Multitape Games

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Abstract. Infinite games have been a valuable tool to characterize classes of real-valued functions. In this paper we present the *basic multitape game* and the *multitape eraser game* and show that both games characterize a class of functions satisfying a certain partition property.

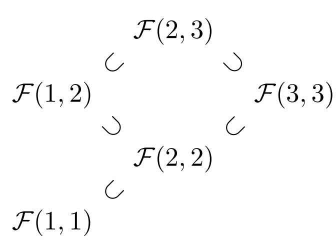
1 Notation and Background

As usual, for sets A and B, ${}^{A}B$ denotes the set of functions from A to B. In particular, ${}^{\omega}\omega$ denotes the set of functions from ω to ω, i.e. ${}^{\omega}\omega$ is the set of ω-length sequences of natural numbers. The notation ${}^{<\omega}A$ denotes the set of finite sequences of elements of A, so that ${}^{<\omega}\omega$ denotes the set of finite sequences of natural numbers. We use ${}^{\leq\omega}\omega$ to denote ${}^{<\omega}\omega \cup {}^{\omega}\omega$. For $s \in {}^{<\omega}\omega$, let $[s] := \{u \in {}^{\omega}\omega : s \subset u\}$. For $x \in {}^{\omega}\omega$ and $k \in \omega$, let $x \to k$ be the sequence x shifted by k places; in other words, $(x \to k)(n) := x(n + k)$.

The *Baire Space* is the set ${}^{\omega}\omega$ with the topology generated by the basic open sets $\{[s] : s \in {}^{<\omega}\omega\}$. As is standard in set theory, we think of elements of ${}^{\omega}\omega$ as being *real numbers*.

A real-valued function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ is **continuous** if the preimage of every open set is open, in other words if $f^{-1}[Y] \in \boldsymbol{\Sigma}^0_1$ for every $Y \in \boldsymbol{\Sigma}^0_1$. For $A \subseteq {}^{\omega}\omega$ and $f : A \to {}^{\omega}\omega$, we say that f is continuous if the preimage of every open set is open in the relative topology of A.

We may consider more general classes (sets) of functions as follows. Define $\mathcal{F}(n, m) := \{f : A \to {}^{\omega}\omega$ such that for every $Y \in \boldsymbol{\Sigma}^0_n$, $f^{-1}[Y] = X \cap A$ for some $X \in \boldsymbol{\Sigma}^0_m\}$. For example, $\mathcal{F}(1, 1)$ denotes the continuous functions and $\mathcal{F}(1, 2)$ denotes the set of functions for which the preimage of every $\boldsymbol{\Sigma}^0_1$ set is a $\boldsymbol{\Sigma}^0_2$ set in the relative topology of A. (The latter are commonly known as *Baire Class 1*.) The following diagram illustrates the classes of functions under consideration in this paper.



We use the notation $\mathcal{F}_{\mathrm{T}}(n,m)$ to denote the set of *total* functions in $\mathcal{F}(n,m)$, i.e. $\mathcal{F}_{\mathrm{T}}(n,m) := \mathcal{F}(n,m) \cap \{f : {}^{\omega}\omega \to {}^{\omega}\omega\}$. For $A \subseteq {}^{\omega}\omega$ and $\boldsymbol{\Gamma}$ a boldface pointclass, a $\boldsymbol{\Gamma}$-partition of A is a pairwise disjoint sequence $\langle A_n : n < \omega\rangle$ such that $A_n = A'_n \cap A$ for some $A'_n \in \boldsymbol{\Gamma}$, and $\bigcup_{n<\omega} A_n = A$. For $\mathcal{F}$ a set of real-valued functions, we define $\mathcal{P}(\boldsymbol{\Gamma}, \mathcal{F}) := \{f : A \to {}^{\omega}\omega$ such that there is a $\boldsymbol{\Gamma}$-partition $\langle A_n : n < \omega\rangle$ of A such that $f{\restriction}A_n \in \mathcal{F}\}$. We use the notation $\mathcal{P}_{\mathrm{T}}(\boldsymbol{\Gamma}, \mathcal{F}) := \mathcal{P}(\boldsymbol{\Gamma}, \mathcal{F}) \cap \{f : {}^{\omega}\omega \to {}^{\omega}\omega\}$. For example, a special case of a well-known theorem of Jayne and Rogers states that a function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ is $\mathcal{F}(2,2)$ if and only if there is a $\boldsymbol{\Pi}^0_1$-partition $\langle A_n : n < \omega\rangle$ of ${}^{\omega}\omega$ such that $f{\restriction}A_n$ is continuous; in our notation, this may be written as $\mathcal{F}_{\mathrm{T}}(2,2) = \mathcal{P}_{\mathrm{T}}(\boldsymbol{\Pi}^0_1, \mathcal{F}(1,1))$.

