Approaches to Independence Friendly Modal Logic

Tero Tulenheimo*
tero.tulenheimo@helsinki.fi
University of Helsinki
Academy of Finland

Merlijn Sevenster
sevenstr@science.uva.nl
ILLC
University of Amsterdam

Abstract

The aim of the present paper is to discuss two different ways of formulating independence friendly (IF) modal logic. In one of them, first presented in [17] modifying ideas introduced in [4], the language of basic modal logic is enriched with the slash notation familiar from IF first-order logics, and the resulting logic is interpreted in terms of games and uniform strategies. The present paper formulates a different approach, by introducing a framework that can be used for formulating various IF modal logics. Within the framework, an IF modal logic is defined by imposing conditions on its structural relationships to other logics, namely a specified modal logic (for instance: basic modal logic), its first-order correspondence language, and IF logic. This framework makes it possible to obtain expressively strong languages that nevertheless enjoy 'nice' properties. In this vein, the so-called 'structurally determined IF modal logic' was introduced in [19]. We compare the logics emerging from these two approaches. More generally, the issue of the *Eigenart* of IF modal logics is addressed.

1 Introduction

Already in the seminal publications on *independence friendly first-order logic* (IF logic) [9, 10, 12, 16], applications were pointed out involving a first-order modal setting. It was argued that the logical form of some natural language sentences is best captured by formulas that allow for *slashing* relative to modal operators – marking certain logical operators as independent of modal operators in whose syntactic scope they nevertheless lie. In the first publications that developed an

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independence friendly modal logic, Bradfield [4] together with Fröschle [5] interpreted the logic's independence indications using a combination of transition systems with *concurrency* and games of *imperfect information*. Tulenheimo [17, 18] together with Hyttinen [15] showed that a reasonable IF modal logic can be defined simply using Hintikka's original idea of implementing logical independence by informational independence in the sense of game theory [8, 9, 10]. To model logical independence, this suffices; it is not necessary to enrich standard modal structures by introducing concurrency as a separate, primitive component of the models. This type of study of IF modal logic serves to attract interest in the larger program of independence-friendliness that investigates the notion of informational independence in logic.

The aims of the present paper are twofold. (1) First, we wish to give a recap of the various logics introduced under the heading 'IF modal logic' whose semantics rely simply on the game-theoretical notion of informational independence, as just explained. (Accordingly, we do not discuss Bradfield's independence friendly modal logics.) The pre-theoretical motivation for all these logics was, when they were introduced, that they would be 'modal analogues' of IF first-order logic – in syntax as well as in semantics.

The logics termed 'independence friendly modal logic' in the relevant research publications [15, 17, 18, 19], differ among themselves both in syntax and in semantics. (The sense of diversity is of course only increased when the logics of [4, 5] are considered as well.) Depending on which precise syntactic restrictions one imposes on the formation of the independence indications, logics with different metalogical properties result. As will turn out, the appropriate properties may diverge strikingly from case to case (cf. Figure 5, *Sect.* 6). The present authors conclude that a framework is desirable in which the various systems can be systematically studied and compared. In this vein, our second aim (2) is to introduce a framework that sheds a unifying light on the IF modal logics introduced so far, and can be used to develop new logics that are both modal and independence friendly. The framework we put forward is determined by three parameters. These parameters are inspired by the standard translation of basic modal logic into first-order logic, and Hintikka's IF procedure that brings us from first-order logic to IF first-order logic.

Although we find our framework a natural environment for studying independence friendliness and modal logic, by no means do we claim that the framework covers all conceivable IF modal logics. Neither do we claim that all logics to be found within this framework are equally interesting. In fact, we regard it as one of the virtues of the framework that within its confines, one can isolate logics some of which enjoy 'nicer' properties than others. From this very perspective, we define the so-called 'structurally determined IF modal logic'. In [19] this logic is shown to combine a number of nice properties: strong expressive power, decidability, and allowing for a compositional semantics.

Let us move on to introduce some basic notions, and fix the plan of the paper.

IF first-order logic. It has been observed by Hintikka, on various occasions, that as a matter of fact, the *syntactical scope* and *(logical) priority scope* of quantifiers coincide in first-order logic. Hintikka [10] points out that there is no general pre-theoretical backing for this assumption, and provocatively refers to it as *Frege's fallacy*. The formulas of IF first-order logic, denoted **IF**, carry the slash '/' as a new item of notation. The slash, as in $\forall y(\exists x/y)\phi$, is to be interpreted in such a way that the occurrence of $\exists x$ is outside the logical priority scope of $\forall y$, although it falls within the syntactical scope of $\forall y$. The formulas of **IF** are generated from the fragment of first-order logic in which every variable is quantified at most once and in which every formula is in *negation normal form*, to be denoted **FO**. Formally, we let **IF** be the smallest superset of **FO** closed under the following condition:

- If $\phi \in$ **IF** and $\exists x$ occurs in ϕ in the syntactic scope of quantifiers among which $Q_1 y_1, \ldots, Q_n y_n$, then the formula resulting from replacing $\exists x$ by $(\exists x/y_1, \ldots, y_n)$ is also in **IF**,

where $Q_i y_i$ stands for $\forall y_i$ or $\exists y_i$. The notion of 'binding a variable' is extended from the usual first-order case by saying that the quantifier $Q_i y_i$ *binds* the occurrence of the variable y_i in $(\exists x/y_1, \ldots, y_n)$, with $1 \leq i \leq n$. We write $\exists x$ rather than $(\exists x/\emptyset)$, if the tuple $y_1, \ldots, y_n$ is empty. One may consider the above rule as specifying an *IF procedure*, producing **IF** from **FO**. In the literature various other IF procedures are put forward, that allow for marking propositional connectives as independent of quantifiers, marking quantifiers as independent of (suitably construed) propositional connectives, and/or marking universal quantifiers as independent of other logical operators. In the current paper, we refrain from considering these options.

Basic modal logic. Formulas of *basic modal logic* (**ML**) are generated from a fixed class **prop** of propositional atoms by the following grammar:

$$\phi ::= p \mid \neg p \mid (\phi \vee \phi) \mid (\phi \wedge \phi) \mid \Diamond\phi \mid \Box\phi,$$

where $p \in$ **prop**. Its semantics is defined relative to *modal structures* and their states, that is, tuples $\mathfrak{M} = (M, R, V)$ and elements $w \in M$, where M is a non-empty domain, R is a binary relation on M termed *accessibility relation*, and V : **prop** $\longrightarrow Pow(M)$ is a *valuation function*. It will be assumed that the clauses recursively associating a truth condition with all **ML**-formulas are familiar. (The reader may consult, e.g., [3, *Sect.* 1.3].) *Polymodal basic modal logic* $\mathbf{ML}_k$ is like **ML**, but involves k modality types, each with its own box $\Box_i$ and diamond $\Diamond_i$. Its semantics is in terms of k-ary modal structures $(M, R_1, \dots, R_k, V)$, having for every modality type i an accessibility relation $R_i \subseteq M^2$ of its own.

Expressive power. If L and L' are two modal logics whose semantics are defined relative to a class $\mathcal{K}$ of modal structures, we say that L is *translatable into* L' over $\mathcal{K}$ (in symbols $L \leq_{\mathcal{K}} L'$), if for every $\phi \in L$, there is $\psi_\phi \in L'$ such that for all modal structures $\mathfrak{M} \in \mathcal{K}$ and all $w \in M$, we have: $\mathfrak{M}, w \models \phi$ if, and only if, $\mathfrak{M}, w \models \psi_\phi$. L' is *more expressive than* L over $\mathcal{K}$, or has *greater expressive power than* L over $\mathcal{K}$ (symbolically $L <_{\mathcal{K}} L'$), if $L \leq_{\mathcal{K}} L'$ but $L' \not\leq_{\mathcal{K}} L$. The logics L and L' have the *same expressive power* over $\mathcal{K}$, or *coincide* over $\mathcal{K}$ (denoted $L =_{\mathcal{K}} L'$), if $L \leq_{\mathcal{K}} L'$ and $L' \leq_{\mathcal{K}} L$. When speaking of the class of *all* modal structures, we suppress the subscript indicating the class altogether, and write simply $L \leq L'$ and so on.

These notions are naturally extended to a comparison between a modal logic and (IF) first-order logic. For every modal structure $\mathfrak{M} = (M, R, V)$ there corresponds, in a canonical way, a first-order structure $\mathfrak{M}^{\mathbf{FO}} = (M, R, \langle V(p) \rangle_{p \in \mathbf{prop}})$, interpreting a binary relation symbol R as the binary relation R, and, for each $p \in \mathbf{prop}$, a unary relation symbol P as the set $V(p)$. Saying, for instance, that L is translatable into **FO**, means that for every $\phi \in L$ there is a first-order formula $\psi_\phi(x)$ of one free variable, x, written in the vocabulary $\{\mathrm{R}, \langle \mathrm{P} \rangle_{p \in \mathbf{prop}}\}$, such that for any modal structure $\mathfrak{M}$ and any $w \in M$, we have: $\mathfrak{M}, w \models \phi$ if, and only if, $\mathfrak{M}^{\mathbf{FO}}, \gamma \models \psi_\phi$, where $\gamma(x) = w$.

Plan of the paper. In *Sections* 2 and 3 we survey two IF modal logics interpreting the slash device in terms of informational independence, referred to as **IFML** and $\mathbf{EIFML}_k$. As an original result we prove Theorem 6, stating that **IFML** cannot be translated into first-order logic. The logics **IFML** and $\mathbf{EIFML}_k$ show that allowing independence friendliness serves to increase the expressive power of a modal logic. However, the definitions of these logics also suggest that many more IF modal logics can be obtained by varying the syntax and the IF procedure applied.

In *Section* 4 we propose a new framework for studying IF modal logics from the IF first-order viewpoint. Essentially, the framework allows for systematically varying the syntax and the IF procedure used in defining an IF modal logic. We discuss at some length a particular logic obtained in this framework, termed ‘structurally determined IF modal logic’, or $\mathbf{IFML}_{\mathbf{SD}}$. (This logic is extensively studied by the authors in [19].) To give a fuller picture of the expressivity of the various IF modal logics discussed in the paper, in *Section* 5 we provide an original negative expressivity result concerning **IFML**, $\mathbf{EIFML}_k$ and $\mathbf{IFML}_{\mathbf{SD}}$, proving that relative to a certain class of trees, the expressive power of all these logics collapses to that of basic modal logic. *Section* 6 serves as a conclusion in which we comment the issue of informational independence in logics, putting forward our conviction that the notion of informational independence not only makes sense with respect to logics other than first-order logic (since it can, for instance, be systematically studied in connection with modal logic) but also enjoys general theoretical interest. In this concluding section also a summary of known results concerning different IF modal logics can be found, as well as a table where some conjectures about them are listed.

Note on notation. If L is a logic for which syntax and semantics is defined, and ϕ is a formula of L, we write $\phi \in L$ to say that ϕ is among the formulas of L. That is, when no confusion may arise, we do not notationally distinguish a logic from its set of formulas. By L-formula we mean formula of L.

2 IF modal logic via independence indications

We wish to introduce a modification of basic modal logic where diamonds may be 'indicated as independent' from any syntactically superordinate modal operators (boxes or diamonds). Such indicating is accomplished by using a notation $(\Diamond/i_1, \ldots, i_k)$, where $i_1, \ldots, i_k$ are positive integers which in a specified, systematic way identify superordinate modal operators. Such syntactic independence indications are semantically interpreted in terms of 'logical dependence': the choice of a state as a semantic value of a diamond $(\Diamond/i_1, \ldots, i_k)$ must not depend on the states interpreting the modal operators identified by the integers $i_1, \ldots, i_k$. Supposing that $(a_1, \ldots, a_n)$ and $(a'_1, \ldots, a'_n)$ are two sequences of choices for modal operators superordinate to $(\Diamond/i_1, \ldots, i_k)$, then if these sequences agree on all choices save for those corresponding to the operators identified by the integers $i_1, \ldots, i_k$, the state chosen for $(\Diamond/i_1, \ldots, i_k)$ must be the same in both cases.

The logic we now proceed to define is dubbed *independence friendly modal logic*. We stress that it carries this name for historical reasons – by no means do we wish to suggest that this logic is *the* IF modal logic. Semantically, IF modal logic will emerge as a proper extension of basic modal logic. This observation increases interest in the study of IF modal logics, for it gives rise to the hope that independence friendliness is a dimension of modal logics that yields more expressive, yet decidable systems. (That entertaining such a hope is not entirely unrealistic can be seen from the decidability results concerning certain specific IF modal logics, cf. [15, 19].) We now turn to defining the syntax and semantics of this logic in detail.

