Finite model theory for partially ordered connectives*

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Abstract

In the present article a study of the finite model theory of Henkin quantifiers with boolean variables [5], a.k.a. *partially ordered connectives* [28], is undertaken. The logic of first-order formulae prefixed by partially ordered connectives, denoted D, is considered on finite structures. D is characterized as a fragment of second-order existential logic Σ^1_1; the formulae of the relevant fragment do not allow existentially quantified variables as arguments of predicate variables. Using this characterization result, D is shown to harbor a strict hierarchy induced by the arity of predicate variables. Further, D is shown to capture NP over linearly ordered structures, and not to be closed under complementation. We conclude with a comparison between the logics D and Σ^1_1 on several metatheoretical properties.

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1 Introduction

Fagin's Theorem [11]—characterizing NP in terms of the expressive power of Σ^1_1 over finite models—reveals an intimate connection between finite model theory and complexity theory. As a methodological consequence it appears that questions and results regarding a complexity class may bear relevance to logic and vice versa. For instance, the complexity theorist's NP = coNP problem can now be shared by the logician working on the $\Sigma^1_1 = \Pi^1_1$ problem.[1] Indeed, logicians working in finite model theory address this problem. By and large they go about by mapping out *fragments* of various logics. A case in point is Fagin's study [12] of the *monadic* fragments of Σ^1_1 and Π^1_1, showing that they do not coincide.

The results in [12] aroused a lot of interest in monadic second-order languages [2, 3, 31], but we are still waiting for methods to separate binary, existential, second-order logic from 3-ary, existential, second-order logic, see [7], or even from binary, universal, second-order logic.

The present paper will be concerned with the finite model theory of languages involving (what we propose to call) *restricted Henkin quantifiers*, also known as *partially ordered connectives*. Henkin quantifiers $\mathsf{H}^n_k \vec{x}\vec{y}$ are objects of the form

$$\begin{pmatrix} \forall x_{11} & \dots & \forall x_{1k} & \exists y_1 \\ \vdots & \ddots & \vdots & \vdots \\ \forall x_{n1} & \dots & \forall x_{nk} & \exists y_n \end{pmatrix} \tag{1}$$

that prefix first-order formulae ϕ. Here and henceforth, a tuple of variables as in $x_{11}, \dots, x_{nk}$ is abbreviated by $\vec{x}$. On suitable structures $\mathfrak{A}$, the formula $\mathsf{H}^n_k \vec{x}\vec{y}\ \phi(\vec{x}, \vec{y})$ is defined to be true iff there are k-ary functions $f_1, \dots, f_n$ on the universe of $\mathfrak{A}$ such that

$$\mathfrak{A} \models \forall \vec{x}\ \phi(\vec{x}, f_1(\vec{x}_1), \dots, f_n(\vec{x}_n)), \tag{2}$$

where $\vec{x}_i = x_{i1}, \dots, x_{ik}$. Partially ordered quantifiers have been widely studied from model theoretic and complexity theoretic points of view (see, e.g., [5, 10, 18, 16, 24, 22, 25, 32]); they have also aroused interest in theoretical linguistics (cf. [4, 20]).

It is a milestone result in the theory of Henkin quantification that the logic obtained by applying Henkin quantifiers to first-order formulae, denoted H, coincides with Σ^1_1, cf. [10, 32]. Referring to Fagin's Theorem, Blass and Gurevich [5, Theorem 1] observed that NP can be characterized in terms of

[1] Solving the NP = coNP problem would be worth the effort: if NP $\neq$ coNP, then P $\neq$ NP.

H as well. In the same publication the authors study what constraints can be imposed on the existentially quantified variables in a Henkin quantifier, such as $y_1, \ldots, y_n$ in (1), without the quantifier losing its power to express NP-complete problems. It was shown that Henkin quantifiers of the form

$$\begin{pmatrix} \forall x_{11} & \ldots & \forall x_{1k} & \exists \alpha_1 \\ \forall x_{21} & \ldots & \forall x_{2k} & \exists \alpha_2 \end{pmatrix}, \tag{3}$$

where α_1 and α_2 range over a fixed two-element domain, cannot express NP-complete problems unless NL = NP. The variables α_1 and α_2 are called *boolean variables*. Because their ranges are restricted to two values, $\exists \alpha_i$ is a 'restricted quantifier,' whence the term 'restricted Henkin quantifier' to describe (3). Blass and Gurevich showed further [5, Theorem 3] that allowing three rows instead of two as in (3), a restricted Henkin quantifier is obtained that admits of expressing NP-complete problems, actually with only one universal quantifier at each row.

The model theory for restricted Henkin quantifiers was taken up by Sandu and Väänänen [28], be it under the name of 'partially ordered connectives' and written in the following format:

$$\begin{pmatrix} \forall x_{11} & \ldots & \forall x_{1k} & \bigvee i_1 \\ \vdots & \ddots & \vdots & \vdots \\ \forall x_{n1} & \ldots & \forall x_{nk} & \bigvee i_n \end{pmatrix}, \tag{4}$$

denoted $\mathsf{D}^n_k \vec{x}\vec{i}$. The usage of the symbol $\bigvee$ reflects the fact that the variables i_j range over a fixed finite domain. In [28] an *Ehrenfeucht-Fraïssé game* for partially ordered connectives is given and it is used to prove non-definability results. Note that there are first-order formulae ϕ that can express NP-complete problems, when prefixed with the partially ordered connective $\mathsf{D}^3_1 \vec{x}\vec{i}$, in virtue of Blass and Gurevich's results. A case in point is 3-colorability of graphs. Other publications on Henkin quantifiers and partially ordered connectives in relation to complexity theory include [16, 19, 22, 23, 27].

In this paper the logic D—the result of applying (4) to first-order formulae with arbitrary k, n—is characterized as a fragment of Σ^1_1. The relevant fragment only allows universally quantified variables to appear as arguments of (existentially quantified) relation variables. As this fragment is rather natural, it may be worthwhile to explore the metatheory of variations of this particular fragment. Using the aforementioned characterization result, we show that (a) D can express a property expressible in $(k+1)$-ary, existential, second-order logic that cannot be expressed in k-ary, existential, second-order logic; and that (b) D captures NP on linearly ordered structures. Using a game-theoretic argument we further show that (c) D is not closed under complementation: it can express 2-COLORABILITY but not its complement. Along the way we prove that the Henkin quantifier $\mathsf{H}^2_1\vec{x}$ is not definable in D and that D is strictly contained in NP. Finally, we state that D has a 0-1 law.[2]

The structure of the paper is similar to that of [29, Chapter 4]. Omitted proofs can be found in that publication.

In Section 2, we introduce the apparatus necessary to get going. In Section 3, D is characterized as a fragment of Σ^1_1. Using this characterization, we prove results (a) and (b) in Section 4. In Section 5, an Ehrenfeucht-Fraïssé game for D is given, and result (c) is established. Section 6 states that D has a 0-1 law, and summarizes the results obtained.

