Exploring the Iterated Update Universe

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Abstract. We investigate the asymptotic properties of the logical system for information update developped by Baltag, Moss and Solecki [2]. We build on the idea of looking at update logics as dynamical systems. We show that every epistemic formula either always holds or is always refuted from certain moment on, in the course of update with factual epistemic events, i.e. events with only propositional prerequisite formulas, or signals. We characterize in terms of a pebble game the class of frames such that iterated update with factual epistemic events built over them gives rise only to finite sets of reachable states. The characterization is nontrivial, and so the 'Finite Evolution Conjecture' (see van Benthem [4]) is refuted. Finally, after giving some basic insights into the dissipative nature of update with general, nonfactual epistemic events, we show the distinctive stabilizing nature of *epistemically ordered* multi-S5 events - events in which agents can be ordered in terms of how much they know.

1. Introduction

Modeling knowledge has always been motivated by matters concerning *knowledge change*. Many issues that propelled interest and investigations in formal epistemic systems, such as problems of communication in multi-agent systems, database quering, reasoning in games or the folklore knowledge puzzles (e.g. Muddy Children), have a great deal to do with the evolution of information or knowledge. The general challenging question is: "How are agents' beliefs - I will ignore the distinction between knowledge and belief in this paper - affected by certain events with clear epistemic content?"

In the static setting probably the most widespread approach to formalizing epistemic situations is that of modal logic and Kripke semantics. For a given set of atomic propositions and a set of agents, each pointed Kripke model - *epistemic state* - encodes information about corresponding facts and agents' knowledge about them, knowledge about this knowledge etc, as well as facts held in common knowledge. Following original contribution by Hintikka [18], this is the traditional way of representing agents' factual knowledge, as well as higher-order knowledge - knowledge about knowledge - in a compact way.

How about modeling change? When we build dynamic events explicitly into the formalism, we arrive at various kinds of Dynamic Epistemic Logic. Plaza [21] allowed events mirroring public announcements to a group of agents, Gerbrandy [13], Gerbrandy and Groeneveld [14],[15] extended it to such cases as revealing information only to a subset of agents, sending messages by an agent to a group of agents as well as all events obtained by the application of PDL constructors. Finally, Balag, Moss and Solecki [2] presented a logic capturing effects of updates with events represented by the structures related to the usual Kripke models, so called *epistemic events*, capable of representing quite intricate epistemic patterns. The work in this paper will concentrate around this last, most flexible approach.

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In the case of each formalism, the semantic part constitutes essentially a *dynamical system*. Epistemic states make up the space of states over which the system "travels", or a *phase space*, and each epistemic event defines a map over it. Each event happening at any state brings us to a new state, representing the resulting epistemic situation. At this state the same or different event may unfold, and so on.

The field of dynamical systems, investigating properties of systems subject to certain dynamical processes - in our case: information updates - is a well established branch of mathematics. The most standard questions that a general theory of dynamical systems deals with concern the asymptotic behavior: how the system behaves in the (time) limit of a dynamical process? In our case those questions would concern the limiting properties of iterated execution of epistemic events, iterated update. For example, when the same epistemic event is carried out iteratively, is the set of epistemic states reachable in such a process finite? Do we reach a fixed point, or perhaps we eventually loop over a finite set of states? How about satisfacion of formulas in the limit of such update? etc.

Incidentally, two probably best known puzzles in the epistemic literature, the Muddy Children and the Coordinated Attack puzzles[1], concern exactly the asymptotic behavior of appropriate dynamical update systems. In the case of the Muddy Children, the problem is whether and when in the course of repeated collective answer to the same question each child in a group will find out whether its forehead is dirty. In the case of the Coordinated Attack, the puzzle arises because after arbitrary number of mutual reassurances on the part of the two generals, they will not achieve Common Knowledge of their willingness to attack at a certain time. Similar method is employed in the logical analysis of a game theoretic solution concept of Iterated Deletion of Dominated Strategies by van Benthem [5]. The set of strategies returned by this solution concept for every normal form game is precisely the set "surviving" certain iterated epistemic update, pruning out at each stage unreasonable strategies.

The first paper to look at the iterated update and limit behavior in the context of various Dynamic Epistemic Logics was van Benthem [4]. It analyzed the iterated public anouncement, which corresponds to the iterated semantic relativization of a model (see section 2.2) and always leads to a fixed point of update, as well as signalled the problems of iterated update with more general epistemic events. Among others, it stated the 'Finite Evolution Conjecture': for any pair of finite epistemic state and epistemic event, only finitely many epistemic states can be reached in the course of iterated update.

In this paper we formalize the notion of the dynamical product update system, based on the product update with epistemic events, as formalized in Baltag *et al.* [2]. The main goal is to investigate the nature of the asymptotic behavior of epistemic update, and in particular to give answer to the 'Finite Evolution Conjecture'.

In chapter 2 we briefly review the stadard epistemic logic, the product update logic and state the extension of bisimulation to update models (see van Eijck, Ruan and Sadzik [11] for more details and proofs). Then, in chapter 3, we introduce the dynamical product update system, some basic concepts for analyzing asymptotic behavior and extend the dynamic epistemic language with limit modalities. Finally, chapters 4 and 5 adress the main issue of asymptotic behavior of product update, for factual (i.e. with only propositional prerequisite formulas, signals) and general epistemic events, respectively. We show that every epistemic formula either always holds or is always refuted in the course of up-

[1] for a textbook presentation see Fagin, Halper, Moses, Vardi [12]. For a DEL treatment of Muddy Children puzzle see Gerbrandy and Groeneveld [15].

date from certain moment on, in the course of update with a factual epistemic event. We characterize in terms of a pebble game the class of frames such that iterated update with factual epistemic events built over them gives rise only to finite sets of reachable states. The characterization is nontrivial, and so we refute the 'Finite Evolution Conjecture'. Finally, after giving some basic insights into the dissipative nature of update with general, nonfactual epistemic events, we show the distinctive stabilizing nature of *epistemically ordered* multi-S5 events - events in which agents can be ordered in terms of how much they know. Epistemic ordering is an "almost" necessary and a sufficient condition on an S5 frame to give rise to finite evolution only. Brief conclusion with several open problems follows.

2. Epistemic States and Epistemic Events

2.1. Epistemic logic. First lets fix the static part of our systems. Following well-trodden path we take a Kripke structure as a formal representation of a static epistemic situation. We do not impose any restrictions on accessibility relations at this point, allowing for a very permissive concept of knowledge.

Definition 1. *Fix once and for all a set of agents I and a countably infinite set of atomic propositions Prop. A* state model *is a structure $M = \{S_M, R^i_M|_{i\in I}, V_M\}$, where*

- *S_M is a nonempty set of possible worlds, or (simple) states;*
- *$R^i_M \subseteq S^2_M|_{i\in I}$ are accessibility relations for each agent;*
- *$V_M : S_M \to Pow(Prop)$ assigns a set of atomic propositions to each state.*

  A pointed state model, *or an* epistemic state *is a state model with a distinguished "actual" state $m \in S_M$, denoted (M, m). Let $\mathcal{M}$ be the set of all epistemic states.*

The most standard language for epistemic logic is the polymodal language with additional group common knowledge modalities.

Definition 2. *The language $\mathcal{L}_{EL-C}$ will be defined recursively:*

$$\mathcal{L}_{EL-C} ::= \ \top \mid p \mid \neg\varphi \mid \varphi \wedge \phi \mid [i]\varphi \mid C_J\varphi \ ,$$

where $p \in$Prop, $i \in I$, $J \subseteq I$.

We use the standard abbreviations of $\perp, \varphi \vee \phi$ and $\langle i\rangle \varphi$.

The satisfaction definition will tie the object language to epistemic states. For a given model M and $J \subset I$ let $J{-}path$ be a sequence $(m^1, i^1, m^2, ..., i^{k-1}, m^k)$, $k > 1$, such that $m^l \in S_M$ and $i^l \in J$ for $l \leq k$.

Definition 3. *We define the* satisfaction *relation holding between epistemic states and formulas in the language $\mathcal{L}_{EL-C}$ recursively:*

$$\begin{array}{lll}
(M,m) \vDash \top & \textit{always}, & \\
(M,m) \vDash p & \textit{iff} & p \in V_M(m), \\
(M,m) \vDash \neg\varphi & \textit{iff} & (M,m) \nvDash \varphi, \\
(M,m) \vDash \varphi \wedge \phi & \textit{iff} & (M,m) \vDash \varphi \text{ and } (M,m) \vDash \phi, \\
(M,m) \vDash [i]\,\varphi & \textit{iff} & (M,n) \vDash \varphi \textit{ for all n s.t. } (m,n) \in R^i_M, \\
(M,m) \vDash C_J\varphi & \textit{iff} & (M,n) \vDash \varphi \textit{ for all n s.t. } \exists J{-}path \\
 & & (m^1, ..., i^{k-1}, m^k) \textit{ with } m^1 = m, m^k = n.
\end{array}$$

2.2. Product Update. Each epistemic state is a well defined model for a *static* epistemic situation. The following epistemic events are the models for *dynamic* epistemic scenarios.

Definition 4. *For a given set of agents I and language $\mathcal{L}$ an $\mathcal{L}-$*update model *is a structure $A = \{S_A, R^i_A|_{i\in I}, p_A\}$, where:*

- *S_A is a nonempty finite set of (simple) events;*
- *$R^i_A \subseteq S^2_A|_{i\in I}$ are accessibility relations for each agent;*
- *$p_A : S_A \to \mathcal{L}$ assigns a* precondition *formula to each event.*

A $\mathcal{L}-$(multi) pointed update model, *or an $\mathcal{L}-$*epistemic event *is an $\mathcal{L}-$update model with a set of distinguished "possibly actual" events, denoted e.g. $(A, \mathbf{a})$.*[2] *$\mathcal{A}_\mathcal{L}$ is the space of all such $\mathcal{L}-$epistemic events.*

$\mathcal{L}-$epistemic events with single distinguished simple events are models for deterministic dynamic epistemic situations. Their interpretation is very much like the interpretation of usual "static" Kripke structures. In particular, they are also tangled in a mesh of epistemic admissibility relations, which reflects underlying uncertainty of the agents. The difference between the two types of models lies in the last element, the assignment. The dynamic simple events are not differentiated by their atomic content, atomic propositions that are true there, as in the case of static models, but by the precondition formulas. Each of them constitutes a condition of the simple event to be carried out - each simple event can happen only in the epistemic states that satisfy the precondition. In other words, it is the information, or a *signal* that the event gives out about the underlying epistemic state. As the dynamic (simple) events are identified with the signals they provide, we abstract away from their true nature, leaving many possibilities open. Most usually they are various kinds of communicative acts. It also follows that we treat solely the events that do not affect the facts - as e.g. openning a window - and only model widely conceived flow of information (see comments in Baltag *et al.* [2] and van Bethem, van Eijck, Kooi [7]).

Contrary to epistemic states, epistemic events are by default nondeterministc. Nondeterministic update with an epistemic event $(A, \mathbf{a})$, will return a set of epistemic states. (we will denote it e.g. $(M, \mathbf{m})$).

Example 5. (Public Announcement) *The first paper on Epistemic Update Logic - Plaza [21] - investigated the epistemic logic extended by the operation of public announcement of a formula to the agents. Semantically, the scenario of the announcement of formula φ can be captured by the following epistemic event:* $(A, a) = (\{S_A, R^i_A|_{i\in I}, p_A\}, a) = (\{\{a\}, \{(a, a)\}|_{i\in I}, \{(a, \varphi)\}\}, a)$:

Figure 1. Public Announcement episemic event

[2]Boldface will always mark that a symbol denotes a set.

