A Note on Some Explicit Modal Logics

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Abstract. Artemov introduced the Logic of Proofs (**LP**) as a logic of explicit proofs. We can also offer an epistemic reading of this formula: "t is a possible justification of ϕ". Motivated, in part, by this epistemic reading, Fitting introduced a Kripke style semantics for **LP** in [8]. In this note, we prove soundness and completeness of some axiom systems which are not covered in [8].

1 Introduction

The Logic of Proofs (**LP**), developed by Artemov [1, 2], is a modal logic of explicit proofs. The original motivation was to answer a long standing question about the provability semantics for **S4** via an interpretation into Peano Arithmetic. The basic idea is to replace statements of the form "*there is* a proof of ϕ" ($\Box\phi$) with "t *is* a proof a ϕ" ($t : \phi$). Recently, Fitting [9], Artemov and Nogina [3] and Artemov [4] have suggested an epistemic interpretation of **LP**. Under an epistemic reading, $t:\phi$ is interpreted as "t is possible reason that an agent knows ϕ" or "t is a possible justification of ϕ". The suggestion for a logic of *explicit* knowledge was first made by van Benthem [6] and is explored in [9]. In this note, we will use both terms "proof" and "reason" interchangeably. Motivated, in part, by this epistemic interpretation, Fitting developed a Kripke style semantics for **LP** in [8]. These models, which we will call **Fitting models**, are essentially a combination of Kripke structures with a semantics for **LP** developed by Mkrtychev [11]. A next natural step is to consider a combined language containing both *implicit* modalities ($\Box\phi$) and *explicit* modalities ($t:\phi$). This line of reasoning is explored in [3, 4].

In this note, I will look at completeness with respect to Fitting models for some axiom systems not covered in [8]. In particular, we introduce semantic conditions which are needed to prove completeness for explicit modal logics that contain the explicit version of the 5 axiom ($\neg\Box\phi \to \Box\neg\Box\phi$). We then obtain completeness results for the explicit version of the axiom systems **K5**, **KD**, **KD45** and **S5**.

2 Syntax and Semantics

Proof polynomials are built from a set of axiom constants $\mathcal{C}$, a set of variables $\mathcal{X}$ and closed under the operations binary operation $+$ and $\cdot$ and the unary operation !. Let $\mathbb{P}_{LP}$ be the set of LP proof polynomials. Proof polynomials are constructed according to the following grammar

$$t := c \mid x \mid t+t \mid t\cdot t \mid !t$$

where $c \in \mathcal{C}$ and $x \in \mathcal{X}$. Let $\mathbb{P}_{LP}$ denote the set of proof polynomials. We quickly discuss the intended interpretations of the above operations. For a in-depth discussion of the intended interpretation the reader is referred to [1]. Proof constants are intended to represent reasons which we do not further analyze, i.e., proofs of logical truths. Proof variables can be understood as implicitly universally quantifying over proof polynomials. The $\cdot$ operator is intended to represent an application of the modus ponens rule. That is, if s is a reason for $\phi \rightarrow \psi$ and t is a reason for ϕ, then $s \cdot t$ is a reason for ψ. The ! operator is a proof checker, i.e., $!t$ is a verification of the proof t. Finally, $+$ is a sort of concatenation of proofs. That is, $s + t$ is a proof for everything that follow from either s or t.

It turns out that in order to deal with the modal logics that contain the 5 axiom, we must introduce a new proof operator. Following, [5], we add a unary operator "?", called a **negative proof checker**, to $\mathbb{P}_{LP}$. Define $\mathbb{P}^*_{LP}$ (extended proof polynomials) to be $\mathbb{P}_{LP}$ closed under the unary operation ?. We will say more about this operation below.

Let At be a countable set of propositional variables. A formula in $\mathcal{L}$ will have the following syntactic form

$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid t{:}\phi$$

where $p \in \mathsf{At}$ and $t \in \mathbb{P}_{LP}$. Let $\mathcal{L}^*$ be the language which is just like $\mathcal{L}$ except each proof polynomial is from $\mathbb{P}^*_{LP}$. Again, the intended interpretation of $t : \phi$ is "t is a proof of ϕ" or "t is a justification of ϕ".

