A GAME FOR THE BOREL FUNCTIONS

BRIAN SEMMES

Abstract. We present an infinite game that characterizes the Borel functions on Baire Space.

1. Introduction

Our base theory is $ZF + AC_\omega(\mathbb{R})$. We use ${}^\omega\omega$ to denote the Baire Space, which is the set of infinite sequences of natural numbers together with the topology generated by the basic open sets $\{[s] : s \in {}^{<\omega}\omega\}$. As usual, a function $f : {}^\omega\omega \to {}^\omega\omega$ is **continuous** if the preimage of every open set is open. The **Borel sets** are the smallest class containing the open sets and closed under complements and countable unions (so also countable intersections), and a function $f : {}^\omega\omega \to {}^\omega\omega$ is a **Borel function** if the preimage of every Borel set is Borel. By a theorem of Lebesgue and Hausdorff, the Borel functions are the smallest class containing the continuous functions and closed under pointwise limits of countable sequences of functions. For further information about the Baire Space and Borel functions, the reader may consult [5] or [6].

In this paper, it will be convenient to define $\mathbf{\Lambda}_{m,n} := \{f : f^{-1}[Y] \in \mathbf{\Sigma}^0_n \text{ for every } Y \in \mathbf{\Sigma}^0_m\}$. For example, $\mathbf{\Lambda}_{1,1}$ denotes the continuous functions and $\mathbf{\Lambda}_{1,2}$ denotes the Baire Class 1 functions. (Baire Class 1 functions are pointwise limits of countable sequences of continuous functions and are precisely those functions for which the preimage of a $\mathbf{\Sigma}^0_1$ set is $\mathbf{\Sigma}^0_2$).

The *Wadge game* was developed by William Wadge in his PhD thesis [10] to characterize the notion of continuous reduction. Given two sets of reals $A, B \subseteq {}^\omega\omega$, A is **Wadge reducible** to B (A $\leq_{\mathrm{W}}$ B) if there is a continuous function f such that $f^{-1}[B] = A$. The Wadge game $G_{\mathrm{W}}(A, B)$ consists of two players and is defined in such a way that Player II has a winning strategy if and only if $A \leq_{\mathrm{W}} B$. The relation $\leq_{\mathrm{W}}$ is reflexive and transitive, so if we define $A \equiv_{\mathrm{W}} B :\Leftrightarrow A \leq_{\mathrm{W}} B \wedge B \leq_{\mathrm{W}} A$, then $\equiv_{\mathrm{W}}$ is an equivalence relation. The equivalence classes of $\equiv_{\mathrm{W}}$ are known as Wadge degrees and have been studied in

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detail by descriptive set theorists. In particular, the Wadge game was very useful in determining the structure of these degrees.

The *Backtrack game* $G_{\mathrm{bt}}(A, B)$, a generalization of the Wadge game, was developed by Robert van Wesep [9]. Using a theorem of John Jayne and Ambrose Rogers, Alessandro Andretta showed that the Backtrack Game characterizes the notion of $\mathbf{\Lambda}_{2,2}$ reduction. (We say that A is $\mathbf{\Lambda}_{m,n}$ reducible to B if there is a $\mathbf{\Lambda}_{m,n}$ function f such that $f^{-1}[B] = A$.) As in the Wadge case, one can define degrees with this notion of reducibility. The structure of these degrees was investigated in [1], and it was shown, among other things, that the determinacy of all Wadge games is equivalent to the determinacy of all Backtrack games. Interestingly, although Wadge determinacy follows easily from AD, it is unknown whether the converse holds.

Further progress was made by Jacques Duparc with the development of the *Eraser game*, which characterizes the Baire Class 1 functions. Intuitively, as the Baire Class 1 functions are countable point-wise limits of continuous functions, we see the Eraser game as expressing the notion of "taking limits." This intuitive idea was extended to develop the new game presented in this paper.

2. The Wadge Game

We begin by reviewing the Wadge game. For our purposes, it will be convenient to drop the A's and B's and define a two-player game $G_{\mathrm{W}}(f)$ in which Player II has a winning strategy if and only if f is continuous. Let $f : {}^{\omega}\omega \to {}^{\omega}\omega$. In the game $G_{\mathrm{W}}(f)$, Player I and Player II alternate moves for ω rounds. Player I plays elements $x_i \in \omega$ and Player II plays elements $y_i \in \omega \cup \{\mathsf{P}\}$. The token P is interpreted to mean "pass."

$$\begin{array}{lccccccccl} \text{I:} & x_0 & & x_1 & & x_2 & & & & x = \langle x_n : n \in \omega \rangle \\ & & & & & & & \cdots & & \\ \text{II:} & & y_0 & & y_1 & & y_2 & & & y = \langle y_n : n \in \omega \rangle \end{array}$$

After ω rounds, Player I has produced a sequence $x \in {}^{\omega}\omega$ and Player II has produced a sequence $y \in {}^{\omega}(\omega \cup \{\mathsf{P}\})$. Informally, if we take the sequence y with the P's removed, then Player II *wins the game* if and only if this sequence is infinite and equal to $f(x)$.

