A CLASSIFICATION OF ORDINAL TOPOLOGIES

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Abstract. Ordinals carry a natural topology induced by their linear order. In this note, we classify the homeomorphism types of all ordinal topologies using the Cantor normal form and the notion of the Cantor–Bendixson rank of a point.

Let $<$ be a linear order on a set X. The *order topology* on X is generated by the subbase of rays of the form $\mathsf{left}_X(b) := \{x \in X : x < b\}$ and $\mathsf{right}_X(a) := \{x \in X : a < x\}$ for $a, b \in X$. Natural examples of order topologies are the standard topologies on $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$. Ordinals provide another source for natural order topologies; we call these *ordinal topologies*. These topologies have many discrete points: if $\xi < \xi + 1 < \alpha$, then $\xi + 1$ is a discrete point and the topological space α splits disjointly into the open sets $\mathsf{left}_\alpha(\xi + 1)$ and $\mathsf{right}_\alpha(\xi)$.

We shall provide a complete homeomorphic invariant for ordinal topologies, using the Cantor normal form and the notion of the Cantor-Bendixson rank. The intuition behind this classification is that ordinal topologies are mainly determined by their limit points.

Recall that every nonzero ordinal α can be written uniquely in Cantor normal form (to base ω) as $\alpha = \omega^{\alpha_0} \cdot k_0 + \cdots + \omega^{\alpha_n} \cdot k_n$, where $\alpha \geq \alpha_0 > \cdots > \alpha_n$ and $0 < k_i < \omega$ for $0 \leq i \leq n$ [Je03, Theorem 2.26]. We define the *limit complexity* of α as $\mathrm{lc}(\alpha) := \alpha_0$, the *coefficient* of α as $\mathrm{c}(\alpha) := k_0$ and the *purity* of α as

$$\mathrm{p}(\alpha) := \begin{cases} 0 & \text{if } \alpha = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) \geq \omega, \text{ and} \\ \omega^{\alpha_n} & \text{otherwise.} \end{cases}$$

We call α a *pure* limit ordinal if $\mathrm{p}(\alpha) = 0$, and an *impure* limit ordinal if $\mathrm{p}(\alpha) > 1$. Of course, α is a successor ordinal iff $\mathrm{p}(\alpha) = 1$.

We shall show that these three data provide a complete homeomorphic invariant for ordinal topologies:

Main Theorem. For any two ordinals α and β, the ordinal topologies on α and β are homeomorphic (written $\alpha \cong \beta$) if and only if

$$\langle \mathrm{lc}(\alpha), \mathrm{c}(\alpha), \mathrm{p}(\alpha) \rangle = \langle \mathrm{lc}(\beta), \mathrm{c}(\beta), \mathrm{p}(\beta) \rangle.$$

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Elementary equivalence of ordinals. Since the topology on ordinals is expressed purely in terms of the order relation, it might be guessed that homeomorphism of ordinals could be related to definability in terms of the first order language $\mathcal{L}_\in$ of set theory. But this is not the case.

To illustrate how homeomorphic ordinals can disagree about $\mathcal{L}_\in$-sentences, consider the following formulas that describe that x is a limit ordinal or limit of limit ordinals, respectively:

$$\begin{array}{rcl} \mathrm{L}(x) & \doteq & \forall y(y \in x \to \exists z(y \in z \wedge z \in x)) \\ \mathrm{L}^2(x) & \doteq & \forall y(y \in x \to \exists z(\mathrm{L}(z) \wedge y \in z \wedge z \in x)) \end{array}$$

Then the ordinals $\omega^2+\omega+1$ and ω^2+1 (which are homeomorphic by the function

$$\pi : \begin{array}{ccll} \xi & \mapsto & \omega+1+\xi & \text{for } \xi \leq \omega^2+1 \\ \omega^2+\xi & \mapsto & \xi & \text{for } \xi \leq \omega+1. \end{array}$$

as proved in our main theorem) do not have the same theories:

$$\begin{array}{rccl} \omega^2+\omega+1 & \models & & \exists x \exists y(x \in y \wedge \mathrm{L}^2(x) \wedge \mathrm{L}(y)), \text{ but} \\ \omega^2+1 & \models & \neg & \exists x \exists y(x \in y \wedge \mathrm{L}^2(x) \wedge \mathrm{L}(y)). \end{array}$$

Also the converse is not true: elementarily equivalent ordinals need not be homeomorphic. For this, we need the following result of Mostowski and Tarski:

Theorem 1 (Mostowski-Tarski)**.** Two ordinals $\alpha < \beta$ are elementarily equivalent if $\alpha \geq \omega^\omega$ and there is some δ such that $\beta = \omega^\omega \cdot \delta + \alpha$. [DoMoTa78, Corollary 44]

Now, ω^ω and $\omega^\omega \cdot 2$ are elementarily equivalent by Theorem 1, but not homeomorphic by our main theorem.