2 Preliminaries

A *vocabulary* τ is a finite set of relation symbols, possibly including the equality symbol. Vocabularies do not contain constant or function symbols. A *finite* τ*-structure* $\mathfrak{A} = \langle A, \langle R^{\mathfrak{A}} \rangle_{R\in\tau} \rangle$ consists of a finite set A, referred to as the *universe of* $\mathfrak{A}$, and interpretations of the relation symbols of τ on A. Here and henceforth, the domain of every structure is finite and for this reason we omit mentioning this. The equality symbol '=' is interpreted as the identity relation. If the only symbol in τ other than '=' is a binary relation symbol 'R', then any τ-structure interpreting 'R' as an irreflexive relation, is called a *digraph* (*directed graph*). If $\mathfrak{G} = \langle G, R^{\mathfrak{G}} \rangle$ is a digraph and $R^{\mathfrak{G}}$ is furthermore symmetric, then $\mathfrak{G}$ is a *graph*. A class relevant to this paper is n-COLORABILITY holding of those finite graphs whose chromatic number is n or less. Conversely, let $\overline{n\text{-COLORABILITY}}$ denote the complement of n-COLORABILITY with respect to the class of all graphs. The binary relation symbol '>' is, by convention, interpreted as an irreflexive linear order. That is, for every structure $\mathfrak{A}$ whose vocabulary contains '>', the relation $>^{\mathfrak{A}}$ is irreflexive, transitive, and connected.

Define an *implicit matrix* τ*-formula* γ as a function of type $\{0,1\}^k \to$ FO(τ), where k is an integer and FO(τ) is first-order logic over τ. Let $\mathrm{D}_k(\tau)$ be the logic with formulae of the form $\mathsf{D}^n_k\vec{x}\vec{i}\ \gamma(\vec{i})(\vec{x})$, for arbitrary n. The notions of *bound* and *free variable* are canonically extended from first-order logic so as to apply to the variables $\vec{i}$ as well. A *sentence* is a formula without free variables. We shall usually omit explicit indication of as many variables

[2]This result was obtained in collaboration with Lauri Hella; its proof will be published elsewhere.

from the formulae as possible without losing readability. In this manner we may write $\mathsf{D}^n_k\gamma$ instead of $\mathsf{D}^n_k\vec{x}\vec{i}\ \gamma(\vec{i})(\vec{x})$. Put $\mathrm{D}(\tau) = \bigcup_k \mathrm{D}_k(\tau)$.

Let $\mathfrak{A}$ be a τ-structure and let $\Gamma = \mathsf{D}^n_k\vec{x}\vec{i}\ \gamma(\vec{i})(\vec{x})$ be a D-formula. Then, $\mathfrak{A} \models \Gamma$ (colloquially pronounced as 'Γ is true on $\mathfrak{A}$') iff there exist functions $f_1, \dots, f_n : A^k \to \{0,1\}$ such that

$$\mathfrak{A} \models \forall\vec{x}\ \gamma(f_1(\vec{x}_1), \dots, f_n(\vec{x}_n))(\vec{x}). \tag{5}$$

Let $\Sigma^1_{1,k}(\tau)$ be the fragment of $\Sigma^1_1(\tau)$ whose relation variables have arity k. If k equals 1, we arrive at *monadic*, existential, second-order logic: $\Sigma^1_{1,1}(\tau) = \mathrm{M}\Sigma^1_1(\tau)$. For the semantics of first-order and second-order logic, we refer the reader to [8].

Let $\mathcal{K}$ be a class of finite τ-structures and let $\mathcal{H}$ be a subclass of $\mathcal{K}$. If Φ and Ψ are τ-sentences for which the satisfaction relation $\models$ is defined, and for every structure $\mathfrak{A}$ from $\mathcal{K}$ we have that $\mathfrak{A} \models \Phi$ iff $\mathfrak{A} \models \Psi$, then Φ and Ψ are said to be *equivalent on* $\mathcal{K}$.

Let $\mathrm{L}(\tau)$ and $\mathrm{L}'(\tau)$ be logics for which $\models$ is defined. Then, $\mathcal{H}$ is *characterized on* $\mathcal{K}$ by an $\mathrm{L}(\tau)$-sentence Φ if for every structure $\mathfrak{A}$ from $\mathcal{K}$ it is the case that $\mathfrak{A}$ sits in $\mathcal{H}$ iff $\mathfrak{A} \models \Phi$. If some of its formulae characterize the class $\mathcal{H}$ on $\mathcal{K}$, then $\mathrm{L}(\tau)$ is said to *characterize* or *express* $\mathcal{H}$ *on* $\mathcal{K}$. We write $\mathrm{L}(\tau) \leq_{\mathcal{K}} \mathrm{L}'(\tau)$ to indicate that for every $\mathrm{L}(\tau)$-formula Φ, there is an $\mathrm{L}'(\tau)$-formula Ψ that is equivalent to Φ on $\mathcal{K}$. The symbols $=_{\mathcal{K}}$ and $<_{\mathcal{K}}$ are defined from $\leq_{\mathcal{K}}$ in the standard way. In case $\mathcal{K}$ is the class of all τ-structures, we omit mentioning it, and suppress the subscript.

Let C be a complexity class [15, 26]. $\mathrm{L}(\tau)$ is said to *capture at least* C over $\mathcal{K}$, if each C-decidable subclass of $\mathcal{K}$ can be expressed by L on $\mathcal{K}$. The *expression complexity* of $\mathrm{L}(\tau)$ on $\mathcal{K}$ is said to be in C, if for every sentence Φ in $\mathrm{L}(\tau)$, the class

$$\{\mathfrak{A} \text{ from } \mathcal{K} \mid \mathfrak{A} \models \Phi\}$$

is decidable in C, relative to some natural encoding of $\mathfrak{A}$, see [21]. Finally, $\mathrm{L}(\tau)$ is said to *capture* C on $\mathcal{K}$, if $\mathrm{L}(\tau)$ captures at least C on $\mathcal{K}$ and the expression complexity of L over $\mathcal{K}$ is in C. Again, if $\mathcal{K}$ is the class of all τ-structures we may omit mentioning $\mathcal{K}$.

By means of a game-theoretic argument we show that D cannot characterize the class of structures with a universe of even cardinality, Even. The latter class, however, is definable by an unrestricted Henkin quantifier.

Proposition 1 *There exists a first-order formula ϕ such that $\mathsf{H}^2_1\ \phi$ characterizes* Even.

Proof. A structure $\mathfrak{A}$ has a universe A with even cardinality iff there exists a function $f : A \to A$ such that for every $a \in A$, $f(f(a)) = a$ and $f(a) \neq a$. The latter condition is expressed by the sentence $\mathsf{H}^2_1 x_1 x_2 y_1 y_2\ \phi$, where $\phi := (x_1 = x_2 \to y_1 = y_2) \wedge (y_1 = x_2 \to y_2 = x_1) \wedge (x_1 \neq y_1)$. □

The reader unfamiliar with Henking quantifiers may find it helpful to write down the truth condition of the sentence $\mathsf{H}^2_1 x_1 x_2 y_1 y_2\ \phi$. This condition asserts the existence of functions f_1 and f_2 such that

$$\mathfrak{A} \models \forall x_1 \forall x_2\ \phi(x_1, x_2, f_1(x_1), f_2(x_2)).$$

Especially it is instructive to realize that in order for the sentence to be true on $\mathfrak{A}$, f_1 and f_2 must be one and the same function.