Formally, define $\theta : {}^{<\omega}(\omega \cup \{\mathsf{P}\}) \to {}^{<\omega}\omega$ by $\theta(\varnothing) := \varnothing$ and

$$\theta(s^\frown\langle z\rangle) := \begin{cases} \theta(s) \text{ if } z = \mathsf{P} \\ \theta(s)^\frown\langle z\rangle \text{ otherwise.} \end{cases}$$

If $x \in {}^{\omega}\omega$ is the play of Player I and $y \in {}^{\omega}(\omega \cup \{\mathsf{P}\})$ is the play of Player II, then Player II wins the game if $\bigcup_{s \subset y} \theta(s) = f(x)$. (Note that in order to have a chance, Player II must play infinitely often in ω.)

A **Wadge strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to {}^{<\omega}(\omega \cup \{\mathsf{P}\})$ such that $\mathrm{lh}(\tau(s)) = \mathrm{lh}(s)$ and $s \subseteq t \Rightarrow \tau(s) \subseteq \tau(t)$. The argument to τ is a finite sequence of moves by Player I and the value of τ is a finite sequence of moves by Player II. With respect to the diagram, if Player II follows τ then $\tau(\langle x_0, \dots, x_k \rangle) = \langle y_0, \dots, y_k \rangle$ and $y = \bigcup_{s \subset x} \tau(s)$.

A Wadge strategy τ for Player II is **winning** if Player II wins the game by following τ, regardless of what Player I plays. In other words, τ is winning if $\bigcup_{s \subset x} \theta(\tau(s)) = f(x)$ for all $x \in {}^{\omega}\omega$.

Theorem 1. *(Wadge) A function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ is continuous $\Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{W}}(f)$.*

Proof.

$\Leftarrow$: Suppose τ is the winning strategy. To show that f is continuous, it suffices to show that the preimage of a basic open set is open. Let $t \in {}^{<\omega}\omega$ and let $X := \bigcup \{[\,s\,] : \theta(\tau(s)) = t\}$. Then X is open and $f^{-1}[\,[\,t\,]\,] = X$.

$\Rightarrow$: Define τ by

$$\tau(s^\frown\langle m \rangle) := \begin{cases} \tau(s)^\frown\langle n \rangle \text{ if } f[\,[\,s^\frown\langle m \rangle\,]\,] \subseteq [\,\theta(\tau(s))^\frown\langle n \rangle\,] \\ \tau(s)^\frown\langle \mathsf{P} \rangle \text{ otherwise.} \end{cases}$$

It is not difficult to check that τ is well-defined and winning for Player II in $G_{\mathrm{W}}(f)$.

□

3. The Backtrack Game

As in the Wadge case, it will be convenient to drop the A's and B's and define a two-player game $G_{\mathrm{B}}(f)$ in which Player II has a winning strategy if and only if f is $\mathbf{\Lambda}_{2,2}$. The Backtrack game is like the Wadge game, except that Player II is given the additional option of erasing his entire output finitely many times. (The rules for Player I remain the same.) Formally, Player II plays elements of $\omega \cup \{\mathsf{P}, \mathsf{B}\}$ with the token P interpreted to mean "pass" and the token B interpreted to mean "backtrack."

Let θ be defined as in the Wadge game, we define the interpretation function $\iota_{\mathrm{B}} : {}^{\omega}(\omega \cup \{\mathsf{P}, \mathsf{B}\}) \to {}^{\leq\omega}\omega$,

$$\iota_{\mathrm{B}}(y) := \begin{cases} \varnothing \text{ if } \forall i\ \exists j \geq i\ y(j) = \mathsf{B} \\ \bigcup \{\theta(s) : (y \restriction i)^\frown s \subset y\} \text{ if } i \text{ is least such that } \forall j \geq i\ y(j) \neq \mathsf{B} \end{cases}$$

If $x \in {}^\omega\omega$ is the play of Player I and $y \in {}^\omega(\omega \cup \{\mathsf{P}, \mathsf{B}\})$ is the play of Player II, then Player II wins the game if $\iota_{\mathrm{B}}(y) = f(x)$. (Note that in order to have a chance, Player II cannot backtrack infinitely often and must play infinitely often in ω.)