Proof of the main theorem. The rest of the paper will be devoted to the proof of the main theorem.

Clearly, the main theorem holds if one of the ordinals is finite: the characteristic data for α are $\langle 0, n, 1\rangle$ if and only if $\alpha = n$ and finite ordinals are homeomorphic if and only if they are the same. From now on, we shall restrict ourselves to infinite ordinals.

Lemma 2. Let X be a topological space and let A and B be two open disjoint subsets of X such that $A \cup B = X$. Then X and $A \oplus B$ are homeomorphic where $\oplus$ denotes the direct sum of topological spaces.

Corollary 3. Let ξ and η be ordinals. If ξ and η are successor ordinals, then $\xi + \eta \cong \eta + \xi$.

Proposition 4. Every infinite ordinal α is homeomorphic to $\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \mathrm{p}(\alpha)$.

Proof. As mentioned, without loss of generality, we have $\alpha \geq \omega$, so $\mathrm{lc}(\alpha) \neq 0$. We consider three cases: α is a pure limit, a successor, or a impure limit ordinal.

If α is a pure limit ordinal, then $\mathrm{p}(\alpha) = 0$ and so $\alpha = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \mathrm{p}(\alpha)$ and we have nothing to show.

If α is a successor ordinal, then without loss of generality, we have $\alpha = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha)+1+\beta+1$ for some $\beta < \omega^{\mathrm{lc}(\alpha)}$. By Corollary 3, we get $\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha)+1+\beta+1 \cong \beta+1+\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha)+1 = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha)+1 = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha)+\mathrm{p}(\alpha)$.

If α is an impure limit ordinal, then $\mathrm{p}(\alpha) \geq \omega$. Therefore,

$$\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \beta + \mathrm{p}(\alpha) = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \beta + 1 + \mathrm{p}(\alpha).$$

We use the same idea as in the successor case, and apply it twice. First split the whole space into two pieces:

$$\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \beta + 1 + \mathrm{p}(\alpha) \cong (\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \beta + 1) \oplus \mathrm{p}(\alpha).$$

Then normalize the first piece: $\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \beta + 1 \cong \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + 1$. Finally, put the pieces back together:

$$(\omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + 1) \oplus \mathrm{p}(\alpha) \cong \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + 1 + \mathrm{p}(\alpha) = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \mathrm{p}(\alpha).$$

□

If $\langle \mathrm{lc}(\alpha), \mathrm{c}(\alpha), \mathrm{p}(\alpha)\rangle = \langle \mathrm{lc}(\beta), \mathrm{c}(\beta), \mathrm{p}(\beta)\rangle$, then using Proposition 4 twice we have

$$\alpha \cong \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \mathrm{p}(\alpha) = \omega^{\mathrm{lc}(\beta)} \cdot \mathrm{c}(\beta) + \mathrm{p}(\beta) \cong \beta.$$

This proves the sufficiency of the invariants.

The *Cantor–Bendixson derivative* of a topological space X is the set X' of all limit points in X [Je03, p. 40]. Using transfinite recursion, the iterated derivatives are defined by

(1) $X^{(0)} := X$,
(2) $X^{(\alpha+1)} := (X^{(\alpha)})'$, and
(3) $X^{\delta} := \bigcap_{\alpha<\delta} X^{(\alpha)}$ for limit ordinals δ.

We define the *Cantor–Bendixson rank* of a point $x \in X$ as

$$\mathrm{CB}_X(x) := \sup\{\alpha \in \mathrm{On}\,;\, x \in X^{(\alpha)}\},$$

where we set $\sup \mathrm{On} := \infty$. Note that the Cantor–Bendixson rank of a point is a homeomorphic invariant.

Furthermore, in the special case of ordinal topologies, if $\beta < \alpha_0 < \alpha_1$, then $\mathrm{CB}_{\alpha_0}(\beta) = \mathrm{CB}_{\alpha_1}(\beta)$. We may therefore drop the subscript in our case. Moreover, if $\alpha = \omega^{\alpha_0} \cdot k_0 + \cdots + \omega^{\alpha_n} \cdot k_n$ in Cantor normal form, then $\mathrm{CB}(\alpha) = \alpha_n$ and in particular $\mathrm{CB}(\alpha) < \infty$.

Lemma 5. Let β be an ordinal, k_0 and k_1 positive natural numbers and $0 < \gamma_0, \gamma_1 < \omega^\beta$.

