detailed treatment of the model theory of infinitary languages can be found in [Ke71] and [Di75].

A *signature* σ is specified by giving a set of constant symbols, and for every natural number $n > 0$ a set of n-ary relation symbols and a set of n-ary function symbols. The symbols of the signature are often called *non-logical* symbols. We assume that all non-logical symbols are distinct and that there is an arity function $\operatorname{arity}\colon \sigma \to \omega$ which assigns to every symbol in the signature its arity. We will never explicitly refer to this function in the text. The syntax of the infinitary language $\mathscr{L}_{\kappa\kappa}$ over a signature σ is defined as follows. Note that we suppress the signature, and write $\mathscr{L}_{\kappa\kappa}$ instead of the more verbose $\mathscr{L}_{\kappa\kappa}(\sigma)$.

Every symbol of σ is a symbol of $\mathscr{L}_{\kappa\kappa}$. Furthermore, $\mathscr{L}_{\kappa\kappa}$ has a stock of *logical* symbols: infinitely many variables v_α $(\alpha < \kappa)$, a symbol $=$ for equality, connectives $\neg$ and $\bigwedge$, an existential quantifier $\exists$ and parentheses $), ($. We take disjunction $\bigvee$, implication $\Rightarrow$ and the universal quantifier $\forall$ to stand for the usual abbreviations.

The *terms* of $\mathscr{L}_{\kappa\kappa}$ refer to elements in a structure. Every variable and every constant symbol of $\mathscr{L}_{\kappa\kappa}$ is defined to be term of $\mathscr{L}_{\kappa\kappa}$. If F is an n-ary function symbol of $\mathscr{L}_{\kappa\kappa}$ and $t_1, \dots, t_n$ are terms of $\mathscr{L}_{\kappa\kappa}$, then $F(t_1, \dots, t_n)$ is a term of $\mathscr{L}_{\kappa\kappa}$. The *atomic formulas* of $\mathscr{L}_{\kappa\kappa}$ are basic statements concerning terms. If t_1 and t_2 are terms of $\mathscr{L}_{\kappa\kappa}$, then $(t_1 = t_2)$ is an atomic formula of $\mathscr{L}_{\kappa\kappa}$. If R is an n-ary relation symbol of $\mathscr{L}_{\kappa\kappa}$ and $t_1, \dots, t_n$ are terms of $\mathscr{L}_{\kappa\kappa}$, then $R(t_1, \dots, t_n)$ is an atomic formula of $\mathscr{L}_{\kappa\kappa}$.

The *formulas* of $\mathscr{L}_{\kappa\kappa}$ are recursively defined as follows. Every atomic formula of $\mathscr{L}_{\kappa\kappa}$ is a formula of $\mathscr{L}_{\kappa\kappa}$. If φ is a formula of $\mathscr{L}_{\kappa\kappa}$, then $(\neg\varphi)$ is a formula of $\mathscr{L}_{\kappa\kappa}$. If Φ is a set of formulas of $\mathscr{L}_{\kappa\kappa}$ of cardinality less than κ, then $\bigwedge \Phi$ is a formula of $\mathscr{L}_{\kappa\kappa}$. If φ is a formula of $\mathscr{L}_{\kappa\kappa}$ and $\langle v_\alpha \colon \alpha < \beta \rangle$ is a sequence of variables of length $\beta < \kappa$, then $(\exists \langle v_\alpha \colon \alpha < \beta \rangle)\varphi$ is a formula of $\mathscr{L}_{\kappa\kappa}$.

The formulas of $\mathscr{L}_{\kappa\omega}$ are those formulas of $\mathscr{L}_{\kappa\kappa}$ where all quantification is over a finite number of variables. Note that while $\mathscr{L}_{\kappa\kappa}$ allows infinitary quantification, every formula can contain only finitely many quantifier alterations.

An $\mathscr{L}_{\kappa\kappa}$-*structure* consists of a nonempty set M called the *domain* or *universe*, and an appropriate constant, relation or function for every symbol of σ, the *interpretation* of the symbol in M. If S is any symbol of σ, we write S^M for the interpretation of S in the structure M. Structures will often be identified with their domains.

A variable is said to occur *bounded* if it occurs inside the scope of a quantifier. A *free* variable is not bounded. An $\mathscr{L}_{\kappa\kappa}$-*sentence* is a formula of $\mathscr{L}_{\kappa\kappa}$ in which every variable occurs bounded.

Before we can define whether an $\mathscr{L}_{\kappa\kappa}$-structure M is a *model* of an $\mathscr{L}_{\kappa\kappa}$-sentence φ, we need to be able to substitute elements of a structure for the free variables in a term or an atomic formula. If $\bar{v} = \langle v_0, v_1, \dots \rangle$ is a sequence of variables, we write $t(\bar{v})$ to indicate that all free variables which occur in the term t are among those in $\bar{v}$. If $\bar{x} = \langle x_0, x_1, \dots \rangle$ is a sequence of elements of M which is as least as long as $\bar{v}$, we let $t^M[\bar{x}]$ denote the element of M to which the term t refers when one substitutes the element x_0 for the variable v_0, the element x_1 for v_1, and so on. Using induction on the complexity of t, substitution is formally defined as follows:

- If t is the variable v_α, then $t^M[\bar{x}]$ is the element x_α.
- If t is the constant symbol c, then $t^M[\bar{x}]$ is the element c^M.
- If t is the term $F(t_1, \dots, t_n)$, then $t^M[\bar{x}]$ is the element $F^M(t_1^M[\bar{x}], \dots, t_1^M[\bar{x}])$.
- If $\varphi(\bar{v})$ is the atomic formula $(t_1(\bar{v}) = t_2(\bar{v}))$, then $\varphi^M[\bar{x}]$ is the atomic formula $(t_1^M[\bar{x}] = t_2^M[\bar{x}])$.
- If $\varphi(\bar{v})$ is the atomic formula $R(t_1, \dots, t_n)$, then $\varphi^M[\bar{x}]$ is the atomic formula $R^M(t_1^M[\bar{x}], \dots, t_n^M[\bar{x}])$.

We will define what it means for an $\mathscr{L}_{\kappa\kappa}$-structure M to be a *model* of an $\mathscr{L}_{\kappa\kappa}$-sentence φ, written $M \models \varphi$, by induction on the complexity of φ. The basis of this induction is formed by the atomic formulas:

- If $\varphi(\bar{v})$ is the atomic formula $(t_1(\bar{v}) = t_2(\bar{v}))$, then $M \models \varphi[\bar{x}]$ if and only if $t_1^M[\bar{x}] = t_2^M[\bar{x}]$.
- If $\varphi(\bar{v})$ is the atomic formula $R(t_1(\bar{v}), \dots, t_n(\bar{v}))$, then $M \models \varphi[\bar{x}]$ if and only if $R^M(t_1^M[\bar{x}], \dots, t_n^M[\bar{x}])$.

Then we define $M \models \varphi$ for $\mathscr{L}_{\kappa\kappa}$-sentences:

- $M \models (\neg\varphi)$ if and only if not $M \models \varphi$.
- If Φ is a set of $\mathscr{L}_{\kappa\kappa}$-sentences of cardinality less than κ, then $M \models \bigwedge \Phi$ if and only if $M \models \varphi$ for every $\varphi \in \Phi$.
- If $\bar{v} = \langle v_0, v_1, \dots \rangle$ is a sequence of variables of length less than κ and $\varphi(\bar{v})$ is a formula of $\mathscr{L}_{\kappa\kappa}$, then $M \models (\exists \bar{v})\, \varphi(\bar{v})$ if and only if there is a sequence $\bar{x} = \langle x_0, x_1, \dots \rangle$ of elements of M such that $M \models \varphi[\bar{x}]$.

A set of $\mathscr{L}_{\kappa\kappa}$-sentences is sometimes called an $\mathscr{L}_{\kappa\kappa}$-*theory* in this text. We say that an $\mathscr{L}_{\kappa\kappa}$-theory Σ is *satisfiable* if there is some $\mathscr{L}_{\kappa\kappa}$-structure M such that $M \models \varphi$ for every $\varphi \in \Sigma$, and κ-*satisfiable* if every subset of Σ of cardinality less than κ is satisfiable.

We will only consider the infinitary languages $\mathscr{L}_{\kappa\omega}$ and $\mathscr{L}_{\kappa\kappa}$. Note that there is no point in considering $\mathscr{L}_{\kappa\lambda}$ for $\lambda > \kappa$: since there are only κ many variables, such a language would contain an excessive powerful quantor.

Formulas of a language can of course be coded by sets. Where finite formulas are coded by finite sets, infinite formulas are coded by infinite sets.

Lemma 2.6. *Let κ be a regular infinite cardinal. Every formula of $\mathscr{L}_{\kappa\kappa}$ can be coded by an element of $\mathbf{V}_\kappa$.*

Proof. Since κ is regular, every formula of $\mathscr{L}_{\kappa\kappa}$ is a string of symbols of length less than κ. Therefore every $\mathscr{L}_{\kappa\kappa}$-formula can be coded by a set hereditarily of cardinality less than κ. Every set hereditarily of cardinality less than κ is a element of $\mathbf{V}_\kappa$. Note that the axiom of choice is not needed for this [Ku80, p. 130–131]. Hence, every formula of $\mathscr{L}_{\kappa\kappa}$ can be regarded as an element of $\mathbf{V}_\kappa$. $\square$

2.3 Filters and Ultrafilters

A *filter* on a nonempty set S is a set $F \subseteq \wp(S)$ with the following properties:

(i). $\wp(S) \in F$.

(ii). If $X, Y \in F$, then $X \cap Y \in F$ (closed under intersections).

(iii). If $X \subseteq Y \subseteq S$ and $X \in F$, then $Y \in F$ (closed under supersets).

If $F \neq \wp(S)$, that is, if $\varnothing \notin F$, then F is called *proper*. We will only consider proper filters.

Since a filter on a set S is an element of $\wp(\wp(S))$, as a subset of $\wp(\wp(S))$ the collection of all filters on S is partially ordered by inclusion $\subseteq$. If $F_1 \subseteq F_2$, then F_2 is said to *extend* F_1. A filter M is *maximal* if for every filter F, if $M \subseteq F$, then $F = M$. A filter F on a set S is an *ultrafilter* if for every $X \subseteq S$, either $X \in F$ or $S \setminus X \in F$.

Lemma 2.7. *A filter F on a nonempty set S is an ultrafilter if and only if F is maximal.*

Proof. Suppose U is an ultrafilter on a nonempty set S which is properly contained in a filter F. Then there is a $X \in F \setminus U$. Since U is an ultrafilter, $S \setminus X \in U$ and therefore $S \setminus X \in F$. Then $X \cap (S \setminus X) = \varnothing \in F$, a contradiction. Hence, an ultrafilter is maximal.

For the converse, suppose F_1 is properly contained in F_2. Then there is an $X \subseteq S$ such that $X \in F_2 \setminus F_1$. Clearly, F_2 cannot contain $S \setminus X \in F_1$, for then $S \setminus X \in F_2$, and F_2 already contains X. Hence, neither $X \in F_1$ nor $S \setminus X \in F_1$, so F_1 is not an ultrafilter. $\square$

A *chain* of filters is a sequence of filters such that

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$$

Lemma 2.8. *If C is a chain of filters on a nonempty set S, then $\bigcup C$ is a filter on S such that $F \subseteq \bigcup C$ for every $F \in C$.*

Proof. Since $\varnothing \notin F$ for every $F \in C$, $\varnothing \notin \bigcup C$ and since $\wp(S) \in F$ for every $F \in C$, $\wp(S) \in \bigcup C$. If $X, Y \in \bigcup C$, then there is a $F \in C$ such that $X, Y \in F$ and thus $X \cap Y \in F$. Hence, $X \cap Y \in \bigcup C$. Similarly, if $X \subseteq Y \subseteq S$ and $X \in \bigcup C$, then for some $F \in C$, $X \in F$ and thus $Y \in F$. Hence, $Y \in \bigcup C$. $\square$

Let S be a nonempty set. A nonempty set $B \subseteq \wp(S)$ is a *filterbase* if $\varnothing \notin B$ and for every $X, Y \in B$ there is a $Z \in B$ such that $Z \subseteq X \cap Y$. The *filter on S generated by B* is defined to be

$$F_B := \{X \subseteq S\colon \text{ There is a } Y \in B \text{ such that } Y \subseteq X\}.$$

A filter F on S is said to be *principal* if there is a $X \subseteq S$ such that $F = F_{\{X\}}$. Every principal ultrafilter is generated by a singleton:

Lemma 2.9. *If U is a principal ultrafilter on a nonempty set S, then there is an $s \in S$ such that $U = \{X \subseteq S \colon \{s\} \subseteq X\} = \{X \subseteq S \colon s \in X\}$.*

Proof. Since U is principal, there is an $G \subseteq S$ such that $U = \{X \subseteq S \colon G \subseteq X\}$. Suppose $s, t \in S$ are two distinct elements of G. Clearly, $G \not\subseteq S \setminus \{s\}$ and $G \not\subseteq S \setminus \{t\}$. Therefore, $S \setminus \{s\} \notin U$ and $S \setminus \{t\} \notin U$. Since U is an ultrafilter, $\{s\} \in U$ and $\{t\} \in U$. But then $\{s\} \cap \{t\} = \varnothing \in U$, a contradiction. Hence, G can have only one element, say $s \in S$. Then $U = \{X \subseteq S \colon \{s\} \subseteq X\} = \{X \subseteq S \colon s \in X\}$. □

Let κ be an infinite cardinal. A filter F is *κ-complete* if for every sequence $\langle X_\alpha \colon \alpha < \beta \rangle$ in F of length $\beta < \kappa$, the intersection

$$\bigcap_{\alpha<\beta} X_\alpha \in F.$$

Clearly, every filter is ω-complete and every principal filter is κ-complete for every $\kappa \geq \omega$.

Lemma 2.10. *An ultrafilter U on a nonempty set S is κ-complete if and only if for every sequence $\langle X_\alpha \colon \alpha < \beta < \kappa \rangle$ in $\wp(S)$ such that $\bigcup_{\alpha<\beta} X_\alpha \in U$, there is an $\alpha < \beta$ such that $X_\alpha \in U$.*

Proof. Let $\langle X_\alpha \colon \alpha < \beta \rangle$ be a sequence in $\wp(S)$ such that $X := \bigcup_{\alpha<\beta} X_\alpha \in U$. Suppose that $X_\alpha \notin U$ for every $\alpha < \beta$. Since U is an ultrafilter, $S \setminus X_\alpha \in U$ for every $\alpha < \beta$. Since U is κ-complete,

$$\bigcap_{\alpha<\beta} S \setminus X_\alpha = S \setminus \bigcup_{\alpha<\beta} X_\alpha = S \setminus X \in U.$$

Both $X \in U$ and $S \setminus X \in U$. Then $X \cap (S \setminus X) = \varnothing \in U$, a contradiction. □

Lemma 2.11. *If $\mathcal{B}$ is a filterbase such that for every sequence $\langle B_\alpha \colon \alpha < \beta < \kappa \rangle$ in $\mathcal{B}$ there is a $B \in \mathcal{B}$ such that $B \subseteq \bigcap_{\alpha<\beta} B_\alpha$, then the filter $F_{\mathcal{B}}$ generated by $\mathcal{B}$ is κ-complete.*

Proof. If $\langle X_\alpha \colon \alpha < \beta < \kappa \rangle$ is a sequence in $F_{\mathcal{B}}$, then for every $\alpha < \beta$ there is a $B_\alpha \in \mathcal{B}$ such that $B_\alpha \subseteq X_\alpha$, since $F_{\mathcal{B}}$ is the filter generated by $\mathcal{B}$. By assumption, there is a $B \in \mathcal{B}$ such that

$$B \subseteq \bigcap_{\alpha<\beta} B_\alpha \subseteq \bigcap_{\alpha<\beta} X_\alpha,$$

and therefore $\bigcap_{\alpha<\beta} X_\alpha \in F_{\mathcal{B}}$. Hence, $F_{\mathcal{B}}$ is κ-complete. □

For a set S and any $G \subseteq \wp(S)$, the *κ-complete filter generated by G* is the filter generated by family of all intersections of length less than κ of elements of G, provided none of these intersections is empty.

A κ-complete nonprincipal ultrafilter on κ is called a *measure on κ*. A cardinal κ is *measurable* if there is a measure on κ.

Lemma 2.12. *Let U be a measure on κ. If $X \in U$, then $|X| = \kappa$.*

Proof. Suppose $X \in U$ such that $|X| < \kappa$. Then $X = \{x_\alpha \colon \alpha < \lambda < \kappa\}$. Since $X = \bigcup_{\alpha<\lambda}\{x_\alpha\} \in U$, $\{x_\alpha\} \in U$ for some $\alpha < \lambda$ by Lemma 2.10. But U is nonprincipal, a contradiction. $\square$

Lemma 2.13. *If there is a measure on κ, then κ is regular.*

Proof. Let U be a measure on κ. Suppose there is a sequence $\langle X_\alpha \colon \alpha < \beta < \kappa\rangle$ in $\wp(\kappa)$ such that $|X_\alpha| < \kappa$ for every $\alpha < \beta$ and $\bigcup_{\alpha<\beta} X_\alpha = \kappa$. Since $\varnothing \notin U$,

$$\varnothing = \kappa \setminus \kappa = \kappa \setminus \bigcup_{\alpha<\gamma} X_\alpha = \bigcap_{\alpha<\gamma} \kappa \setminus X_\alpha \notin U.$$

Suppose $\kappa \setminus X_\alpha \in U$ for every $\alpha < \beta$. Then $\bigcap_{\alpha<\beta} \kappa \setminus X_\alpha \in U$, since U is κ-complete. Hence, there is an $\alpha < \beta$ such that $\kappa \setminus X_\alpha \notin U$. Since U is an ultrafilter, $X_\alpha \in U$. But $|X_\alpha| < \kappa$, while every element of a measure on κ has cardinality κ by Lemma 2.12, a contradiction. $\square$

Lemma 2.14. *Let U be a measure on κ. For every $\alpha < \kappa$, $\{\beta < \kappa \colon \beta \geq \alpha\} \in U$ and $\{\beta < \kappa \colon \beta > \alpha\} \in U$.*

Proof. Let $\alpha < \kappa$. For every $\xi < \kappa$, $\kappa \setminus \{\xi\} \in U$ as U is nonprincipal. Since U is κ-complete,

$$\bigcap_{\xi<\alpha} \kappa \setminus \{\xi\} = \{\beta < \kappa \colon \beta \geq \alpha\} \in U.$$

Since $\kappa \setminus \{\alpha\} \in U$,

$$\kappa \setminus \{\alpha\} \cap \{\beta < \kappa \colon \beta \geq \alpha\} = \{\beta < \kappa \colon \beta > \alpha\} \in U.$$

$\square$

A nonprincipal ultrafilter U on κ cannot be κ^+-complete, since then

$$\bigcap_{\alpha<\kappa} \kappa \setminus \{\alpha\} = \varnothing \in U.$$

But it can satisfy a further closure property. A filter F on an uncountable cardinal $\kappa > \omega$ is *normal* if for every sequence $\langle X_\alpha \colon \alpha < \kappa \rangle$ in F, the diagonal intersection $\triangle_{\alpha<\kappa} X_\alpha$ defined by

$$\triangle_{\alpha<\kappa} X_\alpha := \{\alpha < \kappa \colon \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$$

is an element of F. Let F be a filter on κ. A function $f \colon \kappa \to \kappa$ is said to be *almost everywhere regressive* if $\{\alpha < \kappa \colon f(\alpha) \in \alpha\} \in F$, and *almost everywhere constant* if there is a $\beta < \kappa$ such that $\{\alpha < \kappa \colon f(\alpha) = \beta\} \in U$.

Lemma 2.15. *An ultrafilter U on κ is normal if and only if every almost everywhere regressive function $f \colon \kappa \to \kappa$ is almost everywhere constant.*

Proof. Suppose $f \colon \kappa \to \kappa$ is an almost everywhere regressive function which is not almost everywhere constant. Then for every $\beta < \kappa$, $\{\alpha < \kappa \colon f(\alpha) = \beta\} \notin U$. Since U is an ultrafilter, $X_\beta := \{\alpha < \kappa \colon f(\alpha) \neq \beta\} \in U$ for every $\beta < \kappa$. Since U is normal,

$$\begin{aligned}
\triangle_{\beta<\kappa} X_\beta &= \{\alpha < \kappa \colon \alpha \in \bigcap_{\beta<\alpha} \{\beta < \kappa \colon f(\beta) \neq \alpha\}\} \\
&= \{\alpha < \kappa \colon \alpha \in \{\beta < \kappa \colon f(\beta) \geq \alpha\}\} \\
&= \{\alpha < \kappa \colon f(\alpha) \geq \alpha\} \in U.
\end{aligned}$$

But since $f \colon \kappa \to \kappa$ is almost everywhere regressive, we also have

$$\kappa \setminus \{\alpha \in \kappa \colon f(\alpha) \geq \alpha\} = \{\alpha \in \kappa \colon f(\alpha) < \alpha\} \in U,$$

a contradiction. For the converse, suppose $\langle X_\beta \colon \beta \in \kappa \rangle$ is a sequence in U such that the diagonal intersection $\triangle_{\beta<\kappa} X_\beta \notin U$. Since U is an ultrafilter,

$$X := \kappa \setminus \triangle_{\beta<\kappa} X_\beta = \{\alpha < \kappa \colon \alpha \notin \bigcap_{\beta<\alpha} X_\beta\} = \{\alpha < \kappa \colon \alpha \in \bigcup_{\beta<\alpha} (\kappa \setminus X_\beta)\} \in U.$$

Define a function $f \colon \kappa \to \kappa$ by

$$f(\alpha) := \begin{cases} \min\{\beta < \alpha \colon \alpha \in (\kappa \setminus X_\beta)\} & \text{if } \alpha \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is regressive on $X \in U$, there is an $\beta < \kappa$ such that $\{\alpha < \kappa \colon f(\alpha) = \beta\} \in U$ by assumption. Yet $\{\alpha < \kappa \colon f(\alpha) = \beta\} \subseteq (\kappa \setminus X_\beta) \notin U$, a contradiction. $\square$

For any set S, let $\wp_\kappa(S) := \{X \subseteq S \colon |X| < \kappa\}$. Define for every $s \in S$ the set

$$\hat{s} := \{X \in \wp_\kappa(S) \colon s \in X\}.$$

A filter F on $\wp_\kappa(S)$ is *fine* if $\hat{s} \in F$ for every $s \in S$. We will use the notion $\hat{s}$ often in the context of fine filters.

Lemma 2.16. *Let S be a set such that $|S| \geq \kappa$. Every fine ultrafilter on $\wp_\kappa(S)$ is nonprincipal.*

Proof. Let U be a fine ultrafilter on $\wp_\kappa(S)$ and suppose U is principal. Every principal ultrafilter is generated by a singleton (Lemma 2.9). Therefore, there is a $G \in \wp_\kappa(S)$ such that $X \in U$ if and only if $G \in X$. Because U is fine, the set $\hat{s} := \{X \in \wp_\kappa(S) \colon s \in X\} \in U$ for every $s \in S$. Since G generates U, $G \in \hat{s}$ for every $s \in S$. Hence, $s \in G$ for every $s \in S$. But then $|G| \geq |S| \geq \kappa$, a contradiction as $G \in \wp_\kappa(S)$. $\square$

Let $f \colon S \to T$ be a function between two sets S and T. If F is a filter on S, then the *pushout* $f_*(F)$ is defined by $f_*(F) := \{X \subseteq T \colon f^{-1}[X] \in F\}$.

Lemma 2.17. *If F is a (κ-complete) (ultra)filter on S, then $f_*(F)$ is a (κ-complete) (ultra)filter on T.*

Proof. Since $f^{-1}[\varnothing] = \varnothing \notin F$ and $f^{-1}[T] = S \in F$, $\varnothing \notin f_*(F)$ and $T \in f_*(F)$. If $X \in f_*(T)$ and $X \subseteq Y \subseteq T$, then since $f^{-1}[X] \in F$ and $f^{-1}[X] \subseteq f^{-1}[Y] \subseteq S$, $f^{-1}[Y] \in F$. Hence, $Y \in f_*(F)$. If $X \in f_*(F)$ and $Y \in f_*(F)$, then since $f^{-1}[X \cap Y] = f^{-1}[X] \cap f^{-1}[Y] \in F$, $X \cap Y \in f_*(F)$. Thus, $f_*(F)$ is a filter on T.