Public announcement is the simplest type of communicative acts. See Baltag *et al.* [2] for examples of more involved *Secure announcement to a set of agents, Announcements with a suspicious outsider* or *Announcement with common knowledge of suspicion.* Consider also the following communicative scenario, which is the representation of one round of communication from the Coordinated Attack puzzle (or Electronic Mail Game, see e.g. Fagin, Halpern, Moses, Vardi [12] for more details):

Example 6. *Two divisions of the White army, each headed by a general, occupy two hills overlooking the valley. The valley is taken by the opposing Red army. General X of the White army sends a messanger to general Y only in the case when he wants to attack at dawn the next day, in order to coordinate the attack. General Y sends back a messanger with the acknowledgement once he gets the message. General X sends a messanger along the path a, and general Y sends back a messanger along the path b. Each path might be occupied by the Red army, in which case the message is lost.*

There are 4 (simple) events that might unfold, depending whether X wants to attack at dawn, and which paths are occupied by the Reds: no attack at dawn, attack and first messanger intercepted, attack and first messanger goes through but second one intercepted, attack and communication not hindered.

The following is the update model A for this scenario:

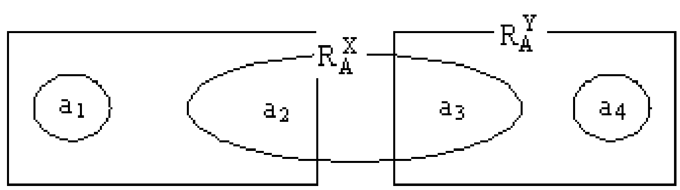

Figure 2. Acknowledged Message Passing frame

$p_A(a_1) = \neg attack$
$p_A(a_2) =$*attack& path_a_taken_by_the_Reds*
$p_A(a_3) =$*attack&¬path_a_taken_by_the_Reds&path_b_taken_by_the_Reds*
$p_A(a_4) =$*attack&¬path_a_taken_by_the_Reds&¬path_b_taken_by_the_Reds*

In the event a_1 X does not attack at dawn and so sends no message. He can clearly distinguish this event from any other. In a_2 the message is sent, but lost on the way to Y, who does not know whether the message was sent in the first place. In a_3 Y gets the message, but his acknowledgement is lost on the way to X, whereas in a_4 the whole process of message sending and acknowledging goes smoothly.

Call the frame *on which the model is based, i.e.* $A=\{S_A, R_A^i|_{i\in I}\}$, *an* Acknowledged Message Passing frame. *We will take a closer look at the properties of update with such frames in section 4.2.*

We dynamify the object language via new kind of modalities - update modalities, indexed by the epistemic events introduced above. Notice in this respect that we demanded the update models to be finite. The intended meaning of a formula $\langle (A,a)\rangle\, \varphi$ is that φ holds after we update with (A,a).

Definition 7. *i) For any language $\mathcal{L}$ interpretable over epistemic states let the language $\mathcal{L}^U$ be defined recursively (jointly with $\mathcal{L}^U-$epistemic events):*

$$\mathcal{L}^U \quad : \quad := \quad "\mathcal{L}" \mid \langle (A, \mathbf{a}) \rangle \varphi \quad ,$$
$$(A, \mathbf{a}) \text{ is an } \mathcal{L}^U-\text{epistemic event.}$$

where "$\mathcal{L}$" is a recursive definition of language $\mathcal{L}$.

ii) For languages $\mathcal{L}$ and $\mathcal{L}'$ interpretable over epistemic states let the language $\mathcal{L}^{U(\mathcal{L}')}$ be defined recursively:

$$\mathcal{L}^{U(\mathcal{L}')} \quad : \quad := \quad "\mathcal{L}" \mid \langle (A, \mathbf{a}) \rangle \varphi \quad ,$$
$$(A, \mathbf{a}) \text{ is an } \mathcal{L}'-\text{epistemic event.}$$

We will also use the usual abbreviations $[(A, \mathbf{a})]\varphi$. As an example, the language $(\mathcal{L}_{EL-C})^U$, abbreviated $\mathcal{L}^U_{EL-C}$, is defined as follows:

$$\mathcal{L}^U_{EL-C} \quad : \quad := \quad \top \mid p \mid \neg\varphi \mid \varphi \wedge \phi \mid [i]\varphi \mid C_J\varphi \mid \langle (A, \mathbf{a}) \rangle \varphi,$$
$$(A, \mathbf{a}) \text{ is an } \mathcal{L}^U_{EL-C}-\text{epistemic event.}$$

Epistemic states describe the static "snapshot" pictures of the epistemic situations, provide truth values for the formulas. Epistemic events denote information updating dynamic epistemic scenarios, each simple event standing not for a complete description of the world, but certain signal about this description. To complete the semantic formalism we need to specify how the epistemic events operate on the epistemic states. If an event $(A, \mathbf{a})$ unfolds at a state (M, m), what is the new resulting epistemic state?

We define the operation of product update which allows us to look at the epistemic events as update maps. It can be taken to represent the uniform way in which the agents merge two kinds of information, stemming from static and dynamic models.

Definition 8. *Suppose that $\mathcal{L}$ is any language interpreted over epistemic states.* Product update operation $\otimes : \mathcal{M} \times \mathcal{A}_{\mathcal{L}} \to Pow(\mathcal{M})$ *is defined as follows:*

$$(M, m) \otimes (A, \mathbf{a}) := \begin{array}{l} \emptyset \text{ if } \{m\} \times \mathbf{a} \cap S_W = \emptyset, \\ (W, \{m\} \times \mathbf{a}) \text{ otherwise,} \end{array}$$

where

- $S_W := \{(n, b) | (M, n) \models p_A(b)\}$,
- $R^i_W := \{((m, b), (n, c)) | (m, n) \in R^i_M$ *and* $(b, c) \in R^i_A\}$,
- $V_W((n, b)) := V_M(n)$.

We will use the mnemonic $M \times A$ for the updated model W above.[3]

[3]For a set of epistemic states $(M, \mathbf{m})$, we will let $(M, \mathbf{m}) \otimes (A, \mathbf{a}) = \bigcup_{(M,m) \in (M,\mathbf{m})} (M, m) \otimes (A, \mathbf{a})$, which is either $(M \times A, \mathbf{m} \times \mathbf{a})$ or $\emptyset$.

Example 9. *Recall the epistemic event of public announcement in example 5. It induces the mapping that simply relativizes epistemic state to the submodel where φ holds:*

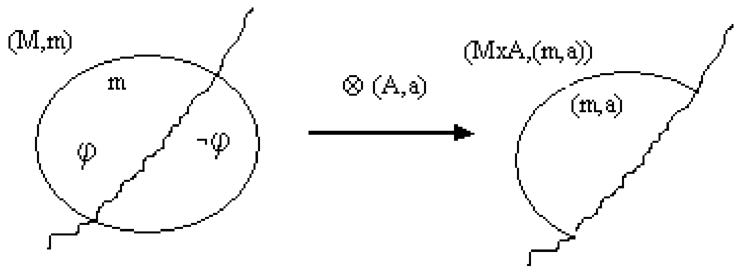

Figure 3. Public Announcement update

The satisfaction definition reflects the intended meaning of update modalities.

Definition 10. *For any language interpreted over epistemic states $\mathcal{L}$ we extend the* satisfaction *relation to formulas in the language $\mathcal{L}^U$ with the clause:*

$$(M, m) \vDash \langle (A, \mathbf{a}) \rangle \varphi \quad \textit{iff} \quad (N, n) \vDash \varphi, \textit{ for some } (N, n) \textit{ in } (M, m) \otimes (A, \mathbf{a}).$$

Baltag *et al.* [2] give a complete axiomatization of the logic for $\mathcal{L}^U_{EL-C}$ induced by the product update, which is copied in the appendix.

2.3. Bisimulation, Update Bisimulation and Equivalence.

The identity relation over $\mathcal{M}$ distinguishes models giving rise to the same modal theories (e.g. in $\mathcal{L}_{EL-C}$). The relation of *bisimulation*, familiar from the modal logic, provides a more appropriate equivalence relation:

Definition 11. *For $M, N \in \mathcal{M}'$ states $m \in M, n \in N$ are* bisimilar *("$m \underline{\leftrightarrow} n$") if the following simulation conditions are verified:*

- *If $m \underline{\leftrightarrow} n$, then $V_M(m) = V_N(n)$;*
- *If $m \underline{\leftrightarrow} n$ and $mR^i_M m'$, then there is n' in N such that $nR^i_N n'$ and $m' \underline{\leftrightarrow} n'$;*
- *Similarly in the opposite direction;*

Two finite epistemic states are bisimilar iff they have the same theories in $\mathcal{L}_{EL-C}$ (see van Benthem [3], Blackburn *et al.* [9]). We will construe $\mathcal{M}$ as a space of bisimulation equivalence classes, identifying each model with the class to which it belongs. *Pointed bisimulation* between $(M, \mathbf{m})$ and $(N, \mathbf{n})$ is a bisimulation joining every $m \in \mathbf{m}$ with some $n \in \mathbf{n}$, and vice versa. We will write $(M, \mathbf{m}) \underline{\leftrightarrow} (N, \mathbf{n})$ if there exists a pointed bisimulation between $(M, \mathbf{m})$ and $(N, \mathbf{n})$.

The following proposition shows that product update is well defined for the space of epistemic states quotiented through bisimulation equivalence relation:

Proposition 12. *[2] Product update $\otimes$ respects bisimulation: If $(M, m) \underline{\leftrightarrow} (N, n)$ then*

$$\textit{either } (M, m) \otimes (A, \mathbf{a}) \underline{\leftrightarrow} (N, n) \otimes (A, \mathbf{a})$$
$$\textit{or } (M, m) \otimes (A, \mathbf{a}) = (N, n) \otimes (A, \mathbf{a}) = \emptyset.$$

Analogously we quotient the set of epistemic events $\mathcal{A}_{\mathcal{L}^U_{EL-C}}$ (understood as the mappings):

Definition 13. *Two epistemic events* $(A, \mathbf{a}), (B, \mathbf{b}) \in \mathcal{A}_{\mathcal{L}^U_{EL-C}}$ *are* update equivalent *("$(A, \mathbf{a}) \approx (B, \mathbf{b})$") if* $\forall (M, m) \in \mathcal{M}$

$$\begin{array}{l} \textit{either } (M, m) \otimes (A, \mathbf{a}) \underline{\leftrightarrow} (M, m) \otimes (B, \mathbf{b}) \\ \textit{or } (M, m) \otimes (A, \mathbf{a}) = (M, m) \otimes (B, \mathbf{b}) = \emptyset. \end{array}$$

Filling in the role that bisimulation filled in in the case of epistemic states and their giving rise to the same modal theories, we will state a characterization of update equivalence for $\mathcal{A}_{\mathcal{L}^U_{EL-C}}$ in terms of *update bisimulation,* a relation between the domains of the respective epistemic events. See the paper by van Eijck *et al.* [11] for more details and proofs.

For given update models A and B let $Q_{A,B}$ be the minimal set of formulas that includes all formulas from the range of p_A and p_B that is closed under subformulas, single negations and the application of the following rule:

$$C_J\varphi \in Q_{A,B} \Rightarrow \bigwedge\nolimits_{j \in J} [j] C_J \varphi \in Q_{A,B}.$$

A *type* from $Q_{A,B}$ is a maximal satisfiable subset of formulas. $T_{A,B}$ will denote the set of all such types, and for a given type t, $\widehat{t}$ will be a formula that is a conjunction of all its elements.