2.1 Definition of the logic

Syntax. The formulas of *Independence friendly (IF) modal logic* (**IFML**) are obtained from those of **ML** by the following rewriting rules:

1. If $\psi \in$ **ML**, then the result of replacing all occurrences of $\Diamond$ in ψ by the symbol $(\Diamond/\emptyset)$ is a formula.

2. If ψ is a formula, $(\Diamond/\emptyset)$ appears in ψ, and $i_1, \ldots, i_n$ is a tuple of positive integers, then the result of replacing that token of $(\Diamond/\emptyset)$ in ψ by the symbol $(\Diamond/i_1, \ldots, i_n)$ is also a formula.

Formulas of **IFML** are precisely the strings generated by the above two rules. By stipulation, we write $\Diamond$ for $(\Diamond/\emptyset)$. Thereby any string that is a formula of **ML**, is in fact also a formula of **IFML**. Note that the input and output of the above rule 2 are identical in the special case that $n := 0$. Examples of **IFML**-formulas are:

$$\Box\Diamond p, \quad \Box(\Diamond/1)p, \quad \Diamond(\Diamond/1)p, \quad \Box(\Diamond/127)\Box(\Diamond/1,2)p,$$
$$(\Box(\Diamond/1)p \wedge \Diamond\Box(\Diamond/1,2)q), \quad \Box(p \vee (\Diamond/1)q), \quad \Box(p \wedge (\Diamond/1)q).$$

It was already pointed out that the role of the integers $i_1, \ldots, i_n$ following a diamond sign, as in $(\Diamond/i_1, \ldots, i_n)$, is to *identify* certain syntactically superordinate modal operators. Which ones? The principle of identification we make use of, is based on the left-linear relation of *syntactic subordination* among tokens of operators $(\vee, \wedge, \Diamond, \Box)$ appearing in formulas $\phi \in$ **IFML**. Relative to a formula ϕ, this relation induces a tree structure, with the unique outmost operator of ϕ at its root, and operators with no subordinate operators at leaves. Hence for any operator-token, the set of its predecessors in this tree structures determines a *linear* order. If $O \in \{\vee, \wedge, \Diamond, \Box\}$ appears in ϕ, it is either itself the unique outmost operator of ϕ, or else there is a unique immediate predecessor O' of O among the operator-tokens to which O is subordinate, and so on. So we may speak of 'the n-th predecessor' of O. We can also restrict attention to *modal* operators preceding O, and enumerate them beginning from the one that is furthest and ending up with the one that is closest. In this way we may speak of 'the n-th modal operator in ϕ among those modal operators that precede O' – hence counting only modal operators and ignoring conjunctions and disjunctions. It is to the numbers identifying the locations of modal operators syntactically superordinate to $(\Diamond/i_1, \ldots, i_n)$ in this latter type of numbering, that the integers $i_1, \ldots, i_n$ refer.

In $\Box(\Diamond/1)\Box(\Diamond/1,2)p$, the numeral 1 in $(\Diamond/1)$ refers to the immediately preceding box, and the numerals 1 and 2 in $(\Diamond/1,2)$ to the first occurrence of $\Box$ *resp.* to the first (and only) occurrence of $(\Diamond/1)$. In $(\Box(\Diamond/1)p \wedge \Diamond\Box(\Diamond/1,2)q)$, the numeral

1 in $(\Diamond/1)$ identifies the box in the left conjunct, whereas the same numeral identifies the outmost diamond of the right conjunct in $(\Diamond/1,2)$. We allow for vacuous identifiers: in $\Box(\Diamond/127)p$ the numeral 127 refers to nothing at all, since there are no 126 or more nested modal operators syntactically preceding $\Box$ in that formula.

In earlier publications on IF modal logic [4, 5, 15, 17, 18], various different identification methods are used for singling out the desired superordinate modal operators. Typically this has been accomplished by introducing an explicit indexing or labeling of tokens of modal operators as a part of the syntax. The possibility of defining the syntax as above shows that such an indexing is by no means a conceptually necessary ingredient of IF modal logics.[1]

The set $Sub[\phi]$ of *subformulas* of a formula $\phi \in$ **IFML** is defined in a straightforward way: $Sub[p] = \{p\}$ and $Sub[\neg p] = \{\neg p\}$; for $\circ \in \{\vee, \wedge\}$: $Sub[(\psi \circ \theta)] = \{(\psi \circ \theta)\} \cup Sub[\psi] \cup Sub[\theta]$; $Sub[\Box\psi] = \{\Box\psi\} \cup Sub[\psi]$; and $Sub[(\Diamond/i_1, \ldots, i_n)\psi] = \{(\Diamond/i_1, \ldots, i_n)\psi\} \cup Sub[\psi]$. A formula $\phi \in$ **IFML** is *closed*, if it contains no vacuous identifiers, i.e. if every $(\Diamond/i_1, \ldots, i_n)$ appearing in ϕ is subordinate to at least $\max\{i_1, \ldots, i_n\}$ nested modal operators in ϕ. Otherwise ϕ is *open*.

Semantics. There may appear in a given formula many *tokens* of the same subformula. (E.g., in $(p \vee p)$ there appear two tokens of the subformula p.) When defining the semantics of an IF logic, one must pay specific attention to this fact, to be able to formulate clauses defining the semantic role of operators with independencies, such as $(\Diamond/i_1, \ldots, i_k)$.

We follow [20] in understanding formulas explicitly as finite *strings of symbols*. Each numeral standing for a positive integer in a formula of **IFML** is counted as a separate symbol (no matter how many digits it has in the chosen presentation), other symbols being propositional atoms,), (, $\neg$, $\vee$, $\wedge$, $\Diamond$, $\Box$, $\emptyset$, the comma and the slash-sign /. The *length* of a string S, in symbols $|S|$, is the number of symbols in S when each symbol is counted as many times as it occurs. The symbols appearing in a formula are *enumerated* with positive integers starting from left to right. For illustration, consider the formula $\phi := \Box(\Diamond/1, 27)(p \vee q)$.

$\Box$	(	$\Diamond$	/	1	,	27	)	(	p	$\vee$	q	)
1	2	3	4	5	6	7	8	9	10	11	12	13

In the special case that the n-th symbol of a string ψ starts itself a string which is a subformula of ψ, we write $\Lambda(\psi, n)$ for that subformula. In the above example, $\Lambda(\phi, 9) = (p \vee q)$ and $\Lambda(\phi, 10) = p$. Every subformula of a formula ψ is of the form $\Lambda(\psi, n)$ for some n, and some subformulas may appear in ψ corresponding

[1] Essentially this way of defining the syntax was suggested to one of the authors (TT) by Balder ten Cate already in December 2002.

to several numbers n. It may further be noted that if ψ is closed and the operator $(\Diamond/i_1,\ldots,i_n)$ appears in ψ, the above enumeration of the symbols in ψ could be used as an alternative way of identifying those modal operators, superordinate to $(\Diamond/i_1,\ldots,i_n)$, that by syntax are identified by the integers $i_1,\ldots,i_n$.

In defining game-theoretical semantics for **IFML**, we adapt the definition that is given in [20] for IF first-order logic. For every formula φ, modal structure $\mathfrak{M}$ and state $w_0 \in M$, a semantic game $G(\varphi,\mathfrak{M},w_0)$ between two players ($\exists$ and $\forall$) is associated by defining the set of its *positions*. We refer to $\exists$ as 'she' and to $\forall$ as 'he'. If $\varsigma = (a_0,\ldots,a_n)$ is a finite sequence, we write $\max(\varsigma)$ for its last member, $\max(\varsigma) := a_n$. If a_{n+1} is any further object, we write $\varsigma^\frown a_{n+1}$ for the extended sequence $(a_1,\ldots,a_n,a_{n+1})$.

Definition 1 (Positions) Positions *are triples (ψ,n,ς), where $\psi = \Lambda(\varphi,n)$ and ς is a finite sequence of elements of M. In the beginning of the game the position is $(\varphi,1,\langle w_0\rangle)$. The following conditions serve to generate the set of all positions of $G(\varphi,\mathfrak{M},w_0)$, with $\mathfrak{M}=(M,R,V)$. They also specify which player makes which type of choice (if any) at a given position.*

1. (*a*) *If (p,n,ς) is a position, then: if* $\max(\varsigma)\in V(p)$, $\exists$ *wins, otherwise* $\forall$ *wins.*
   (*b*) *If $(\neg p,n,\varsigma)$ is a position, then: if* $\max(\varsigma)\notin V(p)$, $\exists$ *wins, else* $\forall$ *wins.*

2. *If $((\psi\vee\phi),n,\varsigma)$ is a position, also $(\psi,n+1,\varsigma)$ and $(\phi,n+2+|\psi|,\varsigma)$ are positions. Player $\exists$ chooses one of these positions at $((\psi\vee\phi),n,\varsigma)$.*

3. *If $((\psi\wedge\phi),n,\varsigma)$ is a position, also $(\psi,n+1,\varsigma)$ and $(\phi,n+2+|\psi|,\varsigma)$ are positions. Player $\forall$ chooses one of these positions at $((\psi\wedge\phi),n,\varsigma)$.*

4. *If $((\Diamond/i_1,\ldots,i_k)\phi,n,\varsigma)$ is a position and $\langle\max(\varsigma),v\rangle\in R$, then*

   $$(\phi, n+2k+3+\sharp(k), \varsigma^\frown v)$$

   is a position, where $\sharp(k)=0$, if $k\geq 1$ and $\sharp(k)=2$, if $k=0$.[2] *If there is at least one such position, player $\exists$ chooses one among them at $((\Diamond/i_1,\ldots,i_k)\phi,n,\varsigma)$. If there is none, player $\forall$ wins.*

5. *If $(\Box\phi,n,\varsigma)$ is a position and $\langle\max(\varsigma),v\rangle\in R$, then $(\phi,n+1,\varsigma^\frown v)$ is a position. If there is at least one such position, player $\forall$ chooses one among them at $(\Box\phi,n,\varsigma)$. If there is none, player $\exists$ wins.* ⊣

[2] If $k\geq 1$, the number $n+2k+3+\sharp(k)=n+2k+3$ is obtained by counting two parentheses, the diamond, the slash, and k numerals together with $k-1$ commas in the independence indication. However, if $k=0$, then $(\Diamond/i_1,\ldots,i_k)=(\Diamond/\emptyset)$, and the correct identifier is $n+2k+3+2=n+5$.

Note that the subformula component ψ in a position (ψ, n, ς) is strictly speaking superfluous, because this subformula is fully determined by the number n: $\psi = \Lambda(\varphi, n)$. It is written down here for clarity of exposition.

The above definition of the set of positions in fact serves to define the game $G(\varphi, \mathfrak{M}, w_0)$. This game is a determined zero-sum game of perfect information. We are not, however, interested in who has a winning strategy in this game. What interests us, instead, is who has a strategy that leads to a win against any sequence of moves by the opponent, *and* satisfies the extra condition of *uniformity*, to be defined shortly. The uniformity requirement will have the consequence that to force the desired outcome, player $\exists$, in particular, must make her choices in a 'universalizable' manner: make the same choice in several 'equivalent' circumstances.[3]

Before we can define the uniformity requirement, let us define the notion of game tree; tell what are strategies of the players; and specify what it means for a player to use a strategy.

Definition 2 (Game tree, play, partial play) *The set of positions of a semantic game $G(\varphi, \mathfrak{M}, w_0)$ determines in a canonical way a tree, to be called the* game tree. *The nodes of the tree are the positions, and its ordering relation is the transitive closure of the relation 'is a successor position of', itself in effect given by the definition of position when telling which are the positions to which a given position gives rise. Any (maximal) branch of the tree represents a possible* play *of the game. Initial segments of plays are called* partial plays. *Sometimes partial plays will be termed* histories *of the game.* ⊣

Definition 3 (Strategy, using a strategy) *A* strategy *of player $\exists$ in semantic game $G(\varphi, \mathfrak{M}, w_0)$ is any finite sequence σ of functions σ_i (called* strategy functions*), defined on the set of all partial plays $(p_0, \ldots, p_{i-1})$ satisfying:*

- *If $p_{i-1} = ((\psi \vee \phi), n, \varsigma)$, then σ tells $\exists$ which formula to pick, that is, $\sigma_i(p_0, \ldots, p_{i-1}) \in \{n + 1, n + 2 + |\psi|\}$. If the strategy gives the lower value, player $\exists$ picks the left-hand formula ψ, otherwise the right-hand formula ϕ.*

- *If $p_{i-1} = ((\Diamond/i_1, \ldots, i_k)\phi, n, \varsigma)$, then σ tells $\exists$, if possible, which element $v \in M$ with $\langle \max(\varsigma), v\rangle \in R$ to pick. Hence $\sigma_i(p_0, \ldots, p_{i-1}) \in M$ and is accessible from $\max(\varsigma)$. If no suitable element exists, σ_i does nothing: $\exists$ gives up!*

[3] The resulting game resembles in many respects games of imperfect information, but strictly speaking is not one. This feature of the semantic games for IF modal logic is discussed in [18, *Subsect.* 2.3.1].