2 Infinite Games

We will consider a variety of *infinite games* in this paper. In each game, there is a set $A \subseteq {}^{\omega}\omega$ and a function $f : A \to {}^{\omega}\omega$. There are two players, Player I and Player II, who alternative moves for ω rounds.

$$\begin{array}{llllllll} \text{I:} & x_0 & & x_1 & & x_2 & & x = \langle x_n : n \in \omega\rangle \\ & & & & & & \cdots & \\ \text{II:} & & y_0 & & y_1 & & y_2 \quad & y = \langle y_n : n \in \omega\rangle \end{array}$$

Player I plays elements $x_i \in \omega$ and Player II plays elements $y_i \in \omega \cup \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$ for some finite set of tokens $\{\mathsf{T}_1, \ldots, \mathsf{T}_k\}$. Informally, each token corresponds to an option that Player II has in the game. At the end of the game, Player I has produced a sequence $x \in {}^{\omega}\omega$ and Player II has produced a sequence $y \in {}^{\omega}(\omega \cup \{\mathsf{T}_1, \ldots, \mathsf{T}_k\})$.

The sequence y is a sequence of natural numbers and tokens – however, the rules of the game will say how to *interpret* y as a sequence (finite or infinite) of natural numbers only. Specifically, for each game we define an interpretation function $\iota : {}^{\omega}(\omega \cup \{\mathsf{T}_1, \ldots, \mathsf{T}_k\}) \to {}^{\leq\omega}\omega$. Player II *wins the game* if $\iota(y) = f(x)$. (If $x \notin A$ then Player II wins automatically.) Note that if $x \in A$, Player II cannot possibly win the game if $\iota(y)$ is finite.

A **strategy for Player II** is a function $\tau : {}^{<\omega}\omega \to {}^{<\omega}(\omega \cup \{\mathsf{T}_1, \ldots, \mathsf{T}_k\})$ such that $\mathrm{lh}(\tau(s)) = \mathrm{lh}(s)$ and $s \subseteq t \Rightarrow \tau(s) \subseteq \tau(t)$. The argument to τ is a finite sequence of moves by Player I and the value of τ is a finite sequence of moves by Player II. Informally, τ tells Player II what to do in the game. With respect to the above diagram, if Player II follows τ then $\tau(\langle x_0, \ldots, x_k\rangle) = \langle y_0, \ldots, y_k\rangle$ and $y = \bigcup_{s \subset x} \tau(s)$.

For a strategy τ for Player II, it is convenient to define $\hat{\tau} : {}^{\omega}\omega \to {}^{\omega}(\omega \cup \{\mathsf{T}_1, \ldots, \mathsf{T}_k\})$,

$$\hat{\tau}(x) := \bigcup_{s \subset x} \tau(s).$$

We then define $\bar{\tau}(x) := \iota(\hat{\tau}(x))$ and say that a strategy τ for Player II is **winning** if $\bar{\tau}(x) = f(x)$ for every $x \in A$. With this paradigm in mind, it will be seen that certain games characterize certain classes of functions. More specifically, for a particular game G, we will see that Player II has a winning strategy in $G(f)$ if and only if f belongs to some particular class.

To get started, we will review the *Wadge*, *Backtrack*, and *Eraser games*.

3 The Wadge Game

We present a slightly modified version of the standard Wadge game (to be consistent with our paradigm). Fix a set $A \subseteq {}^{\omega}\omega$ and a function $f : A \to {}^{\omega}\omega$. The Wadge game $G_{\mathrm{W}}(f)$ has two players, Player I and Player II, who alternate moves for ω rounds. Player I plays elements of ω and Player II plays elements of $\omega \cup \{\mathsf{P}\}$. The token P is interpreted to mean "pass."