The notion of strategy is defined analogously to the Wadge case: a **Backtrack strategy** for Player II is a function $\tau : {}^{<\omega}\omega \to {}^{<\omega}(\omega \cup \{\mathsf{P}, \mathsf{B}\})$ such that $\mathrm{lh}(\tau(s)) = \mathrm{lh}(s)$ and $s \subseteq t \Rightarrow \tau(s) \subseteq \tau(t)$. A Backtrack strategy τ for Player II is **winning** if $\iota_{\mathrm{B}}(\bigcup_{s \subset x} \tau(s)) = f(x)$ for all $x \in {}^\omega\omega$.

Theorem 2. *(Andretta). A function $f : {}^\omega\omega \to {}^\omega\omega$ is $\mathbf{\Lambda}_{2,2} \Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{B}}(f)$.*

The proof uses a theorem of Jayne and Rogers that the $\mathbf{\Lambda}_{2,2}$ functions are precisely those functions f admitting a $\mathbf{\Pi}^0_1$ partition $\langle A_n : n \in \omega \rangle$ such that $f \restriction A_n$ is continuous. Then, using the fact that the same property characterizes the Backtrack functions (functions for which Player II has a winning strategy in $G_{\mathrm{B}}(f)$), the result follows. We will not provide the details here, they may be found in [1] and [4]. An alternative proof of the Jayne-Rogers theorem may be found in [8].

4. The Eraser Game

In the Eraser game $G_{\mathrm{E}}(f)$, Player II plays elements of $\omega \cup \{\mathsf{E}\}$, with the token E interpreted to mean "erase." This option allows Player II to erase his most recent move in ω. We may think of this option as working like the "Delete" key on a keyboard. In contrast with the backtrack option, it is possible for Player II to erase infinitely many times and still play an infinite sequence.

Formally, define $\eta : {}^{<\omega}(\omega \cup \{\mathsf{E}\}) \to {}^{<\omega}\omega$ by $\eta(\varnothing) := \varnothing$ and

$$\eta(s^\frown\langle z \rangle) := \begin{cases} \eta(s)^\frown\langle z \rangle \text{ if } z \in \omega \\ \eta(s) \restriction (\mathrm{lh}(\eta(s)) - 1) \text{ if } z = \mathsf{E} \text{ and } \mathrm{lh}(\eta(s)) > 0 \\ \varnothing \text{ otherwise.} \end{cases}$$

We may then define $\iota_{\mathrm{E}} : {}^\omega(\omega \cup \{\mathsf{E}\}) \to {}^{\leq\omega}\omega$ as

$$\iota_{\mathrm{E}}(y)(n) := m \text{ if } \exists i\ \forall j \geq i\ \eta(y \restriction j)(n) \text{ is defined and equal to } m.$$

If $x \in {}^\omega\omega$ is the play of Player I and $y \in {}^\omega(\omega \cup \{\mathsf{E}\})$ is the play of Player II, then Player II wins the game if $\iota_{\mathrm{E}}(y) = f(x)$. As before, we define an **Eraser strategy** for Player II as a function $\tau : {}^{<\omega}\omega \to {}^{<\omega}(\omega \cup \{\mathsf{E}\})$ such that $\mathrm{lh}(\tau(s)) = \mathrm{lh}(s)$ and $s \subseteq t \Rightarrow \tau(s) \subseteq \tau(t)$. An Eraser strategy τ for Player II is **winning** if $\iota_{\mathrm{E}}(\bigcup_{s \subset x} \tau(s)) = f(x)$ for all $x \in {}^\omega\omega$.

Theorem 3. *(Duparc). A function $f : {}^\omega\omega \to {}^\omega\omega$ is $\mathbf{\Lambda}_{1,2} \Leftrightarrow$ Player II has a winning strategy in the game $G_{\mathrm{E}}(f)$.*

A proof will be given in Section 6.

5. The Tree Game

Let $f : {}^\omega\omega \to {}^\omega\omega$, we present the Tree game $G(f)$. As in our other games, Players I and II alternative moves for ω rounds and Player I plays elements of ω. In the Tree game, however, Player II plays elements of ${}^{<\omega}\omega \times {}^{<\omega}\omega$. In the limit, Player II is required to produce a partial function $\phi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that ϕ is monotone and length-preserving and $\mathrm{dom}(\phi)$ is a tree with a unique infinite branch. The output of Player II is then the infinite sequence (in ${}^\omega\omega$) of values along this branch. Player II *wins the game* if and only if this sequence is equal to $f(x)$, where $x \in {}^\omega\omega$ is the play of Player I.