We now quickly describe the Kripke style of semantics as discussed in [8]. The reader is referred to [8] for motivation and proofs. Suppose that W is a set of states, R is a binary relations on W. A frame is a tuple $\langle W, R\rangle$. A model is a tuple $\mathcal{M} = \langle W, R, \mathcal{E}, V\rangle$, called a **Fitting model**, where $V : \mathsf{At} \rightarrow 2^W$ is a valuation function and $\mathcal{E}$ is an **evidence function**. Formally, an evidence function is any function $\mathcal{E} : W \times \mathbb{P}_{LP} \rightarrow 2^{\mathcal{L}}$ ($\mathcal{E} : W \times \mathbb{P}^*_{LP} \rightarrow \mathcal{L}^*$). We first define truth in a model:

- For $p \in \mathsf{At}$, $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\phi$ iff $\mathcal{M}, w \not\models \phi$
- $\mathcal{M}, w \models \phi \wedge \psi$ iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models t{:}\phi$ iff $\phi \in \mathcal{E}(w, t)$ and for every v, if wRv then $\mathcal{M}, v \models \phi$

We say that ϕ is valid in $\mathcal{M}$ if $\mathcal{M}, w \models \phi$ for every state w, and ϕ valid in a frame $\mathcal{F} = \langle W, R\rangle$ if ϕ is valid in all models based on $\mathcal{F}$. The following properties of $\mathcal{E}$ were discussed in [11, 8].

Monotonicity For all $w, v \in W$, if wRv then for all proof polynomials t, $\mathcal{E}(w, t) \subseteq \mathcal{E}(v, t)$.

Application For all proof polynomials s, t and for each $w \in W$, if $\phi \rightarrow \psi \in \mathcal{E}(w, s)$ and $\phi \in \mathcal{E}(w, t)$ then $\psi \in \mathcal{E}(w, s \cdot t)$

Proof Checker For all proof polynomials t and for each $w \in W$, if $\phi \in \mathcal{E}(w, t)$ then $t{:}\phi \in \mathcal{E}(w, !t)$.

Sum For all proof polynomials s, t and for each $w \in W$, $\mathcal{E}(w, s) \cup \mathcal{E}(w, t) \subseteq \mathcal{E}(w, s{+}t)$.

Various classes of models are discussed in [8]. We describe a few of these classes below.

- $\mathcal{M}$ is an **LP**(**K**) model if $\mathcal{E}$ satisfies the Application and Sum properties (no conditions on R).
- $\mathcal{M}$ is an **LP**(**K4**) model if it is a **LP**(**K**) model, R is transitive, and in addition to Application and Sum $\mathcal{E}$ satisfies Monotonicity and Proof Checker.
- $\mathcal{M}$ is an **LP**, or **LP**(**S4**) model if $\mathcal{M}$ is a **LP**(**K**) model, R is reflexive and transitive, and in addition to Application and Sum, $\mathcal{E}$ satisfies Proof Checker and Monotonicity.

A **constant specification** is a function $CS : \mathcal{C} \rightarrow 2^{\mathcal{L}}$ ($CS : \mathcal{C} \rightarrow 2^{\mathcal{L}^*}$) where $\phi \in CS(c)$ means that ϕ is assigned constant c. Following [8], it is required that any formula assigned to a proof constant with respect to CS must be true at every possible world of every weak **LP** model, i.e., ϕ must be valid. We say a model $\mathcal{M}$ **meets** a constant specification CS provided the following constant condition is satisfied.

Constant Condition For each formula ϕ, with $\phi \in CS(c)$, $\phi \in \mathcal{E}(w, c)$ for each $w \in W$

We say that a formula ϕ is $CS-$**LP** satisfiable if there is a weak **LP** model that meets CS in which ϕ is true at some state. Similarly for CS-valid. Actually, Fitting considers two versions of his semantics (weak models and strong models). The semantics we have defined in this section corresponds to weak models in [8]. However, in the interest of space we will not discuss this distinction. And in fact, given the assumption we will make about the constant specifications in the next section, it turns out that the two versions of the semantics are equivalent. The reader is referred to [8] for more information.

3 Some Logics

Fix a constant specification CS for this section. The following is a list of the axiom schemes and rules which will be discussed in this note. These axioms and rules are easily seen to be explicit versions of some well-know modal axiom schemes. Appendix A lists the relevant modal logics.