Formally, to define the interpretation function we first define $\theta : {}^{<\omega}(\omega \cup \{\mathsf{P}\}) \to {}^{<\omega}\omega$ by $\theta(\varnothing) := \varnothing$ and

$$\theta(s^\frown\langle z\rangle) := \begin{cases} \theta(s) \text{ if } z = \mathsf{P} \\ \theta(s)^\frown\langle z\rangle \text{ otherwise} \end{cases}$$

We then define $\iota_{\mathrm{W}} : {}^{\omega}(\omega \cup \{\mathsf{P}\}) \to {}^{\leq\omega}\omega$, $\iota_{\mathrm{W}}(y) := \bigcup_{s \subset y} \theta(s)$. Letting $x \in {}^{\omega}\omega$ be the infinite play of Player I and $y \in {}^{\omega}(\omega \cup \{\mathsf{P}\})$ be the infinite play of Player II, Player II wins the game if $x \notin A$ or $\iota_{\mathrm{W}}(y) = f(x)$. (Note that in order to have a chance, Player II must play infinitely often in ω if Player I plays $x \in A$.) If τ is a Wadge strategy for Player II, we let $\hat{\tau}(x) := \bigcup_{s \subset x} \tau(s)$, $\bar{\tau}(x) := \iota_{\mathrm{W}}(\hat{\tau}(x))$ and say that τ is **winning** for Player II if $\bar{\tau}(x) = f(x)$ for all $x \in A$.

Examples. Suppose Player II plays the sequence

$$\langle a_0, a_1, \mathsf{P}, a_2, \mathsf{P}, \mathsf{P}, a_3, \dots\rangle,$$

the interpretation will be

$$\langle a_0, a_1, a_2, a_3, \dots\rangle.$$

Suppose Player II plays the sequence

$$\langle a_0, a_1, \mathsf{P}, a_2, \mathsf{P}, \mathsf{P}, \mathsf{P}, \mathsf{P}, \dots\rangle$$

with cofinally many passes, the interpretation will be

$$\langle a_0, a_1, a_2\rangle.$$

Note that Player II cannot win in this case if Player I plays in A.

Theorem 1 *(Wadge). Let $A \subseteq {}^{\omega}\omega$. A function $f : A \to {}^{\omega}\omega$ is continuous $\Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{W}}(f)$.*

Proof. We begin by noting that

$$\bar{\tau}(x) = \bigcup_{s \subset x} \theta(\tau(s)).$$

$\Leftarrow$: Suppose τ is the winning strategy. To show that f is continuous, it suffices to show that the preimage of a basic open set is open in the topology of A. Let $t \in {}^{<\omega}\omega$ and let $X := \bigcup \{[s] : \theta(\tau(s)) = t\}$. It is not difficult to check that $f^{-1}[[t]] = X \cap A$.

$\Rightarrow$: Define τ by

$$\tau(s^\frown\langle m\rangle) := \begin{cases} \tau(s)^\frown\langle n\rangle \text{ if } f[[s^\frown\langle m\rangle]] \subseteq [\theta(\tau(s))^\frown\langle n\rangle] \\ \tau(s)^\frown\langle \mathsf{P}\rangle \text{ otherwise} \end{cases}$$

It is not difficult to check that τ is well-defined and winning for Player II in $G_{\mathrm{W}}(f)$.

□

4 The Backtrack Game

In the backtrack game $G_{\mathrm{B}}(f)$, Player II plays elements of $\omega \cup \{\mathsf{P}, \mathsf{B}\}$. As in the Wadge game, the token P is interpreted to mean "pass." The token B, the "backtrack" option, allows Player II to erase his entire output and start playing a new sequence of natural numbers.

Let ι_{W} be defined as in the Wadge game, we define the interpretation function $\iota_{\mathrm{B}} : {}^{\omega}(\omega \cup \{\mathsf{P}, \mathsf{B}\}) \to {}^{\leq\omega}\omega$,

$$\iota_{\mathrm{B}}(y) := \begin{cases} \varnothing \text{ if } \forall i\ \exists j{>}i\ y(j) = \mathsf{B} \\ \iota_{\mathrm{W}}(y \to i) \text{ if } i \text{ is least such that } \forall j{\geq}i\ y(j) \neq \mathsf{B} \end{cases}$$

We define $\bar{\tau}$ as usual and we say that a Backtrack strategy τ is **winning** if $\bar{\tau} = f(x)$ for all $x \in A$.