More formally, define $\iota : \mathcal{P}({}^{<\omega}\omega \times {}^{<\omega}\omega) \to {}^\omega\omega \cup \{\varnothing\}$ as follows. Let $\phi \subseteq {}^{<\omega}\omega \times {}^{<\omega}\omega$. If ϕ is not a monotone, length-preserving function, or $\mathrm{dom}(\phi)$ is not a tree with a unique infinite branch, let $\iota(\phi) := \varnothing$. Otherwise, let $z \in {}^\omega\omega$ be the unique infinite branch of $\mathrm{dom}(\phi)$ and define $\iota(\phi) := \bigcup_{s \subset z} \phi(s)$. If $x \in {}^\omega\omega$ is the play of Player I and $y \in {}^\omega({}^{<\omega}\omega \times {}^{<\omega}\omega)$ is the play of Player II, then Player II wins the game if and only if $\iota(\bigcup_{n\in\omega} y(n)) = f(x)$.

We define a **Tree strategy** as simply a function $\tau : {}^{<\omega}\omega \to {}^{<\omega}\omega \times {}^{<\omega}\omega$ and say that τ is winning if $\iota(\bigcup_{s\subset x} \tau(s)) = f(x)$ for all $x \in {}^\omega\omega$. Note that if τ is winning, for every $x \in {}^\omega\omega$, τ must produce a monotone, length-preserving function $\phi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $\mathrm{dom}(\phi)$ is a tree. Therefore, if τ plays $\langle s, v\rangle$ at some stage of the game, we may assume that τ has already played $\langle s \restriction n, v \restriction n\rangle$ for every $n \le \mathrm{lh}(s) = \mathrm{lh}(v)$.

Theorem 4. *A function $f : {}^\omega\omega \to {}^\omega\omega$ is Borel $\Leftrightarrow$ Player II has a winning strategy in the game $G(f)$.*

Proof. The main part of the proof is to show that $\mathcal{F} := \{f :$ Player II has a winning strategy in $G(f)\}$ is closed under countable sequences of pointwise limits. Then, since $\mathcal{F}$ contains the continuous functions, this will show every Borel function is in $\mathcal{F}$. For the reverse direction, to show that every function in $\mathcal{F}$ is Borel, a simple complexity argument will suffice.

We begin by showing the closure property, namely that for any sequence $f_n \in \mathcal{F}$, if $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in {}^\omega\omega$, then $f \in \mathcal{F}$. Let f_n and f be given, and let τ_n be a winning strategy for Player II in the game $G(f_n)$. The idea is to amalgamate (or "squash") the strategies τ_n into a single strategy τ for f. There are two difficulties. Firstly, we do not know ahead of time the unique infinite branches that each τ_n will produce. Secondly, we do not know at what rate the f_n's will converge. By rate of convergence, we mean the unique non-decreasing sequence $r \in {}^{\omega}\omega$ where $r(n)$ is least such that for all $m \geq r(n)$, $f_m(x) \restriction n = f_{r(n)}(x) \restriction n$. The idea is that if we knew the infinite branches z_n and the rate of convergence r, we would know what to do. So, we will associate to each finite sequence a finite number of *guesses* about what will happen with the z_n and r. Then, under this association, an infinite sequence will correspond to a unique set of z_n and r (and vice versa), and from this we will be able to construct the amalgamated strategy τ.

To each element of s of ${}^{<\omega}\omega$ we will associate a natural number m and a sequence $\langle s_0, \ldots, s_k\rangle$ satisfying $s_i \in {}^{<\omega}\omega$, $\mathrm{lh}(s_i) = \mathrm{lh}(s)$, and $k = \max(\mathrm{lh}(s), m)$. The natural number m represents the guess that $r(\mathrm{lh}(s)) = m$ and the sequence $\langle s_0, \ldots, s_k\rangle$ represents guesses that $s_i \subset z_i$.