Definition 14. *Let A and B be two update models. Let $\mathcal{R} \subseteq (S_A \times S_B) \times T_{A,B}$ be such (indexed) relation that if $((a, b), t) \in \mathcal{R}$ then:*

- *$p_A(a) \in t$ iff $p_B(b) \in t$;*
- *For every a' and t' such that $aR^i_A a'$, $p_A(a') \in t'$ and $\widehat{t} \wedge \langle i \rangle \widehat{t'}$ satisfiable, there exists b' such that $((a', b'), t') \in \mathcal{R}$;*
- *For every b' and t' such that $bR^i_B b'$,$p_B(b') \in t'$ and $\widehat{t} \wedge \langle i \rangle \widehat{t'}$ satisfiable, there exists a' such that $((a', b'), t') \in \mathcal{R}$.*

For two subsets $\mathbf{a} \subseteq S_A$, $\mathbf{b} \subseteq S_B$ we say that they are update bisimilar *(denoted "$\mathbf{a} \underline{\Leftrightarrow} \mathbf{b}$") if there exists a relation $\mathcal{R}$ fulfilling the above conditions, such that $\forall a \in \mathbf{a}, t \in T_{A,B}.p_A(a) \in t$ we have $((a, b), t) \in \mathcal{R}$ for some $b \in \mathbf{b}$, and similarly vice versa. Epistemic events $(A, \mathbf{a})$ and $(B, \mathbf{b})$ are update bisimilar ("$(A, \mathbf{a}) \underline{\Leftrightarrow} (B, \mathbf{b})$") if $\mathbf{a}$ and $\mathbf{b}$ are.*

Theorem 15. *[11] For any two epistemic events $(A, \mathbf{a}), (B, \mathbf{b}) \in \mathcal{A}_{\mathcal{L}^U_{EL-C}}$ we have:*

$$(A, \mathbf{a}) \approx (B, \mathbf{b}) \quad \textit{iff} \quad (A, \mathbf{a}) \underline{\Leftrightarrow} (B, \mathbf{b}).$$

For the special kinds of update models the definition of update bisimulation simplifies significantly. For any finite $P \subseteq$Prop finite, define the language $\mathcal{L}_{Val,P} = \{\bigwedge_{p \in S} p \wedge \bigwedge_{p \in P \setminus S} \neg p | S \in P\}$. Define also a larger language $\mathcal{L}_{fact}$ as the propositional language based on Prop.

For any update model A and $a' \in S_A$, $\mathbf{a} \subseteq S_A$ we will write $a' R^i_A \mathbf{a}$ when $\forall a \in \mathbf{a}(a' R^i_A a)$.

Definition 16. *Let A and B be two $\mathcal{L}_{fact}-$update models. Let $\underline{\Rightarrow}_f$ be a binary relation between simple events and sets of simple events in those update models such that the following conditions hold:*

- *If $a\underline{\Rightarrow}_f\mathbf{b}$ then $p_A(a) \vDash \bigvee_{b\in\mathbf{b}} p_B(b)$.*
- *If $a\underline{\Rightarrow}_f\mathbf{b}$, $b \in \mathbf{b}$ and $aR^i_A a'$ such that $p_A(a')$ is satisfiable, then $\exists \mathbf{b}' \subseteq S_B$ such that $bR^i_B\mathbf{b}'$ and $a'\underline{\Rightarrow}_f\mathbf{b}'$.*
- *If $a\underline{\Rightarrow}_f\mathbf{b}$, $b \in \mathbf{b}$ and $bR^i_A b'$ such that $p_A(b')$ is satisfiable, then $\exists \mathbf{a}' \in S_A$ such that $aR^i_A\mathbf{a}'$ and $b\prime\underline{\Rightarrow}_f\mathbf{a}'$.*

For two subsets $\mathbf{a} \subseteq S_A$, $\mathbf{b} \subseteq S_B$ let $\mathbf{a}\underline{\Leftrightarrow}_f\mathbf{b}$ iff $\forall a \in \mathbf{a}.p_A(a)$ satisfiable we have $a\underline{\Rightarrow}_f\mathbf{b}'$ for some $\mathbf{b}' \subseteq \mathbf{b}$, and similarly vice versa. For epistemic events $(A, \mathbf{a})$ and $(B, \mathbf{b})$ let $(A, \mathbf{a})\underline{\Leftrightarrow}_f(B, \mathbf{b})$ iff $\mathbf{a}\underline{\Leftrightarrow}_\mathbf{f}\mathbf{b}$ are.

Proposition 17. *[11] i) For any two epistemic events $(A, \mathbf{a}), (B, \mathbf{b}) \in \mathcal{A}_{\mathcal{L}_{Val,P}}$ we have:*

$$(A, \mathbf{a})\underline{\Leftrightarrow}(B, \mathbf{b}) \quad \textit{iff} \quad (A, \mathbf{a})\underline{\leftrightarrow}(B, \mathbf{b}).$$

ii) For any two epistemic events $(A, \mathbf{a}), (B, \mathbf{b}) \in \mathcal{A}_{\mathcal{L}_{fact}}$ we have:

$$(A, \mathbf{a})\underline{\Leftrightarrow}(B, \mathbf{b}) \quad \textit{iff} \quad (A, \mathbf{a})\underline{\Leftrightarrow}_f(B, \mathbf{b}).$$

3. Dynamical Product Update System

3.1. Dynamical System. We continue to regard $\mathcal{M}$ as the space of $\underline{\leftrightarrow}$-equivalence classes, and similarly, making use of the preceding section, $\mathcal{A}_{\mathcal{L}^U_{EL-C}}$ as a space of $\underline{\Leftrightarrow}$-equivalence classes, identifying a model with the class of models $\underline{\leftrightarrow}$- or $\underline{\Leftrightarrow}$-equivalent to it, respectively. Same applies to all relevant subspaces.[4]

Definition 18. ($\mathcal{L}^U_{EL-C}-$) Dynamical Product Update System *is a pair* $(\mathcal{M}, (\mathcal{A}_{\mathcal{L}^U_{EL-C}}, \circ))$, *where the phase space $\mathcal{M}$ is a space of all epistemic states for I and Prop, and $(\mathcal{A}_{\mathcal{L}^U_{EL-C}}, \circ)$ is a semigroup of $\mathcal{L}^U_{EL-C}-$epistemic events. The operation $\circ$ of* action composition, *is defined as follows:*

$$\begin{aligned} \circ \quad &: \quad \mathcal{A}_{\mathcal{L}^U_{EL-C}} \times \mathcal{A}_{\mathcal{L}^U_{EL-C}} \to \mathcal{A}_{\mathcal{L}^U_{EL-C}}, \\ (A, \mathbf{a}) \circ (B, \mathbf{b}) \quad &: \quad = (C, \mathbf{c}), \end{aligned}$$

where

- $S_C := S_A \times S_B$,
- $R^i_C := \{((a, b), (a', b'))|(a, a') \in R^i_A$ and $(b, b') \in R^i_B\}$,
- $p_C((a, b)) := p_A(a) \wedge \langle (A, a)\rangle\, p_B(b)$.

  Finally, each $\mathcal{L}^U_{EL-C}-$epistemic event $(A, \mathbf{a})$ is identified with the mapping $(A, \mathbf{a})$: $\mathcal{M} \to Pow(\mathcal{M})$,

$$(A, \mathbf{a})((M, m)) := (M, m) \otimes (A, \mathbf{a}).$$

[4]In the dynamical systems literature the additional structure over the phase space comes typically in the form of topology over the phase space $\mathcal{M}$. Our treatment is actually equivalent, when we take the topology to be generated by the basis of sets of bisimilar epistemic states. See also section 4.1.

Lemma 19. *[2] For any epistemic events* $(A,\mathbf{a}),(B,\mathbf{b})\in\mathcal{A}_{\mathcal{L}^U_{EL-C}}$ *and* $(M,m)\in\mathcal{M}$ *we have:*

$$(M,m)\otimes((A,\mathbf{a})\circ(B,\mathbf{b}))=((M,m)\otimes(A,\mathbf{a}))\otimes(B,\mathbf{b}).$$

In the composition, we create the epistemic events that consist of sequences of two simple events (happening one after another). The precondition formula of a simple event (a,b) says "$p_A(a)$ is true and after update with (A,a) $p_B(b)$ is true". We easily prove the following lemma:

Lemma 20. $\mathcal{A}_{\mathcal{L}_{Val,P}}$ *,* $\mathcal{A}_{\mathcal{L}_{fact}}$ *and* $\mathcal{A}_{\mathcal{L}^U_{EL-C}}$ *are closed under composition. Moreover,* $\mathcal{A}_{\mathcal{L}_{Val,P}}=\mathcal{A}_{\mathcal{L}^U_{Val,P}}$ and $\mathcal{A}_{\mathcal{L}_{fact}}=\mathcal{A}_{\mathcal{L}^U_{fact}}$.

Proof. It follows through a simple inductive argument and employment of the relevant clauses from the axiom system (see appendix, compare also def. 18):

$$\begin{aligned}\langle(A,a)\rangle\, p &\Leftrightarrow p_A(a)\wedge p\\ \langle(A,a)\rangle\,\neg\varphi &\Leftrightarrow p_A(a)\wedge\neg\langle(A,a)\rangle\,\varphi\\ \langle(A,a)\rangle\,\varphi\wedge\chi &\Leftrightarrow \langle(A,a)\rangle\,\varphi\wedge\langle(A,a)\rangle\,\chi\end{aligned}$$

■

We will also use the following abbreviations:

$$\begin{aligned}(A,\mathbf{a})\circ^0 &:= (Void,v):=(\{v\},\{(v,v)\}|_{i\in I},\{(v,\top)\}\},v),\\ (A,\mathbf{a})\circ^t &:= (A^t,\mathbf{a}^t):=(A,\mathbf{a})\circ((A,\mathbf{a})\circ((A,\mathbf{a})\circ\ldots\circ(A,\mathbf{a}))\ldots)_{t\text{ times}},\quad t\geq 1.\end{aligned}$$

3.2. Asymptotic Behavior. In this subsection we introduce some basic concepts and notation concerning the asymptotic behavior of $(\mathcal{M},(\mathcal{A}_{\mathcal{L}^U_{EL-C}},\circ))$, borrowing from the dynamical system literature (see e.g. Hasselblatt and Katok [17]).

For a given epistemic event $(A,\mathbf{a})$ let $\Omega_{(A,\mathbf{a})}\subset\mathcal{A}_{\mathcal{L}^U_{EL-C}}$ be the semigroup $\{(A,\mathbf{a})\circ^t|t\geq 0\}$ (together with the composition operation '$\circ$'). For a set of epistemic states $(M,\mathbf{m})$ define its $(A,\mathbf{a})$ *-orbit*, $\Omega_{(A,\mathbf{a})}((M,\mathbf{m}))$, as the set of sets of models $\{(M,\mathbf{m})\otimes(A,\mathbf{a})\circ^t|t\geq 0\}$[5], and define the set

$$\bigcap_{T\geq 0}\biguplus_{t\geq T}(M,\mathbf{m})\otimes(A,\mathbf{a})\circ^t$$

of its $(A,\mathbf{a})$*-limit points*, $\omega_{(A,\mathbf{a})}((M,\mathbf{m}))$[6]. Say that $(M,\mathbf{m})$ is $(A,\mathbf{a})$*-recurrent*, if $(M,\mathbf{m})\in\omega_{(A,\mathbf{a})}((M,\mathbf{m}))$. The $(A,\mathbf{a})$*-stationary semigroup* of $(M,\mathbf{m})$ is $G_{(A,\mathbf{a})}((M,\mathbf{m})):=\{(B,\mathbf{b})\in\Omega_{(A,\mathbf{a})}|(M,\mathbf{m})\otimes(B,\mathbf{b})\underline{\leftrightarrow}(M,\mathbf{m})\}$. And lastly, the $(A,\mathbf{a})$*-period* p of $(M,\mathbf{m})$ is such that $(A,\mathbf{a})\circ^p$ is the positive generator of $G_{(A,\mathbf{a})}((M,\mathbf{m}))$.