Examples. Suppose Player II plays a sequence that contains infinitely many B's, then the interpretation will be $\varnothing$ and Player II can only win if Player I plays out of A. Suppose Player II plays the sequence

$$\langle a_0, a_1, \mathsf{B}, a_2, \mathsf{B}, a_3, a_4, \mathsf{P}, a_5, \dots\rangle$$

with two B's only, then the interpretation will be

$$\langle a_3, a_4, a_5, \dots\rangle.$$

Theorem 2 *(Andretta). Let $A \subseteq {}^{\omega}\omega$. A function $f : A \to {}^{\omega}\omega$ is $\mathcal{P}(\boldsymbol{\Pi}^0_1, \mathcal{F}(2,2)) \Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{B}}(f)$.*

From the following theorem of Jayne and Rogers, we conclude that the Backtrack game characterizes the $\mathcal{F}_{\mathrm{T}}(2,2)$ functions, meaning that a total function f is $\mathcal{F}(2,2)$ if and only if Player II has a winning strategy in the game $G_{\mathrm{B}}(f)$.

Theorem 3 *(Jayne, Rogers). A function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ is $\mathcal{F}(2,2) \Leftrightarrow$ there is a $\boldsymbol{\Pi}^0_1$ partition $\langle A_n : n \in \omega \rangle$ of ${}^{\omega}\omega$ such that $f \restriction A_n$ is continuous. In other words, $\mathcal{F}_{\mathrm{T}}(2,2) = \mathcal{P}_{\mathrm{T}}(\boldsymbol{\Pi}^0_1, \mathcal{F}(1,1))$.*

Proof of Theorem 2.

$\Leftarrow$: Assume that τ is winning for Player II in the game $G_{\mathrm{B}}(f)$. Since, in the Baire space, every $\boldsymbol{\Sigma}^0_2$ set is the disjoint union of countably many $\boldsymbol{\Pi}^0_1$ sets, it suffices to give a $\boldsymbol{\Sigma}^0_2$ partition. Let $A_n := \{x \in A :$ on input x, τ backtracks n times$\}$. Then it follows that $f \restriction A_n$ is continuous using Theorem 1. Namely, let τ' be the following Wadge strategy for Player II: "On a scratch tape, run τ until τ has backtracked n times. Then use the remaining output of τ as the output for τ'." It is clear that τ' is winning for Player II in the game $G_{\mathrm{W}}(f \restriction A_n)$, so $f \restriction A_n$ is continuous. Moreover, it is not difficult to see that A_n is $\boldsymbol{\Sigma}^0_2$ (in the relative topology of A). Let $\rho : {}^{<\omega}(\omega \cup \{\mathsf{P}, \mathsf{B}\}) \to \omega$ be defined by $\rho(s) :=$"the number of B's appearing in s". Then the formula

$$\exists i \, \forall j{>}i \, (\rho(\tau(x \restriction j)) = n)$$

witnesses that A_n is $\boldsymbol{\Sigma}^0_2$.

$\Rightarrow$: Let $\langle A_n : n \in \omega \rangle$ be given and let τ_n be a winning strategy for Player II in the game $G_{\mathrm{W}}(f \restriction A_n)$. The following strategy is easily seen to be winning for Player II in the game $G_{\mathrm{B}}(f)$: "Let $i := 0$. Run the Wadge strategy τ_i. If Player I plays out of A_i, then this is known after some finite period since the complement of A_i is open. Use the backtrack option and repeat the process with $i := i + 1$."

$\square$

5 The Eraser Game

In the Eraser game $G_{\mathrm{E}}(f)$, Player II plays elements of $\omega \cup \{\mathsf{E}\}$, with the token E interpreted to mean "erase." This option allows Player II to erase his most recent move in ω. In contrast with the backtrack option, it is possible for Player II to erase infinitely many times and still play an infinite sequence.

To define the interpretation function ι_{E}, we first define $\eta : {}^{<\omega}(\omega \cup \{\mathsf{E}\}) \to {}^{<\omega}\omega$ by recursion. Let $\eta(\varnothing) := \varnothing$ and let

$$\eta(s^\frown\langle z \rangle) := \begin{cases} \eta(s)^\frown\langle z \rangle \text{ if } z \in \omega \\ \eta(s) \restriction (\mathrm{lh}(\eta(s)) - 1) \text{ if } z = \mathsf{E} \text{ and } \mathrm{lh}(\eta(s)) > 0 \\ \varnothing \text{ otherwise} \end{cases}$$

We then define $\iota_{\mathrm{E}} : {}^{\omega}(\omega \cup \{\mathsf{E}\}) \to {}^{\leq\omega}\omega$, $\iota_{\mathrm{E}}(y)(n) := m$ if $\exists i \, \forall j{>}i$, $\eta(y \restriction j)(n)$ is defined and equal to m.