(1) If $\omega^\beta \cdot k_0$ and $\omega^\beta \cdot k_1$ are homeomorphic, then $k_0 = k_1$.
(2) If $\omega^\beta \cdot k_0 + \gamma_0$ and $\omega^\beta \cdot k_1 + \gamma_1$ are homeomorphic, then $k_0 = k_1$.
(3) If $\beta^* > \beta$, then no ordinal $\alpha \geq \omega^{\beta^*}$ is homeomorphic to $\omega^\beta \cdot k_0 + \gamma_0$ or $\omega^\beta \cdot k_0$.

Proof. This is a simple counting argument. We count the number of points ξ with $\mathrm{CB}(\xi) = \beta$: the ordinal $\omega^\beta \cdot k_0$ has $k_0 - 1$ such points, $\omega^\beta \cdot k_1$ has $k_1 - 1$ such points, $\omega^\beta \cdot k_0 + \gamma_0$ has k_0 such points, $\omega^\beta \cdot k_1 + \gamma_1$ has k_1 such points, and $\alpha \geq \omega^{\beta^*}$ has infinitely many such points. □

Lemma 6. Let $\xi < \alpha$ and β be ordinals. Then:

(1) Any infinite $X \subseteq \xi$ has a limit point in α (*viz.*, $\bigcup X \leq \xi < \alpha$).
(2) If $\pi : \alpha \to \beta$ is a homeomorphism and $\zeta \in \alpha$ is a limit point of $Z \subseteq \alpha$, then $\pi[Z] \subseteq \beta$ has a limit point $\pi(\zeta)$.

We shall now generalize the simple idea of Lemma 6 to a technical notion that we shall need in our proof. Let η be a limit ordinal and ζ an arbitrary ordinal. If $C \subseteq \eta$ is a cofinal subset of η, we call a function $S : C \to \zeta$ a **$\langle C, \zeta\rangle$-slope** if $\mathrm{CB}(S(\gamma)) = \gamma$ for all $\gamma \in C$. For $\delta < \eta$, we set $C_\delta := \{\gamma \in C\,;\, \delta < \gamma\}$. An ordinal

$\tau \in \zeta$ is called a **top** of the slope S if for every $\delta < \eta$, the ordinal τ is a limit point of the set $S_\delta := \{S(\gamma)\,;\, \gamma \in C_\delta\}$. As a consequence, $\mathrm{CB}(\tau) = \eta$, and therefore, τ cannot be an element of the slope. Of course, the top of a slope need not be unique: the $\langle \omega, \omega^\omega \cdot 2 + 1\rangle$-slope defined by

$$\begin{aligned} S^0(2n) &:= \omega^{2n} \\ S^0(2n+1) &:= \omega^\omega + \omega^{2n+1} \end{aligned}$$

has both ω^ω and $\omega^\omega \cdot 2$ as tops. Also, the idea of Lemma 6 (1) of taking the union of (the range of) a slope in order to get the top does not work: if $S : \eta\backslash\{0\} \to \zeta$ is any slope and δ is bigger than all elements of the range of S, then S^1 defined by

$$\begin{aligned} S^1(0) &:= \delta + 1 \\ S^1(\xi) &:= S(\xi) \text{ if } \xi \neq 0 \end{aligned}$$

has the property that the union of the range of S^1 is just $\delta + 1$ which is not a top.

For a given $\langle C, \zeta\rangle$-slope S and $\delta < \eta$, we define $\sigma_\delta := \bigcup\{S(\gamma)\,;\, \gamma \in C_\delta\}$. We call a slope **cofinal** if $\delta \mapsto \sigma_\delta$ is a constant function. In that case, we call the constant value of this function the **supremum** of S. Clearly, if $\delta < \delta'$, then $\sigma_\delta \geq \sigma_{\delta'}$; therefore, even if S is not a cofinal slope, there must be some σ and $\delta_0 < \eta$ such that for all $\delta > \delta_0$, we have $\sigma_\delta = \sigma$. Then $\widehat{S} := S{\restriction}C_{\delta_0}$ is a cofinal slope.

Lemma 7. Let η be a limit ordinal, C cofinal in η, and ζ an arbitrary ordinal. If S is a cofinal $\langle C, \zeta\rangle$-slope with supremum $\sigma \in \zeta$, then σ is a top of S.