Suppose F is κ-complete. If $\langle X_\alpha \colon \alpha < \beta < \kappa\rangle$ is a sequence in $\wp(T)$ such that $f^{-1}[X_\alpha] \in F$ for every $\alpha < \kappa$, then since $f^{-1}[\bigcap_{\alpha<\beta} X_\alpha] = \bigcap_{\alpha<\beta} f^{-1}[X_\alpha] \in F$ as F is κ-complete. Thus, $\bigcap_{\alpha<\beta} X_\alpha \in F$ and $f_*(F)$ is κ-complete.

Suppose F is an ultrafilter. If $X \notin f_*(F)$, then $f^{-1}[X] \notin F$. Since F is an ultrafilter on S, $S \setminus f^{-1}[X] \in F$. Since $f^{-1}[T \setminus X] = f^{-1}[T] \setminus f^{-1}[X] = S \setminus X$, $T \setminus X \in f_*(F)$ as required. $\square$

A function $f \colon S \to T$ induces a function $\hat{f} \colon \wp(S) \to \wp(T)$ defined by $\hat{f}(X) := f[X]$.

Lemma 2.18. *Let $f \colon S \to T$ be a surjection. If F is a fine filter on $\wp_\kappa(S)$, then $\hat{f}_*(F)$ is a fine filter on $\wp_\kappa(T)$.*

Proof. Of course, $\hat{f}_*(F)$ is a filter on $\wp_\kappa(T)$ by Lemma 2.17. Let $t \in T$. We have to show that $\{X \subseteq T \colon t \in X\} \in \hat{f}_*(F)$. Since f is surjective, there is an $s \in S$ such that $f(s) = t$. But

$$\hat{f}^{-1}[\{X \subseteq T \colon t \in X\}] = \{X \subseteq S \colon t \in \hat{f}(X)\} = \{X \subseteq S \colon s \in X\} \in F,$$

since F is fine. $\square$

A *measure on* $\wp_\kappa(S)$ is a κ-complete nonprincipal ultrafilter on $\wp_\kappa(S)$. We will be interested in fine measures on $\wp_\kappa(S)$, where $|S| \geq \kappa$.

Lemma 2.19. *Suppose there is a surjection $S \to T$ and $|S| \geq \kappa$. If there is a fine measure on $\wp_\kappa(S)$, then there is fine measure on $\wp_\kappa(T)$.*

Proof. Suppose U is a measure on $\wp_\kappa(S)$ and $f\colon S \to T$ is a surjection. The induced pushout $\hat{f}_*(U)$ is an κ-complete fine ultrafilter on $\wp_\kappa(T)$ by Lemma 2.18 and nonprincipal by Lemma 2.16. □

Proposition 2.20. *If there is a κ-complete nonprincipal fine (normal) ultrafilter on $\wp_\kappa(\lambda)$, then there is a κ-complete nonprincipal fine (normal) ultrafilter on $\wp_\kappa(\alpha)$ for every $\lambda < \alpha < \lambda^+$.*

Proof. Let U be a κ-complete nonprincipal fine (normal) ultrafilter on $\wp_\kappa(\lambda)$. If $\lambda < \alpha < \lambda^+$, then there is a bijection $f\colon \lambda \to \alpha$, which induces a bijection $\hat{f}\colon \wp(\lambda) \to \wp(\alpha)$ defined by $\hat{f}(x) = f[x]$. The pushout $\hat{f}_*(U)$ is a κ-complete fine (normal) ultrafilter on $\wp_\kappa(\lambda)$ by Lemma 2.18. □

Measures on κ are connected to κ-complete nonprincipal ultrafilters ('measures') on $\wp_\kappa(\kappa) = \{X \subseteq \kappa\colon\ |X| < \kappa\}$.

Proposition 2.21. *If F is a filter on κ such that F contains all end-segments, then $F^* := \{X \subseteq \wp_\kappa(\kappa)\colon X \cap \kappa \in F\}$ is a fine filter on $\wp_\kappa(\kappa)$. Furthermore, if F is κ-complete or normal, then F^* is κ-complete or normal.*

Proof. We verify that F^* is a filter on $\wp_\kappa(\kappa)$. Since $\varnothing \cap \kappa = \varnothing \notin F$, we have $\varnothing \notin F^*$. Since $\kappa \cap \kappa = \kappa \in F$, we have $\kappa \in F^*$. If $X \cap \kappa \in F$ and $Y \cap \kappa \in F$, then $(X \cap \kappa) \cap (Y \cap \kappa) = (X \cap Y) \cap \kappa \in F$. Hence, if $X, Y \in F^*$, then $X \cap Y \in F^*$. Finally, if $X \cap \kappa \in F$ and $X \subseteq Y$, then $Y \cap \kappa \in F$ as $X \cap \kappa \subseteq Y \cap \kappa$. Thus, if $X \in F^*$ and $X \subseteq Y$, then $Y \in F^*$.

By definition, F^* is fine if for every $\alpha \in \kappa$ the set $\hat{\alpha} := \{X \in \wp_\kappa(\kappa)\colon \alpha \in X\} \in F^*$. Since $\hat{\alpha} \cap \kappa = \{x \in \kappa\colon \alpha \in x\} \in F$ because F contains all endsgements, the filter F^* is fine. Suppose $\langle X_\alpha\colon \alpha < \beta < \kappa\rangle$ is a sequence in F^*. Since $X_\alpha \cap \kappa \in F$ for every $\alpha < \beta$, $\langle X_\alpha \cap \kappa\colon \alpha < \beta\rangle$ is a sequence in F. If F is κ-complete, then

$$(\bigcap_{\alpha<\beta} X_\alpha) \cap \kappa = \bigcap_{\alpha<\beta} (X_\alpha \cap \kappa) \in F,$$

and therefore $\bigcap_{\alpha<\beta} X_\alpha \in F^*$. Hence, F^* is κ-complete.

If F is a fine filter on κ, then for every $\alpha \in \kappa$,

$$\hat{\alpha} \cap \kappa = \{X \in \wp_\kappa(\kappa)\colon \alpha \in X\} \cap \kappa = \{x \in \kappa\colon \alpha \in x\} \in F.$$

Therefore, $\hat{\alpha} \in F^*$ for every $\alpha \in \kappa$ and hence F^* is a fine filter on $\wp_\kappa(\kappa)$.

If $\langle X_\alpha \colon \alpha < \kappa \rangle$ be a sequence in F^*, then $X_\alpha \cap \kappa \in F$ for every $\alpha < \kappa$. If F is normal, then $\triangle_{\alpha<\kappa}(X_\alpha \cap \kappa) \in F$. Hence, for every sequence $\langle X_\alpha \colon \alpha < \kappa \rangle$ in F^*,

$$\begin{aligned}(\triangle_{\alpha<\kappa} X_\alpha) \cap \kappa &= \{x \in \wp_\kappa(\kappa) \colon x \in \bigcap_{\alpha \in x} X_\alpha\} \cap \kappa \\ &= \{x \in \kappa \colon x \in \bigcap_{\alpha \in x} (X_\alpha \cap \kappa)\} \\ &= \triangle_{\alpha<\kappa}(X_\alpha \cap \kappa) \in F.\end{aligned}$$

Therefore, F^* is normal. □

2.4 Elementary Embeddings

Let M and N be class structures for some language $\mathscr{L}$. A functional class $f \colon M \to N$ is said to be an *elementary embedding of M into N* if for every $\mathscr{L}$-formula $\varphi(v_0, \ldots, v_n)$ and all $x_0, \ldots, x_n \in M$,

$$M \models \varphi(x_0, \ldots, x_n) \text{ if and only if } N \models \varphi(f(x_0), \ldots, f(x_n)).$$

In other words, an elementary embedding preserves all $\mathscr{L}$-formulas between structures. In particular, an elementary embedding preserves equality and is therefore injective.

We will often be interested in elementary embeddings between inner models M and N. Since a set is an ordinal if and only if it is transitive and linearly ordered by $\in$, there is a Σ_0-formula $\varphi(v)$ such that $\varphi(x)$ if and only if $x \in \mathrm{On}$. Therefore, an elementary embedding between inner models maps ordinals to ordinals.

Lemma 2.22. *If $j \colon M \to N$ is an elementary embedding between inner models M and N, then $j(\alpha) \geq \alpha$ for every ordinal $\alpha \in M$.*

Proof. We already know that for every ordinal α, $j(\alpha)$ is an ordinal. Suppose that α is the least ordinal such that $j(\alpha) < \alpha$. Let $\beta := j(\alpha)$. Since $M \models j(\beta) < \alpha$, $N \models j(j(\beta)) < j(\alpha)$ by elementarity of j. Because $j(\alpha) = \beta$, $N \models j(\beta) < \beta$. But then $\beta = j(\alpha) < \alpha$ is an ordinal less than α such that $j(\beta) < \beta$, a contradiction. Therefore, $j(\alpha) \geq \alpha$ for every ordinal $\alpha \in M$. □

Since $0 = \varnothing$, the formula $\forall x(x \notin 0)$ holds. By elementarity of j, this formula also holds of $j(0)$. Thus, $j(0) = \varnothing = 0$, and j fixes 0. This argument shows that an elementary embedding fixes every definable ordinal. We want to show that if j is an elementary embedding between inner models M and N such that $N \subseteq M$ and j is not the identity, then some ordinal has to be moved by j. For this we need the following observation on the rank of sets: There is a formula $\varphi(x, y)$ which holds of x and y if and only if $\text{rank}(x) = y$ [Ka03, p. 5]. Therefore if $M \models \text{rank}(x) = y$, then $N \models \text{rank}(j(x)) = (j(y))$. Substituting, this yields $N \models \text{rank}(j(x)) = j(\text{rank}(x))$. Therefore, $j(\text{rank}(x)) = \text{rank}(j(x))$.

In the next proposition and the rest of the text, we write $j\colon M \prec N$ to denote an elementary embedding $j\colon M \to N$ between inner models M and N such that $N \subseteq M$ and j is not the identity.

Proposition 2.23. *If $j\colon M \prec N$, then there is an ordinal $\delta \in M$ such that $j(\delta) > \delta$.*

Proof. Since $j\colon M \to N$ is not the identity, there is an $x \in M$ of minimal rank $\delta := \text{rank}(x)$ such that $j(x) \neq x$. Note that

$$j(\delta) = j(\text{rank}(x)) = \text{rank}(j(x)) \geq \delta.$$

Suppose that $j(\delta) = \delta$. We first show that $x \subseteq j(x)$. By elementarity of j, $M \models y \in x$ if and only if $N \models j(y) \in j(x)$. Since for every $y \in x$, $\text{rank}(y) < \text{rank}(x)$, $j(y) = y$. Hence, $M \models y \in x$ if and only if $N \models y \in j(x)$. Thus, $x \subseteq j(x)$.

Since $x \neq j(x)$, there is a $z \in j(x) \setminus x$. Because $N \subseteq M$, $z \in M$ and we can consider $j(z)$. Since $\text{rank}(j(x)) = j(\delta) = \delta$ and $z \in j(x)$, $\text{rank}(z) < \delta$. Hence, $j(z) = z$ as x was a set of minimal rank such that $x \neq j(x)$. But then $j(z) = z \in j(x)$ and by elementarity, $z \in x$, a contradiction. $\square$

The least ordinal $\delta \in M$ such that $j(\delta) > \delta$ is called the *critical point* of the elementary embedding. We write $\text{crit}(j) = \delta$ to indicate that δ is the critical point of j. Since every ordinal $\leq \omega$ is definable, the critical point of an elementary embedding is uncountable. The proof of Proposition 2.23 in fact shows that $j(x) = x$ for every set x such that $\text{rank}(x) < \text{crit}(j)$.

Lemma 2.24. *If $j\colon M \prec N$ with $\text{crit}(j) = \kappa$, then $j(x) = x$ for every $x \in (\mathbf{V}_\kappa)^M$, and therefore $(\mathbf{V}_\kappa)^M \subseteq N$.*

Proof. If x is of minimal rank δ such that $j(x) \neq x$ and $\delta = \text{rank}(x) < \kappa$, the proof of Proposition 2.23 shows that $j(\delta) > \delta$. But then $\text{crit}(j) \leq \delta < \kappa$, a contradiction. Since $j(x) = x$ for every $x \in (\mathbf{V}_\kappa)^M$, $(\mathbf{V}_\kappa)^M \subseteq N$. $\square$

In particular if $j\colon \mathbf{V} \prec M$, then $\mathbf{V}_\kappa \subseteq M$.

If there is an elementary embedding of the universe, then its critical point carries a normal measure.

Theorem 2.25. *(Keisler) If $j\colon \mathbf{V} \prec M$ with $\text{crit}(j) = \delta$, then there is a δ-complete nonprincipal normal ultrafilter on δ.*

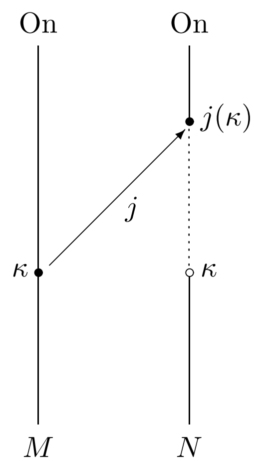


Figure 2.1: If $j\colon M \prec N$ with $\mathrm{crit}(j) = \kappa$, then $j \restriction \mathbf{V}_\kappa^M = \mathrm{id}$.

Proof. We show that $U := \{X \subseteq \delta\colon \delta \in j(X)\}$ is a δ-complete nonprincipal normal ultrafilter on δ. In particular, δ is regular and hence a cardinal.

Since j is elementary, $j(\varnothing) = \varnothing$. Clearly, $\delta \notin \varnothing$, so $\varnothing \notin U$. If $\delta \in j(X)$ and $\delta \in j(Y)$, then $\delta \in j(X) \cap j(Y)$ Since j is injective, $j(X) \cap j(Y) = j(X \cap Y)$. Hence, $X \cap Y \in U$. If $\delta \in j(X)$ and $X \subseteq Y$, then since $j(X) \subseteq j(Y)$, $\delta \in j(Y)$. So, $Y \in U$. For any $X \subseteq \delta$, $\delta \in j(\delta) = j(\delta \setminus X) \cup j(X)$, so $\delta \notin j(X)$ if and only if $\delta \in j(\delta \setminus X)$. Hence, for every $X \subseteq \delta$, either $X \in U$ or $\delta \setminus X \in U$. Finally, since $\delta = \mathrm{crit}(j)$, $j(\alpha) = \alpha$ for every $\alpha < \delta$. Hence, $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$ and $\delta \notin \{\alpha\}$, so $\{\alpha\} \notin U$ for every $\alpha < \delta$. Thus, U is nonprincipal.

We know now that U is a nonprincipal ultrafilter on δ. It remains to be shown that U is δ-complete, and normal.

Suppose $X := \langle X_\alpha\colon \alpha < \gamma\rangle$ is a sequence in U for some $\gamma < \delta$. Since in M, $j(X)$ is a sequence of length $j(\gamma) = \gamma$, $j(X) = \langle j(X_\alpha)\colon \alpha < \gamma\rangle$ and $j(\bigcap X) = \bigcap j(X) = \bigcap_{\alpha<\gamma} j(X_\alpha)$. Hence, $\delta \in j(\bigcap_{\alpha<\gamma} X_\alpha) = \bigcap_{\alpha<\gamma} j(X_\alpha)$. Therefore, U is δ-complete.

By construction $X \in U$ if and only if $\delta \in j(X)$ for every $X \subseteq \delta$. Supose $\langle X_\alpha\colon \alpha < \delta\rangle$ is a sequence in U. Then $\delta \in j(X_\alpha)$ for every $\alpha < \delta$. Let $X :=$

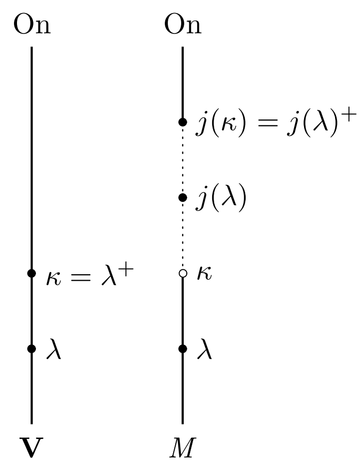
$\triangle_{\alpha<\delta} X_\alpha$ be the diagonal intersection. We compute

$$\begin{aligned} j(X) &= j(\{\alpha < \delta \colon \alpha \in \bigcap_{\beta<\alpha} X_\beta\}) \\ &= \{\alpha < j(\delta) \colon \alpha \in \bigcap_{\beta<\alpha} j(X_\beta)\}. \end{aligned}$$

Since $\delta \in j(X_\beta)$ for every $\beta < \delta$, $\delta \in j(X)$ and therefore $X \in U$. Hence, U is closed under diagonal intersections. □

Proposition 2.26. *If* $j \colon V \prec M$*, then* $\operatorname{crit}(j) = \kappa$ *is weakly inaccessible.*

Proof. There is a measure on κ by Theorem 2.25 and κ is therefore regular. If $\mathbf{V} \models \kappa = \lambda^+$, then $M \models j(\kappa) = j(\lambda^+)$. Since $j(\lambda^+) = j(\lambda)^+$ and $\lambda < \kappa = \operatorname{crit}(j)$, $j(\lambda) = \lambda$. Hence, $M \models j(\kappa) = \lambda^+$. But $(\lambda^+)^M \leq (\lambda^+)^{\mathbf{V}}$. Hence, $j(\kappa) = (\lambda^+)^M \leq \lambda^+ = \kappa$, a contradiction. □



Let $j \colon M \prec N$ with $\operatorname{crit}(j) = \kappa$. Suppose some formula $\varphi(v)$ holds of κ in N. Since $N \models (\exists \alpha < j(\kappa))\, \varphi(\alpha)$, $M \models (\exists \alpha < \kappa)\, \varphi(\alpha)$ by elementarity of j. So, let $\alpha < \kappa$ be such that $M \models \varphi(\alpha)$. By elementarity of j, $N \models \varphi(j(\alpha))$. Since $\alpha < \kappa$, $j(\alpha) = \alpha$ and therefore

$$M \models (\exists \alpha, \beta < j(\kappa))\, (\alpha \neq \beta \wedge \varphi(\alpha) \wedge \varphi(\beta)),$$

and so does $\mathbf{V}$, and so on. Actually, if φ holds of κ in N, then there are κ many cardinals below κ with that property. This phenomenon is called *reflection.*

2.5 Reduced Products and Ultraproducts

Let $\{X_i \colon i \in I\}$ be a family of nonempty (possibly proper) classes indexed by a nonempty set I. The Cartesian product

$$\prod_{i \in I} X_i$$

is the class of all functions f with domain $\mathrm{dom}(f) = I$ such that for every $i \in I$, $f(i) \in X_i$. Note that f is a choice function for the family $\{X_i \colon i \in I\}$. Without the axiom of choice, we cannot prove that every Cartesian product of a family of nonempty sets is nonempty.

If F is a filter on I, we can define a binary relation $=_F$ on the Cartesian product $\prod_{i \in I} X_i$ by

$$f =_F g \text{ if and only if } \{i \in I \colon f(i) = g(i)\} \in F.$$

Lemma 2.27. *The relation $=_F$ is an equivalence relation on $\prod_{i \in I} X_i$.*

Proof. Clearly, $=_F$ is symmetric. Since $\{i \in I \colon f(i) = f(i)\} = I \in F$ for every $f \in \prod_{i \in I} X_i$, $=_F$ is reflexive. If $f =_F g$ and $g =_F h$, then by definition of $=_F$ both $\{i \in I \colon f(i) = g(i)\} \in F$ and $\{i \in I \colon g(i) = h(i)\} \in F$. Therefore, their intersection $\{i \in I \colon f(i) = g(i) = h(i)\} \in F$. Since $\{i \in I \colon f(i) = g(i) = h(i)\}$ is a subset of $\{i \in I \colon f(i) = h(i)\}$, we have that $\{i \in I \colon f(i) = h(i)\} \in F$. Hence, $=_F$ is transitive. $\square$

Although every $f \in \prod_{i \in I} X_i$ is a set since $f \colon I \to \mathrm{ran}(f)$, its equivalence class over F,

$$(f)_F := \{g \in \prod_{i \in I} X_i \colon f =_F g\},$$

may be a proper class. We employ a well known 'trick' invented by Dana Scott, and use only representatives of minimal rank, defining

$$(f)_F^{\min} := \{g \in (f)_F \colon \mathrm{rank}(g) \leq \mathrm{rank}(h) \text{ for every } h \in (f)_F\}.$$

Since $\mathrm{rank}(g) \leq \mathrm{rank}(f)$ for every $g \in (f)_F^{\min}$, the collection $(f)_F^{\min}$ is a subset of $\mathbf{V}_{\mathrm{rank}(f)+1} \in \mathbf{V}$. As f is not necessarily of minimal rank itself, it is possible that $f \notin (f)_F^{\min}$. Nevertheless, we often write $f \in (f)_U^{\min}$ when in fact we need to take a representative of the equivalence class.

We define the quotient of the Cartesian product of the family $\{X_i \colon i \in I\}$ over F to be

$$\prod_{i \in I} X_i / F := \{(f)_F^{\min} \colon f \in \prod_{i \in I} X_i\}.$$

If all of the factors of the Cartesian product are structures for the same language $\mathscr{L}$, we can turn the quotient into an $\mathscr{L}$-structure, as follows.

Let $\{M_i \colon i \in I\}$ be a family of $\mathscr{L}$-structures indexed by a nonempty set I and let F be a filter on I. The *reduced product* of $\{M_i \colon i \in I\}$ modulo F is the $\mathscr{L}$-structure M with domain $\prod_{i\in I} M_i/F$ and the following interpretation of the non-logical symbols of $\mathscr{L}$:

- If c is a constant symbol of $\mathscr{L}$, then the interpretation c^M is the equivalence class $(f)_F^{\min}$ of the function $f \in \prod_{i\in I} M_i$ defined by $f(i) := c^{M_i}$.
- If R is an n-ary relation symbol of $\mathscr{L}$, then the interpretation R^M is that relation on M such that for all $(f_1)_F^{\min}, \dots, (f_n)_F^{\min} \in \prod_{i\in I} M_i/F$,
$$R^M((f_1)_F^{\min}, \dots, (f_n)_F^{\min})$$
if and only if
$$\{i \in I \colon R^{M_i}(f_1(i), \dots, f_n(i))\} \in F.$$
- If F is an n-ary function symbol of $\mathscr{L}$, then the interpretation F^M is that function on M such that for all $(f_1)_F^{\min}, \dots, (f_n)_F^{\min} \in \prod_{i\in I} M_i/F$,
$$F^M((f_1)_F^{\min}, \dots, (f_n)_F^{\min}) = (f)_F^{\min},$$
where $f \in \prod_{i\in I} M_i$ is defined by
$$f(i) := F^{M_i}(f_1(i), \dots, f_n(i)).$$

In order to show that this definition is consistent, we have to check that the interpretation of relation and function symbols in the reduced product depends only on the equivalence classes
$$(f_1)_F^{\min}, \dots, (f_n)_F^{\min} \in \prod_{i\in I} M_i/F,$$
and not on the particular choice of representatives of these equivalence classes.