It is said that $\exists$ has used strategy σ *in a play of* $G(\varphi, \mathfrak{M}, w_0)$*, if in each relevant case* $\exists$ *has made her choice using* σ*. More exactly, player* $\exists$ *has used* σ *in a play* $(p_0, \ldots, p_n)$*, if the following two conditions hold for all* $i < n$*:*

1. *If* $p_{i-1} = ((\psi \vee \phi), m, \varsigma)$ *and* $\sigma_i(p_0, \ldots, p_{i-1}) = m + 1$*, then* $p_i = (\psi, m + 1, \varsigma)$*, whereas if* $\sigma_i(p_0, \ldots, p_{i-1}) = m + 2 + |\psi|$*, then* $p_i = (\phi, m + 2 + |\psi|, \varsigma)$*.*

2. *If* $p_{i-1} = ((\Diamond/i_1, \ldots, i_k)\phi, m, \varsigma)$ *and* $\sigma_i(p_0, \ldots, p_{i-1}) = v$*, then* $p_i = (\phi, m + 2k + 3 + \sharp(k), \varsigma^\frown v)$*.*

The notions of strategy and using a strategy can be analogously defined for player $\forall$*.* $\dashv$

Definition 4 (Uniform strategy, winning strategy) *A strategy* σ *of player* $\exists$ *in semantic game* $G(\varphi, \mathfrak{M}, w_0)$ *is* uniform*, if the following condition holds: Suppose* $p_{i-1} = (\Lambda(\psi, m), m, \varsigma)$ *and* $p'_{i-1} = (\Lambda(\psi, m), m, \varsigma')$ *are two positions arising in the game, when* $\exists$ *has played according to* σ*. Furthermore, assume that*

$$\Lambda(\psi, m) = (\Diamond/i_1, \ldots, i_k)\phi.$$

Then if the sequences ς *and* ς' *agree on their values for all arguments except possibly on* $i_1, \ldots, i_k$*, the strategy* σ *agrees on the positions* $(\Lambda(\psi, m), m, \varsigma)$ *and* $(\Lambda(\psi, m), m, \varsigma')$*, that is to say,* $\sigma_i(p_0, \ldots, p_{i-1}) = \sigma_j(p'_0, \ldots, p'_{i-1})$*.*

A strategy σ *of player* $\exists$ *in game* $G(\varphi, \mathfrak{M}, w_0)$ *is a* winning strategy*, if* σ *is uniform, and player* $\exists$ *wins every play in which she has used the strategy.*

The analogous uniformity condition for strategies of $\forall$ *is vacuous, since by the syntax, there are no operators of the form* $(\Box/i_1, \ldots, i_n)$*. A strategy* σ *of player* $\forall$ *is winning simply if it leads to a win by* $\forall$ *against every sequence of moves by* $\exists$*.* $\dashv$

On the basis of the definition of position, the sequences ς and ς' mentioned in the definition of uniformity indeed necessarily have the same length. That is, there is an initial segment Σ of ω such that ς and ς' both are functions of type $\Sigma \longrightarrow M$. If some or all of the numbers $i_1, \ldots, i_k$ happen to be outside of the domain Σ, the sequences ς and ς' are vacuously uniform in the corresponding arguments.

Truth and falsity of **IFML**-formulas are defined in terms of semantic games:

- ϕ is true in $\mathfrak{M}$ at w (denoted $\mathfrak{M}, w \models^+ \phi$), if there is a w.s. for $\exists$ in $G(\phi, \mathfrak{M}, w)$.

- ϕ is false in $\mathfrak{M}$ at w (denoted $\mathfrak{M}, w \models^- \phi$), if there is a w.s. for $\forall$ in $G(\phi, \mathfrak{M}, w)$.

- ϕ is non-determined in $\mathfrak{M}$ at w (denoted $\mathfrak{M}, w \models^0 \phi$), if game $G(\phi, \mathfrak{M}, w)$ is not determined, i.e. if neither $\mathfrak{M}, w \models^+ \phi$ nor $\mathfrak{M}, w \models^- \phi$.

In what follows we will almost exclusively be interested in *truth* of modal formulas, and we will simply write $\models$ for the relation $\models^+$.

2.2 Expressive power

For an example of evaluating an **IFML**-formula, consider the modal structures $\mathfrak{M}$ and $\mathfrak{N}$ depicted in Figure 1.

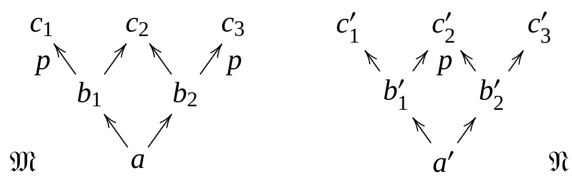

FIGURE 1

The atom p is true in $\mathfrak{M}$ precisely at c_1 and c_3, and in $\mathfrak{N}$ exactly at c'_2. Consider, then, the formula $\phi := \Box(\Diamond/1)p$. We observe three things:

(1) ϕ is not true in $\mathfrak{M}$ at a: There is no winning strategy for $\exists$ in game $G(\phi, \mathfrak{M}, a)$, since a function g inducing a winning strategy would have to satisfy $g(b_1) = g(b_2)$, and if this value was c_1 or c_3, the move would not be in accordance with the game rules if $\forall$'s choice was b_2 *resp.* b_1. On the other hand, if the value was c_2, the resulting plays would be wins for $\forall$, since p is not true at c_2. (As a matter of fact, ϕ is not false in $\mathfrak{M}$ at a either: also for $\forall$ there is no winning strategy in $G(\phi, \mathfrak{M}, a)$. If $\forall$ chooses b_1 (b_2) then by choosing c_1 (*resp.* c_3) $\exists$ generates a play that she wins.)

(2) ϕ is true in $\mathfrak{N}$ at a': The function f defined by the condition $f(b'_1) = f(b'_2) = c'_2$ induces a winning strategy for $\exists$ in $G(\phi, \mathfrak{N}, a')$.

(3) The structures $(\mathfrak{M}, a)$ and $(\mathfrak{N}, a')$ are bisimilar.[4] Hence they are not distinguished by any formula of basic modal logic.

In view of (1), (2) and (3), it follows that **IFML** is not translatable into **ML**. Since **ML** is trivially translatable into **IFML**, we have just established that **IFML** has greater expressive power than basic modal logic:[5]

Theorem 5 **ML** $<$ **IFML**. ⊣

Our main result concerning the expressive power of **IFML** in this paper, Theorem 6, says that this logic is strong enough *not* to admit of a translation into first-order logic. This is in contradistinction to the case of basic modal logic, translatable via the well known standard translation into **FO**, in fact into the 2-variable fragment of **FO**. (For standard translation, see, e.g., [3, *Sect.* 2.4].)

[4] For bisimilarity, see, e.g., [3, *Sect.* 2.2].

[5] This expressivity result was originally proven in [17, Lemma 4].

Theorem 6 **IFML** *is not translatable into* **FO**.

Proof. Let $n \geq 2$ be arbitrary. In what follows, by stipulation $i \oplus 1 := i + 1$, if $i < n$, and $n \oplus 1 := 1$. (Inversely, $i \ominus 1 = j$ means $j \oplus 1 = i$.) Define a modal structure $\mathfrak{M}_n = (M_n, R_n, V_n)$ as follows. The domain M_n consists of five disjoint layers, $L_0 := \{a_1\}$, $L_1 := \{b_1, \ldots, b_n\}$, $L_2 := \{c_1, \ldots, c_n\}$, $L_3 := \{d_1, \ldots, d_n\}$, and $L_4 := \{e_1, \ldots, e_n\}$, related by the accessibility relation $R_n :=$

$$\{(a_1, b_i) : 1 \leq i \leq n\} \cup \{(b_i, c_j) : 1 \leq j \leq n \text{ and } j \leq i \leq j \oplus 1\} \cup$$
$$\{(c_j, d_k) : 1 \leq k \leq n \text{ and } k \leq j \leq k \oplus 1\} \cup \{(d_k, e_{k\oplus 1}) : 1 \leq k \leq n\}.$$

The valuation V_n is empty. In Figure 2, the modal structure $\mathfrak{M}_5$ is depicted.

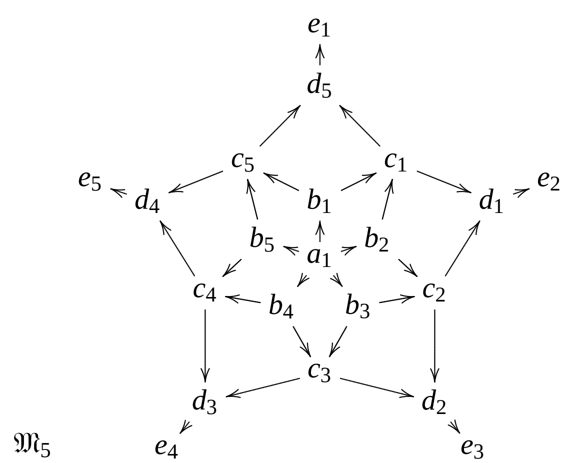

FIGURE 2

Let, then, $\psi := \Box(\Box(\Diamond/2)(\Diamond/1, 3)\top \;\vee\; \Box(\Diamond/2)(\Diamond/1, 3)\top)$.

Claim 7 *For all $n \geq 2$, $\mathfrak{M}_n, a_1 \models \psi$ if, and only if, n is even.*

Proof of the Claim. *From right to left.* Suppose n is even. We define three functions, $f : L_1 \longrightarrow \{\texttt{left},\texttt{right}\}$, $g : \{\texttt{left},\texttt{right}\} \times L_1 \longrightarrow L_3$ and $h : \{\texttt{left},\texttt{right}\} \times L_2 \longrightarrow L_4$.

If ∀'s first move is b_i, define $f(b_i) = \texttt{left}$, if i is odd, and $f(b_i) = \texttt{right}$, otherwise. If ∀ continues by picking out c_j, put $g(\texttt{right}, b_i) = g(\texttt{left}, b_i) = d_{i\ominus 1}$, hence ignoring the information about c_j. Further, if j is odd and $f(b_i) = \texttt{left}$ (and so also i is odd), let $h(f(b_i), c_j) = e_j$, and similarly, if j is even and $f(b_i) = \texttt{right}$ (and so also i is even), let $h(f(b_i), c_j) = e_j$. Otherwise, let $h(f(b_i), c_j) = e_{j\oplus 1}$.

It is immediate that the functions f, g, h serve to define a winning strategy for ∃ in $G(\psi, \mathfrak{M}_n, a_1)$. In particular, when ∃ is supposed to make a choice corresponding to one of the occurrences of $(\Diamond/1, 3)$, she knows by f whether it is the right or the left disjunct that is at stake, and she sees ∀'s move c_j for the second occurrence of □ in that disjunction. She can infer whether ∀'s first choice was b_j or $b_{j\oplus 1}$, because by the evenness of n, the numbers j and $j \oplus 1$ cannot have the same parity, and having used f, ∃ has chosen the left disjunction if, and only if, j is odd. Knowing, then, which of the points b_j or $b_{j\oplus 1}$ ∀ had chosen, ∃ can further infer, by using g, at which point she currently is. But then there is only one point she can choose at all for $(\Diamond/1, 3)$, and this point is as a matter of fact given by h.

From left to right. Assume n is odd, and suppose for contradiction that there is a winning strategy for ∃ in $G(\psi, \mathfrak{M}_n, a_1)$. Let f, g, h be functions as above induced by that winning strategy. Because n is odd, there necessarily are points $b_i, b_{i\oplus 1}$ such that $f(b_i) = f(b_{i\oplus 1})$. Let us w.l.o.g. assume that these points are b_n and b_1, and that $f(b_n) = f(b_1) = \texttt{left}$. Consider, then, the two partial plays where ∀'s pairs of choices are (b_n, c_n) and (b_1, c_n). The function g must yield for $(\Diamond/2)$ the choice d_{n-1} in the former case, and in the latter case the choice d_n. (Because of the uniformity condition, the choice must be the same no matter which successor of b_n *resp.* b_1 player ∀ chooses. In the former case, the options for ∀ are c_n and c_{n-1}, and their only common successor is d_{n-1}. And in the latter case ∀'s options are c_n and c_1, whose only common successor is d_n.)