Using this new jargon we can ask more detailed questions about the asymptotic behavior. For an epistemic state (M,m) and epistemic event $(A,\mathbf{a})$, is the orbit $\Omega_{(A,\mathbf{a})}((M,m))$ finite, meaning that starting at (M,m), updating with $(A,\mathbf{a})$ can lead only to finitely many "nondeterministic" epistemic states (sets of epistemic states)? If so, what is the

[5] Obit $\Omega_{(A,\mathbf{a})}((M,\mathbf{m}))$ corresponds (roughly) to $TREE(M,A)$ in van Benthem [4].

[6] $\biguplus$ stands for disjoint union, and so each limit point is a set of epistemic states.

smallest t such that $(M,m)\otimes(A,\mathbf{a})\circ^t$ is in the set of limit points? What is its period, is it a fixed point? We will say that an orbit $\Omega_{(A,\mathbf{a})}((M,m))$ *(p-)stabilizes (at stage t)* iff it is finite, t is the one described above and p is the period of its recurrent points.

In particular, van Benthem (e.g. [4]) stated the following

Conjecture 21 [Finite Evolution Conjecture]**.** *For any finite epistemic state (M,m) and $\mathcal{L}_{EL-C}-$epistemic event $(A,\mathbf{a})$, the orbit $\Omega_{(A,\mathbf{a})}((M,m))$ stabilizes.*

We will refute the conjecture in section 4.2.

We might also be interested in more uniform properties of updating with $(A,\mathbf{a})$, and ask if the semigroup $\Omega_{(A,\mathbf{a})}$ is finite, and so $\{\{(A,\mathbf{a})\circ^t|s\geq t\},\circ\}$ isomorphic to $\mathbb{Z}_p$, for some p and t, implying that acting of $(A,\mathbf{a})$ on any epistemic state will produce finite orbits only. What is the smallest such t? what is p? Similarly as above, the semigroup $\Omega_{(A,\mathbf{a})}$ or an epistemic event $(A,\mathbf{a})$ *(p-)stabilizes (at stage t)*, iff $\Omega_{(A,\mathbf{a})}$ is finite, p and t are as above.

We can ask those questions on the yet higher level of generality, and the answers that we will try to provide here will be mainly of such nature. Perhaps the stabilization properties are better characterized on the level of frames than particular models. For a given *epistemic frame* $\mathtt{A}$, $\mathtt{A}=\{S_A,\ R^i_A\subseteq S^2_A|_{i\in I}\}$, what are the properties of the epistemic events based on it: do they give rise to finite semigroups i.e. finite orbits when applied to any state model? What if we constrain epistemic events to be factual? Frame $\mathtt{A}$ *(p-) stabilizes (at stage t)* iff for every epistemic event over it, i.e. with domain and accessibility relations as in $\mathtt{A}$, and every (M,m), $\Omega_{(A,\mathbf{a})}((M,m))$ stabilizes, where p and t are the suprema of those indices over all such $\Omega_{(A,\mathbf{a})}((M,m))$. Finally, frame $\mathtt{A}$ *(p-) stabilizes$_f$ (at stage t)* if the definition above is applied only to the $\mathcal{L}_{fact}-$epistemic events.

3.3. Limit modalities.

Accordingly, we extend the dynamic language with the appropriate limit modalities.

Definition 22. *For any language $\mathcal{L}^U$ let the language $\mathcal{L}^{U\to}$ be the extension of $\mathcal{L}^U$ through a recursive application of* limit modalities

$$\mathcal{L}^{U\to}\quad:\;:=\quad "\mathcal{L}^U"\;\big|\;\langle\langle(A,\mathbf{a})^{\to}\rangle\rangle\,\varphi\;\big|\;\langle[(A,\mathbf{a})^{\to}]\rangle\,\varphi\ \ ,$$
$$(A,\mathbf{a})\text{ is an }\mathcal{L}^{U\to}-\text{epistemic event.}$$

Similarly, for any $\mathcal{L}^{U(\mathcal{L}')}$ define the language $\mathcal{L}^{U(\mathcal{L}')\to}$. The definition of satisfiability is extended as follows:

$$\begin{array}{lll}(M,m)\models\langle\langle(A,\mathbf{a})^{\to}\rangle\rangle>\varphi & \Leftrightarrow^{def} & \forall T\exists t\geq T.(M,m)\models\langle(A,\mathbf{a})\circ^t\rangle\,\varphi,\\ (M,m)\models\langle[(A,\mathbf{a})^{\to}]\rangle\,\varphi & \Leftrightarrow^{def} & \forall T\exists t\geq T.(M,m)\models[(A,\mathbf{a})\circ^t]\varphi,\end{array}$$

For example, the formula $\langle\langle(A,\mathbf{a})^{\to}\rangle\rangle\,\varphi$ means: "in the course of iterated nondeterministic update with $(A,\mathbf{a})$ the formula φ is infinitely often satisfied at some of the possible resulting epistemic states". The abbreviations for the duals are standard, with the negation reverting both nested modalities. We have:

$$\begin{array}{lll}(M,m)\models[[(A,\mathbf{a})^{\to}]]\varphi & \text{iff} & \exists T\forall t\geq T.(M,m)\models[(A,\mathbf{a})\circ^t]\varphi,\\ (M,m)\models[\langle(A,\mathbf{a})^{\to}\rangle]\varphi & \text{iff} & \exists T\forall t\geq T.(M,m)\models\langle(A,\mathbf{a})\circ^t\rangle\,\varphi.\end{array}$$

Notice that the formulas:

$$[\langle(A,\mathbf{a})^{\rightarrow}\rangle]\varphi \rightarrow \langle\langle(A,\mathbf{a})^{\rightarrow}\rangle\rangle\,\varphi,$$
$$[[(A,\mathbf{a})^{\rightarrow}]]\varphi \rightarrow \langle[(A,\mathbf{a})^{\rightarrow}]\rangle\,\varphi,$$

are valid over $\mathcal{M}$.

4. Factual Epistemic Events

4.1. Finite evolution of truth value. Our first result illustrates important distinguishing features of factual update (update with $\mathcal{L}_{fact}-$epistemic events). First of all, irrespectively of the problem of finiteness of the orbit, any formula in $\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact})}$ eventually either always holds or is always refuted in the course of update. No other, "looping" recurrency is possible, with, e.g., some formula holding after every second update. We can say that every formula has a definite asymptotic truth value, for a given epistemic state and updating nondeterministic $\mathcal{L}_{fact}-$epistemic event. Moreover, for every formula in $\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact})}$, and so possibly formula including Common Knowledge modalities, the number of updates after which its truth value is fixed forever is a simple function of the size of updating (nondeterministic) epistemic event and the complexity of the formula, with Common Knowledge and a regular knowledge modalities treated on a par. (The results can be interpreted as showing that it is not a gradual build-up of depth of mutual knowledge about some fact that leads to Common Knowledge - see Fagin *et al.* [12]). Factual update is conservative in this sense.

This "smooth" evolution of information is largely due to the property of $\mathcal{L}_{fact}-$epistemic events that in a sequenced model $(A,\mathbf{a})\circ^{n}$ preconditions of (sequenced) simple events don't depend on the order or number of repetitions of single simple events.

Lemma 23. *For any $\mathcal{L}_{fact}-$update model A, for any $(a_1,...,a_t)\in S_{A^t}$ we have*

$$p_{A^s}((a_1,...,a_{t-1},a_t)) = \bigwedge_{a_i\in\{a_1,...,a_t\}} p_A(a_i)$$

Proof. See the proof of lemma 20. ■

Given two $\mathcal{L}_{fact}-$update models A and B and two simple events $a\in A, b\in B$, a is *C'-update similar to* b ("$a\underline{\Rightarrow}_{C'}b$") if the following conditions are verified:

1. If $a\underline{\Rightarrow}_{C'}b$ then $p_A(a)\vDash p_B(b)$;
2. If $a\underline{\Rightarrow}_{C'}b$ and $aR_A^i a'$, then there is $b'\in S_B$ such that $bR_B^i b'$ and $a'\underline{\Rightarrow}_{C'}b'$;
3. If $a\underline{\Rightarrow}_{C'}b$ and $bR_B^i b'$, then there is $a'\in S_A$ such that $aR_A^i a'$ and $b'\underline{\Rightarrow}_{C'}a'$;
4. If $a\underline{\Rightarrow}_{C'}b$, $(a,i^1,a^2,...,i^{k-1},a^k)$ is a J-path in A, then there exists a J-path $(b,i^1,...,i^{k-1},b^k)$ in B such that $\forall_{l=2,...,k}\ p_A(a^l)\vDash p_B(b^l)$ and $a^k\underline{\Rightarrow}_{C'}b^k$;
5. Similarly for a path in B starting with b.

Two sets of simple events are related by C'-update bisimulation ("$\mathbf{a}\underline{\Leftrightarrow}_{C'}\mathbf{b}$") if each simple event with a satisfiable prerequisite formula in either of them is C'-update similar to a simple event in the other. Two epistemic events are C'-update bisimilar if their sets of distinguished simple events are. The definitions are extended to *C'n-update (bi)similarity* ("$a\underline{\Rightarrow}_{C'n}b$", "$\mathbf{a}\underline{\Leftrightarrow}_{C'n}\mathbf{b}$"), in which case we verify only n steps in the definition above.