We define $\bar{\tau}$ as usual and say that an Eraser strategy τ is winning for Player II if $\bar{\tau}(x) = f(x)$ for all $x \in A$.

Examples. Suppose Player II plays the sequence

$$\langle a_0, a_1, a_2, \mathsf{E}, a_3, a_4, \ldots \rangle$$

with one E only, then the interpretation will be

$$\langle a_0, a_1, a_3, a_4, \ldots \rangle.$$

Suppose Player II plays

$$\langle a_0, a_1, a_2, \mathsf{E}, a_3, \mathsf{E}, a_4, \mathsf{E}, a_5, \mathsf{E}, \ldots, a_i, \mathsf{E}, \ldots \rangle$$

then the interpretation will be

$$\langle a_0, a_1 \rangle$$

and Player II can only win the game if Player I plays out of A.

Theorem 4 *(Duparc). Let $A \subseteq {}^\omega\omega$. A function $f : A \to {}^\omega\omega$ is $\mathcal{F}(1,2) \Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{E}}(f)$.*

The proof of Theorem 4 relies on the following topological fact about $\mathcal{F}(1,2)$ functions:

Theorem 5 *A function $f : A \to {}^\omega\omega$ is $\mathcal{F}(1,2) \Leftrightarrow f$ is the limit of a sequence of continuous functions $f_n : A \to {}^\omega\omega$.*

Proof of Theorem 4. By Theorem 5, it suffices to show that f is the limit of a sequence of continuous functions $f_n \Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{E}}(f)$.

$\Rightarrow$: Let f_n be given and let τ_n be a winning strategy in the game $G_{\mathrm{W}}(f_n)$. Let τ be the following eraser strategy for Player II: "Let $x \in A$ be the play of Player I. Let i := 0. On a scratch tape, run τ_i until the first i elements of $\bar{\tau}_i(x)$ are determined. Then output these elements starting from the beginning of the tape, erasing only if necessary. Repeat the process with i := i + 1."

$\Leftarrow$: Let τ be winning for Player II is the game $G_{\mathrm{E}}(f)$. It suffices to give a sequence of Wadge strategies τ_n such that $\bar{\tau}_n(x)$ is infinite for all $x \in A$ and f is the limit of the $\bar{\tau}_n$. Let τ_n be the following: "Let $x \in A$ be the play of Player I. On a scratch tape, determine $\eta(\tau(x{\upharpoonright}n))$ (using the pass option until this is known). Then output $\eta(\tau(x{\upharpoonright}n))$, followed by random moves in ω."

□

6 The Multitape Game

In the Multitape game $G_{\mathrm{M}}(f)$, Player II plays elements of $\omega \cup \{\uparrow, \downarrow\}$. We think of Player II as having countably many *rows* (or *tapes*), which he can select using $\uparrow$ and $\downarrow$. We associate a natural number to each row, with Player II starting at row 0 at the beginning of the game. At any point in the game, if Player II is on row n, he may move to row $n+1$ using $\uparrow$ and row $n-1$ using $\downarrow$. (If he is on row 0, the $\downarrow$ option has no effect.) So, each move in ω occurs on some row. The interpretation is then the infinite sequence on the least row (if it exists) containing an infinite sequence. We refer to this row as the *output row.*

Formally, we define the interpretation function as follows. First, define $\rho : {}^{<\omega}(\omega \cup \{\uparrow, \downarrow\}) \to \omega$ as the "row" function on finite sequences of tokens. Let $\rho(\varnothing) := \varnothing$ and let

$$\rho(s^\frown\langle z\rangle) := \begin{cases} \rho(s) \text{ if } z \in \omega \\ \rho(s) + 1 \text{ if } z = \uparrow \\ \rho(s) - 1 \text{ if } z = \downarrow \text{ and } \rho(s) > 0 \\ 0 \text{ otherwise} \end{cases}$$

On infinite sequences of tokens, we define the partial function $\delta : {}^{\omega}(\omega \cup \{\uparrow, \downarrow\}) \to \omega$, $\delta(x) :=$ the least $n \in \omega$ (if it exists) such that $\forall i\ \exists j{>}i$, $\rho(x \restriction j) = n$ and $x(j-1) \in \omega$. In other words, the value of δ is the output row of an infinite sequence of tokens.