The function h may only use as its arguments the disjunctive choice (which here is `left` in both cases) and ∀'s choice c_n – which likewise is the same for both partial plays, having (b_n, c_n, d_{n-1}) and (b_1, c_n, d_n) as their corresponding respective choices from the model. This means that h will choose the same point e_k in both cases. But which ever point e_k is, the move is possible along the accessibility relation R_n in at most one of the two cases. Hence f, g, h do not induce a winning strategy, contrary to the assumption. ⊣

Claim 8 *For all $n \geq 1$, the first-order structures $\langle \mathfrak{M}^{\mathbf{FO}}_{2^n}, a_1 \rangle$ and $\langle \mathfrak{M}^{\mathbf{FO}}_{2^n+1}, a_1 \rangle$ satisfy exactly the same first-order formulas of one free variable and quantifier rank at most $n + 1$.*

Proof of the Claim. There is a winning strategy for *Duplicator* in the Ehrenfeucht-Fraïssé game $\mathbf{EF}_{n+1}(\mathfrak{M}^{\mathbf{FO}}_{2^n}, a_1; \mathfrak{M}^{\mathbf{FO}}_{2^n+1}, a_1)$: an optimal strategy for *Spoiler* is to choose successively the elements $b_{2^0}, b_{2^1}, \ldots, b_{2^n}$ from the domain of $\mathfrak{M}^{\mathbf{FO}}_{2^n+1}$; let *Duplicator* respond to these choices by the elements $b_{2^0}, b_{2^1}, \ldots, b_{2^n}$, respectively, from the domain of $\mathfrak{M}^{\mathbf{FO}}_{2^n}$. (Should *Spoiler* be allowed an $(n + 2)$-th move, he could choose the element b_{2^n+1} from the domain of $\mathfrak{M}^{\mathbf{FO}}_{2^n+1}$, and to this *Duplicator* would have no response.) ⊣

In view of the two claims, ψ does not admit of translation into **FO**, and so **IFML** is not translatable into first-order logic. ⊣

Thus **IFML** is semantically a much stronger logic than **ML**, in fact, it cannot be translated into **FO**. As yet it is unknown whether the satisfiability and validity problems of **IFML** are decidable.[6] In [15] the decidability of both of these problems was established for the so-called 'IF modal logic of perfect recall' ($\mathbf{IFML_{PR}}$). This logic is a fragment of **IFML**, syntactically restricted in such a way that the semantic games corresponding to its formulas are games of perfect recall. The structurally determined IF modal logic from *Section* 4 is likewise a fragment of **IFML**; like $\mathbf{IFML_{PR}}$, it is more expressive than **ML**; and its satisfiability and validity problems are decidable. The complexity of $\mathbf{IFML_{SD}}$-satisfiability is known to be in PSPACE [19]; by contrast, the exact complexity of $\mathbf{IFML_{PR}}$-satisfiability is an open question (the recursive bound on the size of a finite model of a $\mathbf{IFML_{PR}}$-formula obtained in [15] has the form of tower function w.r.t. the length of the length of the formula, and is hence far from feasible). The validity problems of the logics $\mathbf{IFML_{PR}}$ and $\mathbf{IFML_{SD}}$ are, on the other hand, both known to be decidable in PSPACE.

3 Extended IF modal logic

For one thing, the enterprise of IF logic teaches us that things that are uncontroversial and unproblematic in first-order logic turn out to have intriguing properties when we dare to introduce the slash device. A case in point is the behavior of propositional connectives.

Semantically, the evaluation of conjunction (disjunction) involves a choice between two things: the left and the right conjunct (disjunct). Hence these connectives can be construed as restricted quantifiers. Instead of $(\psi \wedge \chi)$, we may, equivalently, write $\wedge_{i\in\{l,r\}}\phi_i$, given that $\phi_l := \psi$ and $\phi_r := \chi$. Similar observations can be made about the restricted quantifier $\vee_{i\in\{l,r\}}$. It is straightforward to see that in first-order logic, introducing these restricted quantifiers does not yield greater expressive power. E.g., the sentence $\wedge_{i\in\{l,r\}}\exists x\, P_i x$ is simply equivalent to $(\exists x\, P_l x \wedge \exists x\, P_r x)$. The same holds for the result of replacing the usual conjunction and disjunction by the corresponding restricted quantifiers in basic modal logic: what results is just a notational variant of **ML**. For instance, $\wedge_{i\in\{l,r\}}\Diamond_i p$ is equivalent to $(\Diamond_l p \wedge \Diamond_r p)$, where $\Diamond_i$ is the diamond over the accessibility relation R_i.

The restricted quantifiers $\wedge_{i\in\{l,r\}}$ and $\vee_{i\in\{l,r\}}$ were studied in the context of IF logic by Hodges in [13]. A similar move has been made by Tulenheimo with

[6] Note that for IF modal logics, the satisfiability and validity problems are *not* each other's duals. Let '$\neg\psi$' be a shorthand notation for a formula in negation normal form such that: $\exists$ has a w.s. in $G(\neg\psi, \mathfrak{M}, w)$ iff $\forall$ has a w.s. in $G(\psi, \mathfrak{M}, w)$. Choose $\varphi, \mathfrak{M}$ and w so that φ is non-determined in $\mathfrak{M}$ at w. Then $(\varphi \vee \neg\varphi)$ also is non-determined in $\mathfrak{M}$ at w, and thus not valid. Yet $\neg(\varphi \vee \neg\varphi)$ is not satisfiable.

respect to IF modal logic. The resulting logic, called *Extended IF modal logic*, was first introduced in [18], having been originally suggested by Hyttinen (personal communcation). This logic allows marking modal operators as independent even from superordinate conjunctions and disjunctions, when the latter are construed precisely as restricted quantifiers.

To make the *Eigenart* of Extended IF modal logic visible, we assume a polymodal framework. The modal structures considered will have k accessibility relations $R_1, \ldots, R_k$, each corresponding to a modality type of its own. In syntax, the diamonds and boxes carry an index, indicating which accessibility relation is responsible for the semantics of the operator in question. In a polymodal basic modal logic, e.g. the formula $\Box_2\Diamond_7 p$ says that any R_2-successor of the current state has an R_7-successor satisfying the atom p.

Conjunctions and disjunctions are construed as restricted quantifiers ranging over the set $\{l, r\}$. Accordingly, a string $i_1 \ldots i_n \in \{l, r\}^*$ may appear as a subscript of a modal operator syntactically subordinate to n conjunction or disjunction signs, and it is a part of the specification of the syntax to associate the appropriate strings with modality types $1, \ldots, k$. For instance, if $\wedge_{i\in\{l,r\}} \vee_{j\in\{l,r\}} (\Diamond_{ij}/1)\top$ is a formula, the syntax must provide a mapping from the set $\{ll, lr, rl, rr\}$ to the set $\{1, \ldots, k\}$. If the evaluation has proceeded to the subformula $(\Diamond_{ij}/1)\top$, the mapping yields a modality type corresponding to the diamond, depending on which choices among l, r were made earlier, first for $\wedge_{i\in\{l,r\}}$ and then for $\vee_{j\in\{l,r\}}$.

Before we get to the formal underpinnings of this logic, let us consider a nice illustration of its capabilities. Think of Extended IF modal logic with two modality types, one of which is interpreted by means of the identity relation =. Slightly abusing the syntax to make stating the example smoother, consider the formula $\wedge_{i\in\{=,R\}}(\Diamond_i/1)\top$, indicating that there is a state to which $\exists$ can move from the current state w, without knowing which accessibility relation was earlier picked out by $\forall$. One of the accessibility relations being =, $\exists$ must choose w. But this means that in order for this formula to hold in $\mathfrak{M}$ at w, it must also be possible to get from w to w via the relation R. In fact, when evaluated at w, the formula serves to state that $(w, w) \in R$.

Syntax. Write $L(\mathbf{prop})$ for the set of literals, i.e. formulas of the form p or $\neg p$ with $p \in \mathbf{prop}$. *Extended IF modal logic*, $\mathbf{EIFML}_k$, will use k modality types, and its formulas will be strings $O_1 \ldots O_n\gamma(j_1 \ldots j_m)$, where m is the total number of conjunction and disjunction symbols in the prefix $O_1 \ldots O_n$. The components of these strings are explained as follows. First, the strings are associated with a *distribution of modality types* $\mu : \cup_{i\leq m}\{l, r\}^i \longrightarrow \{1, \ldots, k\}$ and a *distribution of literals* $\gamma : \{l, r\}^m \longrightarrow L(\mathbf{prop})$. Second, each O_x is one of the following:

(i) $\wedge_{j_x \in \{l,r\}}$;

(ii) $\vee_{j_x \in \{l,r\}}$;

(iii) $\Box_{j_1 \ldots j_y}$, where y is the number of conjunction and disjunction symbols preceding O_x in the prefix;

(iv) $(\Diamond_{j_1 \ldots j_y}/i_1, \ldots, i_z)$, where y is the number of conjunction and disjunction symbols preceding O_x in the prefix, and $1 \leq i_1, \ldots, i_z \leq x - 1$.

Semantics. The semantics will be defined relative to k-ary modal structures, mentioned in *Section* 1. To formulate the truth conditions for formulas of the logic, a semantic game $G(\varphi, \mathfrak{M}, w_0)$ is associated with each formula φ, k-ary modal structure $\mathfrak{M}$ and state $w_0 \in M$. In the interest of clarity, we will define positions in the game so that they keep explicitly track not only of the states chosen before reaching that position, but also of the conjuncts and disjuncts chosen up to then.

Definition 9 *Let $\varphi := O_1 \ldots O_n \gamma(j_1 \ldots j_m) \in$ $\mathbf{EIFML}_k$. Positions of game $G(\varphi, \mathfrak{M}, w_0)$ are quadruples $(\psi, \ell, \varsigma, \varsigma')$, where $1 \leq \ell \leq n + 1$, $\psi = O_\ell \ldots O_n \gamma(j_1 \ldots j_m)$, $\varsigma : S \longrightarrow M$ and $\varsigma' : S' \longrightarrow \{l, r\}$, where S is the set of those numbers x in $\{1, \ldots, n\}$ for which O_x is a modal operator, and $S' = \{1, \ldots, n\} \backslash S$. (The functions ς, ς' may simply be thought of as sequences of states and sequences of objects l, r, respectively.)*

In the beginning, the position is $(\varphi, 1, \langle w_0 \rangle, \emptyset)$. The following conditions generate the set of all positions of $G(\varphi, \mathfrak{M}, w_0)$, with $\mathfrak{M} = (M, R_1, \ldots, R_k, V)$. They also specify which player makes which type of choice (if any) at a given position.

1. *(a) If $\gamma(\varsigma') = p$ and the position is $(p, n + 1, \varsigma, \varsigma')$, then: if $\max(\varsigma) \in V(p)$, $\exists$ wins, else $\forall$ wins. (b) If $\gamma(\varsigma') = \neg p$ and the position is $(\neg p, n + 1, \varsigma, \varsigma')$, then: if $\max(\varsigma) \notin V(p)$, $\exists$ wins, otherwise $\forall$ wins.*

2. *If the position is $(\vee_{j_x \in \{l,r\}} \phi, \ell, \varsigma, \varsigma')$, then both $(\phi, \ell + 1, \varsigma, \varsigma'^\frown l)$ and $(\phi, \ell + 1, \varsigma, \varsigma'^\frown r)$ are positions. Player $\exists$ chooses one of these positions at $(\vee_{j_x \in \{l,r\}} \phi, \ell, \varsigma, \varsigma')$. The case of $(\wedge_{j_x \in \{l,r\}} \phi, \ell, \varsigma, \varsigma')$ is otherwise similar, but it is player $\forall$ who chooses one of the positions at $(\wedge_{j_x \in \{l,r\}} \phi, \ell, \varsigma, \varsigma')$.*

3. *If $\mu(\varsigma') = j$, and the position is $((\Diamond_j/i_1, \ldots, i_z)\phi, \ell, \varsigma, \varsigma')$, then for every v such that $\langle \max(\varsigma), v \rangle \in R_j$, we have that $(\phi, \ell + 1, \varsigma^\frown v, \varsigma')$ is a position. It is $\exists$ who chooses one of these, or if none exists, there are no further positions and $\forall$ wins. The case of $(\Box_j \phi, \ell, \varsigma, \varsigma')$ is similar, but it is $\forall$ who makes the choice, or, if he can make none, there are no further positions and $\exists$ wins.* $\dashv$

The notions of game tree and (partial) play are defined as in the case of **IFML**. A strategy of $\exists$ is also defined similarly, as a tuple of strategy functions $\sigma = (\sigma_1, \ldots, \sigma_h)$, with one strategy function for each expression of the form $\vee_{j_x \in \{l,r\}}$ or $(\Diamond_{j_1 \ldots, j_y}/i_1, \ldots, i_z)$ in the prefix. If a partial play $(p_0, \ldots, p_{i-1})$ is already produced, such a strategy function makes a choice between the positions each of which is a combinatorially possible next position. The notion of using a strategy is again defined like in the case of **IFML**. A strategy of $\exists$ is winning, if it leads to a win against any sequence of moves by $\forall$, and is, furthermore, uniform in the following sense: if $p_i = (\phi, \ell, \varsigma, \varsigma')$ and $p'_i = (\phi, \ell, \tau, \tau')$ are any two positions arising in the game supposing that $\exists$ has used σ, with

$$\phi = (\Diamond_{j_1 \ldots, j_y}/i_1, \ldots, i_z)\psi,$$

then if the maps $\varsigma \cup \varsigma'$ and $\tau \cup \tau'$ agree on all their values except possibly on $i_1, \ldots, i_z$, then the strategy σ agrees on the positions p_i and p'_i, i.e. maps the sequence of positions leading to p_i to the same element as the sequence of positions leading to p'_i. The notions of strategy, using a strategy and winning strategy are defined analogously for player $\forall$ (keeping in mind that the condition of uniformity is vacuously satisfied by $\forall$'s strategies).