For a formula in $\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact})}$ define its *depth$_C$* in the following manner:

$$\begin{array}{lll} \text{depth}_C(p) & := & 0, \text{ for } p \in \text{Prop}, \\ \text{depth}_C(\neg\varphi) & := & \text{depth}_C(\varphi), \\ \text{depth}_C(\varphi \wedge \psi) & := & \max\{\text{depth}_C(\varphi), \text{depth}_C(\psi)\}, \\ \text{depth}_C(< (A,\mathbf{a}) > \varphi) & := & \text{depth}_C(\varphi), \\ \text{depth}_C(< i > \varphi) = \text{depth}_C(C_J\varphi) & := & \text{depth}_C(\varphi) + 1, \end{array}$$

Lemma 24. *For $\mathcal{L}_{fact}-$epistemic events $(A,\mathbf{a})$ and $(B,\mathbf{b})$, any $(M,m) \in \mathcal{M}$, $\varphi \in \mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact})}$ with depth$_C(\varphi) \leq n$:*[7]

$$\begin{array}{rcl} & & (A,\mathbf{a}) \underline{\Leftrightarrow}_{C'n} (B,\mathbf{b}) \textit{ implies} \\ (M,m) & \vDash & \langle (A,\mathbf{a})\rangle\, \varphi \textit{ iff } (M,m) \vDash \langle (B,\mathbf{b})\rangle\, \varphi. \end{array}$$

Proof. The proof goes inductively over the complexity of φ. In evaluating the formula, the first coordinates of matching states in the producted models are the same. In the case of induction step, when φ is of the form $\langle (C,\mathbf{c})\rangle\, \psi$, we exploit the fact that $(M,m) \vDash \langle (A,\mathbf{a})\rangle \langle (C,\mathbf{c})\rangle\, \psi$ iff $(M,m) \vDash \langle (A,\mathbf{a}) \circ (C,\mathbf{c})\rangle\, \psi$, and also $(A,\mathbf{a}) \underline{\Leftrightarrow}_{C'n} (B,\mathbf{b})$ implies $(A,\mathbf{a}) \circ (C,\mathbf{c}) \underline{\Leftrightarrow}_{C'n} (B,\mathbf{b}) \circ (C,\mathbf{c})$ in the case when all epistemic events are in $\mathcal{A}_{\mathcal{L}_{fact}}$. ■

Clearly, if $(A^s, S_A^s) \underline{\Leftrightarrow}_{C'(n)} (A^t, S_A^t)$, and the defining C'(n)-update simulation is such that from $(a_1, ..., a_s) \underline{\Rightarrow}_{C'(n)} (a'_1, ..., a'_t)$ follows $\{a'_1, ..., a'_t\} \subseteq \{a_1, ..., a_s\}$, then for every $\mathbf{a} \subseteq S_A$ $(A^s, \mathbf{a}^s) \underline{\Leftrightarrow}_{C'(n)} (A^t, \mathbf{a}^t)$. Same holds for update bisimulation.

Lemma 25. *Let A be an $\mathcal{L}_{fact}-$update model, $|S_A| = r$. Then:*

1. *If for any $(a_1, ..., a_s) \in S_{A^s}$, $(a'_1, ..., a'_t) \in S_{A^t}$ we have $\{a'_1, ..., a'_t\} \subseteq \{a_1, ..., a_s\}$, then $(a_1, ..., a_s) \underline{\Rightarrow}_{C'0} (a'_1, ..., a'_t)$.*
2. *If $(a_1, ..., a_t) \in S_{A^t}$ and $(a'_1, ..., a'_t) \in S_{A^t}$ is its permutation, then $(a_1, ..., a_t) \underline{\Leftrightarrow}_{C'} (a'_1, ..., a'_t)$.*
3. *If $(a_1, ..., a_{t+1}) \in S_{A^{t+1}}$ and $(a'_1, ..., a'_t) \in S_{A^t}$ are two sequenced events, first of which has at least $r^n + 1$, $n \geq 0$, repetitions of the same single event in its coordinates, the second one has r^n repetitions of this event and agrees on the number of all the other repetitions, then $(a_1, ..., a_{t+1}) \underline{\Leftrightarrow}_{C'n} (a'_1, ..., a'_t)$.*

Proof. 1. Follows from lemma 23.
2. From the definition of composing epistemic events follows immediately that if $(a_1, ..., a_t)$ is a permutation of $(a'_1, ..., a'_t)$ and $(a_1, ..., a_t) R^i_{A^t} (b_1, ..., b_t)$, then there is $(b'_1, ..., b'_t)$ with $(a'_1, ..., a'_t) R^i_{A^t} (b'_1, ..., b'_t)$ and $(b'_1, ..., b'_t)$ is the same permutation of $(b_1, ..., b_t)$. Application of part 1 proves that first condition of C'-update bisimulation is fulfilled and finishes the proof.
3. The case when $n = 0$ follows from part 1. Suppose $n > 0$. Let $(a_1, ..., a_{t+1})$ and $(a_1, ..., a_t)$ be as in the premise, and assume for now that

$$(a_1, ..., a_{t+1}) = (a_1, ..., a_{t-r^n}, a^*, ..., a^* | r^n + 1 \textit{ repetitions}),$$

and

$$(a'_1, ..., a'_t) = (a_1, ..., a_{t-r^n}, a^*, ..., a^* | r^n \textit{ rep.}).$$

[7] It should be clear that full characterization would require introducing point-set relations as well as constraints on consistency of path-derived formulas, as in update simulation.

We have that $\{a_1,...,a_{t+1}\} = \{a'_1,...,a'_t\}$, and so due to part 1 $p_{A^{t+1}}((a_1,...,a_{t+1})) \vDash p_{A^t}((a'_1,...,a'_t))$, and vice versa.

As for the zigzag clauses of C'n-update simulation, we will prove only path conditions (4 and 5), which contains the proof of direct successor conditions (2 and 3). Take first a path $((a_1,...,a_{t+1}), i_1, (a_1^2,...,a_{t+1}^2), i_2, ..., i_{k-1}, (a_1^k,...,a_{t+1}^k))$ in A^{t+1}. $(a_1^k,...,a_{t+1}^k)$ must have $r^{n-1}+1$ repetitions among its last r^n+1 coordinates – assume for now that it is of the form

$$(a_1^k,...,a_{t-r^{n-1}}^k, a^{k*},...,a^{k*}|r^{n-1}+1\ rep.).$$

The path in A^t will be:

$$\begin{aligned}&((a_1,...,a_{t-r^{n-1}}, a^*,...,a^*|r^{n-1}\ rep.), i_1, (a_1^2,...,a_t^2), i_2, ...\\ &..., i_{k-1}, (a_1^k,...,a_{t-r^{n-1}}^k, a^{k*},...,a^{k*}|r^{n-1}\ rep.)),\end{aligned}$$

the path from A^{t+1} with the last coordinate cut off. The fact that this is indeed a path, i.e. $\forall l < k\ ((a_1^l,...,a_t^l)R_A^{i_l}(a_1^{l+1},...,a_t^{l+1}))$, follows from those conditions on the path in A^{t+1}. The conditions $\forall_{l=1,...,k}\ p_A((a_1^l,...,a_{t+1}^l)) \vDash p_B((a_1^l,...,a_t^l))$ follow from $\forall_{l=1,...,k}$ $\{a_1^l,...,a_{t+1}^l\} \supseteq \{a_1^l,...,a_t^l\}$. The last states of the two paths are C'(n-1)-update bisimilar due to the induction assumption.

In the zag clause proceed analogously: for a path in A^t,

$$((a'_1,...,a'_t), i_1, (a'^2_1,...,a'^2_t), i_2, ..., i_{k-1}, (a'^k_1,...,a'^k_t)),$$

where, as before, we shall assume that $(a'_1,...,a'_t) = (a'_1,...,a'_{t-r^n}, a^*,...,a^*|r^n\ rep.)$, $(a'^k_1,...,a'^k_t) = (a'^k_1,...,a'^k_{t-r^{n-1}}, a^{k*},...,a^{k*}|r^{n-1}\ rep.)$, let the path in A^{t+1} be just

$$\begin{aligned}&((a'_1,..,a'_t,a'_t), i_1, (a'^2_1,...,a'^2_t,a'^2_t), i_2, ..\\ &..., i_{k-1}, (a'^k_1,..,a'^k_t,a'^k_t).\end{aligned}$$

Finally observe that the assumptions on the position of the repetitions in the sequences were done only for the notational convenience and without loss of generality, due to part 2. ■

The following two propositions reap the results of the preceding lemma, concluding this subsection.

Proposition 26. *If $(A,\mathbf{a})$ is a $\mathcal{L}_{fact}$–epistemic event, $|S_A| = r$, then:*

1. $(A^{r^{n+1}+x}, \mathbf{a}^{r^{n+1}+x}) \underline{\Leftrightarrow}_{C'n} (A^{r^{n+1}+y}, \mathbf{a}^{r^{n+1}+y})$, $x, y \geq 0$.
2. *For any finite (M,m) in $\mathcal{M}$ if $\Omega_{(A,\mathbf{a})}((M,m))$ stabilizes, then it 1-stabilizes.*
3. *If $(A,\mathbf{a})$ stabilizes, then it 1-stabilizes.*

Proof. 1. It is enough to prove the claim for $y = x+1, x \geq 0$. The result follows trivially from lemma 25 and remark directly before it, since in the sequence of length $r^{n+1}+1$ consisting of at most r different elements there must be at least one element repeated r^n+1 times.

2. Follows easily from part 1 and lemma 24: any finite epistemic state can be distinguished by a single formula $\varphi \in \mathcal{L}_{EL-C}$ from a finite set of finite epistemic states, none of which is bisimilar to it (see e.g. Blackburn *et al.* [9]).

3. Suppose $(A, \mathbf{a})$ p-stabilizes. For any two $\mathcal{L}_{fact}-$epistemic events $(B, \mathbf{b})$ and $(C, \mathbf{c})$ that are not update bisimilar, we have $(M, m) \otimes (B, \mathbf{b}) \underline{\not\leftrightarrow} (M, m) \otimes (C, \mathbf{c})$ for certain finite (M, m) with $S_M = \{$all valuations over propositional letters appearing in precondition formulas of B and $C\}$, total relations and canonical valuation (see van Eijck *et al.* [11]). Therefore part 2 guarantees that $p = 1$. ■

Proposition 27. *If $(A, \mathbf{a})$ is a $\mathcal{L}_{fact}-$epistemic event, $|S_A| = r$, φ a formula in $\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact})}$, then formulas:*

$$\begin{aligned} \langle\langle (A, \mathbf{a})^{\rightarrow} \rangle\rangle\, \varphi &\leftrightarrow \left\langle (A, \mathbf{a}) \circ^{t} \right\rangle \varphi, \\ \langle\langle (A, \mathbf{a})^{\rightarrow} \rangle\rangle\, \varphi &\leftrightarrow [\langle (A, \mathbf{a})^{\rightarrow} \rangle] \varphi, \end{aligned}$$

and similarly

$$\begin{aligned} \langle [(A, \mathbf{a})^{\rightarrow}] \rangle\, \varphi &\leftrightarrow [(A, \mathbf{a}) \circ^{t}] \varphi, \\ \langle [(A, \mathbf{a})^{\rightarrow}] \rangle\, \varphi &\leftrightarrow [[(A, \mathbf{a})^{\rightarrow}]] \varphi, \end{aligned}$$

where $t \geq r^{depth_C(\varphi)+1}$, are valid over $\mathcal{M}$.

Proof. Follows immediately from 1 in proposition 26 and lemma 24. ■

Remark 28. *All the results from this section are easily extended via appropriate inductive argument from $\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact})}$ to $\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact}^{U})}$ and $\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact}^{U})\rightarrow}$ (in regard to the former - see lemma 20).*

Corollary 29. *$\mathcal{L}_{EL-C}^{U(\mathcal{L}_{fact}^{U})\rightarrow}$ is equally expressive as $\mathcal{L}_{EL-C}$.*

4.2. Finite evolution of epistemic states. Now we would like to see if for factual epistemic events stronger result holds, namely if every orbit is finite. We will analyze the problem of stabilization$_f$ of frames. Nonstabilization of a frame $\mathtt{A}$ will result in an infinite orbit $\Omega_{(A,\mathbf{a})}((M, m))$, where $(A, \mathbf{a})$ is built over $\mathtt{A}$ and whose prerequisite function assigns different atomic proposition to different event, and (M, m) is the canonical model mentioned in the proof of proposition 26, which is finite and has particularly noncomplicated structure: $R_M^i = S_M^2$, for every $i \in I$.

The following crucial lemma reduces the question of stabilization$_f$ of frames to the question of winning strategies in certain pebble game.