3 Characterizing D as a fragment of Σ^1_1

In this section D_k is characterized as a fragment of $\Sigma^1_{1,k}$. In Section 3.1, we provide a translation T from D_k to $\Sigma^1_{1,k}$. Using the technical apparatus from Section 3.1 and the translation T itself, the characterization result is established in Section 3.2.

3.1 Translating D_k into $\Sigma^1_{1,k}$

The translation of D_k into $\Sigma^1_{1,k}$ hinges on the insight that a function $f : A \to \{0, 1\}$ can be mimicked by the set $X = \{\vec{a} \in A^k \mid f(\vec{a}) = 1\}$.

Definition 2 *Let $\vec{x}$ be a string of k variables and let X be a k-ary relation variable. Then, $\langle X, \vec{x}\rangle$ is a* proto-literal *and the formulae $X(\vec{x}), \neg X(\vec{x})$ are the* literals *based on $\langle X, \vec{x}\rangle$. Likewise, if L is a set of proto-literals, then the* set of literals based on L *is defined as*

$$\{X(\vec{x}) \mid \langle X, \vec{x}\rangle \in L\}\ \cup\ \{\neg X(\vec{x}) \mid \langle X, \vec{x}\rangle \in L\}.$$

If Φ is a second-order formula, then

$$L(\Phi) \quad = \quad \{\langle X, x_1, \ldots, x_k\rangle \mid X(x_1, \ldots, x_k) \text{ appears in } \Phi\}$$

is the set of proto-literals of Φ. *Finally, for $\mathsf{D} = \mathsf{D}^n_k \vec{x}_1 \ldots \vec{x}_n i_1 \ldots i_n$, let $L(\mathsf{D})$ be defined as $\{\langle X_j, \vec{x}_j\rangle \mid 1 \leq j \leq n\}$.*

Definition 3 *Let* $L = \{\langle Y_1, \vec{y}_1\rangle, \ldots, \langle Y_m, \vec{y}_m\rangle\}$ *be a set of proto-literals, and let* $\gamma : \{0,1\}^m \to \mathrm{FO}$ *be an implicit matrix formula. Then, the* L-explication *of* γ *is defined as*

$$T_L(\gamma) \;=\; \bigwedge_{i_1 \ldots i_m \in \{0,1\}^m} (\pm_{i_1} Y_1(\vec{y}_1) \wedge \ldots \wedge \pm_{i_m} Y_m(\vec{y}_m) \to \gamma(i_1, \ldots, i_m)(\vec{y})),$$

where $\pm_0 = \neg$ *and* $\pm_1 = \neg\neg$.

The standard translation T *maps every* $\mathrm{D}_k(\tau)$*-formula* $\Gamma = \mathsf{D}_k^n\ \gamma$ *to the* $\Sigma^1_{1,k}(\tau)$*-formula* $T(\Gamma)$*, where*

$$T(\Gamma) \;=\; \exists X_1 \ldots \exists X_n \forall \vec{x}_1 \ldots \forall \vec{x}_n\ T_{L(\mathsf{D}_k^n)}(\gamma)\ .$$

It is straightforward to check that the translation T is adequate:

Proposition 4 *Every* D_k*-sentence* Γ *is equivalent to* $T(\Gamma)$*.* □

3.2 The characterization theorem for D

Prefix classes of Σ^1_1 have been studied extensively. Recall, for instance, the language $\Sigma^1_1(\exists^*\forall^*)$, that is, the fragment of Σ^1_1 with formulae of the form

$$\exists X_1 \ldots \exists X_m \exists y_1 \ldots \exists y_l \forall x_1 \ldots \forall x_n\ \phi, \tag{6}$$

where ϕ is quantifier-free.

In this section D is characterized as a fragment of Σ^1_1, denoted $\Sigma^1_1\heartsuit$. The relevant fragment contains as its subfragment the prefix class $\Sigma^1_1(\forall^*)$, also known as *Strict* NP.

Definition 5 *Let* τ *be a vocabulary. Let* ϕ *be a second-order* τ*-formula. Call* ϕ sober *if in* ϕ *no second-order quantifier appears, and for every relation variable* X*,* $X(x_1, \ldots, x_n)$ *occurring in* ϕ *implies that the variables* $x_1, \ldots, x_n$ *are free in* ϕ*. Let* $\Sigma^1_{1,k}\heartsuit(\tau)$ *be the fragment of* $\Sigma^1_{1,k}(\tau)$*, containing all formulae without free relation variables that are of the form*

$$\exists X_1 \ldots \exists X_m \forall x_1 \ldots \forall x_n\ \phi, \tag{7}$$

where ϕ *is a sober formula and* $X_1, \ldots, X_m$ *are* k*-ary. Finally, put* $\Sigma^1_1\heartsuit(\tau) = \bigcup_k \Sigma^1_{1,k}\heartsuit(\tau)$*.*

Any sober formula is, then, a second-order formula, but only in virtue of the fact that it contains relation variables. If ϕ is a sober formula occurring in a $\Sigma^1_{1,k}\heartsuit(\tau)$-formula as in (7), then there are no existentially quantified variables among the arguments of its relation variables. Since every quantifier-free formula is sober, $\Sigma^1_1(\forall^*)$ is a fragment of $\Sigma^1_1\heartsuit$.

As an example of a $\Sigma^1_1\heartsuit$-formula, consider $\exists X_1 \exists X_2 \exists X_3 \forall x_1 \forall x_2\ (\phi \wedge \phi')$ that characterizes 3-Colorability, where $(\phi \wedge \phi')$ is a sober formula:

$$\phi \;=\; \left(\bigvee_{i\in\{1,2,3\}} X_i(x_1)\right) \wedge \left(\bigwedge_{i\in\{1,2,3\}} \bigwedge_{j\in\{1,2,3\}-\{i\}} \neg(X_i(x_1) \wedge X_j(x_1))\right)$$

$$\phi' \;=\; \left(\bigwedge_{i\in\{1,2,3\}} (X_i(x_1) \wedge X_i(x_2) \rightarrow \neg R(x_1, x_2))\right)\ .$$

Let S_L be the set of literals based on a set of proto-literals L. Call $S \subseteq S_L$ a *maximally consistent subset* of S_L, if S does not contain both a literal and its negation, but adding any literal based on L to S would imply that it contains both a literal and its negation. Put differently, S is a maximally consistent subset of S_L, if for every $\langle X, \vec{x}\rangle \in L$, either $X(\vec{x})$ or $\neg X(\vec{x})$ is in S.

Lemma 6 *Let τ be a vocabulary. Let ϕ be a sober second-order τ-formula and let $L(\phi)$ be the set of proto-literals of ϕ. Then, ϕ is equivalent to a formula of the form*

$$M(\phi) \;=\; \bigwedge_S \left(\bigwedge S \rightarrow \psi_S\right)\ ,$$

where S ranges over the maximally consistent subsets of $S_{L(\phi)}$ and the ψ_S are $\mathrm{FO}(\tau)$*-formulae.* □

The formula $M(\phi)$, as in the statement of Lemma 6, will be called *the explicit matrix formula* of the sober second-order formula ϕ. As stated by clause (2) of the following lemma, a certain fragment of the logic $\Sigma^1_{1,k}\heartsuit$, determined by the syntactic condition $(*)$, can be translated to the logic D_k.