Semantics of **EIFML**$_k$ is, then, simply defined by stipulating that φ is true (false) in $\mathfrak{M}$ at w_0, if there is a winning strategy for $\exists$ (*resp.* $\forall$) in game $G(\varphi, \mathfrak{M}, w_0)$.

About the expressiveness of Extended IF modal logic, we may first note that by the proof of Theorem 6, in particular **EIFML**$_1$ cannot be translated into **FO**:[7]

Theorem 10 *For all $k \geq 1$,* **EIFML**$_k$ *is not translatable into* **FO**. ⊣

Second, it is evident that polymodal **EIFML**$_k$ is more expressive than the polymodal version of **IFML**. E.g., consider evaluating the formula $\wedge_{i \in \{=,R\}} (\Diamond_i/1)\top$ of our earlier example relative to the modal structures $(\{a\}, \{(a,a)\}, \{(a,a)\}, \emptyset)$ and $(\{a,b\}, \{(a,a)\}, \{(a,b)\}, \emptyset)$ with $a \neq b$.

Among the applauded virtues of modal logic are its nice computational properties: e.g., model checking is tractable and satisfiability is decidable. Amusingly, it can be shown that the satisfiability problem of **EIFML**$_k$ with the identity relation is undecidable, cf. unpublished manuscripts by Hyttinen & Tulenheimo, and by Sevenster. This result shows that the power of slashing is considerable even when we import it in modal logic. It is interesting to see whether a similar undecidability result can be achieved without the identity relation and maybe even for **IFML**.

[7] This considerably improves the result proven in [18, Th. 3.3.9], according to which, whenever $k \geq 3$, **EIFML**$_k$ does not have a first-order translation.

It might well turn out that **IFML** proper is decidable, whereas $\mathbf{EIFML}_k$ is undecidable. This would show us – once again – that the particulars of a pre-slash, independence-unfriendly language are sleeping beauties.

4 Structurally determined IF modal logic

The logics **IFML** and $\mathbf{EIFML}_k$ aimed at being modal analogues of IF first-order logic: results of importing the slash device into modal logic and interpreting it so as to produce a modal logic of informational independence. However, as indeed shown by the two languages, the particular independence friendly modal logic that we end up considering depends highly on the syntax used, and on the way independence is introduced. In *Subsection* 4.1, we introduce a new framework in which IF modal logics can be compared and isolated, by tuning three parameters that will be highlighted shortly. Some researchers have objected that there is no *a priori* reason why slashed modal operators would formalize a meaningful notion of independence. This criticism will be revisited in relation to the logics specified within our framework. *Subsection* 4.2 singles out a specific logic by instantiating the three parameters of the framework in a certain way. The logic in question will be a fragment of IF first-order logic; it is denoted by $\mathbf{IF}(ST^2(\mathbf{ML}))$. *Subsection* 4.3 introduces a certain modal-like logic – so-called 'structurally determined IF modal logic' – which is subsequently, in *Subsection* 4.4, shown to characterize $\mathbf{IF}(ST^2(\mathbf{ML}))$. Interestingly, this modal-like logic will have a compositional, 'Tarskian' semantics. Finally, *Subsection* 4.5 discusses some aspects of the expressive power of the structurally determined IF modal logic.

4.1 The framework

The framework we propose essentially isolates 'modal fragments' of IF first-order logic. This approach is partially inspired by current research in modal logic. Namely, although basic modal logic is an extension of propositional logic, nowadays it is usually conceived of as a fragment of first-order logic. Milestone results that brought about this change of perspective include the standard translation, and van Benthem's Theorem [1] which characterizes modal logic as the '*bisimulation invariant*' fragment of first-order logic.

Assuming this perspective, an IF modal logic is obtained by fixing three things: first, a set of strings that are considered as modal formulas; second, a standard translation that maps the former set to a subset of first-order logic; and third, an IF procedure that maps this subset of first-order logic into IF first-order logic. The IF modal logic thus generated is *modal* in that it is originates from a modal logic, and *independence friendly* in that it is a fragment of IF first-order logic, through the appropriate IF procedure.

More precisely, our framework covers all logics that are obtained from a modal logic $\mathcal{ML}$, standard translation ST and IF procedure **IF** as follows:

- Translate every formula from $\mathcal{ML}$ into first-order logic using ST, and obtain the first-order correspondence language of $\mathcal{ML}$, denoted $ST(\mathcal{ML})$.

- Apply to every formula from $ST(\mathcal{ML})$ the IF procedure **IF**, and obtain the set of IF modal formulas constituted by $\mathcal{ML}$, ST, and **IF**, denoted **IF**$(ST(\mathcal{ML}))$.

Observe that there are instantiations of the initial modal logic, the standard translation, and the IF procedure that give rise to meaningless and uninteresting systems. This will happen if the parameters $\mathcal{ML}$, ST and **IF** are incompatible; for instance, if the range of the operation ST is disjoint from the domain of the operation **IF**. But the framework also contains potentially interesting systems. For one thing, as we will see, the (IF first-order correspondence languages of the) logics that were studied earlier under the headings **IFML** and **EIFML**$_k$ can be generated by tuning the parameters of the framework in a specific way (cf. Figure 3 in *Subsect.* 4.2).

We think this framework facilitates finding logics that have 'nice' combinations of properties. It is beyond the scope of the current paper to give a specific sense to the phrase 'nice combination of properties'. But generally a high expressive power combined with a low computational complexity is considered nice. In the context of IF logic, also allowing for a compositional semantics can be appreciated as a desirable property.

Indeed, Cameron and Hodges showed in [6] that *no* compositional semantics exists for IF first-order logic in which the 'interpretation' $|\phi(x)|_{\mathcal{A}}$ of a formula with one free variable is a *subset* of the domain of $\mathcal{A}$; what is more, they even proved that in a compositional semantics for IF first-order logic, $|\phi(x)|_{\mathcal{A}}$ cannot be a subset of $dom(\mathcal{A})^n$, for any $n < \omega$. If by 'Tarskian semantics' we mean a compositional semantics where the interpretation of a formula $|\phi(x_1, \ldots, x_k)|_{\mathcal{A}}$ will be a subset of $dom(\mathcal{A})^m$ for some $m \geq k$, it follows that no Tarskian semantics for IF first-order logic is possible. (On the other hand, Hodges had already proven in [13, 14] that IF first-order logic admits of a compositional semantics where the interpretation $|\phi(x)|_{\mathcal{A}}$ of each formula $\phi(x)$ is a *subset of the powerset* of the domain.) Given this background, being able to show that an IF modal logic can be interpreted in a Tarskian way, signals that the complexity of the full IF logic is tamed in this respect.

As an example of a logic emerging from our framework, we will consider the structurally determined IF modal logic introduced in [19]. The satisfiability and validity problems of this logic are decidable, and it allows for a compositional semantics.

Some researchers have opposed the very idea of an independence friendly modal logic, by insisting that independence is a relation between (syntactically manifest) variables, while none are forthcoming in modal logics. Thus, they have suggested that *a priori* modal operators furnished with independence indications are not necessarily meaningful. Staying within our framework, we need not enter such a discussion. Rather, we can point out that logics generated in our framework are immune to any such criticism, since they are fragments of IF first-order logic.

4.2 Instantiating the three parameters

In this subsection, we will fix the values of the parameters in a certain way. It turns out that the resulting fragment of IF first-order logic admits of a particularly smooth characterization in modal logical terms (see *Subsect.* 4.3). Actually, it is captured by a compositional modal-like language, to be referred to as $\mathbf{IFML_{SD}}$. The concrete cases we wish to consider are these:

- Basic modal logic, **ML**, in *negation normal form*. (It is well known that each basic modal formula has an equivalent form in which the negation-sign appears only as prefixed to a propositional atom.)

- The standard translation $ST^2 : \mathbf{ML} \longrightarrow \mathbf{FO^2}$ of basic modal logic into the 2-variable fragment of first-order logic.

- The IF procedure associating with every first-order formula ϕ the set $\mathbf{IF}(\phi)$ of those IF first-order formulas that are obtained by replacing any number of existential quantifiers $\exists x_k$ appearing in ϕ by the corresponding symbol $(\exists x_k/x_{i_1}, \ldots, x_{i_n})$, provided that: (*a*) in ϕ there appear the universal quantifiers $\forall x_{i_1}, \ldots, \forall x_{i_n}$ superordinate to $\exists x_k$; (*b*) in the formula resulting from the replacement, the variables $x_{i_1}, \ldots, x_{i_n}$ in $(\exists x_k/x_{i_1}, \ldots, x_{i_n})$ become thereby bound by the universal quantifiers $\forall x_{i_1}, \ldots, \forall x_{i_n}$; and (*c*) $x_k \notin \{x_{i_1}, \ldots, x_{i_n}\}$.

Note that clause (*b*) precludes cases like the string $\forall x \exists x (\exists y/x)\phi$, resulting from applying the IF procedure to $\forall x \exists x \exists y \phi$.

Basic modal logic is assumed to be in negation normal form (NNF), to ensure that its first-order correspondence language is in negation normal form as well. This is useful in the present context, since one may safely apply a Hintikka-style IF procedure to extend any fragment of **FO** that is in NNF. Another way in which a basic modal language interesting for the purposes of IF modal logic can be introduced is using the strong prenex normal form (SPNF), by now familiar from

EIFML$_k$: considering formulas $O_1 \dots O_n\gamma(j_1 \dots j_m)$, where each O_x in the prefix is $\wedge_{j_x \in \{l,r\}}$, $\vee_{j_x \in \{l,r\}}$, $\Box_{j_1 \dots j_y}$ or $\Diamond_{j_1 \dots j_y}$ (y being the number of conjunction and disjunction symbols preceding O_x in the prefix), and for all strings $j_1 \dots j_m \in \{0,1\}^m$, we have that $\gamma(j_1 \dots j_m)$ is a literal. As was already noted in *Subsection* 2.2, the standard translation of **ML** into **FO** can be performed using only two variables. Of course, other standard translations mapping **ML** into **FO** are readily available, for instance one that introduces pairwise distinct variables for any quantifiers translating nested modal operators. Finally, as we remarked earlier, various IF procedures have been proposed. The one put forward by us is tailor-made to suit the particulars of the fragment of first-order logic obtained by translating **ML** to **FO** via ST^2.

We stipulate that the two variables of $\mathbf{FO^2}$ are x, y. The *2-variable fragment of IF first-order logic*, denoted $\mathbf{IF^2}$, is then defined as follows. Its formulas are obtained from the formulas of $\mathbf{FO^2}$: Let $\alpha, \beta \in \{x, y\}$ and $\alpha \neq \beta$. If $\phi \in \mathbf{FO^2}$, then the result of replacing any number of occurrences of $\exists\beta$ subordinate to $\forall\alpha$ in ϕ by the symbol $(\exists\beta/\alpha)$ is a formula of $\mathbf{IF^2}$, provided that the variable β in $(\exists\beta/\alpha)$ becomes thereby bound by $\forall\alpha$. So e.g. $\exists x\forall x(\exists y/x)Rxy$ is a formula of $\mathbf{IF^2}$, but $\forall x\exists x(\exists y/x)Rxy$ is not. The semantics of $\mathbf{IF^2}$ is obtained from the semantics of IF first-order logic by applying the stipulation that the variable α mentioned in $(\exists\beta/\alpha)$ is bound by the *closest* universal quantifier $\forall\alpha$ *superordinate* to $(\exists\beta/\alpha)$.