Lemma 30. *Fix an arbitrary finite frame $\mathtt{A}$. The frame $\mathtt{A}$ stabilizes$_f$ at some stage $s<t$, $t > 1$, iff Constructor has a winning strategy in the following game:*

There are t blue and t-1 red stones. Spoiler starts the game by choosing one of the colors and distributes all the stones in this color over the nodes of the frame $\mathtt{A}$. Constructor must distribute the stones in the other color over the nodes that are already covered by Spoiler's stones. Then, in every round Spoiler chooses a color, one of the relations R_A^i, i=1,...,I, and moves the stones in this color to different nodes along the arrows corresponding to this relation, one arrow for each stone at a time. Constructor moves in each round after Spoiler, moving the stones in the other color along the arrows of the chosen relation.

Spoiler wins if after some round the color chosen most recently by Constructor covers at least one node that is not covered by any stone in the color chosen most recently by Spoiler, or if at some point Constructor can't make a response move (if some of his stones are in a blind node). He looses if at some round he himself cannot make a move or Constructor can defend himself arbitrarily long.

Proof. *If Constructor has a winning strategy*: Take any factual update model A based on the frame $\mathtt{A}$. We will show that for $(A, S_A)\circ^t = (A^t, S_A^t)$ and $(A, S_A)\circ^{t-1} = (A^{t-1}, S_A^{t-1})$ we have $S_A^t \underline{\Leftrightarrow} S_A^{t-1}$, with update simulation joining sequenced states such that $(a_1, ..., a_{t/t-1}) \underline{\Rightarrow} (a'_1, ..., a'_{t-1/t})$ implies $\{a'_1, ..., a'_{t-1/t}\} \subseteq \{a_1, ..., a_{t/t-1}\}$ (see the remark before lemma 25).

Take $(a_1, ..., a_t) \in S_A^t$. The winning strategy for Constructor in the pebble game defined above for the frame A provides us with a description of how to create the update simulation zigzag in the following way. First associate with each of the t coordinates of a simple event in S_A^t a single blue stone, and similarly for the coordinates of events in S_A^{t-1} and red stones. Event $(a_1, ..., a_t)$ marks the distribution of blue stones over the nodes of frame $\mathtt{A}$ in which each blue stone "lies" on associated coordinate node. The winning strategy for Constructor prescribes him to distribute t-1 red stones over the nodes in some specific way. This distribution of red stones uniquely marks a simple event $(a'_1, ..., a'_{t-1})$ in S_A^{t-1}: each coordinate has as its value a node of $\mathtt{A}$ on which the corresponding stone lies. We claim that $(a_1, ..., a_t) \underline{\Rightarrow} (a'_1, ..., a'_{t-1})$.

Each choice by Spoiler to prolong the zigzag in the update simulation can be interpreted, similarly as above, as a choice of a color, relation, and moving the stones in this color along the arrows over the frame $\mathtt{A}$. The Constructor's winning strategy in the frame game always provides him with a countermove, which will be translated, as above, into the continuation of the update simulation zigzag in the other model (by a single sequenced simple event, i.e. a singleton set). All we have to do is to check if the condition 1 of the definition of update simulation is fulfilled. Suppose that in the last round Spoiler moved the red stones to a constellation associated with the event $(a^*_1, ..., a^*_{t-1})$ and Constructor responded with a setting of blue stones associated with an event $(a^\circ_1, ..., a^\circ_t)$. From the rules of the game we know that $\{a^\circ_1, ..., a^\circ_t\} \subseteq \{a^*_1, ..., a^*_{t-1}\}$, and so $p_{A^{t-1}}((a^*_1, ..., a^*_{t-1})) \vDash p_{A^t}((a^\circ_1, ..., a^\circ_t))$ (lemma 23). Altogether we get $(a_1, ..., a_t) \underline{\Rightarrow} (a'_1, ..., a'_{t-1})$, with $\{a'_1, ..., a'_{t-1}\} \subseteq \{a_1, ..., a_t\}$.

Exactly the same argument proves that for every $(a_1, ..., a_{t-1})$ in S_A^{t-1} we can find an update similar $(a'_1, ..., a'_t)$ in S_A^t such that $\{a'_1, ..., a'_t\} \subseteq \{a_1, ..., a_{t-1}\}$. This finishes the proof of the first part of the lemma.

If Constructor has no winning strategy: Notice that in this game if Constructor has no winning strategy, then Spoiler has one. (E.g. we can modify the game in such a way that if at the beginning of any turn the position of the stones is the same as at the beginning of any previous turn, then the Constructor wins. Clearly, the solution, in terms of possession of winning strategies, would be the same, since if Spoiler has a winning strategy in the original game, he also has one in which in no possible path of the play the position of the stones is repeated. Application of Zermelo's theorem proves the claim, since the new game is a finite (due to finiteness of frame $\mathtt{A}$) game of perfect information.)

As in the first part of the proof, associate the stones in both colors with coordinates of the states in producted models. Suppose w.l.o.g. that Spoiler's winning strategy prescribes to put the blue stones at the beginning of the game at nodes associated, as in previous part of the proof, with a state $(a_1, ..., a_t)$. Consider an $\mathcal{L}_{fact}-$epistemic event $(\{\mathcal{A}, p_A\}, S_A)$, where p_A simply assigns to each scenario in S_A a different proposition letter. We will show that for every $\mathbf{a}' \subseteq S_A^{t-1}$ it is not the case that $(a_1, ..., a_t) \underline{\Rightarrow} \mathbf{a}'$, which proves that $\mathtt{A}$ does not stabilize$_f$ at stage s<t, since $(\{\mathcal{A}, p_A\}, S_A)$ doesn't.

Due to our choice of p_A, for any b in S_A^{t-1} and any $b^1, ..., b^n$ in S_A^t, if $p_{A^{t-1}}(b) \vDash \bigvee_{i=1,...,n} p_{A^t}(b^i)$, then $p_{A^{t-1}}(b) \vDash p_{A^t}(b^k)$ for some $k \in \{1, ..., n\}$, and similarly for the reversed order of S_A^{t-1} and S_A^t. It follows that if $(a_1, ..., a_t) \underline{\Rightarrow} \mathbf{a}'$ then we can take $\mathbf{a}'$ as well as all the sets of simple events chosen further in the update simulation zigzag to be singletons.

Suppose that, to the contrary, $(a_1, ..., a_t) \underline{\Rightarrow} (a_1', ..., a_{t-1}')$ for some $(a_1', ..., a_{t-1}') \in S_A^{t-1}$. If the clause 1 of the definition of update simulation is to be satisfied, then $\{a_1, ..., a_t\} \supseteq \{a_1', ..., a_{t-1}'\}$, and similarly for all the events reached later by the zigzag. Translated into the language of the pebble game as in the first part of the proof, all this would mean that Spoiler's strategy is not a winning strategy, contradicting our assumption. ■

Example 31. *Consider any $\mathcal{L}_{fact}$−epistemic event $(A, \mathbf{a})$ based on a frame* `A`*, such that for every i, $R_A^i = S_A^2$. We show that $(A, \mathbf{a})$ stabilizes$_f$ at stage 1 by specifying a winning strategy for Constructor in the corresponding pebble game (fig. 4).*

The strategy is very simple. In every round, whatever nodes Spoiler covers by his (one red or two blue) stones, Constructor moves all (one or two) stones in the other color to one of those nodes. Such move by Constructor is always possible due to totality of relations R_A^i in the frame `A`*. The first steps of the pebble game (or update simulation) could look as follows (the abbreviations Co. and Sp. mark in which models Spoiler chooses and Constructor is forced to choose successors):*

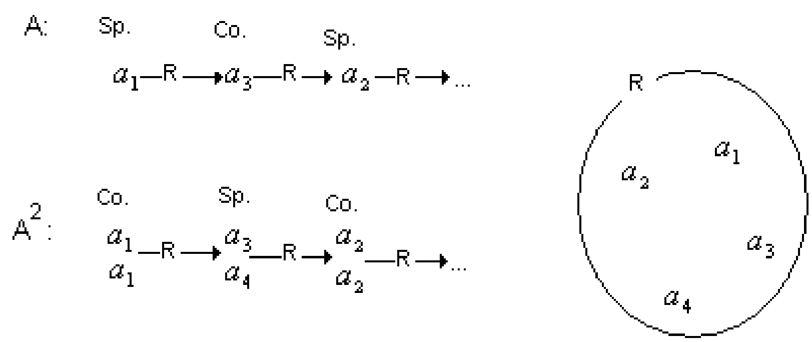

Figure 4. Pebble game/update simulation (left) and frame A (right)

The argument in the above example can be easily extended to a frame where all the relations are the same and build an arbitrary partition of the set S_A. As examples 34 and 35 make clear, the argument breaks down if we allow non-factual precondition formulas or the general, overlapping partitions.

Proposition 32. *Any transitive frame* `A` *with only one relation stabilizes$_f$.*

Proof. [Sketch] We describe a winning strategy in the pebble game for Constructor in the corresponding pebble game. We outline the successive claims of the arguments and leave its verifications to the reader.

First, since frame `A` is transitive, we can assume that it is a finite disjoint union of finite transitive trees, whose nodes are either irreflexive states, or clusters of states that are related to exactly the same states and to each other. This is so, because we can unravel the frame in a standard way, and show that every unravelled $\mathcal{L}_{fact}$−epistemic event built on the original frame is update bisimilar to a corresponding model, where the precondition function assigns the same formulas to the copies of the same state, built on the unraveled frame.

Suppose the frame A has n tree-branches. We show that it stabilizes$_f$ at stage n. The idea is roughly that since Constructor can freely move his stones down along each branch, it is enough if he "guards" with his single stone the lowermost node covered by Spoilers stones in every branch. The fact that nodes can be taken to be whole clusters causes no problems, as example 31 showed.

More precisely, ^{b}index by $\{1, 2, \dots\}$ each of the nodes (irreflexive state or a whole cluster) according to the number of different tree-branches that it is part of. For any distribution of stones chosen by Spoiler/Constructor we inductively $^{s/c}$index by $\{1, 2, \dots\}$ the nodes of the frame as follows:

- start with the lowermost nodes in every branch with at least one of the Spoiler/ Constructor's stones on any of its states. The $^{s/c}$index is min$\{^b$index, number of Spoiler/ Constructor's stones on this node's states$\}$;
- on every node higher on the branch the $^{s/c}$index is min$\{^b$index-(sum of $^{s/c}$index values of all immediate successor nodes of the current node), number of Spoiler/Constructor's stones on this node's states$\}$;

Easy argument shows that the $^{s/c}$number_of_paths_covered, defined as the sum of $^{s/c}$index values over the whole frame, is bounded by n at the start of the pebble game, and is nonincreasing throughout the game.

Similarly, the number of stones required to obtain any given, say, ^{s}indexing of the frame (in the most economical way) is proved to be equal to the ^{s}number_of_the_paths_covered, and Constructor's stones can be put only on the nodes covered by Spoiler's stones. At the start of the game Constructor puts his stones only on the states already covered by the Spoiler in such a way, that ^{c}indexing $\equiv$ ^{s}indexing. From the definition of $^{s/c}$indexing we derive that throughout the game Constructor can counter Spoiler's moves in such a way that ^{c}indexing $\equiv$ ^{s}indexing, and his stones lie only on the states already covered by Spoiler's stones. ∎

Consider probably the simplest example of a frame that does not stabilize$_f$. It shows that the 'Finite Evolution Conjectue' does not hold.