Lemma 7 *Let τ be a vocabulary. Let ϕ be a sober τ-formula containing the k-ary relation variables $X_1, \dots, X_n$, such that*

- *$(*)$ if $X_i(x_1, \dots, x_k)$ and $X_j(x'_1, \dots, x'_k)$ appear in ϕ, then $i \neq j$ or $x_h = x'_h$, for every $1 \leq h \leq k$.*

Then, (1) and (2) hold:

- *(1) There exists an implicit matrix τ-formula γ such that $T_{L(\phi)}(\gamma)$ and ϕ are equivalent.*
- *(2) There exists a $\mathrm{D}_k(\tau)$-formula that is equivalent to $\exists X_1 \dots \exists X_n \forall \vec{x}_1 \dots \forall \vec{x}_n\ \phi$.*

Proof. Let ϕ meet the premise of the lemma and let $L(\phi)$ be the set of proto-literals $\{\langle X_1, \vec{x}_1\rangle, \ldots, \langle X_n, \vec{x}_n\rangle\}$.

(1): Since ϕ is sober, derive from Lemma 6 that ϕ can be rewritten as an explicit matrix formula $M(\phi) = \bigwedge_S(\bigwedge S \to \psi_S)$, where the formulae ψ_S are first-order. Having seen this, all that is to be done is encoding the explicit matrix formula $M(\phi)$ in an implicit matrix formula γ, so that $T_{L(\phi)}(\gamma)$ and $M(\phi)$ are equivalent. This can be done by putting $\gamma(i_1, \ldots, i_n) = \psi_{S_{i_1\ldots i_n}}$, where

$$S_{i_1\ldots i_n} = \{\pm_{i_1} X_1(\vec{x}_1), \ldots, \pm_{i_n} X_n(\vec{x}_n)\},$$

for $\pm_1 = \neg\neg$ and $\pm_0 = \neg$. Following Definition 3, $T_{L(\phi)}(\gamma)$ equals

$$\bigwedge_{i_1\ldots i_n \in \{0,1\}^n} \left(\pm_{i_1} X_1(\vec{x}_1) \wedge \ldots \wedge \pm_{i_n} X_n(\vec{x}_n) \to \gamma(i_1, \ldots, i_n)\right). \qquad (8)$$

Since every string $i_1 \ldots i_n \in \{0,1\}^n$ corresponds thus to a maximally consistent set of proto-literals, and vice versa, (8) is syntactically equivalent to the explicit matrix formula $M(\phi)$. From Lemma 6 it follows that $T_{L(\phi)}(\gamma)$ is equivalent to ϕ.

(2): Consider a $\Sigma^1_{1,k}(\tau)$-formula $\Psi = \exists X_1 \ldots \exists X_n \forall \vec{x}_1 \ldots \forall \vec{x}_n\ \phi$. Then, by clause (1) there exists a matrix formula γ, such that $T_{L(\phi)}(\gamma)$ and ϕ are equivalent. Consider the formula $\Gamma = \mathsf{D}^n_k \vec{x}_1 \ldots \vec{x}_n \vec{i}\ \gamma$ and its standard translation $T(\Gamma)$:

$$\exists X_1 \ldots \exists X_n \forall \vec{x}_1 \ldots \vec{x}_n\ T_{L(\phi)}(\gamma)$$

By $(*)$ it follows that $L(\mathsf{D}^n_k \vec{x}_1 \ldots \vec{x}_n \vec{i}) = L(\phi)$. Hence, $T(\Gamma)$ is syntactically equal to Ψ, and $T(\Gamma)$ is equivalent to Γ in virtue of Proposition 4. □

Theorem 8 *Let τ be a vocabulary including the equality symbol. Then,* $\mathrm{D}_k(\tau) = \Sigma^1_{1,k}\heartsuit(\tau)$*, for every integer k. Hence,* $\mathrm{D}(\tau) = \Sigma^1_1\heartsuit(\tau)$.

Proof. From left to right. This direction follows immediately from the translation T, as it maps every formula in $\mathrm{D}_k(\tau)$ to a formula in $\Sigma^1_{1,k}\heartsuit(\tau)$. For the correctness of the translation T, we refer to Proposition 4.

From right to left. This direction follows from the fact that every $\Sigma^1_{1,k}\heartsuit(\tau)$-formula has an equivalent $\Sigma^1_{1,k}\heartsuit(\tau)$-formula that meets condition $(*)$ of Lemma 7. This is proved by setting up a translation that roughly goes as follows.

Let Φ be an $\Sigma^1_{1,k}\heartsuit(\tau)$-formula in which the k-ary relation variable X appears with precisely the following k-tuples of variables: $\langle y_{11}, \ldots, y_{k1}\rangle, \ldots, \langle y_{1n}, \ldots, y_{kn}\rangle$. That is, if the string $X(z_1, \ldots, z_k)$ appears in Φ, then for some $1 \le i \le n$ and every $1 \le j \le k$, $z_j = y_{ji}$. Note that the symbols y_{ji} and z_j are used as metavariables.

Now, for each unique tuple of variables $\langle y_{1i}, \dots, y_{ki}\rangle$, we replace the string $X(y_{1i}, \dots, y_{ki})$ by $X_i((y_{1i})_i, \dots, (y_{ki})_i)$, where X_i and $(y_{1i})_i, \dots, (y_{ki})_i$ are 'real' variables. For instance, if the metavariable $y_{2,5}$ stands for the variable x, then $(y_{2,5})_5$ stands for the variable x_5.

We repeat this procedure for every relation variable X in Φ that has more than one unique string of variables as argument.

Making use of the equality symbol we ensure that the copies of the variables and the relation variables are assigned the same semantic objects in order for the translation to be equivalent to Φ on any structure. The full proof is given in [29]. $\Box$

We suspect that it is not possible to find a translation from $\Sigma^1_1\heartsuit$ to D in vocabularies that lack the equality symbol. Settling this issue is left for future research, however.

The above characterization of D may speed up discovering interesting properties that it enjoys, for second-order logic happens to be more intensively studied than partially ordered connectives. Now that we have characterized D_k, we can safely conclude that any property expressible in $\Sigma^1_{1,k}\heartsuit(\tau)$ is expressible in $\mathrm{D}_k(\tau)$ as well. Concrete and interesting examples of this mode of research are found in the following section.

4 Applications of the characterization

In this section, two results are obtained using the characterization of D. In Section 4.1, it is shown that for every k there is a vocabulary σ such that $\mathrm{D}_k(\sigma) < \mathrm{D}_{k+1}(\sigma)$. In Section 4.2 it is shown that on linearly ordered structures, $\mathrm{D} = \Sigma^1_1$.

4.1 Strict hierarchy result

Ajtai [1] showed that for every k, there is a vocabulary σ such that $\Sigma^1_{1,k}(\sigma)$ is strictly contained in $\Sigma^1_{1,k+1}(\sigma)$. We will use Ajtai's result to show that for every k there is a σ such that $\mathrm{D}_k(\sigma) < \mathrm{D}_{k+1}(\sigma)$, making use of Theorem 8. Put differently, D contains a strict, arity induced hierarchy, even over finite structures.