Having made the three choices and defined the 2-variable fragment of IF first-order logic, we have determined a fragment of IF first-order logic, to be denoted **IF**(ST^2(**ML**)), consisting of the results of applying the specified IF procedure to the first-order formulas yielded, by the standard translation ST^2, from the formulas of basic modal logic in NNF. The framework of structurally determining a logic is well-suited also for discussing other IF modal logics. The following table lists (the IF first-order correspondence languages of) several IF modal logics studied in the literature, in terms of different instantiations of the three parameters.

	Basic mod. log. in	*Translation into*	*$\exists x_i$ can be indep. of*
L_1	NNF	**FO** in NNF	$\forall x_j$ if betw. $\forall x_j$ and $\exists x_i$, no $\vee$ or $\exists x_k$ appears
L_2	NNF	**FO** in NNF	$\forall x_j$ or $\exists x_j$
L_3	SPNF	**FO** in SPNF	$\forall x_j$, $\exists x_j$, $\wedge$ or $\vee$
L_4	NNF	$\mathbf{FO^2}$ in NNF	$\forall x_j$ if $x_i \neq x_j$ and betw. $\forall x_j$ and $\exists x_i$, $\exists x_j$ does not appear

FIGURE 3. *Logics resulting from different instantiations of the three parameters.*

The logics L_1, L_2, L_3 are the IF first-order correspondence languages of $\mathbf{IFML_{PR}}$, **IFML** *resp.* **EIFML**$_k$, i.e., the canonical translations of these IF modal logics into the suitable formulation of IF first-order logic. L_4 is the logic $\mathbf{IF}(ST^2(\mathbf{ML}))$. The standard translation needed for the logics L_1, L_2 and L_3 introduces distinct quantified variables for any two nested modal operators. The standard translation of L_3 further construes propositional connectives as restricted quantifiers.

4.3 Structurally determined IF modal logic

Our framework generates fragments of IF first-order logic. They may be hard to parse. Therefore we aim at characterizing the logic $\mathbf{IF}(ST^2(\mathbf{ML}))$ in terms of a more transparent, modal machinery. We wish to structurally determine a modal logic – 'IF modal logic' with a modal syntax – by singling it out as the logic $\mathbf{X}$ such that its translation $ST^{\mathbf{IF}}(\mathbf{X})$ into the 2-variable fragment $\mathbf{IF^2}$ of IF first-order logic coincides with the result of applying the IF procedure ($\Downarrow_{\mathrm{IF}}$) to the $\mathbf{FO^2}$-translation of $\mathbf{ML}$:

$$\begin{array}{ccccc}
\mathbf{ML} & \stackrel{ST^2}{\longrightarrow} & ST^2(\mathbf{ML}) & \subset & \mathbf{FO^2} \\
 & & \Downarrow_{\mathrm{IF}} & & \Downarrow_{\mathrm{IF}} \\
\cap & & \mathbf{IF}(ST^2(\mathbf{ML})) & \subset & \mathbf{IF^2} \\
 & & \shortparallel & & \\
\mathbf{X} & \stackrel{ST^{\mathbf{IF}}}{\longrightarrow} & ST^{\mathbf{IF}}(\mathbf{X}) & &
\end{array}$$

In want of better terminology, we will refer to the language $\mathbf{X}$ as 'structurally determined IF modal logic' (and will denote it by '$\mathbf{IFML_{SD}}$').

As will be shown in this subsection, we will be able to find a particularly nice modal-like presentation for the IF modal logic $\mathbf{X}$: a modal-like logic with a compositional semantics. We do *not* wish to suggest that this would be an integral part of our proposed framework of structurally determining modal logics. E.g., for **IFML** and $\mathbf{EIFML}_k$, we do not have such a Tarskian compositional characterization, and neither are we aware of a possibility of obtaining one (except by formulating a non-Tarskian modal 'trump semantics', i.e. by doing to the relevant IF modal logics what Hodges did in [13] for IF first-order logic).

Let us now define the syntax and semantics of a modal-like logic, which turns out to be the logic $\mathbf{X}$ structurally determined above (see Proposition 12).

Syntax. A class of formulas is generated by the two grammars A and B:

$$\begin{array}{lll}
\alpha & ::= & p \mid \neg p \mid (\alpha \vee \alpha) \mid (\alpha \wedge \alpha) \mid \Diamond\alpha \mid \Box\alpha \mid \blacksquare\beta \\
\beta & ::= & \blacklozenge\alpha \mid (\alpha \vee \beta) \mid (\beta \vee \alpha) \mid (\beta \vee \beta) \mid (\alpha \wedge \beta) \mid (\beta \wedge \alpha) \mid (\beta \wedge \beta),
\end{array}$$

where $p \in \mathbf{prop}$. The two grammars generate the formulas of a logic we refer to as $\mathbf{IFML^{\circ}_{SD}}$. The formulas α are said to be *closed*, and the formulas β *open*. If ϕ

is a formula, all tokens of $\blacklozenge$ not subordinate to a token of $\blacksquare$ in ϕ are called *free*. If ϕ is open, all its free tokens of $\blacklozenge$ become *bound by* the outmost token of $\blacksquare$ in $\blacksquare\phi$. For instance, $\blacklozenge p$ is open; and $\blacksquare(\blacklozenge p \vee \blacklozenge q)$ is closed, the two tokens of $\blacklozenge$ being bound by the unique token of $\blacksquare$. We stipulate that the formulas of the *structurally determined IF modal logic*, $\mathbf{IFML_{SD}}$, are the closed formulas of $\mathbf{IFML^{\circ}_{SD}}$ (i.e., the formulas generated by the grammar A). Note that the reason why the grammar B contains both clauses $(\alpha \circ \beta)$ and $(\beta \circ \alpha)$ with $\circ \in \{\wedge, \vee\}$ is simply 'aesthetic': we wish that the conjunctions and disjunctions of formulas of which one is open and the other is closed, can be formed in either order.

The operators $\blacksquare$ and $\blacklozenge$ will be referred to as 'black box' and 'black diamond', and the operators $\Box$ and $\Diamond$ as 'white box' and 'white diamond'. Intuitively, $\blacklozenge$ is the 'independent diamond', and it will by definition be independent precisely of the token of $\blacksquare$ that binds it. For its part, this logic will hence illustrate that the relations 'being bound by' and 'being logically dependent on' need not coincide; this point is made in a more general context by Hintikka [11].

Semantics. For every $\varphi \in \mathbf{IFML^{\circ}_{SD}}$, a satisfaction relation

$$\mathfrak{M}, I, \overline{i}, w \models \phi$$

is defined, where $\mathfrak{M} = (M, R, V)$ is a modal structure and $w \in M$ is a state as usual, and furthermore, $I : \{0,1\}^* \longrightarrow M$ is a *token valuation*, and $\overline{i}$ is a binary string: $\overline{i} \in \{0,1\}^*$. Here, 0 and 1 should be intuitively thought of as the choices *left* and *right*, respectively, made when interpreting propositional connectives $(\wedge, \vee)$. Observe that given a formula ϕ, a binary string $i_1 \dots i_n$ determines a subformula ψ of ϕ, namely the formula yielded by starting to go through, outside-in, the propositional connectives of ϕ and choosing for the j-th connective encountered *left* or *right* according to whether i_j is 0 or 1. The process stops either because there are no more connectives to go or because all the n choices have been made. The determined formula ψ is the subformula reached by the process.

In providing the semantics of formulas of the form $\blacklozenge\varphi$, the token valuation I will be used. The idea is that I will yield states interpreting particular tokens of $\blacklozenge$, the tokens being identified precisely in terms of binary strings $\overline{i} \in \{0,1\}^*$. Hence, in particular, the state interpreting the token of $\blacklozenge$ prefixing $\blacklozenge\varphi$ is determined by I; and in general the valuation I has been chosen already earlier in the evaluation (namely, when interpreting the closest superordinate $\blacksquare$), so that the state to interpret the token of $\blacklozenge$ in question has been determined, as it were, in advance. The truth conditions of literals and formulas of the forms $\Box$ and $\Diamond$ do not make use of the components I and $\overline{i}$ of the models; by contrast, the components I and $\overline{i}$ play a key role in the rest of the clauses:

$$\begin{array}{rll}
\mathfrak{M}, I, \overline{i}, w \models p & \text{iff:} & w \in V(p)\\
\mathfrak{M}, I, \overline{i}, w \models \neg p & \text{iff:} & w \notin V(p)\\
\mathfrak{M}, I, \overline{i}, w \models \Diamond\psi & \text{iff:} & \text{for some } v \text{ with } R(w, v)\text{: } \mathfrak{M}, I, \overline{i}, w \models \psi\\
\mathfrak{M}, I, \overline{i}, w \models \Box\psi & \text{iff:} & \text{for every } v \text{ with } R(w, v)\text{: } \mathfrak{M}, I, \overline{i}, w \models \psi\\
\mathfrak{M}, I, \overline{i}, w \models (\psi \vee \phi) & \text{iff:} & \mathfrak{M}, I, \overline{i0}, w \models \psi \text{ or } \mathfrak{M}, I, \overline{i1}, w \models \phi\\
\mathfrak{M}, I, \overline{i}, w \models (\psi \wedge \phi) & \text{iff:} & \mathfrak{M}, I, \overline{i0}, w \models \psi \text{ and } \mathfrak{M}, I, \overline{i1}, w \models \phi\\
\mathfrak{M}, I, \overline{i}, w \models \blacksquare\phi & \text{iff:} & \text{for some } I' : \{0,1\}^* \longrightarrow M\text{: } \mathfrak{M}, I', \overline{i}, w \models \Box\phi\\
\mathfrak{M}, I, \overline{i}, w \models \blacklozenge\phi & \text{iff:} & R(w, I(\overline{i})) \text{ and } \mathfrak{M}, I, \overline{i}, I(\overline{i}) \models \phi.
\end{array}$$

It should be observed that for the token valuations in $\mathbf{IFML}^{\circ}_{\mathbf{SD}}$-semantics, what really matters are the *free* occurrences of the black diamond in a formula. It can easily be checked that if (relative to an initial string $\overline{i}$) the free occurrences of $\blacklozenge$ in ϕ are those identified by the strings in the set $S \subset \{0,1\}^*$ (which is necessarily finite), then if for *some* I, we have $\mathfrak{M}, I, \overline{i}, w \models \phi$, actually $\mathfrak{M}, I', \overline{i}, w \models \phi$ holds for any I' such that for all $\overline{j} \in S$, $I'(\overline{j}) = I(\overline{j})$. In particular, to keep the satisfaction condition intact we need not have $I'(\overline{i}) = I(\overline{i})$ unless $\overline{i} \in S$. It follows that the semantic clause for the black diamond need not be phrased in terms of quantification over token valuations: to evaluate $\blacksquare\phi$, it suffices to choose a fixed finite number of states: as many states as there are free occurrences of $\blacklozenge$ in ϕ to be interpreted. Let us write $\mathfrak{M}, w \models \phi$ to express the following condition: for all token valuations $I : \{0,1\}^* \longrightarrow M$ and all strings $\overline{i} \in \{0,1\}^*$, we have $\mathfrak{M}, I, \overline{i}, w \models \phi$. Now note that if the $\mathbf{IFML}^{\circ}_{\mathbf{SD}}$-formula ϕ is closed (i.e., if $\phi \in \mathbf{IFML}_{\mathbf{SD}}$) and for *some* I and $\overline{i}$, we have $\mathfrak{M}, I, \overline{i}, w \models \phi$, then actually $\mathfrak{M}, w \models \phi$ holds. Formulas of $\mathbf{IFML}_{\mathbf{SD}}$ are in this respect like *sentences* in first-order logic: if satisfied under one assignment, they are satisfied under all assignments. Their being satisfied is entirely independent of the assignment. By contrast, for open $\mathbf{IFML}^{\circ}_{\mathbf{SD}}$-formulas the token valuations and the binary strings have a crucial relevance. Formally, free tokens of $\blacklozenge$ (as identified by certain strings) bear resemblance to free variables, and the valuations to variable assignments; in the presence of free variables the satisfaction conditions of first-order logic are of course essentially dependent on the assignments.

4.4 Standard translation

In this subsection we show that the modal-like logic $\mathbf{IFML}_{\mathbf{SD}}$ is equally expressive as the logic $\mathbf{IF}(ST^2(\mathbf{ML}))$ specified in *Section* 4.2. This result is interesting, because it shows that this fragment of IF first-order logic indeed can be given a Tarskian semantics, in the sense specified in *Subsection* 4.1, unlike the full IF first-order logic. To be precise, in the compositional semantics we designed for $\mathbf{IFML}^{\circ}_{\mathbf{SD}}$, the interpretation $|\phi|_{\mathfrak{M}}$ of a formula ϕ in a modal structure $\mathfrak{M} = (M, R, V)$ is, in effect, a set of $(n+1)$-tuples of elements of M, where n is the number of *free* *tokens of the black diamond* in ϕ, the remaining member of the tuple simply specifying the state w relative to which the formula is evaluated. Hence the analogue of [6, Cor. 6.2] does not hold for the logic $\mathbf{IFML_{SD}}$: we need not climb to the level of the powerset of the domain to obtain a compositional semantics; for any formula of $\mathbf{IFML^{\circ}_{SD}}$, a fixed Cartesian power of the domain suffices.