Example 33. *Let the the frame* A $= \{S_A, R_A\}$*, be defined as follows (fig. 5):*
$S_A = \{a_1, a_2, a_3, a_4\}$,
$R_A = \{(a_1, a_1), (a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_4)\}$.

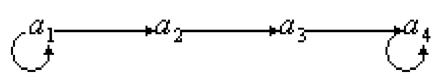

Figure 5. Non-stabilizing$_f$ frame

We will prove that Spoiler has a winning strategy for any t in the pebble game over it, and therefore that the frame A *does not stabilize$_f$. Spoiler puts all the t blue stones on node a_1; to stay in the game Constructor has to do the same with the t-1 red stones. The game proceeds as follows: Spoiler chooses blue stones until there is no red stone left* *on node a_1. In the rounds 1+3i, 2+3i and 3+3i, he moves one blue stone from node a_1 to node a_2, then from a_2 to a_3 and a_3 to a_4 respectively, leaving all the other stones at the reflexive nodes a_1 and a_4. In each 1+3i'th round Constructor also has to detach at least one red stone from node a_1 and move it to node a_2: if he doesn't, then in the next round Spoiler chooses red stones, moves them in any fashion, and Constructor looses, stuck with blue stone at node a_2, which he can move only to node a_3, not covered by any red stone. At latest at the beginning of round 2+3(t-2) we arrive at the situation, in which there is a blue and no red stones at node a_1. In this case Spoiler chooses red stones and moves them in any admissible way. Constructor is stuck with a blue stone at node a_1, which he can move to nodes a_1 or a_2 only, neither of which will stay covered by a red stone at the end of the round. It means that this strategy in fact guarantees victory for Spoiler.*

Similar argument as in the last example shows that the frame $\mathtt{A} = \{\{1, 2, 3, 4, 5\}, R_A\}$, where R_A is the symmetric and reflexive closure of "immediate successor" (in natural numbers) relation also does not stabilize$_f$. On the other hand, transitivity still falls short of characterizing stabilizing$_f$ frames (witness a frame with one state and empty relations). In general, it remains to be shown whether any formula in a basic modal language, or rather some familiar extended version of it, can characterize stabilization$_f$.

So far we dealt only with the case of a single agent, whereas the applications of product update lie mainly in the area of systems of multiple, interacting agents, in modeling various kinds of information exchange. Furthermore, the example of the non-stabilizing$_f$ frame is rather exotic. As it turns out, in the setting of multi-agent update logics, we can maintain the most conservative assumptions of S5-knowledge, and still end up with non-stabilizing$_f$ frames.

Example 34. *Consider the Acknowledged Message-Passing frame* $\mathtt{A}$ *(figure 2). In order to prove non-stabilization$_f$ of* $\mathtt{A}$*, we must provide, for every t, a winning strategy for Spoiler in the corresponding pebble game. The strategy is very similar to the strategy in example 33: Spoiler puts all blue stones on node a_1 and then successively "exhausts" this pile by passing single stones across to node a_4. We leave the details to the reader.*

Our formalism does not allow the precondition formulas that refer to events that have been carried out, as e.g. the success of the most recent message delivery (a_4 above). Accordingly, we are not able to model the full Coordinated Attack scenario, where the generals - precisely - condition sending messages on the success of the most recent message delivery. This result shows that it is not the conditioning on the past events that is responsible for the nonstabilization in the Coordinated Attack.

More generally, it is not difficult to show that any S5 frame on which the formula φ:

$$\langle i \rangle\, [j]p \wedge \langle i \rangle\, [j]\neg p \wedge [i](q \wedge \langle j \rangle\, \neg q)$$

is satisfiable (e.g. at a_2 or a_3 in our frame above), will not stabilize$_f$ and that actually in the case of two-agent update logic, the satisfiability of the above formula fully determines the class of stabilizing$_f$ frames within the class of S5 frames. In the case of update logics with more than two agents, there are other formulas, not implying φ, whose satisfiability also guarantees non-stabilization$_f$. Witness the following one:

$$[i]p \wedge \neg[i]q \wedge [j]q \wedge \neg[j]p \wedge [k](p \wedge q).$$

It's satisfiability corresponds to an "overlap" of the partition cells p_n^i, p_m^j of agents i and j (first four conjuncts), i.e. the condition that two partitions cannot be ordered in terms of coarseness, modulo the auxiliary condition on some third agent's partition (fifth conjunct).

5. General Epistemic events

We have found a convenient way of analyzing stabilization$_f$ of frames in a way of a certain pebble game. However, the constraint of the update models being factual leaves outside the analysis the bulk of epistemic events, where simple events may signal epistemic content of static model. The update model for the epistemic dynamics in the Muddy Children puzzle provides an example.

In general, the stabilization of epistemic events is almost impossible. The modal depth of precondition formulas can grow indefinitely in the course of composing, and so "destabilizes" the semigroup. Accordingly, no frame with at least one reflexive state for any relation stabilizes. Witness the following example.

Example 35. *Consider the $\mathcal{L}_{EL-C}-$epistemic event (A, a) based on the frame* A *consisting of a single reflexive point a, with $p_A(a) = \langle i \rangle \top$. In this case $(A, a)\circ^n$ is the model based on the frame* A *as well, with the prerequisite formula:*

$$p_{A^n}((a, ...a)) = \bigwedge_{k=1,\dots,n} \langle i \rangle^k \top.$$

It follows that neither the epistemic event (A, a) nor frame A *stabilize.*
Let the epistemic state (M, spy) be defined through:

S_M := $\{\mathbb{N} \cup spy\}$, *where $\mathbb{N}$ is the set of all natural numbers,*
R_M^i :≡ $nR_M^i m \; iff \; (n, m \in \mathbb{N} \;, n = m + 1) \vee (n = spy)$,
p_M :≡ $(M, m) \models p \; iff \; m = 2^s$ *for some* $s \in \mathbb{N}$.

We have $(M, spy) \models \langle\langle (A, a)^{\rightarrow} \rangle\rangle \, \varphi_n$, $n > 0$, for the pairwise inconsistent formulas φ_n, $n > 1$:

$$\varphi_n = \langle i \rangle \, ([i]^n \bot \wedge p) \wedge [i] \bigwedge_{k<n} ([i]^k \bot \rightarrow \neg p).$$

The example shows that neither proposition 27 nor the first part of prop. 26 can be generalized to the $\mathcal{L}_{EL-C}-$epistemic events. In order to see that the second point of propositon 26 is also particular to $\mathcal{L}_{fact}-$epistemic events look at the following

Example 36. *Consider an S5 $\mathcal{L}_{EL-C}-$epistemic event (A, a_1) and an epistemic state (M, m) (the accessibility relations are represented by partitions):*

S_A	:=	$\{a_1, a_2\}$,		S_M	:=	$\{x, y\}$,
P_A^i	:=	$\{\{a_1\}, \{a_2\}\}$,		P_M^i	:=	$\{\{x, y\}\}$,
P_A^j	:=	$\{\{a_1, a_2\}\}$,	,	P_M^j	:=	$\{\{x\}, \{y\}\}$,
$p_A(a_1)$	:=	$\langle i \rangle \neg q$,		$V_M(x)$	:=	$\emptyset$,
$p_A(a_2)$	:=	$q \wedge [j] \langle i \rangle \neg q$.		$V_M(y)$	:=	q.

Verify that the models $(M, m) \otimes (A, a_1)\circ^t$ are bisimilar to (M, m) for t odd, and for t even bisimilar to $(MxA, (x, a_1))$:

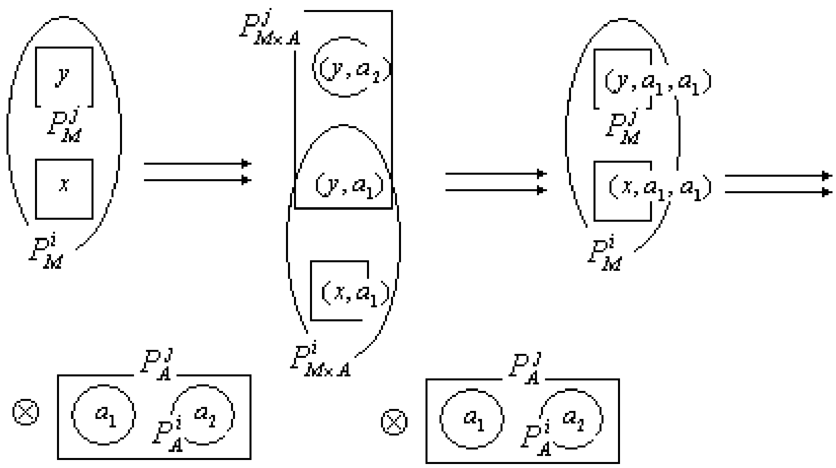

Figure 6. "Looping" product update

We don't know what is in general the status of the last part of proposition 26, i.e. whether there are p-stabilizing $\mathcal{L}_{EL-C}-$epistemic events, with p > 1.

We also cannot hope to generalize corollary 29 (satisfaction problem for $\mathcal{L}^U_{EL-C}$ is clearly decidable - see also Pratt [22] and van Benthem, van Eijck, Kooi [7]):

Proposition 37. *Satisfaction problem for $\mathcal{L}^{U\rightarrow}_{EL-C}$ is Σ^1_1 hard.*

Proof. We refer to Millner, Moss [19], where the instences of the *recurring tiling problem* are encoded in the language with Kleene star operator over (special case of) epistemic events. Fix a recurring domino system $\mathcal{D} = (Dominoes, H, V, d_0)$, where $Dominoes$ is a finite set, $d_0 \in Dominoes$ and $H, V \subseteq Dominoes \times Dominoes$. For any $\mathcal{D}$, Prop$\supseteq$ $Dominoes$ and $I = \{i\}$ define:

$$\begin{array}{rcl}
A & : & = \{\{a\}, \{a, a\}, \{(a, \langle i\rangle \top)\}\}, \\
stalk & : & = [\langle (A, a)^{\rightarrow}\rangle] \langle i\rangle \top, \\
\chi_d & : & = d \wedge [i]\bot, \\
\varphi_{\mathcal{D}} & : & = stalk \wedge C(stalk \rightarrow (\langle i\rangle\, stalk \wedge \neg C\neg(stalk \wedge \langle i\rangle\, \chi_{d_0}))) \wedge \\
{[\langle (A, a)^{\rightarrow}\rangle]}\, C(stalk & \rightarrow & (\langle i\rangle\, (\bigvee_d \chi_d) \wedge \\
 & & \wedge(\neg \bigvee_{\neg H(d,d')} \langle i\rangle\, \chi_d \wedge \langle i\rangle\, \langle i\rangle\, \chi_{d'}) \wedge (\neg \bigvee_{\neg V(d,d')} \langle i\rangle\, \chi_d \wedge \langle i\rangle\, \langle (A, a)\rangle\, \chi_{d'})).
\end{array}$$

The proof that formula $\varphi_{\mathcal{D}} \in \mathcal{L}^{U\rightarrow}_{EL-C}$ is satisfiable iff there exists a proper tiling of $\mathbb{N} \times \mathbb{N}$ by $\mathcal{D}$ follows the proof of theorem 5 in Millner, Moss [19].[8] Finally, Harel [16] showed that deciding whether a recurring domio system has a proper tiling is Σ^1_1-complete. ■

[8] A *proper tiling* of $\mathbb{N} \times \mathbb{N}$ by $\mathcal{D}$ is a function $t : \mathbb{N} \times \mathbb{N} \rightarrow Dominoes$ such that for all $m, n \in \mathbb{N}$:
1) $H(t(n, m), t(n + 1, m))$.
2) $V(t(n, m), t(n, m + 1))$.
3) $t(n, 0) = d_0$ for infinitely many n.