Lemma 2.28. *If $f_1 =_F g_1, \dots, f_n =_F g_n$, then*
$$\{i \in I \colon R^{M_i}(f_1(i), \dots, f_n(i))\} \in F$$
if and only if
$$\{i \in I \colon R^{M_i}(g_1(i), \dots, g_n(i))\} \in F,$$
and $f \in \prod_{i\in I} M_i$ defined by
$$f(i) := F^{M_i}(f_1(i), \dots, f_n(i))$$
is equivalent to $g \in \prod_{i\in I} M_i$ defined by
$$g(i) := F^{M_i}(g_1(i), \dots, g_n(i)).$$

Proof. For every $1 \leq k \leq n$, $f_k =_F g_k$. By definition of the equivalence relation, $\{i \in I \colon f_k(i) = g_k(i)\} \in F$ for every $1 \leq k \leq n$. Therefore, the finite intersection

$$E := \bigcap_{k=1}^{n} \{i \in I \colon f_k(i) = g_k(i)\} \in F.$$

If $i \in E$, then $f_1(i) = g_1(i), \ldots, f_n(i) = g_n(i)$. Hence, if

$$\{i \in I \colon R^{M_i}(f_1(i), \ldots, f_n(i))\} \in F,$$

then

$$\{i \in I \colon R^{M_i}(f_1(i), \ldots, f_n(i))\} \cap E \in F.$$

Since this intersection is a subset of $\{i \in I \colon R^{M_i}(g_1(i), \ldots, g_n(i))\}$,

$$\{i \in I \colon R^{M_i}(g_1(i), \ldots, g_n(i))\} \in F.$$

The same argument with the f_k's and g_k's reversed for $1 \leq k \leq n$ shows that if

$$\{i \in I \colon R^{M_i}(g_1(i), \ldots, g_n(i))\} \in F,$$

then

$$\{i \in I \colon R^{M_i}(f_1(i), \ldots, f_n(i))\} \in F.$$

Therefore, the interpretation of relation symbols in the reduced product is well-defined. In order to show that the functions f and g as defined are equivalent over F, we have to show that $\{i \in I \colon f(i) = g(i)\} \in F$. As before,

$$E := \bigcap_{k=1}^{n} \{i \in I \colon f_k(i) = g_k(i)\} \in F.$$

If $i \in E$, then $f(i) = F^{M_i}(f_1(i), \ldots, f_n(i)) = F^{M_i}(g_1(i), \ldots, g_n(i)) = g(i)$. Hence, $E \subseteq \{i \in I \colon f(i) = g(i)\}$. Since $E \in F$, $\{i \in I \colon f(i) = g(i)\} \in F$. $\square$

It is clear from the definition of the interpretation of a relation symbol R that

$$\prod_{i \in I} M_i/F \models R^M(t_1, \ldots, t_n)$$

if and only if

$$\{i \in I \colon M_i \models R^{M_i}(t_1, \ldots, t_n)\} \in F.$$

In the remainder of this section, we will establish this rule for every $\mathscr{L}_{\kappa\kappa}$-formula for reduced products over a κ-complete ultrafilter. A reduced product over an ultrafilter is called an *ultraproduct*. Łoś (1955) proved this 'fundamental theorem on ultraproducts' in the case of first-order logic. We first state the theorem, then proof it in several lemmas.

Theorem 2.29 (AC)**.** *(Łoś) Let $\prod_{i\in I} M_i/U$ be an ultraproduct of $\mathscr{L}_{\kappa\kappa}$-structures $\{M_i : i \in I\}$ over a κ-complete ultrafilter U on I. For every $\mathscr{L}_{\kappa\kappa}$-formula $\varphi(v_1, \ldots, v_n)$ and all $(f_1)_U^{min}, \ldots, (f_n)_F^{min} \in \prod_{i\in I} M_i/U$,*

$$\prod_{i\in I} M_i/U \models \varphi((f_1)_U^{min}, \ldots, (f_n)_F^{min})$$

if and only if

$$\{i \in I : M_i \models \varphi(f_1(i), \ldots, f_n(i))\} \in U.$$

The proof uses induction on the complexity of the formulas. For notational easy we will only deal with formulas with one free variable. There is no difficulty in generalizing to formulas with more free variables. Furthermore, we sometimes write $\langle f(i) : i \in I\rangle_F$ to denote the equivalence class $(f)_F^{\min}$ of the function $f \in \prod_{i\in I} M_i$. We first have to deal with the terms of $\mathscr{L}$.

Lemma 2.30. *Let $M := \prod_{i\in I} M_i/F$ be the reduced product of $\mathscr{L}$-structures $\{M_i : i \in I\}$ over a filter F on I. For every term $t(v)$ of $\mathscr{L}$ and every $(f)_F^{min} \in M$,*

$$t^M[(f)_F^{min}] = \left\langle t^{M_i}[f(i)] : i \in I)\right\rangle_F .$$

Proof. We use induction on the complexity of the terms. The basis of the induction is formed by the variables and the constants.

If $t(v)$ is the variable v, then

$$t^M[(f)_F^{\min}] = (f)_F^{\min} = \langle f(i) : i \in I\rangle_F = \left\langle t^{M_i}[f(i)] : i \in I\right\rangle_F .$$

If $t(v)$ is the constant symbol c, then $t^M[(f)_F^{\min}]$ is

$$c^M = \left\langle c^{M_i} : i \in I\right\rangle_F = \left\langle t^{M_i}[f(i)] : i \in I\right\rangle_F .$$

There is only one induction step involved, concerning function symbols. Suppose F is an n-ary function symbol of $\mathscr{L}$ and $t_1(v), \ldots, t_n(v)$ are terms of $\mathscr{L}$ which all satisfy the induction hypothesis

$$t_k^M[(f)_F^{\min}] = \left\langle t_k^{M_i}[f(i)] : i \in I\right\rangle_F .$$

If $t(v)$ is the term $F(t_1(v), \ldots, t_n(v))$, then

$$t^M[(f)_F^{\min}] = F^M(t_1^M[(f)_F^{\min}], \ldots, t_n^M[(f)_F^{\min}]),$$

which by definition of the interpretation of function symbols in M is equal to

$$\left\langle F^{M_i}(t_1^{M_i}[f(i)], \ldots, t_n^{M_i}[f(i)]) : i \in I\right\rangle_F ,$$

and this is equal to $\left\langle t^{M_i}[(f)_F^{\min}] : i \in I\right\rangle_F$, as required. □

We now start the proof of Theorem 2.29 by induction on the complexity of the formula $\varphi(v)$. The basis of the induction is the case for atomic formulas. Its proof uses Lemma 2.30 on terms.

Lemma 2.31. *Let $M := \prod_{i \in I} M_i/F$ be the reduced product of $\mathscr{L}$-structures $\{M_i : i \in I\}$ over a filter F on I. For every atomic formula $\varphi(v)$ of $\mathscr{L}$ and every $(f)_F^{min} \in M$, $M \models \varphi[(f)_F^{min}]$ if and only if*

$$\{i \in I : M_i \models \varphi[f(i)]\} \in F.$$

Proof. If R is an n-ary relation symbol of $\mathscr{L}$ and $\varphi(v)$ is the atomic formula $R(t_1(v), \ldots, t_n(v))$, where $t_1(v), \ldots, t_n(v)$ are terms of $\mathscr{L}$, then by definition

$$M \models R(t_1[(f)_F^{\min}, \ldots, t_n[(f)_F^{\min}])$$

if and only if $R^M(t_1^M[(f)_F^{\min}], \ldots, t_n^M[(f)_F^{\min}])$ if and only if

$$\{i \in I : M_i \models R^{M_i}(t_1^{M_i}[f(i)], \ldots, t_n^{M_i}[f(i)])\} \in F.$$

If $\varphi(v)$ is the atomic formula $t_1(v) = t_2(v)$, then $M \models \varphi((f)_F^{\min})$ is equivalent to $M \models t_1[(f)_F^{\min}] = t_2[(f)_F^{\min}]$. $M \models t_1[(f)_F^{\min}] = t_2[(f)_F^{\min}]$ if and only if $t_1^M[(f)_F^{\min}] =_F t_2^M[(f)_F^{\min}]$. By Lemma 2.30, these terms evaluate to $\left\langle t_1^{M_i}[f(i)] : i \in I \right\rangle_F$ and $\left\langle t_1^{M_i}[f(i)] : i \in I \right\rangle_F$, respectively. By the definition of $=_F$, $\left\langle t_1^{M_i}[f(i)] : i \in I \right\rangle_F =_F \left\langle t_1^{M_i}[f(i)] : i \in I \right\rangle_F$ if and only if

$$\{i \in I : t_1^{M_i}[f(i)] = t_2^{M_i}[f(i)]\} \in F,$$

which is equal to

$$\{i \in I : M_i \models t_1^{M_i}[f(i)] = t_2^{M_i}[f(i)]\} = \{i \in I : M_i \models t^{M_i}[f(i)]\} \in F,$$

as required. □

We have established the basis of the induction. There are three inductive steps involved: for infinitary conjunction, negation, and infinitary existential quantification. We will see that we need κ-completeness in the first case, an ultrafilter in the second and the axiom of choice in the third case. The induction hypothesis for an $\mathscr{L}$-formulas $\varphi(v)$ states: for every $(f)_F^{\min} \in M$, $M \models \varphi[(f)_F^{\min}]$ if and only if $\{i \in I : M_i \models \varphi[f(i)]\} \in F$. We start with the inductive step for infinitary conjunction.

Lemma 2.32. *Let $M := \prod_{i \in I} M_i/F$ be a reduced product of $\mathscr{L}_{\kappa\kappa}$-structures $\{M_i : i \in I\}$ over a κ-complete filter F. If $\langle \varphi_\alpha(v) : \alpha < \lambda < \kappa \rangle$ is a sequence of $\mathscr{L}_{\kappa\kappa}$-formulas such that the induction hypothesis holds for every formula in the sequence, then for every $(f)_F^{min} \in M$,*

$$M \models \bigwedge_{\alpha < \lambda} \varphi_\alpha[(f)_F^{min}] \text{ if and only if } \{i \in I : M_i \models \bigwedge_{\alpha < \lambda} \varphi_\alpha[f(i)]\} \in F.$$

Proof. By definition of the semantics, $M \models \bigwedge_{\alpha<\lambda} \varphi_\alpha[(f)_F^{\min}]$ if and only if $M \models \varphi_\alpha[(f)_F^{\min}]$ for every $\alpha < \lambda$. By induction hypothesis, this is the case if and only if $\{i \in I \colon M_i \models \varphi_\alpha[f(i)]\} \in F$. Since F is κ-complete, the intersection

$$I^* := \bigcap_{\alpha<\lambda} \{i \in I \colon M_i \models \varphi_\alpha[f(i)]\} \in F.$$

Since $M_i \models \bigwedge_{\alpha<\lambda} \varphi_\alpha[(f)_F^{\min}]$ if and only if $i \in I^*$, the set $\{i \in I \colon M_i \models \bigwedge_{\alpha<\lambda} \varphi_\alpha\} = I^* \in F$. $\square$

Next is the inductive step for negation. Here we need that the reduced product is over an ultrafilter rather than over a filter.

Lemma 2.33. *Let $M := \prod_{i\in I} M_i/U$ be a reduced product of $\mathscr{L}$-structures $\{M_i \colon i \in I\}$ over an ultrafilter U. If $\varphi(v)$ is an $\mathscr{L}$-formula such that the induction hypothesis holds, then for every $(f)_U^{min} \in M$,*

$$M \models \neg\varphi[(f)_U^{min}]$$

if and only if

$$\{i \in I \colon M_i \models \neg\varphi[f(i)])\} \in U.$$

Proof. By the definition of the semantics, $M \models \neg\varphi[(f)_U^{\min}]$ if and only if it not $M \models \varphi[(f)_U^{\min}]$. By the induction hypothesis, not $M \models \varphi[(f)_U^{\min}]$ if and only if $\{i \in I \colon M_i \models \varphi[f(i)]\} \notin U$. Since U is an ultrafilter, $\{i \in I \colon M_i \models \varphi[f(i)]\} \notin U$ if and only if $\{i \in I \colon \text{ not } M_i \models \varphi[f(i)]\} \in U$. By definition of the semantics, $\{i \in I \colon \text{ not } M_i \models \varphi[f(i)]\} = \{i \in I \colon M_i \models \neg\varphi[f(i)]\} \in U$, as required. $\square$

Finally, we prove the inductive step for infinitary existential quantification. Some form of the axiom of choice is necessary in this step.

Lemma 2.34 (AC)**.** *Let $M := \prod_{i\in I} M_i/F$ be the reduced product of $\mathscr{L}$-structures over a filter F on I. If $\varphi(\langle v_\alpha \colon \alpha < \lambda\rangle)$ is a $\mathscr{L}$-formula such that the induction hypothesis holds, then for every $\left\langle (f_\alpha)_F^{min} \colon \alpha < \lambda \right\rangle \in M$,*

$$M \models (\exists \langle v_\alpha \colon \alpha < \lambda\rangle)\varphi[\langle v_\alpha \colon \alpha < \lambda\rangle, (f)_U^{min}]$$

if and only if

$$\{i \in I \colon M_i \models (\exists \langle v_\alpha \colon \alpha < \lambda\rangle)\varphi[\langle v_\alpha \colon \alpha < \lambda\rangle, f(i)])\} \in U.$$

Proof. By definition of the semantics,

$$M \models (\exists \langle v_\alpha \colon \alpha < \lambda\rangle)\varphi_\alpha[\langle v_\alpha \colon \alpha < \lambda\rangle, (f)_U^{\min}]$$

if and only if there is a sequence $\left\langle (f_\alpha)_F^{\min} \colon \alpha < \lambda \right\rangle$ in M such that

$$M \models \varphi_\alpha[\left\langle (f_\alpha)_F^{\min} \colon \alpha < \lambda \right\rangle, (f)_U^{\min}].$$

By the induction hypothesis, $M \models \varphi_\alpha[\left\langle (f_\alpha)_F^{\min} \colon \alpha < \lambda \right\rangle, (f)_U^{\min}]$ if and only if

$$\{i \in I \colon M_i \models \varphi_\alpha[\langle f_\alpha(i) \colon \alpha < \lambda \rangle, f(i)]\} \in F.$$

Since $\{i \in I \colon M_i \models \varphi_\alpha[\langle f_\alpha(i) \colon \alpha < \lambda \rangle, f(i)]\}$ is a subset of $\{i \in I \colon M_i \models (\exists \langle v_\alpha \colon \alpha < \lambda \rangle)\varphi(\langle v_\alpha \colon \alpha < \lambda \rangle, f(i)]\}$, this latter set is in F. Conversely, if $\{i \in I \colon M_i \models (\exists \langle v_\alpha \colon \alpha < \lambda \rangle)\varphi(\langle v_\alpha \colon \alpha < \lambda \rangle, f(i)\} \in F$, we can use AC to pick functions $\langle f_\alpha \colon \alpha < \lambda \rangle$ such that $\{i \in I \colon M_i \models \varphi(\langle f_\alpha(i) \colon \alpha < \lambda \rangle, f(i)\} \in F$, and reverse the argument. □

This finishes the proof by induction of Theorem 2.29.

2.6 Ultrapowers and Elementary Embeddings

An *ultrapower* of M over U is an ultraproduct of the form $\prod_{i \in I} M/U$, where every factor is the same $\mathscr{L}$-structure M. By Łoś' Theorem 2.29, for every $\mathscr{L}$-sentence φ,

$$\prod_{i \in I} M/U \models \varphi \text{ if and only if } \{i \in I \colon M \models \varphi\} \in U.$$

Since $\{i \in I \colon M \models \varphi\}$ is either the entire index set $I \in U$ when $M \models \varphi$ or the empty set $\varnothing \notin U$ when $M \not\models \varphi$, the displayed statement reduces to

$$\prod_{i \in I} M/U \models \varphi \text{ if and only if } M \models \varphi.$$

In other words, a structure and any ultrapower of it are elementary equivalent. We give an explicit elementary embedding $j \colon M \to \prod_{i \in I} M/U$. When U is κ-complete, this embedding will preserve all $\mathscr{L}_{\kappa\kappa}$-formulas. For any set x, let c_x denote the function with constant value x. The domain of c_x will be defined by the context. For notational ease, we state the following proposition only for formulas with one free variable. Of course, the result easily extends to formulas with any finite number of free variables.

Proposition 2.35 (AC). *If U is a κ-complete ultrafilter on I and M is an $\mathscr{L}_{\kappa\kappa}$-structure, then $j \colon M \to \prod_{i \in I} M/U$ defined by $j(x) := (c_x)_U^{min}$ is an elementary embedding of M into $\prod_{i \in I} M/U$.*

Proof. Let $\varphi(v)$ be an $\mathscr{L}_{\kappa\kappa}$-formula. We have to show that for every $x \in M$, $M \models \varphi(x)$ if and only if $\prod_{i\in I} M/U \models \varphi(j(x))$, which by definition of j is equal to $\prod_{i\in I} M/U \models \varphi((c_x)_U^{\min})$. By Łoś' theorem 2.29, $\prod_{i\in I} M/U \models \varphi((c_x)_U^{\min})$ if and only if

$$\{i \in I \colon M \models \varphi(c_x(i))\} = \{i \in I \colon M \models \varphi(x)\} \in U.$$

But $\{i \in I \colon M \models \varphi(x)\} \in U$ if and only if $\{i \in I \colon M \models \varphi(x)\} = I$, that is, if and only if $M \models \varphi(x)$. □

We can apply the ultrapower construction to get an elementary embedding of the set-theoretic universe $\mathbf{V}$ into an ultrapower $M := \prod_{i\in I} \mathbf{V}/U$. The membership relation $\in^M$ on M is then defined by

$$(f)_U^{\min} \in^M (g)_U^{\min} \text{ if and only if } \{i \in I \colon f(i) \in g(i)\} \in U.$$

Unfortunately, the relation $\in^M$ on $M \subseteq \mathbf{V}$ does often not coincide with the 'real' membership relation $\in$ on $\mathbf{V}$. Mostowski's Collapsing Lemma 2.38 is a technical tool to solve this problem. Possibly, the language $\mathscr{L}$ contains more non-logical symbols than just a binary relation symbol. In order to be able to apply Mostowski's Collapsing Lemma to an ultraproduct $M := \prod_{i\in I} \mathbf{V}/U$ of the universe, we have to show that $\in^M$ is show that $\in^M$ satisfies three requirements: it should be set-like, extensional, and well-founded on M. An relation E is *set-like on* a class C if for every $y \in C$, $\{x \in C \colon x \mathrel{E} y\}$ is a set. Since $(h)_U^{\min} \in^M (f)_U^{\min}$ if and only if $\{i \in I \colon h(i) \in f(i)\} \in U$, the relation $\in^M$ is set-like on M. An relation E is *extensional on* a class C if for all $x, y \in C$

$$\forall z \left[(z \in x \Leftrightarrow z \in y) \Rightarrow x = y\right].$$

Lemma 2.36. *The relation $\in^M$ is extensional on M.*

Proof. Let $(f)_U^{\min}, (g)_U^{\min} \in M$. Suppose for all $(h)_U^{\min} \in M$, $(h)_U^{\min} \in (f)_U^{\min}$ if and only if $(h)_U^{\min} \in (g)_U^{\min}$. We have to show that $(f)_U^{\min}$ and $(g)_U^{\min}$ are equal. By definition, $(f)_U^{\min} =_U (g)_U^{\min}$ if and only if $\{i \in I \colon f(i) = g(i)\} \in U$. If $(f)_U^{\min}$ is empty, then $(g)_U^{\min}$ is empty and therefore $(f)_U^{\min}$ and $(g)_U^{\min}$ are equal. If there is $(h)_U^{\min} \in M$ such that $(h)_U^{\min} \in^M (f)_U^{\min}$, then $(h)_U^{\min} \in^M (g)_U^{\min}$ by assumption. By definition of $\in^M$, both $\{i \in I \colon h(i) = f(i)\} \in U$ and $\{i \in I \colon h(i) = g(i)\} \in U$. Therefore, their intersection $\{i \in I \colon f(i) = h(i) = g(i)\} \in U$. Since this is a subset of $\{i \in I \colon f(i) = g(i)\}$, $\{i \in I \colon f(i) = g(i)\} \in U$. Therefore, $(f)_U^{\min} =_U (g)_U^{\min}$, as required. □

Finally, we will prove that well-foundedness of a relation is preserved when taking ultraproducts over ω_1-complete ultrafilters. For this we use the axiom of dependent choices (DC) and the characterization of well-founded relation it provides (Lemma 2.5).

Proposition 2.37 (DC). *Let $\mathscr{L}$ contain a binary relation symbol R and let M be an $\mathscr{L}$-structure such that the interpretation R^M of R in M is well-founded. The interpretation R^{UP} of R in the ultraproduct $\prod_{i\in I} M/U$ is well-founded if and only if U is ω_1-complete.*

Proof. First, suppose that there is a countable sequence $\langle (f_k)_U^{\min}\colon k\in\omega\rangle$ in the ultraproduct $\prod_{i\in I} M/U$ such that $(f_{k+1})_U^{\min}\ R^{\mathrm{UP}}\ (f_k)_U^{\min}$ for every $k\in\omega$. By definition of the interpretation R^{UP} of R, this means that for every $k\in\omega$,

$$\{i\in I\colon f_{k+1}(i)\ R^M\ f_k(i)\}\in U.$$

Since U is ω_1-complete, the intersection

$$\bigcap_{k\in\omega}\{i\in I\colon f_{k+1}(i)\in f_k(i)\}\in U.$$

In particular, this intersection is nonempty and hence there is an $i\in I$ such that $f_{k+1}(i)\ R^M\ f_k(i)$ for every $k\in\omega$. But this contradictions the well-foundedness of R^M by Lemma 2.4.

Second, suppose that U is not ω_1-complete. Then there is a sequence $\langle X_k\colon k\in\omega\rangle$ in U such that the intersection

$$\bigcap_{k\in\omega} X_k\notin U.$$

We want to show that the interpretation R^{UP} of R is not well-founded on the ultraproduct $\prod_{i\in I} M/U$. Using Lemma 2.5, it is sufficient to find a countable sequence $\langle (f_k)_U^{\min}\colon k\in\omega\rangle$ in $\prod_{i\in I} M/U$ such that for every $k\in\omega$,

$$(f_{k+1})_U^{\min}\ R^{\mathrm{UP}}\ (f_k)_U^{\min}.$$

By definition of R^{UP}, this is the case when for every $k\in\omega$,

$$\{i\in I\colon f_{k+1}(i)\in f_k(i)\}\in U.$$

Define for every $k\in\omega$ the function $f_k\in\prod_{i\in I} M_i$ by

$$f_k(i)=\begin{cases} n-k & \text{if } i\in(\bigcap_{m<n} X_m)\setminus X_n \text{ and } n\geq k,\\ 0 & \text{otherwise.}\end{cases}$$

Note that $f_n(i)=n-k$ for the first $n\geq k\in\omega$ such that $i\notin X_n$. For $k\in\omega$, we have that

$$\bigcap_{m\leq k} X_m\setminus\bigcap_{n\leq\omega} X_n\in U,$$

therefore

$$\bigcap_{m\leq k} X_m\setminus\bigcap_{n\leq\omega} X_n\subseteq\{\xi\in\kappa\colon f_{k+1}(\xi)\in f_k(\xi)\}\in U,$$

and so $\langle (f_n)_U^{\min}\colon n<\omega\rangle$ shows that E_U is not well-founded. $\square$

We can now state Mostowski's Collapsing Lemma.

Lemma 2.38. *(Mostowski's Collapsing Lemma) Let $\mathscr{L}$ be a language containing a binary relation symbol E and let M be a class structure for $\mathscr{L}$. If E is set-like, extensional and well-founded binary, then there is a transitive class T and an unique isomorphism $M \to T$. Here, the binary relation symbol E is interpreted by $\in$ in T.*

Proof. See for example [Je78, p. 88–89]. □

The structure $\langle T, \in, \dots \rangle$ is called the *transitive collapse* of $\langle M, E, \dots \rangle$. To summarize, whenever $M := \prod_{i \in I} \mathbf{V}/U$ is an ultrapower of $\mathbf{V}$ over an ω_1-complete nonprincipal ultrafilter U on I, there is a transitive class T and an isomorphism $\pi\colon \left\langle M, \in^M \right\rangle \to \langle T, \in \rangle$ by Mostowski's Collapsing Lemma 2.38. We denote the image of $(f)_U^{\min} \in M$ under π by $[f]_U := \pi((f)_U^{\min})$ and the structure $\langle T, \in, \dots \rangle$ isomorphic to $\left\langle M, \in^M, \dots \right\rangle$ by $\mathrm{Ult}(\mathbf{V}, U)$. Often, the structure $\mathrm{Ult}(\mathbf{V}, U)$ is also called an ultrapower of $\mathbf{V}$. Clearly, $\mathrm{Ult}(\mathbf{V}, U) \subseteq \mathbf{V}$ is an inner model.