Theorem 9 *Let $k \geq 2$ be an integer and let σ be a vocabulary with at least one k-ary relation symbol P and the linear order symbol $>$. Then, over σ-structures, $\mathrm{D}_{k-1}(\sigma) < \mathrm{D}_k(\sigma)$.*

Proof. From [1] the following can be derived[3]: Let Π_k be the subclass of σ-structures $\mathfrak{A}$ such that

$$\mathfrak{A} \text{ is in } \Pi_k \quad \text{iff} \quad \|P^{\mathfrak{A}}\| \text{ is even.}$$

Then, Π_k is not expressible in $\Sigma^1_{1,k-1}(\sigma)$, but it is expressible in $\Sigma^1_{1,k}(\sigma)$.

To separate D_k from D_{k-1}, we show that Π_k is expressible by a formula of $\mathrm{D}_k(\sigma)$. This suffices, since

$$\mathrm{D}_{k-1} \;=\; \Sigma^1_{1,k-1}\heartsuit \;\leq\; \Sigma^1_{1,k-1}$$

and $\Sigma^1_{1,k-1}$ cannot express Π_k.

We show that $\mathrm{D}_k(\sigma)$ can express Π_k, by giving a $\Sigma^1_{1,k}\heartsuit(\sigma)$-formula Υ_k that expresses Π_k. Intuitively, Υ_k lifts the linear order $>$ (which is a relation among the objects of the universe) to a linear order ψ_k among k-tuples of objects of the universe. With respect to this lifted linear order, Υ_k expresses that there exists a subset Q of k-tuples of objects from the universe of the σ-structure $\mathfrak{A}$ such that:

(1) Q is a subset of $P^{\mathfrak{A}}$.

(2) The ψ_k-minimal k-tuple that is in $P^{\mathfrak{A}}$ is also in Q, and the ψ_k-maximal k-tuple that is in $P^{\mathfrak{A}}$ is not in Q.

(3) If two k-tuples are in $P^{\mathfrak{A}}$ and there is no k-tuple between them (in the ordering constituted by ψ_k) that is in $P^{\mathfrak{A}}$, then exactly one of these k-tuples is in Q.

We define

$$\Upsilon_k \;=\; \exists Q \forall \vec{x} \forall \vec{y}\ (\phi_1 \wedge \phi_2 \wedge \phi_3)\ ,$$

where ϕ_i is the formula that was informally described in clause (i) above and $\vec{x}$ and $\vec{y}$ are strings of k variables. In the light of these descriptions, the following specifications should be self-explanatory:

$$\begin{aligned}
\phi_1 &= Q(\vec{x}) \rightarrow P(\vec{x}) \\
\phi_2 &= (MIN_P(\vec{x}) \rightarrow Q(\vec{x})) \wedge (MAX_P(\vec{x}) \rightarrow \neg Q(\vec{x})) \\
\phi_3 &= NEXT_P(\vec{x},\vec{y}) \rightarrow \neg(Q(\vec{x}) \leftrightarrow Q(\vec{y})),
\end{aligned}$$

[3] The result essentially uses *hypergraphs*, that is, structures interpreting relation symbols of arity ≥ 3. As a consequence, the result does not imply that $\Sigma^1_{1,2}(\tau)$ is strictly weaker than $\Sigma^1_{1,3}(\tau)$, where τ a vocabulary that contains only unary and binary predicates, cf. [7].

where

$$\begin{aligned} MIN_P(\vec{x}) &= \forall \vec{z}\ (P(\vec{z}) \rightarrow (\psi_k(\vec{x},\vec{z}) \vee \vec{z} = \vec{x})) \\ MAX_P(\vec{x}) &= \forall \vec{z}\ (P(\vec{z}) \rightarrow (\psi_k(\vec{z},\vec{x}) \vee \vec{z} = \vec{x})) \\ NEXT_P(\vec{x},\vec{y}) &= P(\vec{x}) \wedge P(\vec{y}) \wedge \psi_k(\vec{x},\vec{y}) \wedge \\ &\quad\ \forall \vec{z}\ (P(\vec{z}) \rightarrow (\psi_k(\vec{z},\vec{x}) \vee \psi_k(\vec{y},\vec{z}) \vee \vec{z} = \vec{x} \vee \vec{y} = \vec{z})) \end{aligned}$$

and the k-dimensional lift of the linear order $>$ is inductively defined as

$$\begin{aligned} \psi_1(x,y) &= x < y \\ \psi_i(x_1,\ldots,x_i,y_1,\ldots,y_i) &= x_i < y_i \vee (x_i = y_i \wedge \psi_{i-1}(x_1,\ldots,x_{i-1},y_1,\ldots,y_{i-1})). \end{aligned}$$

The result follows, since Υ_k is a $\Sigma^1_{1,k}\heartsuit(\sigma)$-formula. $\square$

4.2 On linearly ordered structures $\mathrm{D} = \Sigma^1_1$

In this section, it is shown that on linearly ordered structures, $\mathrm{D} = \Sigma^1_1$. This can be compared to a result we prove in Section 5, namely that on graphs, $\mathrm{D} < \Sigma^1_1$. For the purposes of the present section, we introduce the logic V with sentences of the form

$$\Phi = \begin{pmatrix} \forall x_1 & \ldots & \forall x_k & \exists z \\ \forall y_1 & \ldots & \forall y_k & \bigvee i \in \{0,1\} \end{pmatrix} \gamma(i)(\vec{x},\vec{y},z)\ , \tag{9}$$

such that $\mathfrak{A} \models \Phi$ iff there exists a function $f : A^k \rightarrow A$ and a function $g : A^k \rightarrow \{0,1\}$ where

$$\mathfrak{A} \models \forall\vec{x}\forall\vec{y}\ \gamma(g(\vec{y}))(\vec{x},\vec{y},f(\vec{x}))\ . \tag{10}$$

It was shown by Krynicki [22] that V coincides with Σ^1_1, without restrictions on the vocabulary.

Theorem 10 *On linearly ordered structures,* $\mathrm{D} = \Sigma^1_1$.

Proof. Trivially, $\mathrm{D} \leq \Sigma^1_1$ on arbitrary structures. For the converse direction, in virtue of the result from [22], it suffices to show that for every Φ of the form (9) there is a D-sentence Γ equivalent to Φ on linearly ordered structures. Observe that Φ is equivalent to

$$\exists f \exists X \forall\vec{x}\forall\vec{y}\ (X(\vec{x}) \rightarrow \gamma(1)(\vec{x},\vec{y},f(\vec{x}))\ \wedge\ \neg X(\vec{x}) \rightarrow \gamma(0)(\vec{x},\vec{y},f(\vec{x}))),$$

where X is a k-ary relation variable and f is a k-ary function variable. In the remainder of the proof we show that the function variable f can be mimicked by means of a $2(k{+}1)$-ary relation variable Z. More precisely, we provide a $\Sigma^1_1\heartsuit$-sentence Ψ with second-order quantifiers $\exists Z$ and $\exists X$ that is equivalent to Φ. The sentence Ψ will employ the k-dimensional lift ψ_k of the linear order $>$, from the proof of Theorem 9. The $2k$-ary predicate SUC is defined using ψ_k and contains all $2k$-tuples $\langle\vec{a},\vec{b}\rangle$ such that $\vec{b}$ is the immediate ψ_k-successor of $\vec{a}$.