Example 35 shows that we need to give up infinite state models as the arguments for update, if we want to talk about stabilization. We will say that a frame A *(p-) stabilizes^{fin} (at stage t)* iff for every epistemic event over it and every epistemic state (M, m) with finite domain, $\Omega_{(A,\mathbf{a})}((M,m))$ stabilizes, with p and t being the suprema of corresponding indices over all such orbits.

The last examples in the previous section provided evidence that even quite simple multi S5 frames can lack stabilization$_f$, and therefore also stabilization^{fin} property[9]. The culprit was the "overlapping" of partition members for different relations. More formally, we will say that a frame A is *epistemically ordered,* when for some total ordering $R_A^{i_1} \leq_e R_A^{i_2} \leq_e ... \leq_e R_A^{i_{|I|}}$ of the relations of A it validates the formula:

$$\varphi_{\leq_e} := \bigwedge_{s=2,...,|I|} [i_s]p \rightarrow [i_{s-1}]p$$

It should be clear that if A is a multi S5, then A is epistemically ordered iff there exists a total ordering $\leq_e$ of the set of partitions of A such that if $P_A^i \leq_e P_A^j$ then P_A^i is finer than P_A^j. Epistemical ordering is not only (almost) necessary but also a sufficient condition for stabilization^{fin} of frames in the case of S5 models.

Proposition 38. *Let $(A, \mathbf{a})$ be any $\mathcal{L}_{EL-C}^{U\rightarrow}$−epistemic event over an epistemically ordered multi S5 frame. For any finite state model (M, m) the orbit $\Omega_{(A,\mathbf{a})}((M,m))$ stabilizes.*

Remark 39. *As example 36 shows, those orbits not necessarily 1-stabilize.*

Proof. Fix a model (M, m) with domain S_M, $|S_M| = n$, and $\leq_e$, a total ordering of agents: $i_1 <_e i_2 <_e ... <_e i_{|I|}$. We will consider sets $E_{n,k}$ that are defined inductively:

$$\begin{array}{lllll} E_{n,0} & := & Pow(S_M) & = & \{e_1^0, e_2^0, ..., e_{2^n}^0\}, \\ E_{n,k} & := & Pow(E_{n,k-1}) & = & \{e_1^k, e_2^k, ..., e_{2^{|E_{n,k-1}|}}^k\} \quad , k \in \{1, ..., |I|\}. \end{array}$$

For any $\mathcal{L}_{EL-C}^{U\rightarrow}$−epistemic event $(A, \mathbf{a})$ based on epistemically ordered multi S5 frame ordered by $\leq_e$ assign to its every simple event a an index $(e^0(a), ..., e^{|I|}(a)) \in E_{n,0} \times ... \times E_{n,|I|}$ in the following, inductive way:

$$\begin{array}{lll} e^0(a) & : & = \{m \in S_M | (M, m) \models p_A(a)\}, \\ e^k(a) & : & = \{e^{k-1}(a') | a R_A^{i_k} a'\}. \end{array}$$

Assign to the whole model the set of such indices of its distinguished states. Notice that there are only finitely many such sets of indices. Therefore in order to prove the proposition it is enough to prove the following two claims:

Claim (i) If two $\mathcal{L}_{EL-C}^{U\rightarrow}$−epistemic events $(A, \mathbf{a})$ and $(A', \mathbf{a}')$ based on epistemically ordered S5 frames with the same ordering $\leq_e$ are associated with the same sets of indices, then $(M, m) \otimes (A, \mathbf{a}) \underline{\leftrightarrow} (M, m) \otimes (A', \mathbf{a}')$.

Claim (ii) The set of $\mathcal{L}_{EL-C}^{U\rightarrow}$−epistemic events based on epistemically ordered S5 frames with ordering $\leq_e$ is closed with respect to composition.

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[9] Observe that the canonical model mentioned in the proof of proposition 26 is finite.

Proof Claim (i) The bisimulation joins the states that have the same first, state coordinates and whose second, event coordinates share the same indices. Agreement on the atomic formulas is trivial given that both states exist in the corresponding producted models. This in turn is achieved by the fact that the action coordinates of both states share indices, and so share the first coordinate of the indices, which encode the sets of states in S_M that satisfy its prerequisites.
The proof of the zigzag clauses is a simple consequence of the way we defined the indices $(e^0(a), ..., e^{|I|}(a))$. $\Box$*Claim (i)*

Proof Claim (ii) Let $(A, \mathbf{a})$ and $(A', \mathbf{a}')$ be two $\mathcal{L}^{U\rightarrow}_{EL-C}-$epistemic events based on epistemically ordered S5 frames with ordering $\leq_e$ and consider the $\mathcal{L}^{U\rightarrow}_{EL-C}-$epistemic event $(A, \mathbf{a}) \circ (A', \mathbf{a}')$. Suppose that $k \leq_e j$. We need to show that the frame $\{S_{A\circ A'}, R^i_{A\circ A'}|_{i\in I}\}$ validates the formula

$$[j]p \rightarrow [k]p,$$

and that its relations are reflexive, symmetric and transitive. The first part boils down to showing that for all $(a, b), (a', b') \in S_{A\circ A'}$, $(a, a')R^k_{A\circ A'}(b, b')$ implies $(a, a')R^j_{A\circ A'}(b, b')$, which follows from the validity of $[j]p \rightarrow [k]p$ on frames `A` and `A`':

$$(a, a')R^k_{A\circ A'}(b, b') \Leftrightarrow (a)R^k_A(b) \text{ and } (a')R^k_{A'}(b') \Rightarrow (a)R^j_A(b) \text{ and } (a')R^j_{A'}(b') \Leftrightarrow (a, a')R^j_{A\circ A'}(b, b').$$

Reflexivity, symmetry and transitivity of the relations are proven similarly. $\Box$*Claim (ii)*
$\blacksquare$

Remark 40. *Analogous proof shows that for the special cases of KD45 frames corresponding to* Secure Announcement to a Set of Agents *and* Announcement with a Suspicious Outsider, *and therefore for all the types of communicative events in Baltag* et al. *[2] the claim in the proposition 38 holds.*[10]

6. Conclusion and Further Questions

The paper builds on the idea of looking at the update logics as dynamical systems. The particular logic we took under scrutiny was the product update logic presented by Baltag, Moss and Solecki [2], logic suited for dealing with various dynamic epistemic events and history-independent processes, mainly of the communicative type.

In the course of the paper we proved the particular conservative properties of update with $\mathcal{L}_{fact}-$epistemic events, and also presented a characterization in terms of winning strategies in a certain pebble game of the frames giving rise only to stabilizing $\mathcal{L}_{fact}-$epistemic events. Finally, after providing some basic insights into the dissipative nature of general updates we proved the exceptional role of epistemically ordered S5 frames.

There are plenty of further open questions immediately related to the issues raised in the paper:

- Characterize the *stabilizing*^{fin} frames, i.e. frames giving rise only to epistemic events effecting finite evolution of finite epistemic states. This would be a result corresponding to the pebble game characterization we provided for the case when

[10] The proof uses the analogous method as that of proposition 38, in that we find, for a given epistemic event (and size of epistemic state), appropriate finite sets of structures, and then prove two claims as in the theorem. For example, in the case of Secure Announcement to a Set B of Agents, it is, roughly, the set of structures each of which is a set of structures with one state s reflexive for all types of relations, and other elements reflexive for agents in B and related to s for the other agents.

the epistemic events were constrained to be factual (and arguments not necessarily finite). It is possible that the two classes coincide, proving the evolution of prerequisite formulas immaterial for stabilization, with the determining role of the evolution of producted frame.

- Characterize *1-stabilizing*^{fin} ($\mathcal{L}^U_{EL-C}-$)epistemic events - i.e. ($\mathcal{L}^U_{EL-C}-$)epistemic events for which the uncanny behavior as in the example 36 is excluded. Notice that for $\mathcal{L}_{fact}-$epistemic events only this fixed-point type stabilization is possible.
- Decidability and complexity issues: e.g. is the problem of stabilization$_f$ of a frame, i.e the problem whether for a given frame and some $t \in \mathbb{N}$ Constructor has a winning strategy in the pebble game in lemma 30, decidable?

Another question concerns the extensions of the dynamic language that would allow to model more involved dynamic epistemic situations (see the Coordinated Attack examples 6, 34 and the comments after the latter). In particular, allowing for prerequisite formulas in the richer language we might be able to investigate systems allowig for "non-Markovian" updates, which depend not only on the current state, but also history: past states, perhaps past evens (see also Yap [24]).

The dynamical systems literature furnishes a whole spectrum of typical questions, most of which have a lot of appeal in the setting of information update logic, e.g.:

- For an epistemic state (M, m) characterize its *stabilizer*, i.e. those epistemic events that update (M, m) to itself.

The questions and problems of this guise could also be raised in the settings of different update logics. Perhaps the dynamical, asymptotic properties can provide interesting structural characterizations of such systems.

7. APPENDIX

Sound and Complete Axiomatization of Product Update Logic (Baltag *et al.* [2]):

Basic Axioms

All sentential validities	
($[(A,a)]$ -normality)	$\vdash [(A,a)](\varphi \to \psi) \to ([(A,a)]\varphi \to [(A,a)]\psi)$
($[j]$-normality)	$\vdash [j](\varphi \to \psi) \to ([j]\varphi \to [j]\psi)$
(C_J-normality)	$\vdash C_J(\varphi \to \psi) \to (C_J\varphi \to C_J\psi)$

Update Axioms

(Atomic Permanence)	$\vdash [(A,a)]p \leftrightarrow (p_A(a) \to p)$
(Partial functionality)	$\vdash [(A,a)]\neg\varphi \leftrightarrow (p_A(a) \to \neg[(A,a)]\varphi)$
(Action Knowledge)	$\vdash [(A,a)][j]\varphi \leftrightarrow (p_A(a) \to \bigwedge_{aR^j_A b}[j][(A,b)]\varphi$

Mix Axiom $\vdash C_J\varphi \to \varphi \wedge /\ _{j\in J}[j]C_J\varphi$

Modal Rules

(Modus Ponens)	From $\vdash \varphi$ and $\vdash \varphi \to \psi$ infer $\vdash \psi$
($[(A,a)]$-necessitation)	From $\vdash \varphi$ infer $\vdash [(A,a)]\psi$
($[j]$-necessitation)	From $\vdash \varphi$ infer $\vdash [j]\psi$
(C_J-necessitation)	From $\vdash \varphi$ infer $\vdash [C_J]\psi$

Action Rule

Let for all b s.t.$a \rightarrow_J^* b$ χ_b be sentences, and:

1. $\vdash \chi_b \rightarrow [(A,b)]\psi$
2. If for $j \in J$ we have $b \rightarrow_j c$, then $\vdash (\chi_b \wedge p_A(b)) \rightarrow [j]\chi_c$

From those assumptions infer $\vdash \chi_a \rightarrow [(A,a)]C_J\psi$

It is immediate that in order to deal with the multipointed events we just need to add a reduction axiom (see def. 10):

$$\vdash [(A,\mathbf{a})]\varphi \leftrightarrow \bigwedge_{a\in\mathbf{a}}[(A,a)]\varphi.$$

See also van Benthem *et al.* [7] for a different type of axiomatization, using the language that can express relativizations in the common knowledge formulas.

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