Basic modal logic admits of a translation into the 2-variable fragment of first-order logic. The class of closed $\mathbf{IFML^{\circ}_{SD}}$-formulas, that is, the class $\mathbf{IFML_{SD}}$, has an analogous property. Concretely, the following map $ST_x^{\mathbf{IF}} : \mathbf{IFML_{SD}} \longrightarrow \mathbf{IF^2}$ provides a canonical translation of $\mathbf{IFML_{SD}}$ into the 2-variable fragment of IF first-order logic. For all $\alpha, \beta \in \{x, y\}$ with $\alpha \neq \beta$, define:

$$
\begin{aligned}
ST_\alpha^{\mathbf{IF}}(p) &= P\alpha \\
ST_\alpha^{\mathbf{IF}}(\neg p) &= \neg P\alpha \\
ST_\alpha^{\mathbf{IF}}((\phi \circ \psi)) &= (ST_\alpha^{\mathbf{IF}}(\phi) \circ ST_\alpha^{\mathbf{IF}}(\psi)) \quad \text{ if } \circ \in \{\vee, \wedge\} \\
ST_\alpha^{\mathbf{IF}}(\Diamond\phi) &= \exists\beta(R\alpha\beta \wedge ST_\beta^{\mathbf{IF}}(\phi)) \\
ST_\alpha^{\mathbf{IF}}(\Box\phi) &= \forall\beta(R\alpha\beta \to ST_\beta^{\mathbf{IF}}(\phi)) \\
ST_\alpha^{\mathbf{IF}}(\blacklozenge\phi) &= (\exists\beta/\alpha)(R\alpha\beta \wedge ST_\beta^{\mathbf{IF}}(\phi)) \\
ST_\alpha^{\mathbf{IF}}(\blacksquare\phi) &= ST_\alpha^{\mathbf{IF}}(\Box\phi).
\end{aligned}
$$

Clearly, if ϕ is a closed $\mathbf{IFML^{\circ}_{SD}}$-formula, then $ST_x^{\mathbf{IF}}(\phi)$ is an $\mathbf{IF^2}$-formula with exactly one free variable, x. The mapping $ST_x^{\mathbf{IF}}$ actually provides a translation:

Proposition 11 *For every formula $\phi \in \mathbf{IFML_{SD}}$, modal structure $\mathfrak{M}$, and state w:*

$$\mathfrak{M}, w \models \phi \quad \textit{if, and only if,} \quad \mathfrak{M}^{\mathbf{FO}}, w \models ST_x^{\mathbf{IF}}(\phi).$$

Proof. The proposition can be proven by induction on the structure of closed formulas. Observe that the formulas prefixed with $\blacksquare$ are of the form $\blacksquare\psi$, where ψ is obtained by conjunction and disjunction from closed formulas and formulas of the form $\blacklozenge\theta$, where θ is closed. ⊣

Further, the following 'commutativity' result holds, establishing that $\mathbf{IFML_{SD}}$ actually is the logic $\mathbf{X}$ structurally determined above:

Proposition 12 *Syntactically, $ST_x^{\mathbf{IF}}(\mathbf{IFML_{SD}}) = IF(ST_x^2(\mathbf{ML}))$.*

Proof. *The inclusion from left to right:* Let $\phi \in \mathbf{IFML_{SD}}$ be arbitrary, and let ϕ^- be the $\mathbf{ML}$-formula resulting from ϕ by turning all its black boxes and black diamonds into their white counterparts. Clearly $ST_x^{\mathbf{IF}}(\phi)$ is obtained by the IF procedure from $ST_x^2(\phi^-)$. *The inclusion from right to left:* Let $\psi \in \mathbf{ML}$ be arbitrary, and let ψ^+ be any result of applying the IF procedure to $ST_x^2(\psi)$. Since $ST_x^2(\psi) \in$

$\mathbf{FO^2}$, ψ^+ is a formula of $\mathbf{IF^2}$. Any independence indication appearing in ψ^+ must be in a context of the form $(\exists\alpha/\beta)$, where β is bound by a universal quantifier $\forall\beta$. Let, then, $\psi^\times$ be the result of having turned all those white diamonds $\Diamond$ in ψ black, that correspond to an existential quantifier in $ST^2_x(\psi)$ which has become slashed via the IF procedure leading from ψ to ψ^+; and having also turned all those white boxes $\Box$ in ψ black, that correspond to a universal quantifier in $ST^2_x(\psi)$ binding the slashed variable of some existential slashed quantifier in ψ^+. It is easy to see that $ST^{\mathbf{IF}}_x(\psi^\times)$ is, by syntactical criteria, identical to the formula ψ^+. ⊣

4.5 Expressive power

The expressivity and decidability properties of the logic $\mathbf{IFML_{SD}}$ are extensively studied in [19]. Without entering details concerning the expressive power of $\mathbf{IFML_{SD}}$, let us take an example.

Example 13 *Consider evaluating the closed formula $\phi := \blacksquare(\blacklozenge p \vee \blacklozenge q)$ at the root w of the modal structure $\mathfrak{M}$ depicted in Figure 4:*

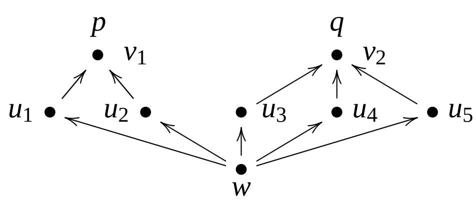

Figure 4

Let I_0 be any token valuation and $\overline{i}$ any binary string. We claim that $\mathfrak{M}, I_0, \overline{i}, w \models \phi$. To see this, choose I so that $I(\overline{i}0) = v_1$ and $I(\overline{i}1) = v_2$. (Choosing a valuation I corresponds to picking out, as it were beforehand, states interpreting the two black diamonds $\blacklozenge$ that can come across later in the evaluation.) It suffices to check that

$$\mathfrak{M}, I, \overline{i}, w \models \Box(\blacklozenge p \ \vee \ \blacklozenge q).$$

For this to hold, it must be possible to partition the set $\{u_1, \ldots, u_5\}$ into two cells (corresponding to the choice left *or* right *for the disjunction symbol), so that if u_j belongs to one of the cells, then $\mathfrak{M}, I, \overline{i}0, u_j \models \blacklozenge p$; and if u_j belongs to the other cell, then $\mathfrak{M}, I, \overline{i}1, u_j \models \blacklozenge q$. Let the cells be $\{u_1, u_2\}$ for* left, *and $\{u_3, u_4, u_5\}$ for* right. *Then the above conditions hold indeed: the former, since $I(\overline{i}0)$ is accessible from all states in the former cell, and p is true at $I(\overline{i}0)$; and the latter because $I(\overline{i}1)$ is accessible from all states in the latter cell, and q is true at $I(\overline{i}1)$.*

The above reasoning shows, then, that $\mathfrak{M}, w \models \phi$. Observe that the formula ϕ can be written in the syntax of **IFML** *as $\Box((\Diamond/1)p \vee (\Diamond/1)q)$.* ⊣

For a further example of what can be expressed in terms of $\mathbf{IFML_{SD}}$, let $n \geq 2$ be arbitrary, and think of the formula $\phi_n :=$

$$\blacksquare(\underbrace{\blacklozenge\top \vee \ldots \vee \blacklozenge\top}_{n-1 \text{ times}}).$$

Evaluated relative to a modal structure $\mathfrak{M} = (M, R, V)$ at a state w, the formula asserts, in effect, that the set $\{v : R(w, v)\}$ can be partitioned into (at most) $n - 1$ cells in such a way that the elements in each cell have a *common successor* along the relation R. Actually the truth condition of ϕ can be expressed by the first-order formula $\phi'_n := \exists z_1 \ldots \exists z_{n-1} \forall y(Rxy \rightarrow (Ryz_1 \vee \ldots \vee Ryz_{n-1}))$. The formula ϕ'_n is in the $(n + 1)$-variable fragment of **FO**. On the other hand, it is not difficult to see (by reference to a pebble game argument[8]) that ϕ'_n is not equivalent to any formula in the n-variable fragment of **FO**. Hence the greater the number n is, the more variables are needed to translate the formula ϕ_n into first-order logic. As a consequence, we may infer the following fact:

Fact 14 *For all $n < \omega$,* $\mathbf{IFML_{SD}} \nleq \mathbf{FO}^n$.

Furthermore, we observe that for all $n \geq 2$, the maximum number of nested modal operators in ϕ_n is 2. Yet whenever $n' > n$, the formulas ϕ_n and $\phi_{n'}$ are not equivalent. So we have:

Fact 15 *For* $\mathbf{IFML_{SD}}$*, it is* not *the case that up to logical equivalence, there are only finitely many formulas of a given modal depth.*

It may be noted that Facts 14 and 15 are in a striking contrast to the case of basic modal logic, which is translatable into $\mathbf{FO^2}$, and has the property that the number of pairwise non-equivalent formulas of any given modal depth is finite. (For the latter fact, see, e.g., [3, Prop. 2.29].) These and other unorthodox properties of the modal-like logic $\mathbf{IFML_{SD}}$ might suggest that it should be of a rather marginal interest as a modal logic; even its status as a modal logic might be thereby called into question. However, it is proven by the present authors in [19] that satisfiability and validity problems of $\mathbf{IFML_{SD}}$ are decidable in PSPACE. Hence this expressive logic shares with basic modal logic a good deal of its nice computational properties. So we see that the distribution of 'desirable' and 'undesirable' properties may, in modal-like logics, be rather surprising. Actually, one of the most interesting negative properties of $\mathbf{IFML_{SD}}$ is its non-translatability into the guarded fragment of first-order logic, proven in [19].

[8] On how to use pebble games $G^n_m(\mathcal{M}, \mathbf{a}, \mathcal{N}, \mathbf{b})$ to characterize equivalence of structures up to quantifier rank $\leq m$ relative to $\mathbf{FO}^n$, see, e.g., [7, pp. 49-50].

5 Collapse of diversity

Two ways to approach independence friendly modal logic have been discussed: one leading from **IFML** to **EIFML**$_k$, proceeding via adding independence indications to modal operators much in the same way as is done in IF first-order logic – the other way being via adjusting several parameters in such a way that an independence friendly logic gets structurally determined. It is fairly evident that the three logics considered here differ in their expressive power. In fact, we have $\mathbf{IFML_{SD}} < \mathbf{IFML} < \mathbf{EIFML}_k$. (For what is known and what is conjectured about the relations of these logics, see Figures 5 and 6 in *Sect.* 6.) By contrast, we now show that in some cases – in fact in cases that are extremely common in modal logical contexts – the expressive powers of these logics coincide.

Let us begin with a couple of definitions. If M is a set and $R \subseteq M^2$ is a binary relation, let us write R^+ for the transitive closure of R, and R^* for the reflexive transitive closure of R. The structure (M, R) is a *tree*, if (i) there is a unique element $r \in M$ such that for all $x \in M$, R^*rx; (ii) every element of M has a unique R-predecessor; and (iii) R is acyclic, i.e. there is no x such that R^+xx. Let us say that a tree is *branching*, if no $x \in M$ has precisely one R-successor (no element has 'out-degree' equal to 1). Hence in a branching tree any element has either no R-successors at all, or at least two R-successors.

A k-ary modal structure $\mathfrak{M} = (M, R_1, \ldots, R_k, V)$ is *(branching and) tree-like*, if the structure $(M, \bigcup_{1 \leq i \leq k} R_i)$ is a (branching) tree. A tree-like k-ary modal structure $\mathfrak{M}$ is *proper*, if for all $x, y \in M$ and all $1 \leq i, j \leq k$:

$$[(x, y) \in R_i \text{ and } (x, y) \in R_j] \implies i = j.$$

That is, in a proper tree-like structure no vertices are connected by more than one relation out of the k available ones. Define $\mathfrak{Tree}_k$ as the class of all proper branching tree-like k-ary modal structures. Note that by virtue of clause (iii) in the definition of tree, all accessibility relations $R_1, \ldots, R_k$ of a structure $\mathfrak{M} \in \mathfrak{Tree}_k$ are irreflexive.