Lemma 2.39 (AC)**.** *If $j\colon \mathbf{V} \prec \mathrm{Ult}(\mathbf{V}, U)$ is an elementary embedding induced by a κ-complete nonprincipal ultrafilter U on $\kappa > \omega$, then $\mathrm{crit}(j) = \kappa$.*

Proof. In order to show that $\mathrm{crit}(j) = \kappa$, we have to show that $j(\alpha) = \alpha$ for every ordinal $\alpha < \kappa$ and that $j(\kappa) > \kappa$.

First, suppose that $\alpha < \kappa$ is the least ordinal such that $\alpha < j(\alpha)$. Let $[f]_U = \alpha$. Then since $[f]_U = \alpha < j(\alpha) = [c_\alpha]_U$, $M \models [f]_U < [c_\alpha]_U$ by elementarity. By Łoś' Theorem 2.29,

$$\{\xi < \kappa\colon f(\xi) < c_\alpha(\xi)\} = \{\xi < \kappa\colon f(\xi) < \alpha\} \in U.$$

Since

$$\{\xi < \kappa\colon f(\xi) < \alpha\} = \bigcup_{\beta < \alpha} \{\xi < \kappa\colon f(\xi) = \beta\} \in U,$$

there is a $\beta < \alpha$ such that $\{\xi < \kappa\colon f(\xi) = \beta\} \in U$ by Lemma 2.10. Because $\{\xi < \kappa\colon f(\xi) = \beta\} = \{\xi < \kappa\colon f(\xi) = c_\beta(\xi)\} \in U$, $M \models [f]_U = [c_\beta]_U$ by Łoś' Theorem 2.29. But then $[f]_U = [c_\beta]_U = j(\beta) = \beta < \alpha$, a contradiction.

Second, we show $\kappa \leq j(\kappa)$. Since U contains all end-segments of κ, for every $\alpha < \kappa$,

$$\{\xi \in \kappa\colon \alpha < \kappa\} = \{\xi \in \kappa\colon c_\alpha(\xi) < \mathrm{id}(\xi)\} \in U$$

Therefore, $M \models [c_\alpha]_U < [\mathrm{id}]_U$ by Łoś' Theorem 2.29. Furthermore,

$$\kappa = \{\xi \in \kappa\colon \xi < \kappa\} = \{\xi \in \kappa\colon \mathrm{id}(\xi) < c_\kappa(\xi)\} \in U.$$

Therefore, $M \models [\mathrm{id}]_U < [c_\kappa]_U$ by Łoś' Theorem 2.29. Combining these inequalities we have that for every $\alpha < \kappa$,

$$M \models [c_\alpha]_U < [c_\kappa]_U.$$

□

Chapter 3

Compact Cardinals

In this chapter we will study four different properties of infinite cardinals. Each of these properties may be used to define strong compactness. In the first section we generalize the compactness theorem of first-order logic to infinitary languages. Then we consider the analogue of the ultrafilter theorem. An ultrafilter proof of the compactness theorem leads us to consider fine measures. Finally, we use these fine measures in the ultrapower construction to obtain elementary embeddings of the universe with a special property.

The last two sections are dedicated to measurable cardinals and a discussion of the relative consistency strength of various forms of infinitary language compactness.

3.1 Language Compactness

A prominent feature of first-order logic is its compactness: a set of first-order sentences has a model if (and only if) every finite subset has a model. 'If any theorem is fundamental in first-order model theory,' the British logician Wilfrid Hodges notes, 'it must surely be the compactness theorem.' [Ho97, p. 124].

In contrast with the finitary language $\mathscr{L}_{\omega\omega}$ of 'ordinary' first-order logic, the usual compactness theorem fails rather badly for the infinitary languages $\mathscr{L}_{\kappa\omega}$ and $\mathscr{L}_{\kappa\kappa}$ in case κ is an uncountable cardinal. For example, if c and c_i $(i < \omega)$ are distinct constant symbols, then the set consisting of the $\mathscr{L}_{\omega_1\omega}$-sentences

$$c \neq c_0, c \neq c_1, c \neq c_2, \dots,$$

$$\bigvee_{i<\omega} c = c_i.$$

cannot have a model, although every proper and therefore every finite subset of it has. [Ho97, p. 127]. This example demonstrates that in order to suitably generalize the compactness theorem to infinitary languages, we have to replace 'finite' with a more appropriate notion of 'small'. Our approach is to read 'finite' as 'of cardinality less than ω' and then substitute κ for ω.

Definition 3.1. Let κ and λ be infinite cardinals, and let $\mathscr{L}_\kappa$ denote either $\mathscr{L}_{\kappa\omega}$ or $\mathscr{L}_{\kappa\kappa}$. The language $\mathscr{L}_\kappa$ is said to be λ-*compact* if every set Σ of $\mathscr{L}_\kappa$-sentences of cardinality λ has a model if (and only if) every subset of Σ of cardinality less than κ has a model.

If $\mathscr{L}_\kappa$ is λ-compact for every $\lambda \geq \kappa$, we say that $\mathscr{L}_\kappa$ is *compact*. With these definitions, the usual compactness theorem for first-order logic is the statement that $\mathscr{L}_{\omega\omega}$ is compact.

If $\omega \leq \lambda < \kappa$, then $\mathscr{L}_{\kappa\omega}$ is λ-compact: if Σ is a set of $\mathscr{L}_{\kappa\omega}$-sentences of cardinality λ such that every $\Phi \subseteq \Sigma$ of cardinality less than κ has a model, then Σ has a model since Σ itself is a subset of Σ of cardinality $\lambda < \kappa$.

We will prove that if κ is a successor or singular cardinal, the infinitary language $\mathscr{L}_{\kappa\omega}$ is not κ-compact: we will give a set of $\mathscr{L}_{\kappa\omega}$-sentences, which has no model although every subset of cardinality less than κ has. In fact, we will use at most κ many nonlogical symbols. Note that these counterexamples must necessarily have cardinality greater than or equal to κ.

Definition 3.2. The language $\mathscr{L}_{\kappa\omega}$ is said to be *weakly* λ-compact if every set of $\mathscr{L}_{\kappa\omega}$-sentences using most κ many nonlogical symbols has a model if every subset of cardinality less than κ has.

Proposition 3.3. *If $\kappa = \lambda^+$ is a successor cardinal, then $\mathscr{L}_{\kappa\omega}$ is not weakly κ-compact.*

Proof. [Dr74, p. 290] Let x_α ($\alpha < \lambda^+$) and y_β ($\beta < \lambda$) be distinct constant symbols. Consider the set Σ consisting of the following $\mathscr{L}_{\kappa\omega}$-sentences:

$$\bigvee_{\beta<\lambda} x_\alpha = y_\beta \quad \text{for every } \alpha < \lambda^+, \text{ and}$$

$$x_\alpha \neq x_{\alpha'} \quad \text{for all } \alpha, \alpha' < \lambda^+, \alpha \neq \alpha'.$$

Suppose $\Psi \subseteq \Sigma$ has cardinality at most $\lambda < \kappa$. There are at most λ distinct constant symbols of the form x_α in Ψ. Hence, we can find suitable interpretations of these constant symbols in λ.

Every subset of Σ of cardinality less than κ is satisfiable. Suppose M is a model of Σ. Since Σ demands different interpretations of the λ^+ many x_α from a set $\{y_\alpha^M : \alpha < \lambda\}$ of cardinality at most λ, there would be a bijection between λ and λ^+, a contradiction. ∎

The next proposition shows that even if κ is a limit cardinal, the language $\mathscr{L}_{\kappa\omega}$ can only be λ-compact for any $\lambda \geq \kappa$ if κ is regular.

Proposition 3.4. *If κ is a singular cardinal, then $\mathscr{L}_{\kappa\omega}$ is not weakly κ-compact.*

Proof. [Je78, p. 384] Let $A \subset \kappa$ be a subset of cardinality less than κ such that $\sup A = \bigcup A = \kappa$. Let c_α be a constant symbol for every $\alpha \leq \kappa$, and let $\prec$ be a binary relation symbol. Consider the following set of $\mathscr{L}_{\kappa\omega}$-sentences: three sentences which state that $\prec$ is a linear order,

$$\forall x\, (x \not\prec x),$$

$$\forall xy\, (x \prec y \vee x = y \vee y \prec x),$$

$$\forall xyz\, (x \prec y \wedge y \prec z \Rightarrow x \prec z),$$

and the following $\mathscr{L}_{\kappa\omega}$-sentences:

$$c_\alpha \prec c_\kappa \quad \text{for every } \alpha < \kappa \tag{3.1}$$

and the single sentence

$$\forall x (\bigvee_{\alpha \in A} x \prec c_\alpha).$$

Note that we use only κ many non-logical symbols and that Σ has cardinality κ. We will show that Σ is κ-satisfiable, but cannot have a model itself.

Suppose $\Phi \subseteq \Sigma$ is a subset of cardinality $\lambda < \kappa$. We have to find a model of Φ. Let M be the structure $\langle \kappa, \in \rangle$. In other words, the domain is κ and interpretation of $\prec$ is $\in$. Clearly, $\in$ is a linear order on κ so the first three sentences are satisfied. We have to find a suitable interpretation of the constant symbols.

Let $X := \{\alpha < \kappa \colon c_\alpha \text{ occurs in } \Phi\}$. Since Φ has cardinality $\lambda < \kappa$, the order type of X is less than $\lambda^+ \leq \kappa$. Let π be the collapsing function. If $\alpha \in X$, interpret c_α by $\pi(\alpha)$, and interpret c_κ by the order type of X. All the sentences of the form (3.1) which are in Φ are satisfied. Since A is cofinal in κ, there is an $\alpha \in A$ such that $c_\kappa \prec c_\alpha$.

Hence, $M \models \Phi$.

If $\alpha \neq \kappa$, interpret c_α by α. Since A is cofinal in κ, for every $x \in \kappa$ there is an $\alpha \in A$ such that $x \in \alpha$, that is,

$$M \models \forall x (\bigvee_{\alpha \in A} x \prec c_\alpha).$$

Suppose towards a contradiction that M is a model for Σ. On the one hand, $c_\alpha^M \prec^M c_\kappa^M$ for every $\alpha < \kappa$. On the other hand, c_κ^M is an element of M and therefore there has to be an $\alpha \in A$ such that $c_\kappa^M \prec^M c_\alpha^M$. This is a contradiction. Therefore, Σ is a set of $\mathscr{L}_{\kappa\omega}$-sentences of cardinality κ, using at most κ non-logical symbols which does not have a model. Since every subset of Σ of cardinality less than κ has a model, $\mathscr{L}_{\kappa\omega}$ is not weakly κ-compact. □

Corollary 3.5. *If $\mathscr{L}_{\kappa\omega}$ is weakly λ-compact for some $\lambda \geq \kappa$, then κ is weakly inaccessible.*

Proof. Clearly, if $\mathscr{L}_{\kappa\omega}$ is weakly λ-compact for some $\lambda \geq \kappa$, then $\mathscr{L}_{\kappa\omega}$ is weakly κ-compact. However, if κ is either a singular cardinal or a successor cardinal, then $\mathscr{L}_{\kappa\omega}$ is not weakly κ-compact by Proposition 3.4 and Proposition 3.3, respectively. Therefore, κ has to be a regular limit cardinal. □

Proposition 3.6. *If there is a $\lambda < \kappa$ such that $\kappa \leq 2^\lambda$, then $\mathscr{L}_{\kappa\omega}$ is not weakly 2^λ-compact.*

Proof. Let c_α and d^i_α be distinct constant symbols for $\alpha < \lambda$ and $i < 2$. Consider the following set Σ which consists of the $\mathscr{L}_{\kappa\omega}$-sentence

$$\bigwedge_{\alpha<\lambda} \left[(c_\alpha = d^0_\alpha \vee c_\alpha = d^1_\alpha) \wedge d^0_\alpha \neq d^1_\alpha\right],$$

plus for every function $f\colon \lambda \to 2$ the $\mathscr{L}_{\kappa\omega}$-sentence

$$\bigvee_{\alpha<\lambda} c_\alpha \neq d^{f(\alpha)}_\alpha. \tag{3.2}$$

This set consists of $2^\lambda \geq \kappa$ sentences. We claim that every subset Φ of cardinality less than κ is satisfiable. Since Φ has cardinality less than $\kappa \leq 2^\lambda$, there has to be a function $g\colon \lambda \to 2$ such that the corresponding sentence of the form (3.2) is not an element of Φ. Take $2 = \{0, 1\}$ as domain. Interpret d^i_α by $i \in 2$ for every $\alpha < \lambda$, and c_α by $d^{g(\alpha)}_\alpha$. For every function $f\colon \lambda \to 2$ different from g there is an $\alpha < \lambda$ such that $f(\alpha) \neq g(\alpha)$. Hence, the sentence of the form (3.2) for f is satisfied. Therefore, every sentence in Φ is satisfied.

Suppose that M is a model of Σ. Since M is a model of the first sentence, for every $\alpha < \lambda$ there is an $i = i(\alpha) < 2$ such that

$$M \models c_\alpha = d^{i(\alpha)}_\alpha.$$

Let f be the function $\lambda \to 2$ defined by $f(\alpha) = i(\alpha)$. On the one hand, we have $c^M_\alpha = (d^{f(\alpha)}_\alpha)^M$ for every $\alpha < \lambda$ by construction of the function f. On the other hand,

$$M \models \bigvee_{\alpha<\lambda} c_\alpha \neq d^{f(\alpha)}_\alpha,$$

and therefore we must have $c^M_\alpha \neq (d^{f(\alpha)}_\alpha)^M$ for some $\alpha < \lambda$, a contradiction.

This shows that while Σ is κ-satisfiable, it has no model. □

Without the axiom of choice, the effect of Proposition 3.6 is limited as trichotomy may not hold. With the trichotomy we can prove that κ is strongly inaccessible when $\mathscr{L}_{\kappa\omega}$ is 2^κ-compact.

Corollary 3.7 (AC). *If $\mathscr{L}_{\kappa\omega}$ is weakly 2^κ-compact, then κ is strongly inaccessible.*

Proof. Suppose $\mathscr{L}_{\kappa\omega}$ is weakly 2^κ-compact. We already know by Corollary 3.5 that κ is weakly inaccessible, and only have to show that $2^\lambda < \kappa$ for every $\lambda < \kappa$.

Let $\lambda < \kappa$. By trichotomy either $2^\lambda < \kappa$ or $\kappa \leq 2^\lambda$ for every $\lambda < \kappa$. We will derive a contradiction in the latter case. Suppose $\kappa \leq 2^\lambda$. Since $\lambda < \kappa$, $2^\lambda < 2^\kappa$. Hence, $\mathscr{L}_{\kappa\omega}$ is weakly 2^λ-compact. But by Proposition 3.6, if $\kappa \leq 2^\lambda$, then $\mathscr{L}_{\kappa\omega}$ is not weakly 2^λ-compact. Therefore, it has to be the case that $2^\lambda < \kappa$, as required. ☐

Hence in ZFC a cardinal κ is strongly inaccessible if the language $\mathscr{L}_{\kappa\omega}$ is weakly 2^κ-compact. According to [Ka03, p. 37], William Boos [Bo76] has shown that the weak κ-compactness of $\mathscr{L}_{\kappa\omega}$ does not entail inaccessibility in ZFC.

3.2 Extension of Filters

We have noted in Section 2.3 that the collection of all filters on a set is partially ordered by inclusion, and the maximal filters in this order are precisely the ultrafilters (Lemma 2.7). Since every chain of filters has an upper bound (Lemma 2.8), we can use Zorn's lemma to easily prove the following theorem.

Theorem 3.8 (AC). *(Ultrafilter theorem) Every filter can be extended to an ultrafilter.*

Proof. Recall that Zorn's lemma (an equivalent of the axiom of choice) is the statement that if every chain in a partial order has an upper bound, then the partial order has a maximal element. By the remarks above, the theorem follows immediately. ☐

Just as we did with the notion of compactness we can try to generalize this result to arbitrary infinite cardinals. We read the ultrafilter theorem as 'every ω-complete filter can be extended to an ω-complete ultrafilter' and then replace ω by κ.

Definition 3.9. An infinite cardinal κ has the *extension property* if every κ-complete filter can be extended to a κ-complete ultrafilter.

Note that the ultrafilter theorem is the statement that ω has the extension property. In the proof of the next result, the use of Zorn's lemma is replaced by a compactness argument. This will provide an analogue of the ultrafilter theorem to uncountable cardinals.

Theorem 3.10. *(Keisler–Tarski, 1964) Let κ be a regular infinite cardinal. If $\mathscr{L}_{\kappa\omega}$ is compact, then every κ-complete filter can be extended to a κ-complete ultrafilter.*

Proof. Let F be a κ-complete filter on a set S. Using a constant symbol $\dot{X}$ for every $X \subseteq S$, let Σ' be the set of all $\mathscr{L}_{\kappa\omega}$-sentences true in the structure $M = \langle S \cup \wp(S), \in, X \rangle_{X \subseteq S}$. Let $\dot{c}$ be a new constant and let Σ be Σ' together with the sentence $\dot{c} \in \dot{X}$ for every $X \in F$. First we show that every subset of Σ of cardinality less than κ has a model. Then Σ has a model by compactness. We use this model to define a κ-complete ultrafilter extending F.

Suppose $\Phi \subseteq \Sigma$ is a subset of cardinality less than κ. Since Φ has cardinality less than κ, the set $A := \{X : X \text{ occurs in } \Phi\}$ has cardinality less than κ, too. Since F is κ-complete, there is an element $c \in S$ such that

$$c \in \bigcap_{X \in A} X \neq \varnothing.$$

Therefore, $M = \langle S \cup \wp(S), \in, X, c \rangle_{X \subseteq S} \models \Phi$. Since every subset of Σ of cardinality less than κ has a model, Σ itself has some model N by compactness of $\mathscr{L}_{\kappa\omega}$. Define $U \subseteq \wp(S)$ by

$$X \in U \text{ if and only if } N \models \dot{c} \in \dot{X}.$$

We verify that U is a κ-complete ultrafilter extending F. Of course U extends F because for every $X \in F$ the sentence $\dot{c} \in \dot{X}$ is in $\Sigma \setminus \Sigma'$. Since F is nonempty and U extends F, U is nonempty. The sentences of Σ' are needed to prove that U is a κ-complete ultrafilter. Since $N \models \forall x(x \notin \dot{\varnothing})$, we have $M \models \forall x(x \notin \dot{\varnothing})$. Therefore not $M \models \dot{c} \in \dot{\varnothing}$, thus $\varnothing \notin U$. From similar arguments we have that U is closed under taking supersets, since $N \models \forall x(x \in \dot{X} \Rightarrow x \in \dot{Y})$ for every $X \subseteq Y \subseteq S$. Furthermore, U is κ-complete (and hence closed under finite intersections) because for every $A \subseteq \wp(S)$ such that $|A| < \kappa$ we have that

$$N \models \forall x(\bigwedge_{X \in A} x \in \dot{X} \Rightarrow x \in \dot{I}),$$

where $I := (\bigcap_{X \in A} X) \subseteq S$. $\square$

Theorem 3.10 shows that the ultrafilter theorem is a ZF-consequence of the compactness theorem. While Zorn's lemma is equivalent to the axiom of choice [RR85], the compactness theorem and the ultrafilter theorem are equivalent to the Boolean prime ideal theorem, which is known to be weaker than the axiom of choice [Je78]. We will be concerned with the converse of Theorem 3.10 in the next section.

We turn now to a local version of the extension property. There are (at least) three possible candidates for this. We will introduce the following one here, and two more in later sections.

Definition 3.11. An infinite cardinal κ has the *restricted λ-extension property* if every κ-complete filter F on a set S such that $|\wp(S)| = \lambda$ can be extended to a κ-complete ultrafilter.

This local extension property is a consequence of the local compactness property. For this we need a more explicit proof than for Theorem 3.10.

Proposition 3.12. *If $\mathscr{L}_{\kappa\omega}$ is λ-compact, then κ has the restricted λ-extension property.*

Proof. Let F be a κ-complete filter on a set S such that $|\wp(S)| = \lambda$. Let c_X be a constant symbol for every $X \subseteq S$ (λ many symbols), and let $\dot{U}$ be a unary predicate symbol. Let Σ be the set of the following $\mathscr{L}_{\kappa\omega}$-sentences:

$$\neg\,\dot{U}(c_\varnothing), \tag{3.3}$$

$$\dot{U}(c_X) \vee \dot{U}(c_{S\setminus X}) \quad \text{for every } X \subseteq S, \tag{3.4}$$

$$\dot{U}(c_X) \Rightarrow \dot{U}(c_Y) \quad \text{for every } X \subseteq Y \subseteq S, \tag{3.5}$$

$$\dot{U}(c_X) \quad \text{for every } X \in F, \tag{3.6}$$

$$\bigwedge_{X\in A} \dot{U}(c_X) \Rightarrow \dot{U}(c_{\bigcap A}) \quad \text{for every } A \subseteq \wp(S) \text{ such that } |A| < \kappa. \tag{3.7}$$

We want to show that Σ is κ-satisfiable. Suppose that $\langle \varphi_\alpha \colon \alpha < \mu < \kappa\rangle$ is a sequence in Σ. We have to find an $\mathscr{L}_{\kappa\omega}$-structure which is a model for these sentences. We take as domain $\wp(S)$ and for every $X \subseteq S$ we interpret the constant symbol c_X by X. We have to find a suitable interpretation U of the unary predicate symbol $\dot{U}$. We do this by 'approximating' U in stages U_ξ for $\xi < \mu$. At each stage ξ, U_ξ will be a κ-complete filter on $\wp(S)$ and $\langle \wp(S), X, U_\xi\rangle_{X\subseteq S}$ a model for the sentences in $\{\varphi_\alpha \colon \alpha < \xi\}$, and $U_\xi \subseteq U_\zeta$ for $\xi < \zeta$.

We start with $U_0 := F$ and run through the list $\langle \varphi_\alpha \colon \alpha < \mu < \kappa\rangle$ of sentences. At limit stages δ we take $U_\delta := \bigcup_{\alpha<\delta} U_\alpha$. Note that this a κ-complete filter on $\wp(S)$ since $U_\xi \subseteq U_\zeta$ for every $\xi \leq \zeta < \delta$. Suppose we are at a successor stage ξ. If φ_ξ is a sentence of the form 3.3, 3.5, 3.6 or 3.7, we can take $U_{\xi+1} = U_\xi$ as φ_ξ is satisfied by the fact that U_ξ is a κ-complete filter on $\wp(S)$. The only difficulty arises when φ_ξ is a sentence of the form 3.4:

$$\dot{U}(c_X) \vee \dot{U}(c_{S\setminus X})$$

and neither $X \in U_\xi$ nor $S \setminus X \in U_\xi$ already. We claim we can always add either X or $S \setminus X$ to U_ξ and take $U_{\xi+1}$ to be the filter generated by this. Suppose $\langle X_\alpha \colon \alpha < \mu < \kappa\rangle$ is a sequence in U_ξ such that $(\bigcap_{\alpha<\mu} X_\alpha) \cap X = \varnothing$

and $(\bigcap_{\alpha<\mu} X_\alpha) \cap (S \setminus X) = \varnothing$. Then since $S = (S \setminus X) \cup X$, we have

$$\begin{aligned} S \cap (\bigcap_{\alpha<\mu} X_\alpha) &= [(S \setminus X) \cup X] \cap (\bigcap_{\alpha<\mu} X_\alpha) \\ &= \left[(S \setminus X) \cap \bigcap_{\alpha<\mu} X_\alpha\right] \cup \left[X \cap \bigcap_{\alpha<\mu} X_\alpha\right] \\ &= \varnothing \cup \varnothing = \varnothing \end{aligned}$$

But then $\bigcap_{\alpha<\mu} X_\alpha = \varnothing$, a contradiction. So we can either add X or $S \setminus X$ and keep κ-completeness. If both sets are egible, just add the one least in the well-ordering of $\wp(S)$.

To finish the proof, let M be a model of Σ by λ-compactness of $\mathscr{L}_{\kappa\omega}$. Define $U \subseteq \wp(S)$ by

$$X \in U \text{ if and only if } M \models \dot{U}(c_X).$$

It is straight-forward to check that U is a κ-complete ultrafilter on S such that $F \subseteq U$. $\square$

3.3 Fine Measures on $\wp_\kappa(\lambda)$

Recall that a filter F on $\wp_\kappa(S)$ is said to be *fine* if for every $s \in S$, the set

$$\hat{s} := \{X \in \wp_\kappa(S) \colon s \in X\}$$

is an element of F. Fine κ-complete nonprincipal ultrafilters on $\wp_\kappa(S)$ are used together with Łoś' theorem in an ultrapower proof of the compactness theorem for the infinitary language $\mathscr{L}_{\kappa\kappa}$.