Intuitively, in Ψ the relation variable Z will be defined so that on an arbitrary linearly ordered structure $\mathfrak{A}$,

(1) Z is a linear order among $(k+1)$-tuples of the universe of $\mathfrak{A}$; and

(2) for all $\vec{a},\vec{b} \in A^k$, if $\psi_k(\vec{a},\vec{b})$, then for all $a', b' \in A$, $Z(\vec{a},a',\vec{b},b')$.

Thus, per k-tuple $\vec{a}$ one can associate an *$\vec{a}$-interval* of A^{k+1}-objects, such that for two k-tuples $\vec{a}$ and $\vec{b}$, if $\psi_k(\vec{a},\vec{b})$ then every object in the $\vec{a}$-interval precedes every tuple in the $\vec{b}$-interval in the ordering imposed by Z.

Let $\vec{a} \in A^k$ and let $a' \in A$. If for all $a'' \in A$ it is the case that $Z(\vec{a},a',\vec{a},a'')$, then a' is called the *Z-minimal object* of $\vec{a}$. In the same vein, call a' the *Z-maximal object* of $\vec{a}$, if for all $a'' \in A$ we have that $Z(\vec{a},a'',\vec{a},a')$.

Although Z is a relation, it will be used to the effect of a k-ary function f_Z by letting $f_Z(\vec{a})$ be the Z-minimal object of $\vec{a}$. But—for reasons that will become clear in due course—if $\vec{a}$ is the ψ_k-minimal tuple, then $f_Z(\vec{a})$ is the Z-maximal object of $\vec{a}$.

For instance, consider the following ordering Z of $\{1,2,3\}^2$, observing the 1-, 2-, and 3-interval:

$$\underbrace{\langle 1,2\rangle\ Z\ \langle 1,3\rangle\ Z\ \langle 1,1\rangle}_{\text{1-interval}}\ Z\ \underbrace{\langle 2,2\rangle\ Z\ \langle 2,1\rangle\ Z\ \langle 2,3\rangle}_{\text{2-interval}}\ Z\ \underbrace{\langle 3,1\rangle\ Z\ \langle 3,3\rangle\ Z\ \langle 3,1\rangle}_{\text{3-interval}}.$$

Then, Z gives rise to the function f_Z, such that

$$\begin{aligned} f_Z(1) &= 1 \\ f_Z(2) &= 2 \\ f_Z(3) &= 1. \end{aligned}$$

In the implementation of Z, the Z-minimal object of $\vec{a}$ will be recognized as the object a' such that there exists a tuple $\vec{b}$ and an object b' where $SUC(\vec{b},\vec{a})$ and $Z(\vec{b},b',\vec{a},a')$. If $\vec{a}$ is the ψ_k-minimal tuple, then it cannot be recognized in this manner, since there is no $\vec{b}$ such that $SUC(\vec{b},\vec{a})$. It is for this reason that if $\vec{a}$ is the ψ_k-minimal tuple, then $f_Z(\vec{a})$ is the Z-maximal object of $\vec{a}$. The Z-maximal object of $\vec{a}$ is recognized as the object a' such that there exists $\vec{b}$ and b' such that $SUC(\vec{a},\vec{b})$ and $Z(\vec{a},a',\vec{b},b')$.

Let Ψ be the following sentence:

$$\exists Z \exists X \forall \vec{x} \forall \vec{y} \forall \vec{z} \forall u \forall u' \forall u'' \quad \begin{array}{l} \text{“}Z \text{ is a linear order of } (k{+}1)\text{-tuples”} \ \wedge \\ \psi_k(\vec{x},\vec{y}) \rightarrow Z(\vec{x},u,\vec{y},u') \ \wedge \\ (X(\vec{x}) \rightarrow \delta(1)) \ \wedge \ (\neg X(\vec{x}) \rightarrow \delta(0)), \end{array}$$

where “Z is a linear order of $(k{+}1)$-tuples” abbreviates the conjunction of

$$\begin{array}{l} \neg Z(\vec{x},u,\vec{x},u) \\ Z(\vec{x},u,\vec{y},u') \vee (x_1 = y_1 \wedge \ldots \wedge x_k = y_k \wedge u = u') \vee Z(\vec{y},u',\vec{x},u) \\ Z(\vec{x},u,\vec{y},u') \wedge Z(\vec{y},u',\vec{z},u'') \rightarrow Z(\vec{x},u,\vec{z},u'') \end{array}$$

and $\delta(i)$, for $i \in \{0,1\}$, abbreviates the conjunction of

$$\begin{array}{l} \neg \mathit{MIN}(\vec{y}) \wedge \mathit{SUC}(\vec{z},u'',\vec{y},u') \wedge Z(\vec{z},u'',\vec{y},u') \rightarrow \gamma(i)(\vec{x},\vec{y},u') \\ \mathit{MIN}(\vec{y}) \wedge \mathit{SUC}(\vec{y},u',\vec{z},u'') \wedge Z(\vec{y},u',\vec{z},u'') \rightarrow \gamma(i)(\vec{x},\vec{y},u'). \end{array}$$

In $\delta(i)$, MIN is the predicate that holds only of the ψ_k-minimal tuple. It is left to the reader to check that Ψ is indeed equivalent to Φ.

To prove that there is a D-sentence that is equivalent to Φ on linearly ordered structures, it suffices—in virtue of Theorem 8—to show that Ψ is a $\Sigma^1_1\heartsuit$-formula. To this end observe that one can define ψ_k, SUC, and MIN using only the binary relation symbol $>$. So in particular it follows that these predicates can be defined without the help of relation variables. Finally, observe that each argument of the relation variables Z and X is quantified by one of the universal quantifiers in the block $\forall \vec{x} \forall \vec{y} \forall \vec{z} \forall u \forall u' \forall u''$. □

By Theorem 10, D *captures* NP on linearly ordered structures, adopting the terminology from descriptive complexity theory.[4] Combining Theorems 9 and 10, it can be noted that if σ is a vocabulary containing a linear order symbol and a further predicate symbol of each arity, then the sequence $\langle \mathrm{D}_k(\sigma) : k < \omega \rangle$ of logics, evaluated on linearly ordered structures, approaches Σ^1_1 as a limit, in the sense that for any Σ^1_1-sentence there is an equivalent $\mathrm{D}_k(\sigma)$-sentence, for some $k < \omega$.

[4] Barnaby Martin made us aware of the fact that $\Sigma^1_1(\forall^*)$ captures NP on the class of structures that interpret, amongst others, the symbols $<$, $+$, and $\times$. Theorem 10 does not follow from this fact, since in order to define the numeric predicates $+$ and $\times$, one needs existentially quantified variables as arguments of relation variables. For a short discussion of these matters, the reader is referred to [21, pp. 117–8].

5 Ehrenfeucht-Fraïssé game for D

Ehrenfeucht-Fraïssé games or *model comparison games* are usually employed to prove that some property is not definable in a certain logic. These games were first introduced for first-order logic in [9, 13].