We define this association as a function $\rho : {}^{<\omega}\omega \to \omega$ and a function $\sigma : {}^{<\omega}\omega \to {}^{<\omega}({}^{<\omega}\omega)$. Let $\rho(\varnothing) := 0$, $\sigma(\varnothing) := \langle\varnothing\rangle$, and suppose that $\rho(s) = m$ and $\sigma(s) := \langle s_0, \ldots, s_k\rangle$ have been defined. For each $m' \geq m$, let $k'(m') = \max(m', \mathrm{lh}(s) + 1)$ and let

$$G(m') := \{\langle t_0, \ldots, t_{k'(m')}\rangle : \mathrm{lh}(t_i) = \mathrm{lh}(s)+1 \text{ and } \forall i \; 0 \leq i \leq k \Rightarrow s_i \subset t_i\}.$$

Let

$$G = \bigcup_{\substack{m' \geq m \\ \bar{t} \in G(m')}} \langle m', \bar{t}\,\rangle$$

and let $\beta : \omega \to G$ be an enumeration (without repetition) of G. For each j, if $\beta(j) = \langle m', \bar{t}\,\rangle$, define $\rho(s^\frown\langle j\rangle) = m'$ and $\sigma(s^\frown\langle j\rangle) := \bar{t}$.

Intuitively, we want the guesses we make on a successor of s to consistently extend the guesses we made on s. Hence the $m' \geq m$ condition: if we've already guessed that $r(\mathrm{lh}(s)) = m$ then it is inconsistent to guess that $r(\mathrm{lh}(s) + 1) < m$. Similarly, if we've already guessed that $s_i \subset z_i$, then it is inconsistent to guess that $t_i \subset z_i$ if s_i and t_i are incompatible (hence the $s_i \subset t_i$ condition).

We proceed with the definition of τ. At each round of the game, we consider certain sequences $s \in {}^{<\omega}\omega$ to be *activated*. Informally, s is activated if it looks like the guesses $\sigma(s)$ might be correct (given the behavior of the τ_i) and are consistent with the guesses we have made along s about the rate of convergence. More formally, if $\sigma(s) = \langle s_0, \ldots, s_k\rangle$ then we say that s is *activated* if

- $\forall i \; 0 \leq i \leq k \Rightarrow \tau_i$ has played $\langle s_i, v_i\rangle$,
- $\forall n \leq \mathrm{lh}(s) \; \rho(s \restriction n) > 0 \Rightarrow v_{\rho(s\restriction n)} \restriction n \neq v_{\rho(s\restriction n)-1} \restriction n$, and
- $\forall n \leq \mathrm{lh}(s) \; \forall i \; \rho(s \restriction n) \leq i \leq k \Rightarrow v_{\rho(s\restriction n)} \restriction n = v_i \restriction n$.

Note that the v_i's are unique (since τ_i must produce a function) and $\mathrm{lh}(v_i) = \mathrm{lh}(s_i) = \mathrm{lh}(s)$. To understand the first condition, recall that s_i

represents the guess that $s_i \subset z_i$ where z_i is the infinite branch produced by τ_i. Intuitively, if τ_i hasn't played s_i yet, we are not yet interested in this guess. For the second condition, recall that $\rho(s \upharpoonright n) = m$ is the guess that $r(n) = m$. In words, this is the guess that the sequence of functions converges on the first n digits precisely at the mth function. If our guess for the mth function and our guess for the $(m-1)$th function agree on the first n digits, then the guess m is too big, given our other guesses. Similarly, for any $n \leq \mathrm{lh}(s)$, the guess $\rho(s \upharpoonright n) = m$ is too small if, for some $i > m$, our guess for the mth function and the ith function disagree on the first n digits.

Note that once a sequence is activated, it will remain so. Since we can assume that τ_i has played $\langle s_i, v_i\rangle$ means that τ_i has already played $\langle s_i \upharpoonright n, v_i \upharpoonright n\rangle$ for all $n \leq \mathrm{lh}(s)$, it follows that once s is activated, all $s' \subseteq s$ are activated as well.

The strategy τ proceeds as follows. For an activated sequence s as above, τ *resolves* s by playing $\langle s, v_{\rho(s)}\rangle$. By cycling through the sequences in ${}^{<\omega}\omega$ in the appropriate way, τ can ensure that, in the limit, every sequence that is activated is resolved. These will be the only moves that τ plays, so this completes the definition of τ.

It remains to show that τ is winning in the game $G(f)$. It is not difficult to check that τ produces a monotone, length-preserving function ϕ such that $\mathrm{dom}(\phi)$ is a tree, so it remains to be shown that τ produces a unique infinite branch along which the value is $f(x)$. On input x, let r be the rate of convergence and let z_i be the infinite branches produced by τ_i. It is clear that there is a unique $z \in {}^{\omega}\omega$ such that for all $s \subset z$, $\rho(s) = r(\mathrm{lh}(s))$ and $\sigma(s) = \langle s_0, \ldots, s_k\rangle$ with $s_i \subset z_i$. In other words, z is the unique infinite sequence along which every guess is correct. It is not difficult to see that every $s \subset z$ will be activated at some stage, and moreover, τ will resolve every such s by playing $\langle s, f(x) \upharpoonright \mathrm{lh}(s)\rangle$.