Proposition 3.13 (AC)**.** *If there is a fine measure on $\wp_\kappa(\lambda)$, then $\mathscr{L}_{\kappa\kappa}$ is λ-compact.*

Proof. Let Σ be a set of $\mathscr{L}_{\kappa\kappa}$-sentences of cardinality λ such that every subset $\Phi \subseteq \Sigma$ of cardinality less than κ has a model. If there is a κ-complete fine ultrafilter on $\wp_\kappa(\lambda)$, then there also is a κ-complete fine ultrafilter U on $\wp_\kappa(\Sigma)$. For every $\Phi \in \wp_\kappa(\Sigma)$ there is a some $\mathscr{L}_{\kappa\kappa}$-structure M_Φ such that $M_\Phi \models \Phi$. We will use Łoś' theorem 2.29 to prove that the ultraproduct

$$M := \prod_{\Phi \in \wp_\kappa(\Sigma)} M_\Phi / U$$

is a model for Σ. To this end, let $\varphi \in \Sigma$ be an arbitrary $\mathscr{L}_{\kappa\kappa}$-sentence from Σ. By Łoś' theorem 2.29, $M \models \varphi$ if and only if $\{\Phi \in \wp_\kappa(\Sigma) \colon M_\Phi \models \varphi\} \in U$. Since $M_\Phi \models \varphi$ whenever $\varphi \in \Phi$,

$$\{\Phi \in \wp_\kappa(\Sigma) \colon \varphi \in \Phi\} \subseteq \{\Phi \in \wp_\kappa(\Sigma) \colon M_\Phi \models \varphi\}.$$

Since U is fine, $\{\Phi \in \wp_\kappa(\Sigma) \colon \varphi \in \Phi\} \in U$, and therefore

$$\{\Phi \in \wp_\kappa(\Sigma) \colon M_\Phi \models \varphi\} \in U.$$

Hence, $M \models \varphi$ by Łoś' theorem 2.29. Since $\varphi \in \Sigma$ wss arbitrary, $M \models \Sigma$. □

Proposition 3.14. *If κ has the λ-extension property, then there is a fine measure on $\wp_\kappa(\lambda)$.*

Proof. Let F be the κ-complete filter generated by $\{\hat{\alpha} \subseteq \wp_\kappa(\lambda) \colon \alpha < \lambda\}$, and let U be a κ-complete ultrafilter extending F. Clearly, U is fine. Suppose U is principal. Since a principal ultrafilter is generated by a singleton, there is a $y \in \wp_\kappa(\lambda)$ such that U is generated by $\{\{y\}\}$. Hence, for every $\alpha < \lambda$, $\{y\} \subseteq \{x \in \wp_\kappa(\lambda) \colon \alpha \in x\}$, that is, $y \in \{x \in \wp_\kappa(\lambda) \colon \alpha \in x\}$. Hence, for every $\alpha < \lambda$, $\alpha \in y$. But then $y \in \wp_\kappa(\lambda)$ cannot have cardinality $< \kappa$, a contradiction. Therefore, U is a κ-complete nonprincipal fine ultrafilter on $\wp_\kappa(\lambda)$. □

$\mathscr{L}_{\kappa\kappa}$ is compact is not a stronger requirement that the compactness of $\mathscr{L}_{\kappa\omega}$ in ZFC.

Corollary 3.15 (AC)**.** *If $\mathscr{L}_{\kappa\omega}$ is compact, then $\mathscr{L}_{\kappa\kappa}$ is compact.*

Proof. If $\mathscr{L}_{\kappa\omega}$ is compact, then every κ-complete filter can be extended to a κ-complete ultrafilter (Theorem 3.10) and in particular every κ-complete filter generated by at most λ sets for any $\lambda \geq \kappa$. Hence, there is a fine κ-complete nonprincipal ultrafilter on $\wp_\kappa(\lambda)$ and therefore $\mathscr{L}_{\kappa\kappa}$ is λ-compact. Since $\mathscr{L}_{\kappa\kappa}$ is λ-compact for every $\lambda \geq \kappa$, $\mathscr{L}_{\kappa\kappa}$ is compact. □

Since every $\mathscr{L}_{\kappa\omega}$-sentence is clearly an $\mathscr{L}_{\kappa\kappa}$-sentence, $\mathscr{L}_{\kappa\omega}$ is λ-compact whenever $\mathscr{L}_{\kappa\kappa}$ is λ-compact. Therefore, the compactness of $\mathscr{L}_{\kappa\omega}$ and $\mathscr{L}_{\kappa\kappa}$ are ZFC-equivalent.

Corollary 3.16 (AC)**.** *Let κ be a regular infinite cardinal. $\mathscr{L}_{\kappa\omega}$ is compact if and only if $\mathscr{L}_{\kappa\kappa}$ is compact.*

3.4 λ-Covering Elementary Embeddings

We have developed the basics of elementary embeddings of the universe $\mathbf{V}$ in Section 2.4. In this section we consider elementary embeddings with an additional property, and their critical points.

Definition 3.17. Let $\kappa \leq \lambda$ be infinite cardinals. An elementary embedding $j \colon \mathbf{V} \prec M$ with $\operatorname{crit}(j) = \kappa$ is said to be λ-*covering* if for every $X \subseteq M$ such that $|X| \leq \lambda$ in $\mathbf{V}$, there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

We say that an infinite cardinal has the λ-*embedding property* if it is the critical point of a λ-covering elementary embedding.

First we show that if κ has the λ-embedding property, then κ has the λ-extension property. We have already connected the λ-extension property of κ to the existence of fine measures on $\wp_\kappa(\lambda)$ in Section 3.3. With the axiom of choice these measures induce an elementary embedding and we shall show that these are indeed λ-covering.

Theorem 3.18. *If κ has the λ-embedding property, then κ has the λ-extension property.*

Proof. Let $j\colon \mathbf{V} \prec M$ be an elementary embedding with $\operatorname{crit}(j) = \kappa$ such that for every $X \subseteq M$ with $|X| \leq \lambda$ in $\mathbf{V}$, there is a $Y \in M$ with $X \subseteq Y$ and $M \models |Y| < j(\kappa)$. We have to show that every κ-complete filter generated at most λ sets can be extended to a κ-complete ultrafilter.

Suppose F is a κ-complete filter on some set S such that F is generated by $G \subseteq \wp(S)$ with $|G| \leq \lambda$ in $\mathbf{V}$. Since $j\colon \mathbf{V} \prec M$ is λ-covering, there is a $Y \in M$ with $j[G] \subseteq Y$ and $M \models |Y| < j(\kappa)$. Because $j(F)$ is a $j(\kappa)$-complete filter in M, there is a $c \in M$ such that

$$c \in \bigcap (j(F) \cap Y).$$

One can verify that $U := \{X \subseteq S\colon c \in j(X)\}$ is a κ-complete nonprincipal ultrafilter on S using the same arguments as in the proof of Theorem 2.25. Since c is an element of every generator of the κ-complete filter F, c is an element of every set in F. Therefore, U extends F. □

Recall the ultrafilter theorem 3.8 provable in ZFC: every ω-complete filter can be extended to an ω-complete ultrafilter. Hence, the infinite cardinal ω has the global extension property. But since ω is a definable ordinal it cannot be the critical point of an elementary embedding. Therefore, the global extension property for ω does not imply the global embedding property.

If κ has the λ-extension property, then there is a measure on $\wp_\kappa(\lambda)$ by Proposition 3.14. These measures induce λ-covering elementary embeddings.

Theorem 3.19 (AC)**.** *If U is a fine measure on $\wp_\kappa(\lambda)$, then the elementary embedding $j\colon \mathbf{V} \prec M$ induced by U is λ-covering.*

Proof. Suppose $X \subseteq M$ such that $|X| \leq \lambda$ in $\mathbf{V}$. Then there are functions $f_\alpha\colon \wp_\kappa(S) \to \mathbf{V}$ for $\alpha < \lambda$ such that $X = \{[f_\alpha]\colon \alpha < \lambda\}$. Define $F\colon \wp_\kappa(\lambda) \to \mathbf{V}$ by $F(x) := \{f_\alpha(x)\colon \alpha \in x\}$ and let $Y := [F]_U$. We claim that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

To see that $X \subseteq Y = [F]$, let $[f_\xi] \in X$. By Łoś' Theorem 2.29, $M \models [f_\xi] \in [F]$ if and only if $\{x \in \wp_\kappa(\lambda)\colon f_\xi(x) \in F(x)\} \in U$. Since $F(x) = \{f_\alpha(x)\colon \alpha \in x\}$, $f_\xi(x) \in F(x)$ if and only if $\xi \in x$. Therefore,

$$\{x \in \wp_\kappa(\lambda)\colon f_\xi(x) \in F(x)\} = \{x \in \wp_\kappa(\lambda)\colon \xi \in x\} = \hat{\xi}.$$

Since U is fine, $\hat{\xi} \in U$ for every $\xi < \lambda$. Hence, $M \models [f_\xi] \in [F]$. If $[f_\xi] \in [F]$ in M, then $[f_\xi] \in [F]$ in $\mathbf{V}$.

To see that $M \models |Y| < j(\kappa)$, note that since $Y = [F]$ and $j(\kappa) = [c_\kappa]$, $M \models |Y| < j(\kappa)$ if and only if

$$\{x \in \wp_\kappa(\lambda)\colon\ |F(x)| < c_\kappa(x)\} = \{x \in \wp_\kappa(\lambda)\colon\ |F(x)| < \kappa\} \in U.$$

by Łoś' Theorem 2.29. Since for every $x \in \wp_\kappa(\lambda)$,

$$|F(x)| = |\{f_\alpha(x)\colon \alpha \in x\}| \leq |x| < \kappa,$$

we have $\{x \in \wp_\kappa(\lambda)\colon\ |F(x)| < \kappa\} = \wp_\kappa(\lambda)$, which is an element of U. □

Corollary 3.20 (AC)**.** *If U is a fine measure on $\wp_\kappa(\lambda)$, then the critical point of the elementary embedding $j\colon \mathbf{V} \prec M$ induced by U is $\operatorname{crit}(j) = \kappa$.*

Proof. We show that $\kappa < j(\kappa)$ and that $j(\alpha) = \alpha$ for every $\alpha < \kappa$. Since $\kappa \subseteq M$ such that $|\kappa| = \kappa \leq \lambda$ in $\mathbf{V}$, there is a $Y \in M$ such that $\kappa \subseteq Y$ and $M \models |Y| < j(\kappa)$ by Theorem 3.19. As $\kappa \subseteq Y$, $\kappa = |\kappa| \leq |Y| < j(\kappa)$.

Suppose $\alpha < \kappa$ is the least ordinal such that $j(\alpha) > \alpha$. Since $\alpha < \kappa$, $\alpha \in M$. Let $\alpha = [f]_U$. Since $[f]_U = \alpha < j_U(\alpha) = [c_\alpha]_U$, $M \models [f]_U < [c_\alpha]_U$ by elementarity of j_U. By Łoś' Theorem 2.29,

$$\{x \in \wp_\kappa(\lambda)\colon f(x) < c_\alpha(x)\} = \{x \in \wp_\kappa(\lambda)\colon f(x) < \alpha\} \in U.$$

We can write $\{x \in \wp_\kappa(\lambda)\colon f(x) < \alpha\}$ as a disjoint union of α sets:

$$\{x \in \wp_\kappa(\lambda)\colon f(x) < \alpha\} = \bigcup_{\beta<\alpha} \{x \in \wp_\kappa(\lambda)\colon f(x) = \beta\}.$$

By Lemma 2.10, there is a $\beta < \alpha$ such that $\{x \in \wp_\kappa(\lambda)\colon f(x) = \beta\} \in U$. Since

$$\{x \in \wp_\kappa(\lambda)\colon f(x) = \beta\} = \{x \in \wp_\kappa(\lambda)\colon f(x) = c_\beta(x)\} \in U,$$

Łoś' Theorem 2.29 yields $M \models [f]_U = [c_\beta]_U$. As $\alpha < \kappa$ was the least ordinal such that $j_U(\alpha) > \alpha$ and $\beta < \alpha$, $j_U(\beta) = [c_\beta]_U = \beta$. But then

$$\alpha = [f]_U = [c_\beta]_U = j_U(\beta) = \beta,$$

while $\beta < \alpha$, a contradiction. □

We now use a λ-covering elementary embedding with critical point κ to prove that the infinitary language $\mathscr{L}_{\kappa\kappa}$ is λ-compact.

Theorem 3.21. *Let κ be a regular cardinal. If κ has the λ-embedding property, then κ has the λ-compactness property.*

Proof. Let $j\colon \mathbf{V} \prec M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$ such that for every $X \subseteq M$ with $|X| \leq \lambda$ in $\mathbf{V}$, there is a $Y \in M$ with $X \subseteq Y$ and $M \models |Y| < j(\kappa)$. We have to show that every κ-satisfiable set of $\mathscr{L}_{\kappa\kappa}$-sentences of cardinality λ has a model.

Because κ is the critical point of an elementary embedding, there is a measure on κ by Theorem 2.25. Therefore κ is regular by Lemma 2.13. Since κ is regular, every formula of $\mathscr{L}_{\kappa\kappa}$ can be coded by an element of $\mathbf{V}_\kappa$ by Lemma 2.6. Because $\mathbf{V}_\kappa \subseteq M$ by Lemma 2.24, this means that every formula of $\mathscr{L}_{\kappa\kappa}$ can be coded by an element of M.

So if Σ is any κ-satisfiable set of $\mathscr{L}_{\kappa\kappa}$-sentences of cardinality λ, we may assume that $\Sigma \subseteq M$. Because $j\colon \mathbf{V} \prec M$ is λ-covering, there is a $Y \in M$ such that $\Sigma \subseteq Y$ and $M \models |Y| < j(\kappa)$.

Since $\mathbf{V} \models$ 'Σ is κ-satisfiable', we have by elementary of j that

$$M \models \text{‘}j(\Sigma) \text{ is } j(\kappa)\text{-satisfiable'}.$$

Define a subset $S \in M$ such that $x \in S \Leftrightarrow x \in Y \wedge x \in j(\Sigma)$. Then $S \subseteq Y$ and since $M \models |Y| < j(\kappa)$, we have $M \models |S| \leq |Y|$ as $S \subseteq Y$. Hence, there is a model of S in M. Since $j[\Sigma] \subseteq S$ is also a model of $j[\Sigma]$ in M. But a model for $j[\Sigma]$ is like a model for Σ except that the nonlogical symbols are possibly renamed. Therefore, Σ has a model in M and therefore a model in $\mathbf{V}$.

Since this argument shows that every κ-satisfiable set of $\mathscr{L}_{\kappa\kappa}$-sentences of cardinality λ has a model, the infinitary language $\mathscr{L}_{\kappa\kappa}$ is λ-compact. □

It would be nice to have a converse to Theorem 3.21. Using the compactness of the infinitary language $\mathscr{L}_{\kappa\kappa}$, we can try and define an elementary embedding $j\colon \mathbf{V} \prec M$ with critical point $\mathrm{crit}(j) = \kappa$. In order to define an embedding we need a symbol for every element in the domain of the embedding. Consequently, we cannot prove a full converse, but only the existence of set-sized embeddings.

Since the embedding should leave all ordinals less than κ fixed and move κ, we use constant symbols c_α for every $\alpha \leq \kappa$ to ensure $j(\kappa) > \kappa$. Furthermore it is necessary that the binary relation on M is well-founded. In the presence of the Principle of Dependent Choice, well-foundedness is characterized by the nonexistence of countably infinite descending chains (Lemma 2.5). In order to capture this property with an infinitary language, we have really need to use countable quantification. Hence, the assumption that $\mathscr{L}_{\kappa\omega_1}$ is compact, rather than $\mathscr{L}_{\kappa\omega}$. Of course in ZFC this is not really a stronger assumption for uncountable cardinals κ: if $\mathscr{L}_{\kappa\omega}$ is compact, then $\mathscr{L}_{\kappa\kappa} \supseteq \mathscr{L}_{\kappa\omega_1}$ is also compact by Corollary 3.15.

Proposition 3.22 (DC)**.** *Let κ be a regular uncountable cardinal. If $\mathscr{L}_{\kappa\omega_1}$ is strongly compact, then there is transitive set M such that $\mathbf{V}_{\kappa+1} \cap \mathrm{On} \subseteq M$ and a $j\colon \langle \mathbf{V}_{\kappa+1}, \in\rangle \prec \langle M, \in\rangle$ such that $j(\alpha) = \alpha$ for every $\alpha < \kappa$ and $j(\kappa) > \kappa$.*

Proof. Consider the $\mathscr{L}_{\kappa\omega_1}$-language consisting of a binary relation symbol $\dot{\in}$, a constant symbol $\dot{x}$ for every $x \in \mathbf{V}_{\kappa+1}$, and a constant symbol $\dot{c}_\alpha$ for every $\alpha \leq \kappa$. Let Σ be the $\mathscr{L}_{\kappa\omega_1}$-theory of the structure

$$\langle \mathbf{V}_{\kappa+1}, \in, x \rangle_{x \in \mathbf{V}_{\kappa+1}}$$

together with the sentences

$$\dot{c}_\alpha \in \dot{\kappa} \quad \text{for every } \alpha \leq \kappa, \tag{3.8}$$

$$\dot{c}_\alpha \in \dot{c}_\beta \quad \text{for every } \alpha < \beta \leq \kappa. \tag{3.9}$$

We want to show that Σ is κ-satisfiable. It is sufficient to be able to find suitable interpretations in $\langle \mathbf{V}_{\kappa+1}, \in, x \rangle_{x \in \mathbf{V}_{\kappa+1}}$ for every set C of less than κ many constant symbols of the form $\dot{c}_\alpha$. Since κ is regular, $\sup(C) < \kappa$ and therefore the $\dot{c}_\alpha$'s in C can be interpreted by elements of κ.

Because Σ is κ-satisfiable, it has a model by the compactness of $\mathscr{L}_{\kappa\omega_1}$. Hence, there is a structure

$$\langle D, E, \breve{x}, \breve{c}_\alpha \rangle_{x \in \mathbf{V}_\kappa, \alpha \leq \kappa}$$

where E interprets the binary relation $\dot{\in}$, $\breve{x}$ is the interpretation of $\dot{x}$ for every $x \in \mathbf{V}_{\kappa+1}$ and $\breve{c}_\alpha$ is the interpretation of c_α for every $\alpha \leq \kappa$.

The relation E is set-like and extensional, because $\in$ has these properties on $\mathbf{V}_{\kappa+1}$. Since well-foundedness is expressible in $\mathscr{L}_{\kappa\omega_1}$, E is well-founded on M. Taking a transitive collapse by Mostowski's Collapsing Lemma 2.38, Σ has a model of the form

$$\langle M, \in, [x], [c_\alpha] \rangle_{x \in \mathbf{V}_{\kappa+1}, \alpha \leq \kappa}$$

We claim that the map $j \colon \mathbf{V}_{\kappa+1} \to M$ defined by $j(x) := [x]$ is elementary between the structures $\langle \mathbf{V}_{\kappa+1}, \in \rangle$ and $\langle M, \in \rangle$. Suppose $\varphi(v_0, \ldots, v_n)$ is a formula of set theory. We have to show that for every $x_0, \ldots, x_n \in \mathbf{V}_{\kappa+1}$,

$$\langle \mathbf{V}_{\kappa+1}, \in \rangle \models \varphi(x_0, \ldots, x_n) \text{ if and only if } \langle M, \in \rangle \models \varphi([x_0], \ldots, [x_n]).$$

Using the constants $\dot{x}_0, \ldots, \dot{x}_n$, the *formula* $\varphi(x_0, \ldots, x_n)$ corresponds to the *sentence* $\varphi(\dot{x}_0, \ldots, \dot{x}_n)$. If $\langle \mathbf{V}_{\kappa+1}, \in \rangle \models \varphi(x_0, \ldots, x_n)$, then Σ contains the sentence $\varphi(\dot{x}_0, \ldots, \dot{x}_n)$. Since $\langle M, \in, [x], [c_\alpha] \rangle_{x \in \mathbf{V}_{\kappa+1}, \alpha \leq \kappa}$ is a model for Σ, it is in particular a model for $\varphi(\dot{x}_0, \ldots, \dot{x}_n)$. Hence, $M \models \varphi([x_0], \ldots, [x_n])$. Conversely, if $M \models \varphi([x_0], \ldots, [x_n])$, then we must have that $\mathbf{V}_{\kappa+1} \models \varphi(x_0, \ldots, x_n)$. For suppose not, then Σ contains the sentence $\neg\varphi(\dot{x}_0, \ldots, \dot{x}_n)$ and therefore $M \models \neg\varphi([x_0], \ldots, [x_n])$, a contradiction.

To finish the proof we have to show that every ordinal $< \kappa$ is fixed and κ is moved. For every $\alpha < \kappa$, the $\mathscr{L}_{\kappa\omega_1}$-sentence

$$\forall x \, [x \in \dot{\alpha} \Leftrightarrow \bigvee_{\beta < \alpha} (x = \dot{\beta})]$$

holds in $\langle \mathbf{V}_{\kappa+1}, \in, x\rangle_{x\in\mathbf{V}_{\kappa+1}}$. Therefore, Σ contains this $\mathscr{L}_{\kappa\omega_1}$-sentence and

$$\langle M, \in, [x], [c_\alpha]\rangle_{x\in\mathbf{V}_\kappa, \alpha\leq\kappa}$$

is a model for it. But then by induction on α, $j(\alpha) = \alpha$ for every $\alpha < \kappa$. In other words, every ordinal $< \kappa$ is fixed by j. To see that κ is moved, note that $j(\kappa) = [\kappa]$ is the interpretation of $\dot{\kappa}$. The sentences of the form (3.8) ensure $[c_\alpha] \in [\kappa]$ for every $\alpha \leq \kappa$. Hence, $\{[c_\alpha] \colon \alpha \leq \kappa\}$ is a subset of $j(\kappa)$ of order type $\kappa + 1 > \kappa$. Therefore, $\kappa < j(\kappa)$. ☐

3.5 Strength of Infinitary Language Compactness

The purpose of this section is to gauge in ZFC the consistency strength of some forms of compactness of infinitary language $\mathscr{L}_{\kappa\kappa}$. For this analysis we will introduce a compactness property of $\mathscr{L}_{\kappa\kappa}$ which is ZFC-equivalent to measurability, which Chang and Keisler named 'medium compactness' [CK77, p. 198].

Definition 3.23. Let $\mathscr{L}_\kappa$ denote either $\mathscr{L}_{\kappa\omega}$ or $\mathscr{L}_{\kappa\kappa}$. The language $\mathscr{L}_\kappa$ is said to be *medium compact* if the following property holds: If $\langle \Sigma_\alpha \colon \alpha < \kappa\rangle$ is a sequence of sets of $\mathscr{L}_\kappa$-sentences such that for every $\beta < \kappa$ the union $\bigcup_{\alpha<\beta} \Sigma_\alpha$ has a model, then $\bigcup_{\alpha<\kappa} \Sigma_\alpha$ has a model.

We first show how to use a measure on κ in an ultraproduct proof of the medium compactness of $\mathscr{L}_{\kappa\kappa}$ similar in spirit as the proof of Proposition 3.13. Using Łoś' theorem 2.29, this proof needs the axiom of choice.