Let the *quantifier rank* of a first-order formula be its maximum number of nested quantifiers. Let m be an integer. If $\mathfrak{A}$ and $\mathfrak{B}$ are τ-structures, $\vec{a}^{\mathfrak{A}} = \langle a_1^{\mathfrak{A}}, \ldots, a_r^{\mathfrak{A}} \rangle \in A^r$, and $\vec{b}^{\mathfrak{B}} = \langle b_1^{\mathfrak{B}}, \ldots, b_r^{\mathfrak{B}} \rangle \in B^r$, then the *$m$-round Ehrenfeucht-Fraïssé game on the structures $\mathfrak{A}$ and $\mathfrak{B}$*, denoted by

$$EF_m^{\mathrm{FO}}(\langle \mathfrak{A}, \vec{a}^{\mathfrak{A}} \rangle, \langle \mathfrak{B}, \vec{b}^{\mathfrak{B}} \rangle) \ ,$$

is an m-round game proceeding as specified below. There are two players, Spoiler and Duplicator. During the ith round, Spoiler first chooses a structure $\mathfrak{A}$ (or $\mathfrak{B}$) and an element called c_i (or d_i) from the domain of the chosen structure. Duplicator replies by choosing an element d_i (or c_i) from the domain of the other structure $\mathfrak{B}$ (or $\mathfrak{A}$). Duplicator wins the play $\langle \langle c_1, d_1 \rangle, \ldots, \langle c_m, d_m \rangle \rangle$, if the relation

$$\{ \langle a_i^{\mathfrak{A}}, b_i^{\mathfrak{B}} \rangle \mid 1 \leq i \leq r \} \cup \{ \langle c_i, d_i \rangle \mid 1 \leq i \leq m \} \qquad (11)$$

is a *partial isomorphism* between $\mathfrak{A}$ and $\mathfrak{B}$; otherwise, Spoiler wins the play. If against any sequence of moves by Spoiler, Duplicator is able to make her moves so as to win the resulting play, Duplicator is said to have a *winning strategy in $EF_m^{\mathrm{FO}}(\langle \mathfrak{A}, \vec{a}^{\mathfrak{A}} \rangle, \langle \mathfrak{B}, \vec{b}^{\mathfrak{B}} \rangle)$*. The notion of winning strategy for Spoiler is defined analogously. By the Gale-Stewart Theorem [14], Ehrenfeucht-Fraïssé games are determined; that is, precisely one of the players has a winning strategy. The usefulness of these games is established in the following seminal result.

Theorem 11 ([9, 13]) *For every integer m, the following are equivalent:*

- *$\langle \mathfrak{A}, \vec{a}^{\mathfrak{A}} \rangle$ and $\langle \mathfrak{B}, \vec{b}^{\mathfrak{B}} \rangle$ satisfy the same first-order formulae $\varphi(x_1, \ldots, x_r)$ of quantifier rank at most m.*
- *Duplicator has a winning strategy in $EF_m^{\mathrm{FO}}(\langle \mathfrak{A}, \vec{a}^{\mathfrak{A}} \rangle, \langle \mathfrak{B}, \vec{b}^{\mathfrak{B}} \rangle)$.* □

Readers unfamiliar with these games may find it helpful to consult [8], and [12, 21] for similar games for $\mathrm{M}\Sigma_1^1$.

The notion of quantifier rank is extended to implicit matrix formulae as follows: $qr(\gamma) = \max\{ qr(\gamma(\vec{i})) \mid \vec{i} \in \{0,1\}^k \}$, for γ of type $\{0,1\}^k \to \mathrm{FO}$.

The model comparison game for D has two phases: a *watercoloring phase* and a *first-order phase.* Let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures and let m be an integer.

Then, the *m-round, watercolor* D*-Ehrenfeucht-Fraïssé game on the structures* $\mathfrak{A}$ *and* $\mathfrak{B}$, denoted as

$$EF^{\mathrm{D}}_{m,n,k}(\mathfrak{A}, \mathfrak{B}) \ ,$$

is an $(m+1)$-round game proceeding as follows: First we have the watercoloring phase. Spoiler picks out for every $1 \leq i \leq n$ a subset A_i from A^k. Duplicator picks out a subset B_i of B^k, for every $1 \leq i \leq n$. Next, Spoiler chooses a tuple $\vec{b}_i^{\mathfrak{B}} \in B^k$, for every $1 \leq i \leq n$, and Duplicator replies by choosing a tuple $\vec{a}_i^{\mathfrak{A}} \in A^k$. If for every $1 \leq i \leq n$ the selected tuples satisfy $\vec{a}_i^{\mathfrak{A}} \in A_i$ iff $\vec{b}_i^{\mathfrak{B}} \in B_i$, then the game proceeds to the first-order phase as $EF^{\mathrm{FO}}_m(\langle \mathfrak{A}, \vec{a}^{\mathfrak{A}} \rangle, \langle \mathfrak{B}, \vec{b}^{\mathfrak{B}} \rangle)$; otherwise, Duplicator loses right away.

It is interesting to note that in the first-order Ehrenfeucht-Fraïssé game that is started up after the watercolor phase, the actual colorings are immaterial. The watercolors fade away quickly, so to say.

Proposition 12 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* τ*-structures, and let* k, n *be integers. Let* $\Gamma = \mathsf{D}^n_k\ \gamma$ *be any* D_k*-sentence with* $qr(\gamma) \leq m$*. Then, the first assertion implies the second:*

- *Duplicator has a winning strategy in* $EF^{\mathrm{D}}_{m,n,k}(\mathfrak{A}, \mathfrak{B})$*.*
- $\mathfrak{A} \models \Gamma$ *implies* $\mathfrak{B} \models \Gamma$*.*

Hence, if the first assertion holds for arbitrary k, n*, the second assertion holds for every* D*-formula* Γ*, where* $qr(\Gamma) \leq m$*.*

Proof. The game is a simple adaptation of the one presented in [28]. □

Fagin [12] showed that the monadic fragments of Σ^1_1 and Π^1_1 do not coincide, as the latter harbors CONNECTED but the former does not. Thus we say that $\mathrm{M}\Sigma^1_1$ *is not closed under complementation.*

Using the model comparison games for D, it can be shown that D is not closed under complementation either. This result may be interesting, because $\mathrm{D} = \Sigma^1_1\heartsuit$ is a fragment of Σ^1_1 that is not bounded by the arity of the relation variables, and has a non-empty intersection with k-ary, existential, second-order logic, for arbitrary k (cf. Theorem 9). Clearly, these properties are not enjoyed by $\mathrm{M}\Sigma^1_1$.

For any two τ-structures $\mathfrak{A}$ and $\mathfrak{B}$ with non-intersecting universes, let $\mathfrak{A} \cup \mathfrak{B}$ denote the τ-structure with universe $A \cup B$ and $R^{\mathfrak{A} \cup \mathfrak{B}} = R^{\mathfrak{A}} \cup R^{\mathfrak{B}}$, for any $R \in \tau$.