It remains to be shown that z is the only infinite branch produced by τ. Let $z' \neq z$, it will be shown that there is an initial segment of z' that is never activated. Let z'_i be the infinite branches encoded by z'. If $z'_i \neq z_i$ for some i, then there is an $s \subset z'_i$ such that τ_i will never play $\langle s, v\rangle$ for any v. (Otherwise, τ_i would produce two infinite branches, a contradiction.) Therefore, s will never be activated.

Assume $z'_i = z_i$ for all i, so it must be the case that $\rho(s) \neq r(\mathrm{lh}(s))$ for some $s \subset z'_i$. If $\rho(s) > r(\mathrm{lh}(s))$, then s will never be activated since the guess $\rho(s)$ is too big. If $\rho(s) < r(\mathrm{lh}(s))$, then there is an i such that $i > \rho(s)$ and $f_{\rho(s)}(x) \upharpoonright \mathrm{lh}(s) \neq f_i(x) \upharpoonright \mathrm{lh}(s)$. Pick t with $s \subseteq t \subset z'$ such that $\sigma(t) = \langle t_0, \ldots, t_k\rangle$ with $i \leq k$. Then t is never activated, since t_i witnesses that the guess $\rho(s)$ was too small.

This completes the proof of the closure property.

For the reverse direction, that every function in $\mathcal{F}$ is Borel, let $f \in \mathcal{F}$ and let τ be a winning Tree strategy for Player II in the game $G(f)$. It suffices to show that the preimage of a basic open set $[u]$ is $\mathbf{\Sigma}^1_1$:

$$\exists z \in {}^\omega\omega\ \exists i\ \tau(x \restriction i) = \langle z \restriction \mathrm{lh}(u), u\rangle \wedge$$
$$\forall n\ \exists i\ \tau(x \restriction i) = \langle z \restriction n, v\rangle.$$

□

6. The Eraser Revisited

If we consider the Tree game and add additional requirements for Player II, we may obtain games that are equivalent to the Wadge, Backtrack, and Eraser games. For the Wadge case, we require Player II to produce a function $\phi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ such that $\mathrm{dom}(\phi)$ is linear, e.g. for all $s, t \in \mathrm{dom}(\phi)$, $s \subseteq t$ or $t \subseteq s$. It is immediate that this linear Tree game is equivalent to the Wadge game and vice versa. (For one direction, to simulate the passing option, Player II may simply play the pair $\langle \varnothing, \varnothing\rangle$.)

Moreover, if we require $\mathrm{dom}(\phi)$ to be finitely branching, e.g. that the set $\{s \in \mathrm{dom}(\phi) : \mathrm{lh}(s) = n\}$ is finite for every n, the resulting game is equivalent to the Eraser game. We argue as follows: let τ be a finitely branching Tree strategy that is winning for some f. We may assume that the only duplicate moves played by τ are $\langle \varnothing, \varnothing\rangle$. The following Eraser strategy is winning in $G_{\mathrm{E}}(f)$: "When τ plays a pair $\langle s, v\rangle$ of non-empty sequences, put v on the output tape, erasing only when necessary." For the other direction, simply note that if an Eraser strategy τ_{E} produces an infinite sequence (which it must do in order to be winning), at any finite depth it can only use the eraser finitely many times. Thus, it is easy to construct a finitely branching Tree strategy that is equivalent to τ_{E}.

Following a suggestion of Benedikt Löwe, we may prove Theorem 3 by simply inspecting the proof of Theorem 4.

Proof of Theorem 3. Suppose $f \in \mathbf{\Lambda}_{1,2}$, it must be shown that there is a finitely branching Tree strategy τ that is winning in $G(f)$. Since f is Baire Class 1, it is the pointwise limit of a countable sequence of continuous functions $\langle f_n : n \in \omega\rangle$. For each n, let τ_n be a linear Tree strategy that wins $G(f_n)$ and let τ as in the proof of Theorem 4. We already know from the proof that τ is winning in $G(f)$, it just needs to be shown that τ is finitely branching. It suffices to show, for any activated sequence s, only finitely many successors $t = s^\frown\langle j\rangle$ of s can be activated. Let $s \in {}^{<\omega}\omega$ be activated and let z_i be the infinite branches produced by τ_i. Since each τ_i is linear, we need only concern ourselves with those successors t of s such that $\sigma(t) = \langle t_0, \ldots, t_k \rangle$ with $t_i \subset z_i$. If $\rho(t) > r(\mathrm{lh}(t))$ then t will never be activated. Therefore, we only need to consider those successors t such that the guesses $\sigma(t)$ are correct and the guess $\rho(t)$ is either correct or too small... but there are only finitely many of these.