Proposition 3.24 (AC)**.** *If there is a measure on κ, then the infinitary language $\mathscr{L}_{\kappa\kappa}$ is medium compact.*

Proof. Let U be a measure on κ. Suppose $\langle \Sigma_\alpha \colon \alpha < \kappa\rangle$ is a sequence of sets of $\mathscr{L}_{\kappa\kappa}$-sentences such that for every $\beta < \kappa$ the union $\bigcup_{\alpha<\beta} \Sigma_\alpha$ has a model, say M_β. We will show that

$$\prod_{\beta<\kappa} M_\beta/U \models \bigcup_{\alpha<\kappa} \Sigma_\alpha.$$

Let $\varphi \in \bigcup_{\alpha<\kappa} \Sigma_\alpha$. Then there is a $\gamma < \kappa$ such that $\varphi \in \Sigma_\gamma$. Hence, for every $\beta \geq \gamma$ we have that $\varphi \in \bigcup_{\alpha<\beta} \Sigma_\alpha$ and therefore $M_\beta \models \varphi$. Thus,

$$\{\beta < \kappa \colon \beta \geq \gamma\} \subseteq \{\beta < \kappa \colon M_\beta \models \varphi\}.$$

Since U contains all end-segments of κ by Lemma 2.14, $\{\beta < \kappa \colon \beta \geq \gamma\} \in U$ and therefore $\{\beta < \kappa \colon M_\beta \models \varphi\} \in U$. Hence, by Łoś' theorem 2.29,

$$\prod_{\beta<\kappa} M_\beta/U \models \varphi.$$

Because $\varphi \in \bigcup_{\alpha<\kappa} \Sigma_\alpha$ was arbitrary,

$$\prod_{\beta<\kappa} M_\beta/U \models \bigcup_{\alpha<\kappa} \Sigma_\alpha.$$

$\square$

The converse of the previous proposition is derivable in ZF via a curtailed extension property. Compare this property with the one in Theorem 3.27.

Proposition 3.25. *If $\mathscr{L}_{\kappa\omega}$ is medium compact, then every κ-complete filter on κ generated by at most κ sets can be extended to a κ-complete ultrafilter.*

Proof. Let F be a κ-complete filter on κ such that F is generated by at most κ sets. Let $\{G_\alpha \subseteq \kappa \colon \alpha < \kappa\}$ be a family of generators for F.

Consider the $\mathscr{L}_{\kappa\omega}$-language which consists of a unary predicate $\dot{X}$ for every $X \subseteq \kappa$ and a constant symbol $\dot{c}$. For every $\gamma < \kappa$ let Σ_γ be the $\mathscr{L}_{\kappa\omega}$-theory of the structure $\langle \kappa, X \rangle_{X\subseteq\kappa}$ together with the sentence $\dot{G}_\alpha(\dot{c})$ for every $\alpha < \kappa$.

We check that for every $\beta < \kappa$, the union $\bigcup_{\alpha<\beta} \Sigma_\alpha$ has a model. Since $\bigcup_{\alpha<\beta} \Sigma_\alpha = \Sigma_\beta$ for every $\beta < \kappa$, it is sufficient to find a model for Σ_β for every $\beta < \kappa$. For every $\alpha < \beta$ the sentence $\dot{G}_\alpha(\dot{c})$ is in the set Σ_β. Since F is κ-complete, there is an element $c \in \kappa$ such that $c \in \bigcap_{\alpha<\beta} G_\alpha$. Hence, $\langle \kappa, X, c \rangle_{X\subseteq\kappa}$ is a model for Σ_β.

Let M be a model for $\bigcup_{\alpha<\kappa} \Sigma_\alpha$ by medium compactness. Define

$$U := \{X \subseteq \kappa \colon M \models \dot{X}(\dot{c})\}.$$

Then U is a κ-complete ultrafilter on κ extending F, which may be verified by similar arguments as in the proof of Theorem 3.10. $\square$

Corollary 3.26 (AC)**.** *If every κ-complete filter on κ generated by at most κ sets can be extended to a κ-complete ultrafilter, then there is a fine measure on κ.*

Proof. The collection $\{\hat{\alpha} \subseteq \wp(\kappa) \colon \alpha < \kappa\}$ generates a κ-complete filter F on κ. Any κ-complete ultrafilter U extending F is clearly fine. Furthermore, U has to be nonprincipal by Lemma 2.16. $\square$

Combining Proposition 3.24 and Corollary 3.26 we see that in ZFC there is a fine measure on κ if and only if $\mathscr{L}_{\kappa\kappa}$ is medium compact. Furthermore, if κ is measurable then $\mathscr{L}_{\kappa\kappa}$ is weakly compact [Ka03, p. 38] and κ is in fact the κth cardinal with this property [Ka03, p. 55]. Hence, the consistency strength of

$$\mathsf{ZFC} + \text{‘there is a } \kappa \text{ such that } \mathscr{L}_{\kappa\kappa} \text{ is medium compact’}$$

is strictly greater than that of

$$\mathsf{ZFC} + \text{‘there is a } \kappa \text{ such that } \mathscr{L}_{\kappa\kappa} \text{ is weakly compact’.}$$

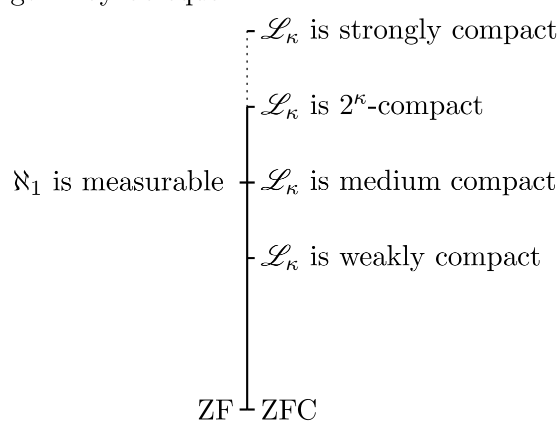
If $\mathscr{L}_{\kappa\kappa}$ is 2^κ-compact every κ-complete filter generated by at most 2^κ sets can be extended to a κ-complete ultrafilter by Proposition 3.12. Since every filter on κ is generated by at most 2^κ many sets, it follows that if $\mathscr{L}_{\kappa\kappa}$ is 2^κ-compact, then every κ-complete filter on κ can be extended to a κ-complete ultrafilter. The existence of uncountable cardinals with this filter extension property has a fairly high consistency strength.

Theorem 3.27 (AC)**.** *(Kunen, 1971) If there is an uncountable cardinal κ such that every κ-complete filter on κ can be extended to a κ-complete ultrafilter, then for every λ there is an inner model of* ZFC *with λ measurable cardinals.*

Proof. See for example [Je78, p. 401–405]. □

In ZFC we have that if κ is measurable, then $\mathscr{L}_{\kappa\kappa}$ is medium compact. Hence, the consistency strength of ZFC + ‘there is a κ such that $\mathscr{L}_{\kappa\kappa}$ is 2^κ-compact’ is strictly greater consistency strength than that of ZFC + ‘there is a κ such that $\mathscr{L}_{\kappa\kappa}$ is medium compact’.

The consistency strength of various forms is gauged in the diagram below. The consistency strength increases from the bottom upwards. A solid line indicates the consistency strength is known to be strictly larger, a dotted line that the consistency strength may be equal.



We quickly summarize the arguments behind this diagram. Note that we reason in ZFC from the bottom to the top. Measurability of κ is equivalent to medium compactness of $\mathscr{L}_{\kappa\kappa}$ (Proposition 3.24 and Corollary 3.26). A measurable cardinal κ is the κth weakly compact cardinal [Ka03, p. 55]. If $\mathscr{L}_{\kappa\kappa}$ is 2^κ-compact, then every κ-complete filter on κ can be extended to a κ-complete ultrafilter. Therefore, the 2^κ-compactness of $\mathscr{L}_{\kappa\kappa}$ implies the consistency of inner models with many measurables (Theorem 3.27). Finally, if $\mathscr{L}_{\kappa\kappa}$ is strongly compact then it is obviously 2^κ-compact. However, we do not know if the consistency strength of the former is strictly greater than that of the latter.

Chapter 4

Determinacy

Mycielski and Steinhaus introduced the axiom of determinacy (AD) in 1962. Solovay used AD in 1968 to prove that there is a measure on ω_1. Since then numerous results have shown that under AD 'small' cardinals have properties normally ascribed to 'large' cardinals.

In this chapter we will look at connections between AD and some of the different notions of compactness from the previous chapter. Our focus will be mainly on ω_1. Since this is a successor cardinal, it cannot have the λ-covering embedding property or the λ-compactness property for any $\lambda \geq \omega_1$. Hence, we will concentrate on the fine measure property and the extension property.

After the preliminary definitions of infinite games and strategies in Section 4.1 we consider choice under AD. While AD contradicts AC, DC is consistent with AD. This leads to a limitive result on the consequences AD can have regarding fine measures. Basic results on filters under AD yield the failure of the ultrafilter theorem and hence the compactness theorem under AD.

We consider the extension of ω_1-complete filters in Section 4.4 and the fine measure property of ω_1 in Section 4.5. Finally, we look at the consistency strength of the axiom 'there is a κ with a fine measure on $\wp_\kappa(\kappa^+)$ in Section 4.6.

4.1 Infinite Games and Strategies

Let X be a nonempty set and let $A \subseteq {}^\omega X$. We define the infinite game $G_X(A)$ of perfect information on X with pay-off set A as follows: The game is played by two players, denoted by **I** and **II**. Player **I** begins the game by choosing $x_0 \in X$, then player **II** chooses $x_1 \in X$; player **I** then chooses $x_2 \in X$, and player **II** chooses $x_3 \in X$, and so on. The resulting sequence $x = \langle x_i : i \in \omega \rangle \in {}^\omega X$ is called a *play* of the game:

$$\begin{array}{llllll} \mathbf{I} & x_0 & & x_2 & & \dots \\ \mathbf{II} & & x_1 & & x_3 & \dots \end{array}$$

Player **I** wins the game $G_X(A)$ if and only if $x \in A$. For $x \in {}^{\omega}X$, we define $x_{\mathbf{I}}, x_{\mathbf{II}} \in {}^{\omega}X$ by $x_{\mathbf{I}}(i) := x(2i)$ and $x_{\mathbf{II}}(i) := x(2i+1)$. Thus, the moves of player **I** are enumerated by the sequence $x_{\mathbf{I}}$, and those of player **II** by $x_{\mathbf{II}}$:

$$\begin{array}{llllllll} \mathbf{I} & x_{\mathbf{I}}(0) & & x_{\mathbf{I}}(1) & & \dots & x_{\mathbf{I}} \\ \mathbf{II} & & x_{\mathbf{II}}(0) & & x_{\mathbf{II}}(1) & \dots & x_{\mathbf{II}} \end{array}$$

A *strategy for player* **I** *in games on* X is a function

$$\sigma\colon \bigcup_{i\in\omega} {}^{2i}X \to X.$$

Similarly, a *strategy for player* **II** *in games on* X is a function

$$\tau\colon \bigcup_{i\in\omega} {}^{2i+1}X \to X.$$

Lemma 4.1. *Every strategy for player* ***I (II)*** *in games on* ω *can be coded by a real.*

Proof. Let $\sigma\colon \bigcup_{i\in\omega} {}^{2i}\omega \to \omega$ be a strategy for player **I**. We can extend σ to all finite sequences in ω by defining $\sigma(s) := 0$ for every finite sequence $s \in \bigcup_{i\in\omega} {}^{2i+1}\omega$. Let $f\colon \omega \to {}^{<\omega}\omega$ be a surjection and let $g\colon {}^{<\omega}\omega \to \omega$ be an injection such that their composition $f \circ g$ is equal to the identity map id_ω on ω.

The surjection $f\colon \omega \to {}^{<\omega}\omega$ can be used to code the strategy σ into a real: define $\hat{\sigma}\colon \omega \to \omega$ by

$$\hat{\sigma}(i) := \sigma(f(i))$$

for every $i \in \omega$. We can use the injection $g\colon {}^{<\omega}\omega \to \omega$ to decode the strategy σ from the real $\hat{\sigma}\colon \omega \to \omega$. If $s \in \mathbb{R}$, then

$$\hat{\sigma}(g(s)) = \sigma(f(g(s)) = \sigma(s).$$

In a similar way, a strategy τ for player **II** can be coded by a real $\hat{\tau}$ using f, and decoded again using g. $\square$

Let σ be a stategy for player **I** for games on X. A play according to σ is a play of the form

$$\begin{array}{llllll} \mathbf{I} & \sigma(\varnothing) & & \sigma(\langle\sigma(\varnothing), y_0\rangle) & & \dots \\ \mathbf{II} & & y_0 & & y_1 & \dots \end{array}$$

When the moves of player **II** are enumerated by $y = \langle y_i \colon i \in \omega\rangle \in {}^{\omega}X$, we denote this play by $\sigma * y$. A strategy σ is a *winning strategy for player* **I** if for every $y \in {}^{\omega}X$, $\sigma * y \in A$.

Similarly, if τ is a strategy for player **II**, then a play according to τ is a play of the form

$$\begin{array}{llllll} \mathbf{I} & x_0 & & x_2 & & \dots \\ \mathbf{II} & & \tau(\langle x_0\rangle & & \tau(\langle x_0, \tau(\langle x_0\rangle), x_2\rangle) & \dots \end{array}$$

When the moves of player **I** are enumerated by $x = \langle x_i \colon i \in \omega\rangle \in {}^\omega X$, we denote this play by $x * \tau$. A strategy τ is a *winning strategy for player* **II** if for every $x \in {}^\omega X$, $x * \tau \notin A$.

We say that a player *wins* the game $G_X(A)$ if he has a winning strategy. A game $G_X(A)$ is *determined* if one of the players has a winning strategy. Note that player **I** and **II** cannot both have a winning strategy. A set of reals $A \subseteq \mathbb{R}$ is said to be determined if the game $G_\omega(A)$ is determined. Mycielski and Steinhaus (1962) proposed the following axiom of determinacy (AD):

Every subset of $\mathbb{R}$ is determined.

4.2 Determinacy and Choice

Nine years before Mycielski and Steinhaus proposed the axiom of determinacy, Gale and Stewart [GS53] had proven the determinacy of open and closed subsets of the reals. They had also used a well-ordering of the reals to construct a game which is not determined.

Theorem 4.2 ($\mathsf{AC}(\mathbb{R})$)**.** *(Gale–Stewart, 1953) There is an $A \subseteq \mathbb{R}$ such that $G_\omega(A)$ is not determined.*

Proof. Every strategy for a player can be coded by a real by Lemma 4.1. By $\mathsf{AC}(\mathbb{R})$, the set of strategies for a player can be well-ordered. Therefore, there are $|\mathbb{R}| = 2^\omega$ strategies. Let $\langle \sigma_\alpha \colon \alpha < 2^\omega\rangle$ and $\langle \tau_\alpha \colon \alpha < 2^\omega\rangle$ enumerate all of the strategies for player **I** and for player **II**, respectively. We will construct disjoint sets $A, B \subseteq \mathbb{R}$ such that

(i). For every strategy τ for player **II**, there is an $x \in \mathbb{R}$ such that $x * \tau \in A$, and

(ii). For every strategy σ for player **I**, there is an $y \in \mathbb{R}$ such that $\sigma * y \in B \subseteq \mathbb{R} \setminus A$.

Neither player **I** nor player **II** can have a winning strategy for the game $G_\omega(A)$: Player **I** cannot have a winning strategy $\sigma = \sigma_\alpha$ for some $\alpha < 2^\omega$, since there is a $y_\alpha \in \mathbb{R}$ such that $\sigma_\alpha * y_\alpha \in B \nsubseteq A$. Similarly, player **II** cannot have a winning strategy $\tau = \tau_\alpha$ for some $\alpha < 2^\omega$, since there is an $x_\alpha \in \mathbb{R}$ such that $x_\alpha * \tau_\alpha \in A$.

For the construction of A and B, we recursively choose $a_\xi, b_\xi \in \mathbb{R}$ for every $\xi < 2^\omega$ as follows. Assume that at stage ξ, a_α and b_α have been chosen for every $\alpha < \xi$. Choose b_ξ so that $b_\xi = \sigma_\xi * y$ for some $y \in \mathbb{R}$, but ensure $b_\xi \notin \{a_\alpha \colon \alpha < \xi\}$. This is possible, because the cardinality of $\{\sigma_\xi * y \colon y \in \mathbb{R}\}$ is

2^ω since the function $y \mapsto \sigma_\alpha * y$ is injective for every $\alpha < 2^\omega$. Similarly, choose a_ξ so that $a_\xi = x * \tau_\xi$, yet $a_\xi \notin \{b_\alpha \colon \alpha < \xi\}$. The resulting sets $A := \{a_\alpha \colon \alpha < 2^\omega\}$ and $B := \{b_\alpha \colon \alpha < 2^\omega\}$ are disjoint by construction. □

Although AD contradicts the axiom of choice, the weaker countable axiom of choice for nonempty sets of reals $\mathsf{AC}_\omega(\mathbb{R})$ is a consequence of AD.

Proposition 4.3 (AD)**.** *(Swierczkowski, Mycielski, Scott) Every countable family of nonempty sets of reals has a choice function.*

Proof. Let $\{X_i \colon i \in \omega\}$ be a countable family of nonempty sets of reals. Define $A \subseteq \mathbb{R}$ by

$$x \in A \text{ if and only if } x_{\mathbf{II}} \notin X_{x(0)}.$$

Clearly, player **I** cannot have a winning strategy for the game $G(A)$. Hence, player **II** must have some winning strategy τ by AD. Define a function $f \colon \omega \to \mathbb{R}$ by

$$f(i) := (\langle i, 0, 0, \ldots \rangle * \tau)_{\mathbf{II}}.$$

Since τ is a winning strategy for player **II**, $f(i) \in X_i$ for every $i \in \omega$. In other words, f is a choice function for the countable family $\{X_i \colon i \in \omega\}$. □

A consequence of $\mathsf{AC}_\omega(\mathbb{R})$ is the regularity of ω_1 (Proposition 2.3). More choice is possible under AD. Kechris [Ke84] provided an elegant model for $\mathsf{AD} + \mathsf{DC}$, while Solovay [So78] demonstrated the independence of DC from AD.

Theorem 4.4 (AD)**.** *(Kechris, 1984)* $\mathbf{L}(\mathbb{R})$ *is a model of* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$. □

We mention some results in order to examine the question: can $\mathsf{ZF} + \mathsf{AD}$ prove that there is a cardinal κ such that for every $\lambda \geq \kappa$ there is a fine measure on $\wp_\kappa(\lambda)$? We know that such cardinals are strongly inaccessible in ZFC, and therefore ZFC cannot prove the existence of these cardinal. In 1966, Vopěnka and Hrbáček established the transcendence of a these cardinals over $\mathbf{L}(x)$ for any set x in ZFC, using the axiom of choice in an ultrapower construction, see for example [Ka03, p. 51]. Spector [Sp91] improved upon this result using his technique of extended ultrapowers, which needs only DC.

Theorem 4.5 (DC)**.** *(Spector, 1991 [Sp91]) If there is a cardinal κ such that for every $\lambda \geq \kappa$ there is a fine measure on $\wp_\kappa(\lambda)$, then $\mathbf{V} \neq \mathbf{L}(x)$ for any set x.* □

Once the result of Vopěnka and Hrbáček is available in the context of DC, one can easily derive a limit on the consequences of AD regarding strong compactness.

Corollary 4.6. *The existence of a cardinal κ such that for every $\lambda \geq \kappa$ there is a fine measure on $\wp_\kappa(\lambda)$ cannot be a consequence of* AD.

Proof. By Kechris' Theorem 4.4, $\mathbf{L}(\mathbb{R})$ is a model of $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$. Since $\mathbf{V} = \mathbf{L}(\mathbb{R})$ in this model, there cannot be a strongly compact cardinal by Spector's Theorem 4.5. Therefore, $\mathbf{L}(\mathbb{R})$ is a model of $\mathsf{ZF}+\mathsf{AD}+$ 'there is no strongly compact cardinal'. But then the existence of a strongly compact cardinal cannot be a consequence of AD. ☐

4.3 Ultrafilters and Compactness

As a consequence of the axiom of determinacy, there are no nonprincipal ultrafilter on ω.

Proposition 4.7. *If there exists a nonprincipal ultrafilter on ω, then there is an $A \subseteq \mathbb{R}$ such that the game $G_\omega(A)$ is not determined.*

Proof. Let U be a nonprincipal ultrafilter on ω. Consider a game $G(U)$ in which the players choose finite sequences of natural numbers, instead of natural numbers as in games of the form $G_\omega(A)$. Suppose that $\langle s_i \colon i \in \omega\rangle \in {}^{\omega}({}^{<\omega}\omega)$ is a play of this game $G(U)$:

$$\begin{array}{llllll} \mathbf{I} & s_0 & & s_2 & & \ldots \\ \mathbf{II} & & s_1 & & s_3 & \ldots \end{array}$$

We stipulate that if there is an $n \in \omega$ such that

$$(s_n \cap \bigcup_{i<n} s_i) \neq \varnothing,$$

the player first to make such a move loses. Otherwise, player $\mathbf{I}$ wins if and only if

$$\bigcup_{i\in\omega} s_{2i} \in U.$$

Since the set ${}^{<\omega}\omega$ of all finite sequences of natural numbers is countable, there is some $A \subseteq \mathbb{R}$ such that the game $G(U)$ is equivalent to the game $G_\omega(A)$. If $G_\omega(A)$ is determined, then so is $G(U)$. We will prove that $G(U)$ cannot be determined if U is an ultrafilter. We will do this by transforming a winning strategy for one player into a winning strategy for his opponent.

Case 1. Suppose σ is a winning strategy for player $\mathbf{I}$ in the game $G(U)$. Let τ_σ be the strategy for player $\mathbf{II}$ defined by $\tau_\sigma(\langle s_0\rangle) := \sigma(\varnothing) \setminus s_0$, and for $i > 0$ by

$$\tau_\sigma(\langle s_0, \ldots, s_{2i}\rangle) := \sigma(\langle s_1, \ldots, s_{2i}\rangle) \setminus s_0.$$

A play $s = \langle s_i \colon i \in \omega\rangle$ according to τ_σ is of the form

$$\begin{array}{llllll} \mathbf{I} & s_0 & & s_2 & & \ldots \\ \mathbf{II} & & s_1 := \sigma(\varnothing) \setminus s_0 & & s_3 := \sigma(\langle s_1, s_2\rangle) \setminus s_0 & \ldots \end{array}$$

Since σ is a winning strategy for player **I**,

$$s_0 \cup \bigcup_{i\in\omega} s_{2i+1} \in U.$$

As U is nonprincipal, we have that $\bigcup_{i\in\omega} s_{2i+1} \in U$ and since U is an ultrafilter, this means that

$$\omega \setminus \bigcup_{i\in\omega} s_{2i+1} \notin U$$

Since the moves are disjoint, for every $n \in \omega$, $s_{\mathbf{I}}(n) \subseteq \omega \setminus \bigcup_{i\in\omega} s_{2i+1}$. Therefore, $\bigcup s_{\mathbf{I}} = \bigcup_{i\in\omega} s_{2i} \notin U$ and hence τ_σ is a winning strategy for player **II**.

Case 2. Suppose τ is a winning strategy for player **II**. First, we modify τ to the strategy $\bar{\tau}$ for player **II** defined by

$$\bar{\tau}(\langle s_0, \ldots, s_{2i}\rangle) := \begin{cases} \tau(\langle s_0, \ldots, s_{2i}\rangle) \cup \{i\} & \text{if } i \notin \bigcup_{j\leq 2i} s_j, \text{ and} \\ \tau(\langle s_0, \ldots, s_{2i}\rangle) & \text{otherwise.} \end{cases}$$

Then $\bar{\tau}$ is a winning strategy for player **II** because τ is a winning strategy, and for every $s = \langle s_i \colon i \in \omega\rangle$,

$$\bigcup_{i\in\omega} (s_{\mathbf{I}} * \bar{\tau})_i = \omega.$$

When $s = s_{\mathbf{I}} * \bar{\tau}$ is a play of the game where the moves of player **I** are disjoint,

$$\bigcup_{i\in\omega} s_{2i} \notin U$$

because $\bar{\tau}$ is a winning strategy for player **II**. Since U is an ultrafilter,

$$\bigcup_{i\in\omega} s_{2i+1} = \omega \setminus \bigcup_{i\in\omega} s_{2i} \in U.$$

Define a strategy σ_τ for player **I** by

$$\sigma_\tau(\langle s_0, \ldots, s_{2i-1}\rangle) := \bar{\tau}(\langle \varnothing, s_0, \ldots, s_{2i-1}\rangle).$$

If $s := \sigma_\tau * y = \langle s_i \colon i \in \omega\rangle$ is a play according to σ_τ, then

$$s' := \langle 0, \varnothing\rangle \cup \bigcup_{i\in\omega} \{\langle i+1, s_i\rangle\}$$

is a play according to τ. Since τ is a winning strategy for player **II**,

$$\bigcup_{i\in\omega} s_{2i} = \bigcup_{i\in\omega} s'_{2i+1} \in U,$$

and therefore σ_τ is a winning strategy for **I**. $\square$

Consider the Fréchet filter of all cofinite subsets of ω, defined by

$$F := \{X \subseteq \omega \colon |\omega \setminus X| < \omega\}.$$

Since $\omega \setminus \{n\} \in F$ for every $n \in \omega$, any ultrafilter extending it has to be nonprincipal. Yet in $\mathsf{ZF} + \mathsf{AD}$ there are no nonprincipal ultrafilters on ω by Proposition 4.7 and therefore the Fréchet filter on ω cannot be extended to an ultrafilter. The ultrafilter theorem does not hold in $\mathsf{ZF} + \mathsf{AD}$. As the the compactness theorem for first–order logic implies the ultrafilter theorem, $\mathscr{L}_{\omega\omega}$ is not compact in $\mathsf{ZF} + \mathsf{AD}$.