We next define $\zeta_n : {}^{<\omega}(\omega \cup \{\uparrow, \downarrow\}) \to {}^{<\omega}\omega$ by recursion. Let $\zeta_n(\varnothing) := \varnothing$ and let

$$\zeta_n(s^\frown\langle z\rangle) := \begin{cases} \zeta_n(s)^\frown\langle z\rangle \text{ if } z \in \omega \text{ and } \rho(s) = n \\ \zeta_n(s) \text{ otherwise} \end{cases}$$

In words, given a finite sequence of tokens, ζ_n is the output on the nth row. We may then define the interpretation function $\iota_{\mathrm{M}} : {}^{\omega}(\omega \cup \{\uparrow, \downarrow\}) \to {}^{\leq\omega}\omega$ by

$$\iota_{\mathrm{M}}(y) := \bigcup_{n\in\omega} \zeta_{\delta(y)}(y \restriction n) \text{ if } \delta(y) \text{ is defined.}$$

If $\delta(y)$ is undefined, we define $\iota_{\mathrm{M}}(y) := \varnothing$.

We define $\bar{\tau}$ as usual and say that an Multitape strategy τ is winning for Player II if $\bar{\tau}(x) = f(x)$ for all $x \in A$.

Examples. Suppose Player II plays the sequence

$$\langle a_0, \uparrow, a_1, \uparrow, a_2, \uparrow, a_3, \uparrow, \dots, a_i, \uparrow, \dots\rangle,$$

then the interpretation will be $\varnothing$ since Player II has only played a single element on each row. Suppose Player II plays a sequence

$$\langle a_0, a_1, \uparrow, a_2, \uparrow, a_3, a_4, \downarrow, a_5, a_6, \dots\rangle$$

such that the output row is row 1, then the interpretation will be

$$\langle a_2, a_5, a_6, \dots\rangle.$$

Theorem 6 *(Andretta, S.). Let $A \subseteq {}^{\omega}\omega$. A function $f : A \to {}^{\omega}\omega$ is $\mathcal{P}(\boldsymbol{\Pi}^0_2, \mathcal{F}(1,1))$ $\Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{M}}(f)$.*

Proof.

$\Leftarrow$: Let τ be the winning multitape strategy for Player II. Since every $\boldsymbol{\Sigma}^0_3$ set in the Baire Space is the disjoint union of $\boldsymbol{\Pi}^0_2$ sets, it suffices to give a $\boldsymbol{\Sigma}^0_3$ partition.

We define $A_n := \{x \in A : \delta(\hat{\tau}(x)) = n\}$. Note that $\delta(\hat{\tau}(x))$ is always defined if $x \in A$ since τ is winning. Thus $\langle A_n : n < \omega \rangle$ is indeed a partition of A. The following formula witnesses that A_n is $\boldsymbol{\Sigma}^0_3$ (in the relative topology):

$$\exists i\, \forall j{>}i\, [\rho(\tau(x{\restriction}j)) < n \Rightarrow \tau(x{\restriction}j)(j{-}1) \in \{\uparrow, \downarrow\}] \wedge \\ \forall i\, \exists j{>}i\, [\rho(\tau(x{\restriction}j)) = n \ \wedge \ \tau(x{\restriction}j)(j{-}1) \in \omega)].$$

In words, this formula says "there exists a round in the game after which Player II does not play an element of ω on a row less than n, and Player II plays infinitely many elements of ω on row n."

Furthermore, $f{\restriction}A_n$ is continuous. The following strategy is winning in $G_{\mathrm{W}}(f{\restriction}A_n)$: "Run τ on a scratch tape. Copy the output from row n, using the pass option when necessary."