We will prove that all logics **IFML**, **EIFML**$_k$ and $\mathbf{IFML_{SD}}$ coincide with basic modal logic (and hence with each other) relative to the class $\mathfrak{Tree}_k$. We begin by considering **EIFML**$_k$. If $\phi = O_1 \ldots O_n\gamma \in \mathbf{EIFML}_k$, $O_{z+1} = (\Diamond_{j_1 \ldots j_y}/i_1, \ldots, i_m)$ and $[x, z] \subseteq \{i_1, \ldots, i_m\}$, we say that the operator O_{z+1} involves *independence of a continuous block of predecessors*. This terminology is reasonable, since by assumption O_{z+1} is indicated as independent from its immediate predecessor O_z, from the predecessor O_{z-1} of O_z and so on, (at least) until O_x. (The smallest number in the list $i_1, \ldots, i_m$ may well be smaller than x, while its greatest number must be z, given that the interval $[x, z]$ is contained in $\{i_1, \ldots, i_m\}$.) In what follows, we will rewrite any operator $(\Diamond_{j_1 \ldots j_y}/i_1, \ldots, i_{m+m'})$, as given by the syntax, in the form

$(\Diamond_{j_1 \dots j_y}/i_1, \dots, i_m, i'_1, \dots, i'_{m'})$, where the integers $i_1, \dots, i_m$ refer by stipulation to modal operators, and the integers $i'_1, \dots, i'_{m'}$ to propositional connectives.

Lemma 16 **(a)** *If* $\phi \in \mathbf{EIFML}_k$*, let* ϕ^- *be the result of replacing all independent diamonds* $(\Diamond_{j_1 \dots j_y}/i_1, \dots, i_m, i'_1, \dots, i'_{m'})$ *in* ϕ *by the corresponding diamond* $(\Diamond_{j_1 \dots j_y}/i'_1, \dots, i'_{m'})$ *involving no independencies of modal operators. Relative to* $\mathfrak{Tree}_k$*,* ϕ *is equivalent to* ϕ^-*.* **(b)** *If no diamonds in* $\phi \in \mathbf{EIFML}_k$ *contain independencies of modal operators, let* ϕ^- *be the result of replacing all independent diamonds* $(\Diamond_{j_1 \dots j_y}/i'_1, \dots, i'_{m'})$ *in* ϕ *by the simple diamond* $\Diamond_{j_1 \dots j_y}$*. Relative to* $\mathfrak{Tree}_k$*,* ϕ *is equivalent to* ϕ^-*.*

Proof. **(a)** Let $\mathfrak{M} \in \mathfrak{Tree}_k$ and $w \in M$, and assume that $\mathfrak{M}, w \models \phi$. Suppose ϕ contains a diamond $O_{z+1} = (\Diamond_{j_1 \dots j_y}/i_1, \dots, i_m, i'_1, \dots, i'_{m'})$ involving independence of a continuous block of predecessors $O_x, \dots, O_z$ such that $O_x = \Box_{j_1 \dots j_x}$. But this means that the strategy function σ_{z+1} corresponding to O_{z+1} given by ∃'s winning strategy (which exists by assumption) satisfies:

$$\sigma_{z+1}(p_0, p_1, \dots, p_z) = \sigma_{z+1}(p_0, p'_1 \dots, p'_z),$$

where the sequence of choices from the domain associated with p_x is $\varsigma = (w, w_1, \dots, w_r)$ and the one associated with p'_x is $\varsigma' = (w, w'_1, \dots, w'_r)$, and $w_r \neq w'_r$. (This is because $\mathfrak{M} \in \mathfrak{Tree}_k$ is branching and so in the two plays ∀ has chosen pairwise incomparable and hence distinct states when choosing for the box O_x.) But this is impossible, since in a tree no distinct nodes can have a common successor and so σ_{z+1} cannot be a strategy function involved in a winning strategy.

If the longest possible continuous block of predecessors of O_{z+1} contains only diamonds, then O_{z+1} may trivially be replaced by $(\Diamond_{j_1 \dots j_y}/i'_1, \dots, i'_{m'})$.

Finally, if ϕ contains no operator involving independence of a continuous block of predecessors, then all operators $(\Diamond_{j_1 \dots j_y}/i_1, \dots, i_m, i'_1, \dots, i_{m'})$ in ϕ satisfy: either the list $i_1, \dots, i_m$ is empty, or subordinate to the closest box (if any) identified by an integer in the list, there is a modal operator superordinate to the diamond and not identified by any integer in the list. Hence, if w_r is the most recent choice from the domain made before arriving at the position where ∃ must make a choice for the diamond $(\Diamond_{j_1 \dots j_y}/i_1, \dots, i_m, i'_1, \dots, i_{m'})$, then, to put it intuitively, ∃'s move for the diamond is allowed to depend on w_r. But there is a uniquely determined path in the tree-like structure $\mathfrak{M}$ leading from w to w_r, whence ∃ can infer all previous choices made in the relevant partial play. Hence the diamond $(\Diamond_{j_1 \dots j_y}/i_1, \dots, i_m, i'_1, \dots, i_{m'})$ may, without changing the truth condition, be replaced by $(\Diamond_{j_1 \dots j_y}/i'_1, \dots, i'_{m'})$.

(b) Let μ be the distribution of modality types associated with ϕ, and suppose that $\mathfrak{M}, w \models \phi$. Consider a diamond $(\Diamond_{j_1 \dots j_y}/i'_1, \dots, i'_{m'})$ appearing in ϕ. (If none exists, there is nothing to prove.) If the indicated independencies from conjunctions correspond, as determined by μ, to the requirement of reaching one state along several accessibility relations, then $\exists$'s winning strategy in $G(\phi, \mathfrak{M}, w)$ will choose such a state. But this is impossible, because $\mathfrak{M}$ is a proper tree-like structure and hence no such state exists. On the other hand, if the indicated independencies from conjunctions correspond to making a choice along one and the same accessibility relation irrespective of what the choices for those conjunctions were, then the independent diamond can be replaced by the simple diamond. Finally, if in the diamond considered there are only independencies from disjunctions, the formula says the same as the result of replacing the independent diamond with a simple diamond. $\dashv$

Recall that $\mathbf{ML}_k$ stands for the polymodal basic modal logic, evaluated relative to k-ary modal structures. We are in a position to prove:

Theorem 17 **(a)** *For all* $k \geq 1$, $\mathbf{EIFML}_k$ *coincides with* $\mathbf{ML}_k$ *over* $\mathfrak{Tree}_k$. **(b)** *Both* **IFML** *and* $\mathbf{IFML_{SD}}$ *coincide with* **ML** *over* $\mathfrak{Tree}_1$.

Proof. Statement (a) follows by Lemma 16; and statement (b) by an argument exactly like the one presented for item (a) in the proof of Lemma 16. $\dashv$

The class of tree-like structures is omnipresent in modal logic. In particular, any **ML**-formula that has a model at all, has a tree-like model. (Cf., e.g., [3, Prop. 2.15].) The class $\mathfrak{Tree}_k$ discerned above is quite representative a subclass of all tree-like models from the viewpoint of basic modal logic. (It is not difficult to see that any satisfiable polymodal formula $\phi \in \mathbf{ML}_k$ is satisfied in a structure $\mathfrak{M} \in \mathfrak{Tree}_k$.) Hence it is of interest to see that the additional expressive power of the IF modal languages discussed in the present paper does *not* lie in their capacity to distinguish such tree-like models. Relative to $\mathfrak{Tree}_k$ the three logics do not exceed what already their common core, basic modal logic, is able to express.

There is, at least tentatively, a positive methodological side to our negative expressivity result. Namely, one can propose to turn the tables and suggest that a result such as Theorem 17 points to a feature that *any* IF version of basic modal logic should exhibit.[9] That is, this type of results can be used in assessing the general question as to the 'nature' of IF modal logics. From this perspective, indeed it seems reasonable to require that IF modal logics of the appropriate kind precisely *should* coincide with basic modal logic on the class of trees discussed; if a logic does not, it cannot be properly called an IF modal logic in the sense intended. This systematic idea alone brings some order in the manifold of different logics that could conceivably be termed IF modal logics. However, it must be noted that deciding the precise characteristics of a family of logics, such as IF modal logics, is

[9] We are indebted to the anonymous referee for pointing out this positive side of our negative result.

bound to leave some room for discussion; the same holds for the acceptance criteria of any exclusive club of logics – modal, first-order, or what not. What is more, when applying the framework introduced in *Section* 4, we need not choose a fragment of basic modal logic as the class of modal formulas we start with. Choosing for instance basic tense logic, or first-order modal logic, will likewise result in a system that can, in a generic sense, be termed an IF modal logic. And such logics need not satisfy any specific conditions that may be necessary for IF modal logics corresponding to some different choice of input modal formulas; for instance, there is in general no reason why they should meet the conditions that IF modal logics emerging from basic modal logic actually satisfy.

6 Concluding remarks

In this paper we aimed to discuss two different ways of formulating independence friendly modal logic. To achieve this, we began by surveying and further studying IF modal logics of one of these two kinds, i.e. those obtained from basic modal logic by introducing a suitably interpreted slash device to the syntax (*Sects.* 2 and 3). Now one respect in which modal logic differs from first-order logic is that syntactically, modal operators do not carry variables, whereas quantifiers do. When subject to suitable syntactic restrictions, these variables can easily be employed in referring to particular tokens of quantifiers, whereas no similar syntactic mechanism is available in standard modal logic. This is why in the approaches such as those discussed in *Sections* 2 and 3, one must introduce an identification method by means of which to single out those tokens of modal operators from whose logical (priority) scope one wishes to exempt, say, a given diamond. On conceptual grounds one might find introducing such identification methods into the syntax less than fortunate. One could argue that independence is a relation between (syntactically manifest) variables, and suggest that since modal syntax does not offer any such variables, it does not really make sense to attempt formulating an IF modal logic. According to such a viewpoint, adding for instance indices to modal operators to make reference to tokens of such operators possible, would be an *ad hoc* move from the perspective of what modal logic is about.

The present authors do not share the ideas on which such a critique is based; we hold independence to be a relation between tokens of logical operators, not first and foremost between syntactically manifest variables. However, in *Section* 4, we hope to have made it clear that even if independence was considered precisely as a relation between variables, an independence friendly modal logic analogous to IF first-order logic can be defined by considering *fragments of IF first-order logic*. This is the framework of the IF modal logics of the second kind considered in the present paper. The particular fragment to which we gave attention, the result of taking the standard translation of **ML** into $\mathbf{FO^2}$, and applying a certain IF procedure to the resulting class of first-order formulas, turned out to be even of further interest. Namely, we found a modal-like logic $\mathbf{IFML_{SD}}$, with a compositional semantics, which actually captures the relevant subfragment of (the 2-variable fragment of) IF first-order logic.

The following table lists the main results known about the various logics discussed in the present paper. $\mathbf{EIFML_=}$ stands for the polymodal $\mathbf{EIFML}_k$, one of whose accessibility relations is rigidly interpreted as equality.

L	*Expressivity*	*Satisfiability*	*Validity*
ML	$L < \mathbf{FO^2}$	PSPACE	PSPACE
$\mathbf{IFML_{PR}}$	$\mathbf{ML} < L < \mathbf{FO^3}$, $\mathbf{IFML_{SD}} \nleq L < \mathbf{IFML}$	$\leq$ SUPEREXP	PSPACE
IFML	$L \nleq \mathbf{FO}$	?	?
$\mathbf{EIFML}_k$	$\mathbf{IFML} < L \nleq \mathbf{FO}$	?	?
$\mathbf{EIFML_=}$	$L \nleq \mathbf{FO}$	undecidable	?
$\mathbf{IFML_{SD}}$	$L \nleq \mathbf{FO}^n$, $L < \mathbf{FO}$, $L < \mathbf{IFML}$	PSPACE	PSPACE

FIGURE 5. *Known results on (IF) modal logics.*

In the table below, some conjectures about the various IF logics are presented. The conjecture to the effect that $\mathbf{IFML_{PR}}$ cannot be translated into $\mathbf{IFML_{SD}}$ holds fairly obviously, but the requisite tool called for by the standard proof technique (viz. an appropriate bisimulation relation) has not as yet been formulated in the literature.

L	*Expressivity*	*Validity*
IFML		PSPACE
$\mathbf{EIFML}_k$		PSPACE
$\mathbf{EIFML_=}$		PSPACE
$\mathbf{IFML_{SD}}$	$\mathbf{IFML_{PR}} \nleq L$	

FIGURE 6. *Conjectures on IF modal logics.*

Although we feel that the logics discussed and studied in the present paper are interesting in their own right, we think that more generally, they help to see the interest of the grand program of independence-friendliness in logic – that is, to repair Frege's fallacy also outside of first-order logic.

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