One way to get a nonprincipal ultrafilter on ω is to construct one using an ultrafilter which is not ω_1-complete.

Proposition 4.8. *If there exists an ultrafilter which is not ω_1-complete, then there exists a nonprincipal ultrafilter on ω.*

Proof. Let U be an ultrafilter on some set S such that U is not ω_1-complete. Then there is a countable sequence $\langle X_i \colon i \in \omega\rangle$ in U such that the intersection

$$\bigcap_{i\in\omega} X_i \notin U.$$

Let $f \colon S \to \omega$ be the function such that $f(x) := 0$ if $x \notin \bigcup_{i\in\omega}(S \setminus X_i)$, and

$$f(x) := \min\{n+1 \colon x \in (S \setminus X_n) \text{ and } x \notin \bigcup_{i<n}(S \setminus X_i)\}$$

if $x \in \bigcup_{i\in\omega}(S \setminus X_i)$. The pushout $f_*(U) := \{X \subseteq \omega \colon f^{-1}[X] \in U\}$ is an ultrafilter on ω by Lemma 2.17. We still have to show that it is nonprincipal. Suppose that $f_*(U)$ is principal. Since a principal ultrafilter is generated by a singleton (Lemma 2.9), there is an $n \in \omega$ such that $\{n\} \in f_*(U)$, that is, $f^{-1}[\{n\}] = f^{-1}(n) \in U$. But

$$\begin{aligned} f^{-1}(0) &= \{x \in S \colon x \notin \bigcup_{i\in\omega}(S \setminus X_i)\} \\ &= \{x \in S \colon x \in S \setminus \bigcup_{i\in\omega}(S \setminus X_i)\} \\ &= \{x \in S \colon x \in \bigcap_{i\in\omega} X_i\} = \bigcap_{i\in\omega} X_i \notin U, \end{aligned}$$

and for $n > 0$, then

$$f^{-1}(n) \subseteq S \setminus X_n \notin U.$$

Therefore, $f_*(U)$ has to be a nonprincipal ultrafilter on ω. $\square$

Corollary 4.9 (AD). *Every ultrafilter is ω_1-complete.*

Proof. Suppose there exists an ultrafilter which is not ω_1-complete. Then there is a nonprincipal ultrafilter on ω by Proposition 4.8, and therefore a nondetermined game of the form $G_\omega(A)$ by Proposition 4.7. Of course, every game of the form $G_\omega(A)$ is determined by the axiom of determinacy. □

4.4 Extension of Filters

The axiom of determinacy implies three well-known regularity properties for sets of reals: every set of reals is Lebesgue measurable, has the Baire property, and the perfect set property (Mycielski–Swierczkowski, 1964; Mazur, Banach; Davis, 1964). 'This was the main incentive behind the formulation of AD.' [Ka03, p. 377].

Bernstein's early analysis (1908) of the connection between the perfect set property and the Continuum Problem had revealed that if $\omega_1 \leq 2^{\aleph_0}$, then there is a set of reals without the perfect set property. Mycielski (1964) therefore concluded that $\omega_1 \not\leq 2^{\aleph_0}$ under AD: there is no uncountable well-orderable set of reals. In particular, $\wp(\omega)$ is not well-orderable. As a consequence, no infinite powerset is wellorderable.

Proposition 4.10. *If there is an infinite powerset which is wellorderable, then $\wp(\omega)$ is wellorderable.*

Proof. Let S be a set such that $\wp(S)$ is infinite and wellorderable. Clearly, S cannot be finite since the powerset of a finite set is finite. Hence, S has to be infinite, which means that there is an injection $\omega \to S$. This injection lifts to an injection $\wp(\omega) \to \wp(S)$. Since $\wp(S)$ is wellorderable by assumption, $\wp(\omega) \subseteq \wp(S)$ inherits this wellorder. □

This bears directly on the restricted λ-extension property for infinite cardinals under AD: since there is no set S such that $|\wp(S)| = \lambda$ for some $\lambda \geq \omega$, there are also no filters on such sets. Trivially, every filter on such a set can be extended to an ultrafilter. Therefore, the axiom of determinacy implies that every infinite cardinal has the restricted λ-extension property for every $\lambda \geq \omega$. In other words, without the axiom of choice the restricted extension property loses its strength.

Donald Martin (1968) defined a filter on the Turing degrees and proved that under the axiom of determinacy this filter is an ultrafilter. We present Martin's construction in the slightly more general setting of an abstract equivalence relation on $\mathbb{R}$ satisfying certain properties. The reader familiar with the basics of computability theory may verify that Turing reducibility is such a relation.

Let $\preccurlyeq$ be a reflexive and transitive relation on $\mathbb{R}$. The binary relation $\equiv$ on $\mathbb{R}$ defined by

$$x \equiv y \text{ if and only if } x \preccurlyeq y \text{ and } y \preccurlyeq x$$

is easily seen to be an equivalence relation. We call the equivalence classes under $\equiv$ *degrees*, and write

$$[x]_{\equiv} := \{y \in \mathbb{R} \colon x \equiv y\}$$

for the degree of $x \in \mathbb{R}$. The quotient set of all degrees is denoted by

$$\mathbf{D}_{\equiv} := \{[x]_{\equiv} \colon x \in \mathbb{R}\} = \mathbb{R}/{\equiv}\ .$$

Note that $\preccurlyeq$ lifts to an order on the degrees by defining for every $\mathbf{d}, \mathbf{e} \in \mathbf{D}_{\equiv}$

$$\mathbf{d} \preccurlyeq \mathbf{e} \text{ if and only if there are } x \in \mathbf{d}, y \in \mathbf{e} \text{ such that } x \preccurlyeq y.$$

The *cone over* a degree $\mathbf{d} \in \mathbf{D}$ with respect to $\preccurlyeq$ is the set

$$C(\mathbf{d}) := \{\mathbf{e} \in \mathbf{D} \colon \mathbf{d} \preccurlyeq \mathbf{e}\}.$$

Define $\mathbf{M} \subseteq \wp(\mathbf{D})$ by

$$X \in \mathbf{M} \text{ if and only if } (\exists \mathbf{d} \in \mathbf{D})\,(C(\mathbf{d}) \subseteq X).$$

Since $\preccurlyeq$ is reflexive, for every $\mathbf{d} \in \mathbf{D}$, we have that $\mathbf{d} \in C(\mathbf{d}) \neq \varnothing$. Hence, no cone is a subset of the empty set and therefore $\varnothing \notin \mathbf{M}$. Furthermore, $\mathbf{D}$ contains every cone and therefore $\mathbf{D} \in \mathbf{M}$. If $X \subseteq Y \subseteq \mathbf{D}$, then if X contains a cone, so does Y. Hence, if $X \in \mathbf{M}$, then $Y \in \mathbf{M}$. So, $\mathbf{M}$ is proper, contains $\mathbf{D}$, and is closed under taking supersets. We need an additional property for $\preccurlyeq$ to conclude that M is closed under taking intersections.

We say that $\preccurlyeq$ on $\mathbb{R}$ is *upwards closed* if for every $x, y \in \mathbb{R}$ there is an $z \in \mathbb{R}$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$.

Lemma 4.11. *Suppose that $\preccurlyeq$ is upwards closed. Then $\mathbf{M}$ is a filter on $\mathbf{D}$.*

Proof. The preceding remarks show that $\mathbf{M}$ is proper and closed under taking supersets. We have to show that if $X, Y \in \mathbf{M}$, then $X \cap Y \in \mathbf{M}$. Suppose $X \in \mathbf{M}$ and $Y \in \mathbf{M}$. Then there are $x, y \in \mathbb{R}$ such that $C([x]_{\equiv}) \subseteq X$ and $C([y]_{\equiv}) \subseteq Y$. Let $z \in \mathbb{R}$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$. Clearly, $C([z]_{\equiv}) \subseteq X \cap Y$ and hence $X \cap Y \in \mathbf{M}$. □

Thus, $\mathbf{M}$ is a filter on $\mathbf{D}$. From a similar argument, we can conclude using $\mathsf{AC}_{\omega}(\mathbb{R})$ that $\mathbf{M}$ is ω_1-complete if $\preccurlyeq$ on $\mathbb{R}$ is *countably upwards closed*: if for every countable subset $\{x_i \in \mathbb{R} \colon i \in \omega\}$ of $\mathbb{R}$ there is an $x \in \mathbb{R}$ such that $x_i \preccurlyeq x$ for every $i \in \omega$.

Lemma 4.12 ($\mathsf{AC}_\omega(\mathbb{R})$)**.** *Suppose $\preccurlyeq$ on $\mathbb{R}$ is countably upwards closed. Then $\mathbf{M}$ is ω_1-complete.*

Proof. Suppose $\langle X_i \colon i \in \omega\rangle$ is a sequence in $\mathbf{M}$. Using $\mathsf{AC}_\omega(\mathbb{R})$, choose for every $i \in \omega$ an $x_i \in \mathbb{R}$ such that $C([x_i]_\equiv) \subseteq X_i$. There is an $x \in \mathbb{R}$ such that $x_i \preccurlyeq x$ for every $i \in \omega$. Then for every $i \in \omega$, $C([x]_\equiv) \subseteq X_i$ and thus $C([x]_\equiv) \subseteq \bigcap_{i\in\omega} X_i$. Therefore, $\bigcap_{i\in\omega} X_i \in \mathbf{M}$. This shows that M is ω_1-complete. $\square$

To summarize, if $\preccurlyeq$ is a reflexive and transitive relation on $\mathbb{R}$ such that $\preccurlyeq$ is countably upwards closed, then $\mathbf{M} \subseteq \wp(\mathbf{D})$ defined by

$$X \in \mathbf{M} \text{ if and only if } (\exists \mathbf{d} \in \mathbf{D})\,(C(\mathbf{d}) \subseteq X)$$

is a ω_1-complete filter on $\mathbf{D}$, the degree structure induced by $\preccurlyeq$. Martin used the axiom of determinacy to prove that every set either contains a cone or is disjoint from a cone. For this we need the relation $\preccurlyeq$ to be compatible with the $*$-operation, as follows: say that $\preccurlyeq$ is *compatible with* $*$ if for every $x, y \in \mathbb{R}$ with $x \preccurlyeq y$, we have $(x * y) \preccurlyeq y$ and $y \preccurlyeq (x * y)$, and $(y * x) \preccurlyeq x$ and $x \preccurlyeq (y * x)$.

Theorem 4.13 (AD)**.** *(Martin, 1968) Suppose that $\preccurlyeq$ is a reflexive and transitive relation on $\mathbb{R}$ which is countably upwards closed and compatible with $*$. Then $\mathbf{M}$ is an ω_1-complete ultrafilter on $\mathbf{D}$.*

Proof. We know that $\mathbf{M}$ is an ω_1-complete filter on $\mathbf{D}$ by Lemmas 4.11 and 4.12. We need to prove that for every $X \subseteq \mathbf{D}$ either $X \in \mathbf{M}$ or $\mathbf{D} \setminus X \in \mathbf{M}$.

Let $X \subseteq \mathbf{D}$. Since every $\mathbf{d} \in \mathbf{D}$ is a subset of $\mathbb{R}$, we have that $\bigcup X \subseteq \mathbb{R}$. Hence, the game $G_\omega(\bigcup X)$ is determined by AD. Either player $\mathbf{I}$ or player $\mathbf{II}$ has a winning strategy for this game. A winning strategy can be considered an element of $\mathbb{R}$ through a coding of ${}^{<\omega}\omega$. We will prove that the cone over the degree of the winning strategy is either contained in X (if player $\mathbf{I}$ wins) or in $\mathbf{D} \setminus X$ (if player $\mathbf{II}$ wins). There are two cases: either player $\mathbf{I}$ wins or player $\mathbf{II}$ wins.

Case 1. Suppose σ is a winning strategy for player $\mathbf{I}$ in $G_\omega(\bigcup X)$. Let $\mathbf{d} := [\sigma]_\equiv$. We claim that the cone over $\mathbf{d}$ is contained in X, that is,

$$C(\mathbf{d}) = \{\mathbf{e} \in \mathbf{D} \colon \mathbf{d} \preccurlyeq \mathbf{e}\} \subseteq X.$$

Suppose $\mathbf{e} \in C(\mathbf{d})$. Since $\mathbf{d} \preccurlyeq \mathbf{e}$, there are $x, y \in \mathbb{R}$ with $[x]_\equiv = \mathbf{d} = [\sigma]_\equiv$ and $[y]_\equiv = \mathbf{e}$ such that $x \preccurlyeq y$, that is, $\sigma \preccurlyeq y$. Since $\preccurlyeq$ is compatible with $*$, we have

$$\sigma * y \preccurlyeq y \text{ and } y \preccurlyeq (\sigma * y),$$

and therefore $[\sigma * y]_\equiv = [y]_\equiv$. Because σ is a winning strategy for player $\mathbf{I}$, $\sigma * y \in \bigcup X$. That is, $[\sigma * y]_\equiv \in X$ and thus $\mathbf{e} = [y]_\equiv = [\sigma * y]_\equiv \in X$. Hence, $C(\mathbf{d}) \subseteq X$.

Case 2. Suppose player **II** has a winning strategy τ for $G_\omega(\bigcup X)$. We claim that the cone over $\mathbf{d} = [\tau]_\equiv$ is contained in $\mathbf{D} \setminus X$, that is,

$$C(\mathbf{d}) = \{\mathbf{e} \in \mathbf{D} \colon \mathbf{d} \preccurlyeq \mathbf{e}\} \subseteq \mathbf{D} \setminus X.$$

Suppose $\mathbf{e} \in C(\mathbf{d})$. Since $\mathbf{d} \preccurlyeq \mathbf{e}$, there are $x, y \in \mathbb{R}$ with $[x]_\equiv = \mathbf{d} = [\tau]_\equiv$ and $[y]_\equiv = \mathbf{e}$ such that $x \preccurlyeq y$, that is, $\tau \preccurlyeq y$. Since $\preccurlyeq$ is compatible with $*$, we have

$$y * \tau \preccurlyeq y \text{ and } y \preccurlyeq (y * \tau),$$

and therefore $[y * \tau]_\equiv = [\tau]_\equiv$. Because τ is a winning strategy for player **II**, $y * \tau \notin \bigcup X$. That is, $[y * \tau]_\equiv \notin X$ and thus $\mathbf{e} = [y]_\equiv = [y * \tau]_\equiv \notin X$. Hence, $C(\mathbf{d}) \subseteq \mathbf{D} \setminus X$.

Therefore $\mathbf{M}_T$ is an ultrafilter. □

We can push it out to get a measure on ω_1 using the following result.

Lemma 4.14 (AD)**.** *For every $x \in \mathbb{R}$, the cardinal ω_1 is inaccessible in $\mathbf{L}[x]$.*

Proof. See [Ka03, p. 379]. □

Corollary 4.15 (AD)**.** *There is an ω_1-complete nonprincipal ultrafilter on ω_1.*

Proof. If for every $x, y \in \mathbb{R}$ with $x \equiv y$,

$$\omega_1^{\mathbf{L}[x]} = \omega_1^{\mathbf{L}[y]}$$

and this is a countable ordinal, then we can define a function $f \colon \mathbf{D} \to \omega_1$ by

$$f([x]_\equiv) := \omega_1^{\mathbf{L}[x]}.$$

Since Martin's measure $\mathbf{M}_T$ is an ultrafilter on $\mathbf{D}_T$ by Theorem 4.13, its pushout

$$f_*(\mathbf{M}_T) := \{X \subseteq \omega_1 \colon f^{-1}[X] \in \mathbf{M}_T\}$$

is an ω_1-complete ultrafilter on ω_1 by Lemma 2.17. To complete the proof, note that $f_*(\mathbf{M}_T)$ is nonprincipal as for every $\alpha < \omega_1$, there is a $b \in \mathbb{R}$ such that $\alpha < \omega_1^{\mathbf{L}[b]}$, so $f^{-1}(\{\alpha\}) \notin \mathbf{M}_T$. □

Before we can use Martin's measure to show that certain ω_1-complete filters can be extended to ω_1-complete ultrafilter, we need a technical tool. Moschovakis [Mo70] defined the ordinal

$$\Theta := \sup\{\xi \in \mathrm{On} \colon \text{there is a surjection } \mathbb{R} \to \xi\}$$

and proved the following useful result.

Theorem 4.16 (AD). *(Moschovakis, 1970) If there is a surjection $\mathbb{R} \to \xi$, then there is a surjection $\mathbb{R} \to \wp(\xi)$.*

Proof. See for example [Ka03, p. 397–398]. $\square$

Let $\preccurlyeq$ be a reflexive and transitive binary relation on $\mathbb{R}$. We say that every degree has countably many predecessors if for every $\mathbf{d} \in \mathbf{D}$ the set $\{\mathbf{e} \in \mathbf{D} \colon \mathbf{e} \preccurlyeq \mathbf{d}\}$ is countable.

Theorem 4.17 (AD). *(Kunen) Let $\preccurlyeq$ be a reflexive and transitive binary relation on $\mathbb{R}$ which is countably upwards closed, compatible with $*$ and such that every element has countably many predecessors. Then every ω_1-complete filter on any $\lambda < \Theta$ can be extended to an ω_1-complete ultrafilter.*

Proof. Let F be an ω_1-complete filter on $\lambda < \Theta$. There exists a surjection $g \colon \mathbb{R} \to \wp(\lambda)$ by Moschovakis' Theorem 4.16. Since for every $\mathbf{d} \in \mathbf{D}$ the set $\{\mathbf{e} \in \mathbf{D} \colon \mathbf{e} \preccurlyeq \mathbf{d}\}$ is countable, the set $\{g(x) \in F \colon x \in \mathbb{R} \text{ and } [x]_\equiv \preccurlyeq \mathbf{d}\}$ is countable. Therefore the intersection

$$\bigcap \{g(x) \in F \colon x \in \mathbb{R} \text{ and } [x]_\equiv \preccurlyeq \mathbf{d}\}$$

is nonempty as it is a countable intersection of elements of the ω_1-complete filter F. Therefore, we can define a function $f \colon \mathbf{D} \to \lambda$ on the degrees by

$$f(\mathbf{d}) := \text{least ordinal in } \bigcap \{g(x) \in F \colon x \in \mathbb{R} \text{ and } [x]_\equiv \preccurlyeq \mathbf{d}\}.$$

Since $\mathbf{M}$ is an ultrafilter by Martin's Theorem 4.13, its pull-back

$$f_*(\mathbf{M}) := \{X \subseteq \lambda \colon f^{-1}[X] \in \mathbf{M}\}$$

is an ω_1-complete ultrafilter by Lemma 2.17. To prove that it extends F, we have to show that for every $X \subseteq \lambda$ such that $X \in F$, $f^{-1}[X]$ is an element of $\mathbf{M}$, that is, that $f^{-1}[X]$ contains a cone. Suppose $X \in F$. Since $g \colon \mathbb{R} \to \wp(\lambda)$ is surjective, there is an $x \in \mathbb{R}$ such that $g(x) = X$. Consider the cone $C = C([x]_\equiv) = \{\mathbf{d} \in \mathbf{D} \colon [x]_\equiv \preccurlyeq \mathbf{d}\}$. If $\mathbf{d} \in C$, then $f(\mathbf{d}) \in X$. Thus, $C \subseteq f^{-1}[X]$ is a cone in the pre-image of X, as required. $\square$

4.5 A Fine Measure on $\wp_{\omega_1}(\mathbb{R})$

The goal of this section is to construct a fine measure on $\wp_{\omega_1}(\mathbb{R})$ using the axiom of determinacy AD. Once we have a fine measure on $\wp_{\omega_1}(\mathbb{R})$ we can push it out to get a fine measure on $\wp_{\omega_1}(S)$ for every surjective image S of $\mathbb{R}$. In particular, AD proves the existence of a fine measure on $\wp_{\omega_1}(\alpha)$ for every $\alpha < \Theta$. This construction is due to Solovay [So78].

Every real $x \in \mathbb{R}$ codes a countable set of reals $\{(x)_i \colon i \in \omega\}$, where for every $i \in \omega$ the real $(x)_i$ is defined by $(x)_i(j) := x(\ulcorner i, j \urcorner)$. Let $X \subseteq \wp_{\omega_1}(\mathbb{R})$ be a collection of countable sets of reals. We define the Solovay game $G(X)$ as follows. The players choose natural numbers:

$$\begin{array}{llllll} \mathbf{I} & x_0 & & x_2 & & \ldots \\ \mathbf{II} & & x_1 & & x_3 & \ldots \end{array}$$

Player **II** wins the game $G(X)$ if and only if

$$\{(x_{\mathbf{I}})_i \colon i \in \omega\} \cup \{(x_{\mathbf{II}})_i \colon i \in \omega\} \in X.$$

Define $U \subseteq \wp(\wp_{\omega_1}(\mathbb{R}))$ by

$$X \in U \text{ if and only if player } \mathbf{II} \text{ has a winning strategy for } G(X).$$

Of course, U is the fine measure on $\wp_{\omega_1}(\mathbb{R})$ that we want to construct. Only closure under intersections is hard to prove, we briefly state the arguments for the other filter properties: Player **II** never wins the game $G(\varnothing)$ and always wins the game $G(\wp_{\omega_1}(\mathbb{R}))$. Therefore, $\varnothing \notin U$ and $\wp_{\omega_1}(\mathbb{R}) \in U$. If $X \subseteq Y \subseteq \wp_{\omega_1}(\mathbb{R})$, then every winning strategy for player **II** in $G(X)$ is also a winning strategy for player **II** in $G(Y)$. Hence, if $X \in U$, then $Y \in U$. By definition, U is fine if for every $x \in \mathbb{R}$, the set $\hat{x} := \{X \in \wp_{\omega_1}(\mathbb{R}) \colon x \in X\} \in U$. We have to find a winning strategy for player **II** in $G(\hat{x})$. Player **II** wins $G(\hat{x})$ if

$$x \in \{(x_{\mathbf{I}})_i \colon i \in \omega\} \cup \{(x_{\mathbf{II}})_i \colon i \in \omega\}.$$

Therefore, any stratey for player **II** such that $(x_{\mathbf{II}})_0 = x$ will do.

We now first show that U is closed under intersections of two filter elements. Closure under countable intersections will follow from more or less the same proof. We need to show that if player **II** wins the game $G(X_0)$ and the game $G(X_1)$, then player **II** also wins the game $G(X_0 \cap X_1)$. Let τ_0 and τ_1 be winning strategies for player **II** in $G(X_0)$ and $G(X_1)$, respectively. In the proof these strategies are used to play two auxiliary games $G(X_0)$ and $G(X_1)$. Player **II** plays according to the strategy τ_0 or τ_1, respectively. Since these are winning strategies for the respective games, the countable set of reals coded by the play of these games will be an element of the respective pay-off set. In the game $G(X_0 \cap X_1)$ player **II** uses the moves in the auxiliary games. The idea of the proof is to make sure that the countable set of reals coded by the play of $G(X_0 \cap X_1)$, $G(X_0)$, and $G(X_1)$ is the same. Then this will be an element of both X_0 and X_1, and therefore of $X_0 \cap X_1$.