$\Rightarrow$: Let A_n be given. Since A_n is $\boldsymbol{\Pi}^0_2$, there are $\boldsymbol{\Pi}^0_2$ formulas $\chi_n(x)$ (possibly with extra parameters) such that $x \in A_n \Leftrightarrow \chi_n(x)$. Since χ_n is $\boldsymbol{\Pi}^0_2$, we have that

$$\chi_n(x) \equiv \forall i\ \exists j\ \psi_n(x, i, j)$$

where ψ_n is a basic formula. Note that for any i and j we can check, using only a finite initial segment of x, whether $\psi_n(x, i, j)$ is true. Let $x \in {}^{\omega}\omega$ be the infinite play of Player I. We are ready to define the multitape strategy τ for Player II. For each row n, there will be two counters, i_n and j_n, which are initialized to 0. At each stage of the game, Player II is considered to be *working* on a row n. We say that Player II works on a row n for one step if he does the following:

Given i_n and j_n, try to determine whether the formula $\psi_n(x, i_n, j_n)$ is true. Since only a finite initial segment of x is known, it may be impossible to do this, in which case do nothing. If the formula is true, run the Wadge strategy σ_n for one step on row n. Increment the i_n counter by 1, and reset the j_n counter to 0. If the formula is false, increment the j_n counter by 1.

In the obvious way, Player II can work on every row n for infinitely many steps. Since $\langle A_n \in \omega \rangle$ is a partition, Player I will play into exactly one A_n. This means that exactly one of the formulas $\chi_n(x)$ will be true, which means that Player II will only play infinitely often (namely, the Wadge strategy σ_n) on row n. Then $\bar{\tau}(x) = \bar{\sigma}_n(x) = f{\restriction}A_n(x) = f(x)$, so τ is winning.

□

7 The Multitape Eraser Game

The Multitape Eraser game $G_{\mathrm{ME}}(f)$ is like the multitape game, except that Player II is given the additional option of erasing. So, Player II plays elements of $\omega \cup \{\uparrow, \downarrow, \mathsf{E}\}$. The output of Player II is the sequence on the least row (if it exists) on which Player II plays infinitely often in $\omega \cup \{\mathsf{E}\}$. Note that this sequence may not necessarily be infinite.

The definition of the interpretation function is similar to the case of the Multitape game. For convenience, we reuse some of the variables.

Define $\rho : {}^{<\omega}(\omega \cup \{\uparrow, \downarrow, \mathsf{E}\}) \to \omega$ as the "row" function on finite sequences of tokens. Let $\rho(\varnothing) := \varnothing$ and let

$$\rho(s^\frown\langle z\rangle) := \begin{cases} \rho(s) \text{ if } z \in \omega \cup \{\mathsf{E}\} \\ \rho(s)+1 \text{ if } z = \uparrow \\ \rho(s)-1 \text{ if } z = \downarrow \text{ and } \rho(s) > 0 \\ 0 \text{ otherwise} \end{cases}$$

On infinite sequences of tokens, define the partial function $\delta : {}^{\omega}(\omega \cup \{\uparrow, \downarrow, \mathsf{E}\}) \to \omega$, $\delta(x) :=$ the least $n \in \omega$ (if it exists) such that $\forall i\ \exists j{>}i$, $\rho(x{\restriction}j) = n$ and $x(j-1) \in \omega \cup \{\mathsf{E}\}$. In other words, the value of δ is the output row of an infinite sequence of tokens.

We next define $\zeta_n : {}^{<\omega}(\omega\cup\{\uparrow, \downarrow, \mathsf{E}\}) \to {}^{<\omega}(\omega\cup\{\mathsf{E}\})$ by recursion. Let $\zeta_n(\varnothing) := \varnothing$ and let

$$\zeta_n(s^\frown\langle z\rangle) := \begin{cases} \zeta_n(s)^\frown\langle z\rangle \text{ if } z \in \omega \cup \{\mathsf{E}\} \text{ and } \rho(s) = n \\ \zeta_n(s) \text{ otherwise} \end{cases}$$

In words, given a finite sequence of tokens, ζ_n is the output (with the E's still present) on the nth row. We may then define the interpretation function $\iota_{\mathrm{ME}} : {}^{\omega}(\omega \cup \{\uparrow, \downarrow, \mathsf{E}\}) \to {}^{\leq\omega}\omega$ by

$$\iota_{\mathrm{ME}}(y) := \iota_{\mathrm{E}}(\bigcup_{n\in\omega} \zeta_{\delta(y)}(y{\restriction}n)) \text{ if } \delta(y) \text{ is defined.}$$

If $\delta(y)$ is undefined, then we define $\iota_{\mathrm{ME}}(y) := \varnothing$.