EK	$s{:}(\phi \rightarrow \psi) \rightarrow (t{:}\phi \rightarrow s \cdot t{:}\psi)$
ET	$t{:}\phi \rightarrow \phi$
$E4$	$t{:}\phi \rightarrow !t{:}t{:}\phi$
$E5$	$\neg t{:}\phi \rightarrow ?t{:}\neg t{:}\phi$
ED	$t{:}\bot \rightarrow \bot$
Sum	$s{:}\phi \rightarrow (s+t){:}\phi$ and $s{:}\phi \rightarrow (t+s){:}\phi$
AS	Infer $c{:}\phi$, where c is an axiom constant, and $\phi \in CS(c)$

The Logic of Proofs (**LP**) contains the axiom schemes PC, EK, ET, $E4$, Sum and the rule AS. We say ϕ is CS-derivable, or there is a CS-**LP** proof, if ϕ there is a derivation in **LP** using axiom specification CS. An axiom specification CS is called **axiomatically appropriate** if it is exactly instances of axiom schemes which are assigned proof constants. In this note, we will restrict attention to constant specifications which are axiomatically appropriate. Axiom $E5$ was introduced in [5] and will be discussed below. Axiom ED was introduced by Breznhev [7] is easily seen to be the explicit version of the D axiom ($\Box\bot \rightarrow \bot$). This axiom says that t is never a reason for a contradiction. We will use the following notation, if Λ is a modal logic, then $\mathbf{LP}(\Lambda)$ is the explicit version of Λ. So, the logic **LP** can be written as $\mathbf{LP}(\mathbf{S4})$. This notation suggests a the modal logic **S4** and its explicit version **LP** are somehow related. In [1, 2], Artemov shows that there is, in fact, an important connection between **LP** and **S4**. There are two directions to this connection. The first is that the "forgetful projection" of **LP** is **S4**. That is, define a map $\cdot^\circ$ that is the identity on propositional variables, commutes with propositional connectives and for any proof polynomial t, $(t\!:\!\phi)^\circ = \Box\phi^\circ$. Then it is not too difficult to see that if ϕ is derivable in **LP**, then ϕ° is derivable in **S4**. One of the most interesting facts about **LP** is that Artemov showed the converse. That is, there is a "realization" algorithm that will turn any theorem of **S4** into a theorem of **LP** where all negative occurrences of boxes are replaced by proof variables. In [7] proves a realization theorem for the logics $\mathbf{K}, \mathbf{KD}, \mathbf{KD4}$ and **T**. Artemov, Kazakov and Shapiro [5] provide a realization theorem for **S5**. The explicit version of these logics are given below. For the logics without the 4 axiom scheme, the AS rule must be replaced by the following rule.

AS^* Infer $c\!:\!\phi$, where c is an axiom constant,
and $\phi \in CS(c)$ or inferable using AS^*.

The following are the axiom systems are discussed in this note.

- $\mathbf{LP}(\mathbf{K})$ contains PC, EK, Sum and AS^*
- $\mathbf{LP}(\mathbf{KD})$ contains all axiom schemes and rules of $\mathbf{LP}(\mathbf{K})$ plus the axiom scheme $t\!:\!\bot \rightarrow \bot$.
- $\mathbf{LP}(\mathbf{KD4})$ contains EK, Sum, $E4$ and AS.
- $\mathbf{LP}(\mathbf{K5})$ contains EK, Sum, $E5$ and AS^*.
- $\mathbf{LP}(\mathbf{KD45})$ is $\mathbf{LP}(\mathbf{KD4})$ together with the $E5$ axiom scheme.
- $\mathbf{LP}(\mathbf{S5})$ contains all the axiom schemes and rules of $\mathbf{LP}(\mathbf{S4})$ plus the axiom scheme $E5$.

One more fact about the above axiom systems is needed for the completeness proofs. Notice that the CS-AS axiom is essentially the explicit version of a restricted form of the necessitation rule. That is, necessitation can only be applied to instances of axiom schemes[1]. In fact, Artemov showed that there is an explicit version of full necessitation in **LP** [1], called **explicit necessitation**. That is ϕ is CS-derivable in **LP**, then there is a proof polynomial t (which does not contain variables), such that $t\!:\!\phi$ is CS-derivable in **LP**. It was shown in [7] and [5] that each axiom system presented in this section all satisfy explicit necessitation.