For the other direction, suppose τ is a finitely branching Tree strategy that wins $G(f)$. Again, we assume that the only duplicate moves played by τ are $\langle \varnothing, \varnothing \rangle$. To show that $f \in \mathbf{\Lambda}_{1,2}$ it suffices to show that the preimage of a basic open set [u] is $\mathbf{\Sigma}^0_2$:

$$\exists s \in {}^{<\omega}\omega \; \exists i \; \tau(x \restriction i) = \langle s, u \rangle$$
$$\exists j \; \forall k \geq j \; \tau(x \restriction k) = \langle \varnothing, \varnothing \rangle \;\vee\; (\tau(x \restriction k) = \langle t, v \rangle \;\wedge\; s \subseteq t).$$

□

Thus, we have shown that the finitely branching Tree game characterizes the $\mathbf{\Lambda}_{1,2}$ functions. Following the suggestion of Löwe once more, we may continue this process and obtain a game for the $\mathbf{\Lambda}_{1,3}$ functions. We require Player II to produce a function ϕ with the following property: for each s, $\{t \in \mathrm{dom}(\phi) : s \subset t\}$ is infinite $\Rightarrow s \subset z$, where z is the infinite branch of $\mathrm{dom}(\phi)$. In other words, if Player II extends a sequence s infinitely often (not counting duplicates), then s is an initial segment of the infinite branch. If τ is a Tree strategy for Player II that meets this requirement, we call τ a $\mathbf{\Lambda}_{1,3}$ strategy.

Theorem 5. *A function $f : {}^{\omega}\omega \to {}^{\omega}\omega$ is $\mathbf{\Lambda}_{1,3}$ $\Leftrightarrow$ Player II has a winning $\mathbf{\Lambda}_{1,3}$ strategy in the game $G(f)$.*

Proof. For the $\Rightarrow$ direction, since f is $\mathbf{\Lambda}_{1,3}$ it is the pointwise limit of a countable sequence of $\mathbf{\Lambda}_{1,2}$ functions $\langle f_n : n \in \omega \rangle$. For each n, let τ_n be a finitely branching Tree strategy that is winning in the game $G(f_n)$. As before, we let τ as in the proof of Theorem 4, except that a stricter definition of activation is required. We say that s is *activated* if (1) s is activated in the original sense and (2) if $\sigma(s) = \langle s_0, \ldots, s_k \rangle$ and $\rho(s) = m$, then τ_i has extended s_i m times for all i, $0 \leq i \leq k$. In other words, the second condition says that $\{t \supseteq s_i : \tau_i \text{ has played } \langle t, v \rangle \text{ for some } v\}$ has at least m elements. It is clear that the proof of Theorem 4 works with this stronger version of activation, so let τ be given by the (new) proof. We know from the proof that τ is winning in the game $G(f)$, so we only need to show that τ satisfies the $\mathbf{\Lambda}_{1,3}$ requirement. It suffices to show that for any activated sequence s, if s encodes an incorrect guess then only finitely many successors of s are activated. (Then the tree $\{t \in \mathrm{dom}(\phi) : t \subseteq s \vee s \subseteq t\}$ is finitely branching, so s is not extended infinitely many times by τ.) Let $s \in {}^{<\omega}\omega$ be an activated sequence that encodes an incorrect guess. Let z_i be the infinite branches produced by τ_i and let r be the rate of convergence. Since s encodes an incorrect guess, there are two cases to consider: (1) $\sigma(s) = \langle s_0, \dots s_k \rangle$ with $s_i \not\subset z_i$ for some i, (2) $\rho(s') < r(\mathrm{lh}(s'))$ for some $s' \subseteq s$. (Since s is activated, if we assume (1) doesn't hold then $\rho(s') > r(\mathrm{lh}(s'))$ is impossible.)