We define $\bar{\tau}$ as usual and say that an Multitape Eraser strategy τ is winning for Player II if $\bar{\tau}(x) = f(x)$ for all $x \in A$.

Example. Suppose Player II plays a sequence

$$\langle a_0, a_1, \uparrow, a_2, a_3, a_4, \mathsf{E}, a_5, \mathsf{E}, a_6, \mathsf{E}, \ldots, a_i, \mathsf{E}, \ldots\rangle$$

such that the output row is row 1, then the interpretation will be

$$\langle a_2, a_3\rangle.$$

Note that in this case, Player II can only win the game if Player I plays out of A.

Theorem 7 *Let $A \subseteq {}^{\omega}\omega$. A function $f : A \to {}^{\omega}\omega$ is $\mathcal{P}(\boldsymbol{\Pi}^0_2, \mathcal{F}(1,2)) \Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{ME}}(f)$.*

Proof.

$\Leftarrow$: Let τ be the winning multitape strategy for Player II. We define $A_n := \{x \in A : \delta(\hat{\tau}(x)) = n\}$. As in the proof of Theorem 6, it follows that A_n is $\boldsymbol{\Sigma}^0_3$.

Furthermore, $f{\restriction}A_n$ is $\mathcal{F}(1,2)$. The following strategy is winning in $G_{\mathrm{E}}(f{\restriction}A_n)$: "Run τ on a scratch tape. Copy the output from row n, using the Eraser option when necessary."

$\Rightarrow$: As in Theorem 6, with "Wadge strategy" replaced by "Eraser strategy." □

8 Future Directions

We finish by noting several problems that are open (at least, as far as this author is aware). We state without proof the following:

Observation 8 *Let* $m, n \geq 1$. *Then* $\mathcal{F}(n,m) \subseteq \mathcal{F}(n+1, m+1)$. *Therefore,* $\mathcal{F}(n,m) \subseteq \mathcal{F}(n+k, m+k)$ *for any* $k \geq 0$.

The following fact is also easy to show:

Observation 9 *Let* $m, n \geq 2$. *Then* $\mathcal{P}(\boldsymbol{\Pi}^0_{m-1}, \mathcal{F}(1, m-n+1)) \subseteq \mathcal{F}(n,m)$.

Proof. Let $f : A \to {}^\omega\omega$ in $\mathcal{P}(\boldsymbol{\Pi}^0_{m-1}, \mathcal{F}(1, m-n+1))$ and let $\langle A_i : i < \omega \rangle$ be the partition. Let $Y \in \boldsymbol{\Sigma}^0_n$ and $Y_j \in \boldsymbol{\Pi}^0_{n-1}$ such that $Y = \bigcup_j Y_j$. It follows that

$$\begin{aligned} f^{-1}[Y] =& \bigcup_i (f{\restriction}A_i)^{-1}[Y] \\ =& \bigcup_i \bigcup_j (f{\restriction}A_i)^{-1}[Y_j] \\ =& \bigcup_i \bigcup_j A \cap X_{ij}, \text{ where } X_{ij} \in \boldsymbol{\Pi}^0_{m-1} \\ =& A \cap X, \text{ where } X \in \boldsymbol{\Sigma}^0_m. \end{aligned}$$

For the second to last equality, note that $f{\restriction}A_i \in \mathcal{F}(n-1, m-1)$ by Observation 8 (take $k = n-2$). □

Open Problem 10 *For which* m *and* n *is*

$$\mathcal{F}_{\mathrm{T}}(n,m) = \mathcal{P}_{\mathrm{T}}(\boldsymbol{\Pi}^0_{m-1}, \mathcal{F}(1, m-n+1))?$$

In particular, the statement for $n = m = 2$ is Theorem 3. If the statement were to hold for $n = m = 3$, then the Multitape game would characterize the total $\mathcal{F}(3,3)$ functions. Similarly, if the statement were to hold for $n = 2$ and $m = 3$, then the Multitape Eraser game would characterize the total $\mathcal{F}(2,3)$ functions.

Open Problem 11 *Is the inclusion in Observation* 8 *always proper?*

Open Problem 12 *For* $n \geq 2$ *and* $m \geq 1$, *is* $\mathcal{F}(n,m)$ *properly contained in* $\mathcal{F}(n-1, m)$*?*

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