[1] Actually, this follows because we are assuming that are constant specifications are axiomatically appropriate.

4 The $E5$ Axiom

Moving on to the $E5$ axiom. The motivation for this operation can be found in [5]. The difficulty in coming up with an appropriate explicit version of the 5 axiom ($\neg\Box\phi \to \Box\neg\Box\phi$) is that the antecedent contains *negative* information about a reason, or proof. The approach taken in [5] is to use an alternative axiomatization of **S5**, which gives an explicit version of **S5** admitting an arithmetical interpretation. The approach taken in this paper is to assume the existence of an operation on proofs such that given a proof t, $?t$ is a proof of each formula that t does not prove. In particular, we must assume that $?t$ is an infinite conclusion proof. The exact connection between the approach of this paper and Artemov et al. is a topic for further study.

The $E5$ axiom says that if t is not a reason for ϕ, then $?t$ is a reason for this fact. The following two conditions are needed for logics containing this axiom scheme.

Anti-monotonicity If $\phi \notin \mathcal{E}(w,t)$ and wRv, then $\phi \notin \mathcal{E}(v,t)$
Negative Proof Checker If there is a v such that wRv and $\mathcal{M}, v \models \neg\phi$ or $\phi \notin \mathcal{E}(w,t)$, then $\neg t\!:\!\phi \in \mathcal{E}(w, ?t)$

Anti-monotonicity says that if t is not a reason for ϕ in w, then it will not be a reason for ϕ in any accessible world. Of course, both Monotonicity and Anti-monotonicity together imply that if wRv, then $\mathcal{E}(w,t) = \mathcal{E}(v,t)$. We now show that these two conditions (together with assuming that R is Euclidean) are exactly what is needed to prove the validity of the $E5$ axiom.

Lemma 1. *If $\mathcal{M} = \langle W, R, \mathcal{E}, V\rangle$ is a Fitting model where $\mathcal{E}$ satisfies anti-monotonicity and negative proof checker and R is Euclidean, then $\neg t\!:\!\phi \to ?t\!:\!\neg t\!:\!\phi$ is valid in $\mathcal{M}$.*

Proof. Suppose that $\mathcal{M} = \langle W, R, \mathcal{E}, V\rangle$, $\mathcal{E}$ satisfies anti-monotonicity and negative proof checker and R is Euclidean. Suppose that $\mathcal{M}, w \models \neg t\!:\!\phi$. Then either $\phi \notin \mathcal{E}(w,t)$ or there exists a v with wRv and $\mathcal{M}, v \models \neg\phi$. By the negative proof checker, we have that $\neg t : \phi \in \mathcal{E}(w, ?t)$. Thus we need only show that for all v if wRv, then $\mathcal{M}, v \models \neg t : \phi$. Suppose that $\phi \notin \mathcal{E}(w,t)$ and let v be an arbitrary state with wRv. By negative proof checker, $\phi \notin \mathcal{E}(v,t)$. Hence, $\mathcal{M}, v \models \neg t : \phi$. Alternatively, suppose that v_0 is a state such that wRv_0 and $\mathcal{M}, v_0 \models \neg\phi$. Then if v is any state with wRv, since R is Euclidean, vRv_0. Hence, $\mathcal{M}, v \models \neg t\!:\!\phi$. $\Box$

We say that $\mathcal{M}$ is a model for **LP**(**K5**) if $\mathcal{E}$ satisfies anti-monotonicity and negative proof checker and R is Euclidean. Models for the other logics such as **LP**(**S5**) or **LP**(**KD45**) are defined as expected. For example, $\mathcal{M}$ is a **LP**(**S5**) model if R is an equivalence relation and $\mathcal{E}$ satisfies Sum, Application, Proof Checker, Negative Proof Checker, Monotonicity and Anti-monotonicity (and hence $\mathcal{E}$ is constant on an equivalence class of states).

5 Completeness

Fitting showed in [8] that **LP** is sound and complete with respect to the class of **LP** models. In this section we describe Fitting's completeness proof and show how to adapt it in order to prove completeness of **LP**(**KD**) and **LP**(**K5**).