The countable set of reals coded by the play of a Solovay game is determined by the bijection $\ulcorner \cdot, \cdot \urcorner \colon \omega \times \omega \to \omega$. Define 'inverse' functions $r \colon \omega \to \omega$ and

$b\colon \omega \to \omega$ such that $\ulcorner r(n), b(n) \urcorner = n$ for every $n \in \omega$. The nth move of a player corresponds to the $b(n)$th bit of the $r(n)$th real under his control.

Let $x^0_{\mathbf{II}}$ and $x^1_{\mathbf{II}}$ enumerate the moves of player **II** in $G(X_0)$ and $G(X_1)$, respectively. Define the strategy τ for player **II** in the game $G(X_0 \cap X_1)$ as follows. Suppose player **II** has to play his nth move. If $r(n) = 2m$, play the $b(n)$th bit of the mth real controlled by **II** in $G(X_0)$, and if $r(n) = 2m+1$, play the $b(n)$th bit of the mth real controlled by **II** in $G(X_1)$. More formally, τ is defined by

$$x_{\mathbf{II}}(n) := \begin{cases} x^0_{\mathbf{II}}(\ulcorner m, b(n) \urcorner) & \text{if } r(n) = 2m, \text{ and} \\ x^1_{\mathbf{II}}(\ulcorner m, b(n) \urcorner) & \text{if } r(n) = 2m+1. \end{cases}$$

Of course, for $i \in 2$ we let $x^i_{\mathbf{II}}(\ulcorner m, b \urcorner) = \tau_i(\langle x_0, x_1, \dots, x_{\ulcorner m,b \urcorner} \rangle)$ as we want player **II** to play according to the strategy τ_i in $G(X_i)$. Hence, we have to define the moves of the first player in $G(X_i)$. We need not only to code the reals coded by $x_{\mathbf{I}} = \langle x_0, x_2, \dots \rangle$ of player **II**, but also those moves of the second player in the other auxilary game. Therefore, for $i \in 2$ and $i \neq j \in 2$, let

$$x^i_{\mathbf{I}}(n) := \begin{cases} x_{\mathbf{I}}(\ulcorner m, b(n) \urcorner) & \text{if } r(n) = 2m, \text{ and} \\ x^i_{\mathbf{II}}(\ulcorner m, b(n) \urcorner) & \text{if } r(n) = 2m+1. \end{cases}$$

We have to prove that the strategy τ is well-defined, and that indeed the countable set of reals coded by the three games is the same.

Lemma 4.18. *The strategy τ is well-defined.*

Proof. The possible problem with the definition of the strategy τ is that the moves of the first player in an auxiliary game are not defined as they could depend on moves of the second player that have not been defined yet.

Suppose player **II** is to play his nth move. We may assume that $x_{\mathbf{I}}(k)$ is defined for every $k \leq n$, and that $x_{\mathbf{II}}(k)$ is defined for every $k < n$. These are the moves of the players played before the nth move of player **II**. There are two possible cases: either $r(n) = 2m$ or $r(n) = 2m+1$. We will give the details of the former case, the latter follows using similar arguments.

Player **II** needs to find move $x_{\mathbf{II}}(\ulcorner m, b(n) \urcorner)$ in the auxiliary game $G(X_0)$. In order to do so, all moves $x^0_{\mathbf{I}}(k)$ of the first player need to be defined for $k \leq \ulcorner m, b(n) \urcorner$. Let $k \leq \ulcorner m, b(n) \urcorner$. Again there are two cases.

Case 1. If $r(k) = 2\ell$, then $x^0_{\mathbf{I}}(k) := x_{\mathbf{I}}(\ulcorner \ell, b(k) \urcorner)$. Since $\ell < 2\ell$,

$$\ulcorner \ell, b(k) \urcorner < \ulcorner 2\ell, b(k) \urcorner = \ulcorner r(k), b(k) \urcorner = k \leq n.$$

Therefore $j := \ulcorner \ell, b(k) \urcorner < k \leq n$, and by assumption $x^0_{\mathbf{I}}(j)$ is defined for $j \leq n$.

Case 2. If $r(k) = 2\ell+1$, then $x^0_{\mathbf{I}}(k) := x^1_{\mathbf{II}}(\ulcorner \ell, b(k) \urcorner)$. Since $\ell < 2\ell < 2\ell+1$,

$$\ulcorner \ell, b(k) \urcorner < \ulcorner 2\ell, b(k) \urcorner < \ulcorner 2\ell+1, b(k) \urcorner = \ulcorner r(k), b(k) \urcorner = k \leq n.$$

Therefore, $j := \ulcorner \ell, b(k) \urcorner < k \leq n$, and by assumption $x^0_{\mathbf{I}}(k)$ is defined for $j \leq n$. □

Finally, to finish the proof that U is a fine filter we want to show that the countable set of reals coded by a play of $G(X_0 \cap X_1)$ according to the strategy τ is the same as the countable set of reals coded by the play of the auxiliary games $G(X_0)$ and $G(X_1)$. Then this countable set of reals is an element of both X_0 and X_1, and therefore of $X_0 \cap X_1$. Hence, τ is a winning strategy.

Lemma 4.19. *The countable set of reals coded by a play of $G(X_0 \cap X_1)$ according to the strategy τ is the same as the countable set of reals coded by the play of the auxiliary games $G(X_0)$ and $G(X_1)$.*

Proof. Let $x_{\mathbf{I}}$ and $x_{\mathbf{II}}$ enumerate the moves of player **I** and **II** in $G(X_0 \cap X_1)$, respectively. Let $r \in \omega$ be arbitrarily and let $i \in 2$. The rth real controlled by player **I** is $(x_{\mathbf{I}})_r = \langle x_{\mathbf{I}}(\ulcorner r, 0 \urcorner), x_{\mathbf{I}}(\ulcorner r, 1 \urcorner), \dots \rangle$, and is coded by the $(2r)$th real controlled by the first player in $G(X_i)$. We have for every $b \in \omega$,

$$(x^i_{\mathbf{I}})_{2r}(b) = x^i_{\mathbf{I}}(\ulcorner 2r, b \urcorner) = x_{\mathbf{I}}(\ulcorner r, b \urcorner) = (x_{\mathbf{I}})_r(b).$$

For the rth real controlled by player **II** in $G(X_0 \cap X_1)$, there are two (very similar) cases.

Case 1. If $r = 2m$, then this real is coded by the mth real controlled by the second player in $G(X_0)$. We have for every $b \in \omega$,

$$(x^0_{\mathbf{II}})_m(b) = x^0_{\mathbf{II}}(\ulcorner m, b \urcorner) = x_{\mathbf{II}}(\ulcorner 2m, b \urcorner) = (x_{\mathbf{II}})_2 m(b) = (x_{\mathbf{II}})_r(b).$$

Case 2. If $r = 2m + 1$, then this real is coded by the mth real controlled by the second player in $G(X_1)$. We have for every $b \in \omega$,

$$(x^1_{\mathbf{II}})_m(b) = x^1_{\mathbf{II}}(\ulcorner m, b \urcorner) = x_{\mathbf{II}}(\ulcorner 2m + 1, b \urcorner) = (x_{\mathbf{II}})_{2m+1}(b) = (x_{\mathbf{II}})_r(b).$$

□

We have finished the proof that U is a fine filter on $\wp_{\omega_1}(\mathbb{R})$.

Proposition 4.20. *The set U is a fine filter on $\wp_{\omega_1}(\mathbb{R})$.* □

We could now use this idea of playing auxiliary games to prove that U is ω_1-complete, using $\mathsf{AC}_\omega(\mathbb{R})$. This is a straight-forward but messy generalization. Since we are mainly interested in this filter in the context of AD, we will take an easy way out. The axiom of determinacy AD implies that U is an ultrafilter, and we know already that every ultrafilter is ω_1-complete under AD.

Theorem 4.21 (AD)**.** *(Solovay, 1978) If $X \subseteq \wp_{\omega_1}(\mathbb{R})$ such that $X \notin U$, then $\wp_{\omega_1}(\mathbb{R}) \setminus X \in U$.*

Proof. Suppose $X \subseteq \wp_{\omega_1}(\mathbb{R})$ such that $X \notin U$. By definition of U, this means that player **II** does not have a winning strategy for $G(X)$. Since for every $X \subseteq \wp_{\omega_1}(\mathbb{R})$ there is an $A \subseteq \mathbb{R}$ such that the game $G(X)$ is equivalent to the game $G_\omega(A)$, AD implies that $G(X)$ is determined for every $X \subseteq \wp_{\omega_1}(\mathbb{R})$. Therefore, player **I** must have a winning strategy σ in $G(X)$. We will use this strategy to define a winning strategy τ for player **II** in $G(\wp_{\omega_1}(\mathbb{R}) \setminus X)$.

Player **I** begins the game $G(\wp_{\omega_1}(\mathbb{R}) \setminus X)$ by playing $x_0 := x_{\mathbf{I}}(0)$. Player **II** just ignores this move and responds with $\sigma(\varnothing)$. Then player **I** plays $x_2 := x_{\mathbf{I}}(1)$, and **II** responds with $\sigma(\langle \sigma(\varnothing), x_0 \rangle)$, and so on:

$$
\begin{array}{cccccccc}
\mathbf{I} & x_0 & & x_2 & & x_4 & & \ldots \\
\mathbf{II} & & \sigma(\varnothing) & & \sigma(\langle \sigma(\varnothing), x_0 \rangle) & & \sigma(\langle \sigma(\varnothing), x_0, \sigma(\langle \varnothing, x_0 \rangle), x_2 \rangle) & \ldots
\end{array}
$$

The outcome of this play of the game $G(\wp_{\omega_1}(\mathbb{R}) \setminus X)$ is exactly the same as the outcome of the following play of the game $G(X)$, where the players have switched sides:

$$
\begin{array}{ccccccc}
\mathbf{I} & \sigma(\varnothing) & & \sigma(\langle \sigma(\varnothing), x_0 \rangle) & & \sigma(\langle \sigma(\varnothing), x_0, \sigma(\langle \varnothing, x_0 \rangle), x_2 \rangle) & \ldots \\
\mathbf{II} & & x_0 & & x_2 & & \ldots
\end{array}
$$

Notice that in this game player **I** plays according to the strategy σ. Since σ is a winning strategy for player **I** in the game $G(X)$, the outcome of the play will *not* be in X and is therefore an element of $\wp_{\omega_1}(\mathbb{R}) \setminus X$. Thus, the strategy τ for player **II** in the game $G(\wp_{\omega_1}(\mathbb{R}) \setminus X)$ defined by $\tau(\langle x_0 \rangle) := \sigma(\varnothing)$ and for $n \geq 1$ by

$$\tau(\langle x_0, x_1, \ldots, x_n \rangle) := \sigma(\langle x_1, x_2, \ldots, x_n \rangle)$$

is a winning strategy for player **II** for the game $G(\wp_{\omega_1}(\mathbb{R}) \setminus X)$. Hence, if $X \notin U$, then $\wp_{\omega_1}(\mathbb{R}) \setminus X \in U$ as required. $\square$

Combining Corollary 4.9, Proposition 4.20, and Theorem 4.21 under AD, we see that U is the desired fine measure on $\wp_{\omega_1}(\mathbb{R})$. Once we have a fine measure U on $\wp_{\omega_1}(\mathbb{R})$, we can push U out to a fine measure on every surjective image of $\mathbb{R}$ by Lemma 2.19. In particular, there is a fine measure on $\wp_{\omega_1}(\alpha)$ for every $\omega_1 \leq \alpha < \Theta$.

Corollary 4.22 (AD)**.** *There is a fine measure on $\wp_{\omega_1}(\mathbb{R})$ and therefore also on $\wp_{\omega_1}(\alpha)$ for every $\omega_1 \leq \alpha < \Theta$.* $\square$

We cannot use AD to prove that for every $\lambda \geq \omega_1$ there is a fine measure on $\wp_{\omega_1}(\lambda)$ by Corollary 4.6. In fact, there is no fine measure on $\wp_{\omega_1}(\Theta)$.

Theorem 4.23 (DC)**.** *(Spector [Sp91]) If $\kappa < \Theta$, then there is no fine measure on $\wp_\kappa(\Theta)$.* $\square$

In a sense, the result of Corollary 4.22 is optimal. Using the stronger axiom of real determinacy $\mathsf{AD}_{\mathbb{R}}$, Solovay showed that there is a *normal* measure on $\wp_{\omega_1}(\mathbb{R})$ and therefore on $\wp_{\omega_1}(\alpha)$ for every $\omega_1 \leq \alpha < \Theta$. Woodin used $\mathsf{AD}_{\mathbb{R}}$ to show that these normal measures are unique.

4.6 Consistency Strength and the Axiom of Choice

The relative consistency strength of measurability does not depend on the presence or absence of the axiom of choice. Jech [Je68] has shown that ZFC+'there is a measurable cardinal' and ZF+'there is a measure on ω_1' are equiconsistent.

One of several ZFC-equivalent notions of measurability of κ is 'there is a fine measure on $\wp_\kappa(\kappa)$'. As a natural next step, one might consider the axiom

'there is a κ such that $\wp_\kappa(\kappa^+)$ carries a fine measure.'

Let SFM denote this axiom. In the rest of this section we will outline an argument to prove that $\mathsf{ZFC} + \mathsf{SFM}$ and $\mathsf{ZF} + \mathsf{SFM}$ are not equiconsistent. The argument here is analogous to an argument that can be found in [Bo02].

Since the consistency of $\mathsf{ZFC} + \mathsf{SFM}$ clearly implies the consistency of $\mathsf{ZF} + \mathsf{SFM}$, we want to show that the consistency strength of $\mathsf{ZF} + \mathsf{SFM}$ is strictly less than the consistency strength of $\mathsf{ZFC} + \mathsf{SFM}$. The argument consists of three steps. First we outline an argument that $\mathsf{ZFC} + \mathsf{SFM}$ implies $\mathrm{Con}(\mathsf{ZF} + \mathsf{AD})$. Second, we remark that $\mathsf{ZF} + \mathsf{AD} \vdash \mathsf{SFM}$. Finally, we use Gödel's second incompleteness result to finish the argument.

After Solovay reached the same conclusion from the existence of a cardinal κ such that there is a normal measure on $\wp_\kappa(\kappa^+)$, Gregory [SRK78] proved that $\mathsf{ZFC} + \mathsf{SFM}$ implies the failure of $\square_\kappa$, a combinatorial principle introduced by Jensen.

Schimmerling and Zeman [SZ01] have shown that the failure of $\square_\kappa$ entails the existence of a nontame mouse (See for example [LS98, p. 35]). The existence of a nontame mouse implies the consistency of ZFC + 'there are ω Woodin cardinals'.

Finally, Woodin [WMH∞] proved that ZFC + 'there are ω Woodin cardinals' and $\mathsf{ZF} + \mathsf{AD}$ are equiconsistent. To summarize, we have the following string of implications:

$$\begin{tikzcd}
\mathsf{ZFC} + \mathsf{SFM} \arrow[d, "\text{Gregory}"] \\
\mathsf{ZFC} + \text{`there is a } \kappa \text{ such that } \square_\kappa \text{ fails'} \arrow[d, "\text{Schimmerling--Zeman}"] \\
\mathsf{ZFC} + \text{`there is a nontame mouse'} \arrow[d] \\
\mathrm{Con}(\mathsf{ZFC} + \text{`there are } \omega \text{ Woodin cardinals'}) \arrow[d, leftrightarrow, "\text{Woodin}"] \\
\mathrm{Con}(\mathsf{ZF} + \mathsf{AD})
\end{tikzcd}$$

Using Solovay's ultrafilter on $\wp_{\omega_1}(\mathbb{R})$ we have proved in Section 4.5 that under AD there is a fine measure on $\wp_{\omega_1}(\lambda)$ for every $\lambda < \Theta$. Since under AD there is a surjection $\mathbb{R} \to \omega_2$, there is in particular a fine measure on $\wp_{\omega_1}(\omega_2)$. Hence, $\mathsf{ZF} + \mathsf{AD} \vdash \mathsf{SFM}$.

We apply Gödel's second incompleteness theorem to conclude that the consistency strength of $\mathsf{ZF} + \mathsf{SFM}$ is strictly less than the consistency strength of $\mathsf{ZFC} + \mathsf{SFM}$. Suppose that $\mathrm{Con}(\mathsf{ZF} + \mathsf{SFM})$ would imply $\mathrm{Con}(\mathsf{ZFC} + \mathsf{SFM})$. Then the following string of implications would hold:

$$\begin{tikzcd}
\mathsf{ZFC} + \mathsf{SFM} \arrow[d] \\
\mathrm{Con}(\mathsf{ZF} + \mathsf{AD}) \arrow[d] \\
\mathrm{Con}(\mathsf{ZF} + \mathsf{AD} + \mathsf{SFM}) \arrow[d] \\
\mathrm{Con}(\mathsf{ZF} + \mathsf{SFM}) \arrow[d] \\
\mathrm{Con}(\mathsf{ZFC} + \mathsf{SFM})
\end{tikzcd}$$

which would contradict Gödel's second incompleteness theorem that no recursive extension of ZF can prove its own consistency. Hence, the consistency strength of $\mathsf{ZF} + \mathsf{SFM}$ has to be strictly less than the consistency strength of $\mathsf{ZFC} + \mathsf{SFM}$. In other words, in contrast to the axiom

$$(\exists\kappa)\ (\text{`there is a fine measure on } \wp_\kappa(\kappa)\text{'}),$$

the consistency strength of

$$(\exists\kappa)\ (\text{`there is a fine measure on } \wp_\kappa(\kappa^+)\text{'})$$

does depend on the presence of the axiom of choice.

Chapter 5

Conclusions

We have studied four different definitions related to the notion 'κ is a compact cardinal'. These definitions were related to the compactness of the infinitary language $\mathscr{L}_{\kappa\kappa}$ (Section 3.1), the possibility of extending κ-complete filters to ultrafilters while retaining their completeness (Section 3.2), the existence of fine measures on $\wp_\kappa(\lambda)$ (Section 3.3), and being the critical point of a λ-compact elementary embedding (Section 3.4).

For each of these properties we can consider both a global version and a local one. In case of the extension property there are three local versions. These restrict the filters which can be extended to those that are not 'too large'. The λ-extension property is concerned with filters generated by at most λ sets, the restricted λ-extension property with filters on a set S such that $|\wp_\kappa(S)| = \lambda$. We discussed a third extension property in Chapter 4: the extension of filters on λ.

The following diagram presents the structure of implications between the local forms of these definitions. A solid arrow indicates an implication provable in ZF, a dotted arrow indicates an implication provable in ZFC.

$$\begin{tikzcd}
 & \begin{array}{c}\lambda\text{-covering}\\ \text{embedding}\end{array} \arrow[dl, "3.18"'] \arrow[dr, "3.21"] & \\
\lambda\text{-extension} \arrow[r, "3.14"] \arrow[dr] & \begin{array}{c}\text{fine measure}\\ \text{on } \wp_\kappa(\lambda)\end{array} \arrow[u, dashed, "3.19"] \arrow[r, dashed, "3.13"] & \lambda\text{-compactness} \arrow[dl, "3.12"] \\
 & \begin{array}{c}\text{restricted}\\ \lambda\text{-extension}\end{array} &
\end{tikzcd}$$

We have tried to find answers to the following two questions: Which of these implications cannot be reversed? Is the use of the axiom of choice necessary to prove the implications (3.13) and (3.19)?

First of all, in $\mathsf{ZF}+\mathsf{AD}$ every infinite cardinal has the restricted λ-extension property for every λ. Since none of the other properties is trivial (they imply regularity or weak inaccessibility, for example), it follows that restricted λ-extension cannot imply any of the others. Furthermore, since ω_1 is accessible, it can neither be a critical point of an elementary embedding nor can $\mathscr{L}_{\omega_1\omega}$ be λ-compact for any $\lambda \geq \omega_1$. But under AD there is a fine measure on $\wp_{\omega_1}(\lambda)$ for every $\lambda \leq \Theta$. We draw a similar diagram as above, but this time showing the properties of $\kappa = \omega_1$ under AD:

$$\begin{tikzcd}
& \text{No} \arrow[dl] \arrow[dr] & \\
? \arrow[r] \arrow[dr] & \text{Yes} \arrow[u, dashed] \arrow[r, dashed] & \text{No} \arrow[dl] \\
& \text{Yes} &
\end{tikzcd}$$

There cannot be an arrow from 'yes' to 'no'. Hence, the dotted arrows really are implications in ZFC and not ZF. Unfortunately, we do not know if AD implies that ω_1 has the λ-extension property for any $\lambda \geq \omega_1$. A result of Kunen (Theorem 4.17) does show that every ω_1-complete filter on $\lambda < \Theta$ can be extended to an ω_1-complete ultrafilter but this needs the restriction to filters on $\lambda < \Theta$.

Besides the local forms of the definitions we can also consider their global analogues. For the embedding property and the fine measure property, these are just the universally quantified local properties. In case of the filter extension property,

$$(\forall\lambda)\,(\kappa \text{ has the } \lambda\text{-extension property})$$

and

$$(\forall\lambda)\,(\kappa \text{ has the restricted } \lambda\text{-extension property})$$

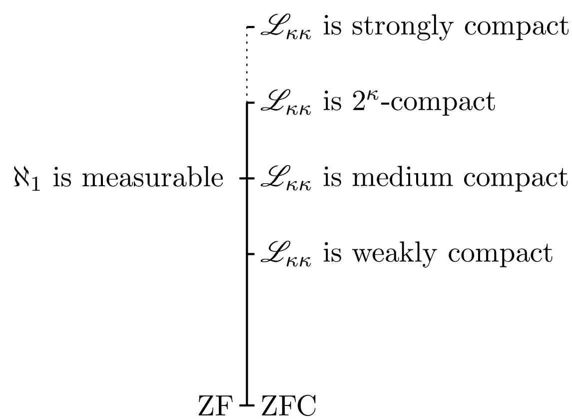
are equivalent. With the axiom of choice, these are equivalent to the property 'every κ-complete filter can be extended to a κ-complete ultrafilter' which we will call the extension property. Without AC the extension property for κ may be stronger than $(\forall\lambda)\,(\kappa$ has the λ-extension property$)$.

Since we were able to derive the extension property for κ from the compactness of the infinitary language $\mathscr{L}_{\kappa\omega}$, the structure of implications between the global forms of the definitions simplifies to the following diagram:

$$\begin{tikzcd}
\text{covering embeddings} \arrow[r, "3.21"] & \text{language compactness} \arrow[d, "3.10"] \\
\text{fine measures} \arrow[u, dashed, "3.19"] & \text{extension property} \arrow[l, "3.14"']
\end{tikzcd}$$

The diagram above displays the well known equivalent definitions of a strongly compact cardinal. Note that AD not does answer the question whether AC is necessary for the implication between the global version of the fine measure property and the global version of the embedding property. In fact we know that the existence of a cardinal with the global fine measure property cannot be a consequence of AD (Corollary 4.6).

We gauged the consistency strength of various local forms of infinitary language compactness in ZFC using some well known results on measurable cardinals. The diagram below was discussed in Section 3.5. The consistency strength increases from the bottom to the top. A solid line means that we know the consistency strength is strictly larger, a dotted line indicates that it could be that the consistency strength is equal.



Finally, we outlined in Section 4.6 an argument using AD that the consistency strength of

$$\mathsf{ZF} + \text{`there is } \kappa \text{ with a fine measure on } \wp_\kappa(\kappa^+)\text{'}$$

is strictly less than the consistency strength of

$$\mathsf{ZFC} + \text{`there is } \kappa \text{ with a fine measure on } \wp_\kappa(\kappa^+)\text{'}.$$

This is very different than for example the case of measurable cardinals.

Appendix A

Questions

1. Let κ be an uncountable cardinal. Are $\mathsf{ZFC}+$ 'there is a κ such that $\mathscr{L}_{\kappa\kappa}$ is 2^κ-compact' and $\mathsf{ZFC}+$ 'there is a κ such that $\mathscr{L}_{\kappa\kappa}$ is strongly compact' equiconsistent?

2. (AD) Can every ω_1-complete filter generated by at most ω_2 sets be extended to an ω_1-complete ultrafilter?

3. (AD) Can every ω_2-complete filter on $\lambda < \Theta$ be extended to an ω_2-complete ultrafilter?

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