If we are in the first case, then let i be such that $s_i \not\subset z_i$. Since τ_i is finitely branching, there are finitely many t_j such that $s_i \subset t_j$, $\mathrm{lh}(t_j) = \mathrm{lh}(s) + 1$, and τ_i plays $\langle t_j, v \rangle$ for some v. Since each $t_j \not\subset z_i$, it may be extended only finitely many times by τ_i. Let $m \geq i$ be strictly greater than the number of times any t_j is extended, it follows that a successor t of s cannot be activated if $\rho(t) \geq m$. Thus we need only consider those t such that $\rho(t) < m$, but only finitely many of these may be activated since the τ_i are finitely branching.

In the second case, we assume that the guesses $\sigma(s) = \langle s_0, \dots, s_k \rangle$ are correct. Let $s' \subseteq s$ such that $\rho(s') < r(\mathrm{lh}(s'))$. Since the guess $\rho(s')$ is too small, there is a $k' > \rho(s')$ such that $f_{k'}(x) \restriction \mathrm{lh}(s') \neq f_{\rho(s')}(x) \restriction \mathrm{lh}(s')$. (Note that $k' > k$ since s is activated.) Since $\tau_{k'}$ is finitely branching, there are finitely many t_j of length $\mathrm{lh}(s) + 1$ such that $\tau_{k'}$ plays $\langle t_j, v \rangle$ for some v. Let j' be unique such that $t_{j'} \subset z_{k'}$. So, if $j \neq j'$, then t_j is extended only finitely many times by $\tau_{k'}$. Let $m \geq k'$ be strictly greater that the number of times any t_j is extended for $j \neq j'$. Then a successor t of s can never be activated if $\rho(t) \geq m$. Namely, if $\sigma(t)(k') = t_j$ with $j \neq j'$, then $\sigma(t)$ can never be activated by the choice of m. If $\sigma(t)(k') = t_{j'}$, in other words if the guess $\sigma(t)(k')$ is correct, then t can not be activated by choice of k'.

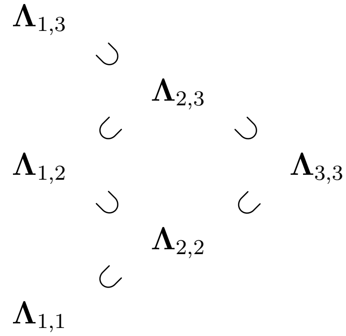
For the other direction, suppose τ is a $\mathbf{\Lambda}_{1,3}$ strategy that wins $G(f)$. Again, we assume that the only duplicate moves played by τ are $\langle \varnothing, \varnothing \rangle$. To show that $f \in \mathbf{\Lambda}_{1,3}$ it suffices to show that the preimage of a basic open set [u] is $\mathbf{\Sigma}^0_3$:

$$\begin{aligned} &\exists s \in {}^{<\omega}\omega \; \exists i \; \tau(x \restriction i) = \langle s, u \rangle \\ &\forall j \; \exists k \geq j \; (\tau(x \restriction k) = \langle t, v \rangle \;\wedge\; s \subseteq t). \end{aligned}$$

□

7. Future Directions

The original goal of the author was to find a game that characterizes the $\mathbf{\Lambda}_{3,3}$ functions. Interestingly, to characterize these functions seems to be a more difficult problem than to characterize the Borel functions. The following picture is useful:



In the previous section, we presented games characterizing the $\mathbf{\Lambda}_{1,1}$, $\mathbf{\Lambda}_{1,2}$, and $\mathbf{\Lambda}_{1,3}$ functions. We also mentioned that there is a Tree game that is equivalent to the Backtrack game. For this, we require Player II to produce a function $\phi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ with the property that there is an $n \in \omega$ such that for all $s, t \in \mathrm{dom}(\phi)$, $\mathrm{lh}(s), \mathrm{lh}(t) \geq n \Rightarrow s \subseteq t$ or $t \subseteq s$. In other words, we require that the tree produced by Player II is linear (the $\mathbf{\Lambda}_{1,1}$ requirement) after some finite depth. It is not difficult to prove that this game is equivalent to the Backtrack game. Thus, by Theorem 2, this game characterizes the $\mathbf{\Lambda}_{2,2}$ functions.

Future work will be directed towards extending these results to the $\mathbf{\Lambda}_{3,3}$ case. In particular, we would like to find a Tree game characterizing $\mathbf{\Lambda}_{3,3}$ and determine whether the analogous partition property holds. Namely, we would like to know if a function f is $\mathbf{\Lambda}_{3,3}$ if and only if there is a $\mathbf{\Pi}^0_2$ partition $\langle A_n : n \in \omega \rangle$ such that $f \restriction A_n$ is continuous.

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