Let Λ be a logic, we will construct a canonical model $\mathcal{M} = \langle W, R, \mathcal{E}, V \rangle$ based on Λ as follows. Let $W = \{\Gamma \mid \Gamma \text{ is a } \Lambda\text{-maximally consistent set}\}$ and $\Gamma^{\#} = \{\phi \mid t{:}\phi \in \Gamma\}$. Then we say $\Gamma R \Delta$ iff $\Gamma^{\#} \subseteq \Delta$. The evidence function is defined as $\mathcal{E}(\Gamma, t) = \{\phi \mid t : \phi \in \Gamma\}$. We define the valuation function as usual, $p \in V(\Gamma)$ iff $p \in \Gamma$. Now we have the following facts.

Theorem 1 (Truth Lemma). *For any formula $\phi \in \mathcal{L}$,*

$$\mathcal{M}, \Gamma \models \phi \textit{ iff } \phi \in \Gamma$$

Proof. The proof is by induction on ϕ. For the propositional variables, the result holds by definition. The boolean connectives are obvious. Suppose that $\mathcal{M}, \Gamma \models t{:}\phi$, then $\phi \in \mathcal{E}(\Gamma, t)$. Hence by construction of $\mathcal{E}$, $t : \phi \in \Gamma$. Suppose $t : \phi \in \Gamma$. Then $\phi \in \Gamma^{\#}$ and so $\phi \in \Delta$ for each Δ with $\Gamma R \Delta$. Thus by the induction hypothesis, if $\Gamma R \Delta$, then $\mathcal{M}, \Delta \models \phi$. Furthermore, $\phi \in \mathcal{E}(\Gamma, t)$. Hence, $\mathcal{M}, \Gamma \models t{:}\phi$. □

Given the above truth lemma, proving completeness of a logic Λ is reduced to showing that the canonical model for Λ is in fact a model for Λ. In particular, the above lemma immediately gives us a completeness proof for **LP**(**K**).

Theorem 2. *If ϕ is CS-derivable in* **LP**(**K**)*, then ϕ is CS-***LP**(**K**) *valid.*

The proofs of the following Lemmas can be found in Appendix B.

Lemma 2. *If Λ contains the axiom scheme ED, then in any canonical model based on Λ R is serial.*

Lemma 3. *If Λ contains the axiom scheme $E5$, in any canonical model based on Λ, R is Euclidean and $\mathcal{E}$ satisfies both anti-monotonicity and negative proof checker.*

The above Lemmas together with the Truth Lemma can be used to show the two main soundness and completeness theorems of this paper. The proofs are standard and so will be omitted.

Theorem 3. *Let CS be an axiomatically appropriate constant specification, then ϕ has a* **LP**(**KD**) *axiomatic proof using CS if and only if ϕ is valid in any* **LP**(**KD**) *model.*

Theorem 4. *Let CS be an axiomatically appropriate constant specification, then ϕ has a* **LP**(**K5**) *axiomatic proof using CS if and only if ϕ is valid in any* **LP**(**K5**) *model.*

These proofs can easily be adapted to show completeness for **LP**(**KD5**), **LP**(**KD45**) and **LP**(**S5**).

6 Conclusions and Further Work

In this short note, we proved soundness and completeness for some axiom systems not covered by Fitting in [8]. In [3], Artemov and Nogina consider systems with both implicit modalities ($\Box$) and explicit modalities ($t : \phi$). The proofs in this section can easily be adapted to work in this setting. We briefly discuss some future work.

- Develop and tableau system for **LP**(**S5**). An tableau system for **LP** was introduced by Renee in [12] and is discussed in [8].
- In [8], Fitting proves the realization theorem first proved by Artemov [1, 2] by a semantical argument. We hope to find a similar proof with respect to **S5** (a proof-theoretic argument is offered in [5]).
- In [10], Fitting introduces a quantified version of **LP**. It should not be too difficult to adapt the results from this note to the semantics presented in [10].

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A Some Modal Logics

We remind the reader of a number of well-known modal axiom systems.