Proof. Fix $\delta < \eta$ and fix a neighbourhood $B := \mathsf{right}_\zeta(\vartheta) \cap \mathsf{left}_\zeta(\sigma + 1)$ of σ (so $\vartheta < \sigma$). Since $\sigma_\delta = \bigcup\{S(\gamma)\,;\, \gamma \in C_\delta\} = \sigma$, there is some $\delta < \gamma \in C$ such that $S(\gamma) > \vartheta$. □

Clearly, if π is a homeomorphism between ζ and ζ' and S is a $\langle C, \zeta\rangle$-slope, then $\pi \circ S$ is a $\langle C, \zeta'\rangle$-slope, and by Lemma 6 (2), if $\tau \in \zeta$ is a top of S, then $\pi(\tau)$ is a top of $\pi \circ S$.

Proposition 8. If $\alpha \cong \beta$, then $\langle \mathrm{lc}(\alpha), \mathrm{c}(\alpha), \mathrm{p}(\alpha)\rangle = \langle \mathrm{lc}(\beta), \mathrm{c}(\beta), \mathrm{p}(\beta)\rangle$.

Proof. It suffices by Proposition 4 to consider the situation where

$$\alpha = \omega^{\mathrm{lc}(\alpha)} \cdot \mathrm{c}(\alpha) + \mathrm{p}(\alpha) \cong \omega^{\mathrm{lc}(\beta)} \cdot \mathrm{c}(\beta) + \mathrm{p}(\beta) = \beta.$$

Lemma 5 immediately shows that $\mathrm{lc}(\alpha) = \mathrm{lc}(\beta)$ and $\mathrm{c}(\alpha) = \mathrm{c}(\beta)$. We are left to show that $\mathrm{p}(\alpha) = \mathrm{p}(\beta)$. Assume towards a contradiction that $\omega^{\alpha^*} = \mathrm{p}(\alpha) > \mathrm{p}(\beta) = \omega^{\beta^*}$. Let us write $\alpha = \widehat{\alpha} + \omega^{\alpha^*}$ and $\beta = \widehat{\beta} + \omega^{\beta^*}$.

Case 1. $\alpha^* = \gamma + 1$. Then $\beta^* \leq \gamma$, and thus ω^{β^*} does not contain any points of Cantor-Bendixson rank γ. Therefore all such points in β must be below $\widehat{\beta}$. As a consequence, by Lemma 6 (1), every infinite set of points of Cantor-Bendixson rank γ in β has a limit point.

Now in α, consider the set $\{\widehat{\alpha} + \omega^\gamma \cdot n\,;\, n \in \omega\}$. This is an infinite set of points of Cantor-Bendixson rank γ without a limit point in α. Lemma 6 (2) shows that α and β cannot be isomorphic.

Case 2. α^* is a limit ordinal. We use our notion of slope here. Obviously, the function $S^*(\xi) := \widehat{\alpha} + \omega^\xi$ is an $\langle \alpha^*, \alpha\rangle$-slope with the property that for any $\delta < \alpha^*$, the slope $S^*{\restriction}C_\delta$ does not have a top (in α).

If α and β are homeomorphic by some function π, then $\pi \circ S^*$ is a $\langle \alpha^*, \beta \rangle$-slope. Let $\widehat{S} = \pi \circ S^* \upharpoonright C_\delta$ be a cofinal subslope of $\pi(S^*)$. Note that since $\beta^* < \alpha^*$, the range of $\widehat{S}$ is contained in $\widehat{\beta}$. Therefore, by Lemma 7, $\widehat{S}$ has a top in $\widehat{\beta} + 1 < \beta$. But then $\pi^{-1} \circ \widehat{S} = S^* \upharpoonright C_\delta$ has a top in α contradicting the definition of S^*. $\square$

Classification up to Borel isomorphism. The simple classification of ordinal topologies suggests asking the same question for the Borel structure of ordinals. The order topology on an ordinal α generates a Borel σ-algebra. A bijection between ordinals is a **Borel isomorphism** if the both the preimage and the image of every Borel set is Borel. Can we give a simple classification of the Borel isomorphism types of ordinals?

The second author had posed this question in 2001. In the spring of 2006, it was solved by Su Gao and Steve Jackson. The following is the result of Gao and Jackson (unpublished): If α is an ordinal then define b_α as follows: The ordinal $\mathrm{b}_\alpha := 0$ if $\mathrm{Card}(\alpha)$ is singular. If $\mathrm{Card}(\alpha) = \kappa$ is a regular cardinal, then let b_α be the largest cardinal $\lambda \leq \kappa$ such that $\kappa \cdot \lambda \leq \alpha$. Then the pair

$$\langle \mathrm{Card}(\alpha), \mathrm{b}_\alpha \rangle$$

is a Borel isomorphic invariant of the ordinal, *i.e.*, α and β are Borel isomorphic if and only if $\langle \mathrm{Card}(\alpha), \mathrm{b}_\alpha \rangle = \langle \mathrm{Card}(\beta), \mathrm{b}_\beta \rangle$.

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