Theorem 13 $\overline{\text{2-COLORABILITY}}$ *cannot be expressed in* D*. Hence,* D *is not closed under complementation.*

Proof. For contradiction, suppose $\overline{\text{2-COLORABILITY}}$ were characterizable in D. So there would be a sentence in D that characterizes $\overline{\text{2-COLORABILITY}}$, say Γ. This sentence Γ would have a partially ordered connective with dimensions k, n prefixing an implicit matrix τ-formula of quantifier rank m. Now if we are able to find structures $\mathfrak{A}$ and $\mathfrak{B}$ such that (i) $\mathfrak{A}$ is not 2-colorable but $\mathfrak{B}$ is 2-colorable, and (ii) Duplicator has a winning strategy in $EF^{\mathrm{D}}_{m,n,k}(\mathfrak{A}, \mathfrak{B})$, we may reason as follows: Since Γ is supposed to characterize $\overline{\text{2-COLORABILITY}}$, we derive from (i) that $\mathfrak{A} \models \Gamma$ and $\mathfrak{B} \not\models \Gamma$. But from (ii) and $\mathfrak{A} \models \Gamma$ it follows by Proposition 12, that $\mathfrak{B} \models \Gamma$. A contradiction. So if such structures $\mathfrak{A}$ and $\mathfrak{B}$ are found for all m, k, n, we may conclude that no sentence Γ exists in D that expresses $\overline{\text{2-COLORABILITY}}$.

It remains to be shown that for arbitrary m, k, n, there indeed exist graphs $\mathfrak{A}$ and $\mathfrak{B}$ meeting (i) and (ii). To this end, fix integers m, k, n and consider the graphs $\mathfrak{C}$ and $\mathfrak{D}$, where

$$\begin{aligned} C &= \{c_1, \ldots, c_N\} \\ R^{\mathfrak{C}} &= \text{the symmetric closure of } \{\langle c_i, c_{i+1}\rangle \mid 1 \leq i \leq N-1\} \cup \{\langle c_N, c_1\rangle\} \\ D &= \{d_1, \ldots, d_{N+1}\} \\ R^{\mathfrak{D}} &= \text{the symmetric closure of } \{\langle d_i, d_{i+1}\rangle \mid 1 \leq i \leq N\} \cup \{\langle d_{N+1}, d_1\rangle\} \end{aligned}$$

and $N = 2^{m+k\cdot n}$. So $\mathfrak{C}$ and $\mathfrak{D}$ are cycles of even and odd length, respectively. A cycle is 2-colorable iff it is of even length, hence $\mathfrak{D}$ is not 2-colorable, whereas $\mathfrak{C}$ is. Obviously, the structure $\mathfrak{C} \cup \mathfrak{D}$ is not 2-colorable either.

Let us proceed to show that Duplicator has a winning strategy in $EF^{\mathrm{D}}_{m,n,k}(\mathfrak{C} \cup \mathfrak{D}, \mathfrak{C})$. Suppose Spoiler selects, for every $1 \leq i \leq n$, a set $X_i \subseteq (C \cup D)^k$. Let Duplicator respond with X_i restricted to $\mathfrak{C}$, that is, with $Y_i = X_i \cap C^k$, for every $1 \leq i \leq n$. Suppose Spoiler selects the tuple $\vec{c}_i^{\,\mathfrak{C}} \in C^k$, for every $1 \leq i \leq n$. Let Duplicator respond by simply copying these tuples on $(C \cup D)^k$, that is, setting $\vec{c}_i^{\,\mathfrak{C}\cup\mathfrak{D}} = \vec{c}_i^{\,\mathfrak{C}}$. The game advances to the first-order phase, since obviously $\vec{c}^{\,i} \in X_i$ iff $\vec{c}^{\,i} \in Y_i$. A standard argument (cf. [8, p. 23]) suffices to show that Duplicator has a winning strategy in

$$EF^{\mathrm{FO}}_m(\langle \mathfrak{C} \cup \mathfrak{D}, \vec{c}_1^{\,\mathfrak{C}\cup\mathfrak{D}}, \ldots, \vec{c}_n^{\,\mathfrak{C}\cup\mathfrak{D}}\rangle, \langle \mathfrak{C}, \vec{c}_1^{\,\mathfrak{C}}, \ldots, \vec{c}_n^{\,\mathfrak{C}}\rangle).$$

As noted in Section 1, D characterizes the class of 3-colorable graphs. In the same way it characterizes 2-COLORABILITY. It was just shown that the complement of this class is not expressible in D. Therefore, D is not closed under complementation. □

In the proof of Theorem 13, the universe of $\mathfrak{C}$ has an even cardinality but that of $\mathfrak{D}$ does not have. Thus:

Corollary 14 *The class* EVEN *is not characterizable in* D. □

By contrast, in Proposition 1 it was shown that EVEN is characterizable by a sentence of the form $\mathsf{H}^2_1\ \phi$. So already the simplest Henkin quantifier not definable in first-order logic, fails to be definable in D as well. Since EVEN is obviously characterizable in binary Σ^1_1 and $\Sigma^1_1 = \mathrm{H}$, the following result ensues:

Corollary 15 *On graphs,* $\mathrm{D} < \Sigma^1_1$. □

6 Discussion

It is interesting to compare partially ordered connectives with Henkin quantifiers by comparing the properties of the logics D and H. Since the latter is equivalent to Σ^1_1, we might just as well compare D with Σ^1_1.

To increase the value of the comparison, we cite a result from an unpublished manuscript by Lauri Hella and the present authors, concerning 0-1 laws. It is well-known that first-order logic has a 0-1 law, but Σ^1_1 does not have one. In fact, Σ^1_1's capability to express EVEN is a witness of this fact. If a logic has a 0-1 law, it is said to be *unable to count*. For a textbook treatment of 0-1 laws consult [8].

Theorem 16 D *has a 0-1 law.*

Proof. The result follows as a corollary to (a simple extension of) a result from [6], and the observation that for every D-formula Γ, if Γ holds on $\mathfrak{A}$ then, Γ holds on every substructure of $\mathfrak{A}$.[5] □

The following table gives an overview of the finite model theory of D in comparison to Σ^1_1.

	Σ^1_1	D
Able to express NP-c. properties	yes	yes, [5]
Captures NP	yes, [12]	no, Cor. 15
Captures NP over lin. o. structures	yes	yes, Th. 10
Closed under complementation	iff NP = coNP	no, Th. 13
0-1 law	no	yes

[5]This argument was pointed out to us by Lauri Hella, whom we gratefully acknowledge.

Considering this table it is seen that D exhibits an interesting mix of 'strong' and 'weak' properties, relative to Σ^1_1. D is strong, because it can express NP-complete properties and captures NP over linearly ordered structures. On the other hand, D does not capture NP over arbitrary structures and it has a 0-1 law.

First and foremost, our results apply to the logic D. But of course they apply to $\Sigma^1_1\heartsuit$ as well. As we pointed out earlier, $\Sigma^1_1\heartsuit$ loosens the restrictions defining $\Sigma^1_1(\forall^*)$ (i.e., strict NP), by moving from quantifier-free to sober formulae. We consider it worthwhile to explore what are the properties of other sober prefix classes of Σ^1_1. That is, to compare the properties of $\Sigma^1_1(r)$ and its sober counterpart, where $\Sigma^1_1(r)$ contains all formulae of the form

$$\exists X_1 \ldots \exists X_m Q_1 x_1 \ldots Q_n x_n \; \phi,$$

where $Q_1, \ldots, Q_n$ is a string accepted by the regular expression r. For instance, it would be very interesting to see whether for every class $\Sigma^1_1(r)$ it is the case that it can define NP-complete problems iff its 'soberized' counterpart can. Such results would put the results from [17] in a broader perspective.

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