PC	An appropriate axiomatization of propositional calculus
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
T	$\Box\phi \to \phi$
4	$\Box\phi \to \Box\Box\phi$
5	$\neg\Box\phi \to \Box\neg\Box\phi$
D	$\Box\phi \to \Diamond\phi$
N	From ϕ infer $\Box\phi$
MP	From ϕ and $\phi \to \psi$ infer ψ

The modal logic **K** contains the axiom schemes PC and K and the rules N and MP, **K5** is **K** together with the 5 axiom, **KD** is **K** with the axiom scheme D, **KD45** is **KD** with the axiom schemes 4 and 5, and **S5** is **K** with the axiom schemes T, 4 and 5.

B Proofs

Lemma 4. *If Λ contains the axiom scheme ED, then in any canonical model based on Λ R is serial.*

Proof. Let $\Gamma \in W$ be a maximally consistent set. We will show that the set $\{\phi \mid t : \phi \in \Gamma\}$ is a consistent set. Suppose not. That is there are $\phi_1, \ldots, \phi_n$ with proof polynomials $t_1, \ldots, t_n$ such that for each $i = 1, \ldots, n$, $t_i : \phi_i \in \Gamma$ where $(\phi_1 \wedge \cdots \wedge \phi_n) \to \bot$. Then using explicit necessitation there is a proof polynomial t such that $t : (\phi_1 \wedge \cdots \wedge \phi_n) \to \bot$ is provable. Using standard **LP** reasoning, there is a proof polynomial $f(t, t_1, \ldots, t_n)$ such that $(t_1 : \phi_1 \wedge \cdots t_n : \phi_n) \to f(t, t_1, \ldots, t_n) : \bot$ is provable. Hence $f(t, t_1, \ldots, t_n) : \bot \in \Gamma$, and so $\bot \in \Gamma$ using ED. Contradiction. □

Lemma 5. *If Λ contains the axiom scheme $E5$, in any canonical model based on Λ, R is Euclidean and $\mathcal{E}$ satisfies both anti-monotonicity and negative proof checker.*

Proof. Suppose Λ contains the $E5$ axiom scheme.

– We must show $\mathcal{E}$ is anti-monotonic. Suppose that $\phi \notin \mathcal{E}(\Gamma, t)$, $\Gamma^\# \subseteq \Delta$ and $\phi \in \mathcal{E}(\Delta, t)$. Then $t : \phi \notin \Gamma$ but $t : \phi \in \Delta$. Since Γ is maximal, $\neg t : \phi \in \Gamma$ and so using axiom 5, $?t : \neg t : \phi \in \Gamma$. Therefore, $\neg t : \phi \in \Delta$, which is a contradiction. So, $\mathcal{E}$ is anti-monotonic.
– We must show $\mathcal{E}$ satisfies negative proof checker. First suppose that $\phi \notin \mathcal{E}(\Gamma, t)$. Then, $t : \phi \notin \Gamma$, so $\neg t : \phi \in \Gamma$. Hence using the $E5$ axiom, $?t : \neg t : \phi \in \Gamma$. Hence $\neg t : \phi \in \mathcal{E}(\Gamma, ?t)$. Suppose that there is a Δ such that $\Gamma R \Delta$ and $\mathcal{M}, \Delta \models \neg\phi$. Using the truth lemma, $\neg\phi \in \Delta$. Suppose that $\neg t : \phi \notin \mathcal{E}(\Gamma, ?t)$. Then $?t : \neg t : \phi \notin \Gamma$. Therefore $t : \phi \in \Gamma$. Hence $\phi \in \Gamma^\# \subseteq \Delta$. Contradiction. So, $\mathcal{E}$ satisfies negative proof checker.
– To show R is Euclidean, suppose that $\Gamma, \Delta, \Delta' \in W$ and $\Gamma R \Delta$ and $\Gamma R \Delta'$. We must show that $\Delta^\# \subseteq \Delta'$. Let $\phi \in \Delta^\#$. Suppose that $\phi \notin \Delta$. Then $\phi \notin \Gamma^\#$. Hence for each proof polynomial, $t : \phi \notin \Gamma$ and so for each proof polynomial t, $\neg t : \phi \in \Gamma$. Now we have $t_0 : \phi \in \Delta$ for some proof polynomial t_0 (since $\phi \in \Delta^\#$). We have $\neg t_0 : \phi \in \Gamma$. Therefore using $E5$, $?t : \neg t_0 : \phi \in \Gamma$. And so, $\neg t_0 : \phi \in \Delta$. Contradiction, so $\phi \in \Delta'$. □