ESSENTIALLY Σ_1 FORMULAE IN ΣL

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ABSTRACT. The essentially Σ_1 formulae of ΣL are exactly those which are provably equivalent to a disjunction of conjunctions of $\Box$ and Σ_1 formulae.

1. Introduction and preliminaries

1.1. Fix some recursively enumerable theory of arithmetic T extending Peano arithmetic (PA). Let A be a formula of $\mathcal{L}_{\Sigma \mathrm{L}}$, the propositional modal language containing modalities $\Box$ and Σ_1. An arithmetical realization $*\colon \mathcal{L}_{\Sigma \mathrm{L}} \to \mathcal{L}_T$ of modal $\mathcal{L}_{\Sigma \mathrm{L}}$-formulae in the arithmetical theory T is a translation which assigns sentences of T to propositional letters, commutes with Booleans and translates $\Box A$ and $\Sigma_1 A$ to 'there is a T-proof for A^*' and 'A^* is equivalent-in-T to a Σ_1-formula' respectively. We say A is *essentially* Σ_1 *with respect to* T if for every arithmetical realization $*\colon \mathcal{L}_{\Sigma \mathrm{L}} \to \mathcal{L}_T$, A^* is (T-equivalent to) a Σ_1 formula of T (note the difference between this metalogical property and the intended interpretation of modal formulae $\Sigma_1 A$). In standard provability logic GL, the essentially Σ_1 formulae are exactly those which are equivalent to a disjunction of boxed formulae. Below we will prove an analogous result for the logic ΣL. In §1.2 we will motivate the classification of essentially Σ_1 formulae. In §1.3 we briefly touch upon ΣL and ΣILM, the modal logics we are concerned with, before we prove the theorem proper in §2. Final remarks, historical notes and acknowledgements can be found in §3.

1.2. One way to think of the classification of the essentially Σ_1 formulae of a modal logic L (with language $\mathcal{L}$) with respect to an arithmetical theory T is to liken it to arithmetical completeness: if

$$\{A \in \mathcal{L} \mid \forall * \left(T \vdash A^*\right)\} = \{A \in \mathcal{L} \mid \vdash_{\mathrm{L}} A\},$$

we say that L is arithmetically complete (and sound) with respect to T. On the left hand side we have a set of (modal) formulae characterized arithmetically and on the right hand side we have a set of formulae characterized modally. Similarly, we may arithmetically define a set

$$\{A \in \mathcal{L} \mid \forall * \left(A^* \text{ is } (T\text{-equivalent to) a } \Sigma_1 \text{ formula of } T\right)\},$$

and wonder if we can define it modally. This question was first asked as a conjecture by Guaspari for L=R and T=PA in [5] and first solved by Visser for L=GL in [8]. The question has later been answered for several other logics (including R), see §3 for a brief overview.

1.3. ΣL is an extension of GL, and the language of ΣL is that of GL extended with a unary modality Σ_1. The intended arithmetical interpretation of the modal formula $\Sigma_1 A$ is 'the interpretation of A is T-equivalent to a Σ_1 formula'. It happens that for our present purposes, we are not at all interested in the modal semantics of ΣL, but those of its bigger sibling ΣILM instead. The language of ΣILM has three modalites; a unary $\Box$ (provability) and Σ_1 (being a Σ_1 formula) and a binary $\rhd$ with several possible interpretations, see below. ΣILM frames are ILM-frames with

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an extended forcing relation. An ILM frame is a triple $\langle W, R, S\rangle$, where $\langle W, R\rangle$ is a Kripke-frame for GL (i.e. R is transitive and conversely well-founded) and S is a ternary relation; we usually write uS_wv for $(w, u, v) \in S$, treating S as a collection of binary relations indexed by the set of worlds. For all $w \in W$, S_w is required to be reflexive and transitive. (Abusing language we will sometimes write S when we mean the binary relation $\bigcup_{w\in W} S_w$.) The $\Box$ modality is interpreted using the relation R:

$$M, w \Vdash \Box A \text{ iff } \forall w' \text{ s.t. } wRw',\ w' \Vdash A.$$

Things are more complicated for $\rhd$:

$$M, w \Vdash A \rhd B \text{ iff } \forall w' \text{ s.t. } wRw' \text{ and } w' \Vdash A, \text{ there is } w'' \text{ s.t. } w'S_ww'' \text{ and } w'' \Vdash B.$$

Finally, the forcing relation for fomulae of the form $\Sigma_1 A$, which is the main novelty, is

$$M, w \Vdash \Sigma_1 A \text{ iff } \forall u, v, w' \text{ s.t. } w(R \cup S)^*w' \text{ and } uS_{w'}v,\ M, u \Vdash A \Rightarrow M, v \Vdash A.$$

The reason we presently introduce ΣILM is that it has been proven to be arithmetically complete with respect to (r.e. theories extending) PA, with $A \rhd B$ being interpreted as 'PA + B is Π_1-conservative over PA + A'; in other words, ΣILM is (contains) the logic of Π_1-conservativity over PA. We will make use of this below by negating Σ_1 formulae, thus making them Π_1.

The idea to extend GL with an operator for Σ_1-ness is due to Japaridze, who introduced a logic of provability extended with modalities for Boolean combinations of Σ_n formulae for all $n \in \mathbb{N}$ (see [1]). For a proper exposition about ΣL and ΣILM we refer the reader to Goris ([3]), where one will also find both modal and arithmetical soundness and completeness results. For an introduction to GL and ILM, one may consult Japaridze and De Jongh's [2].

2. The theorem proper

We define

$$\mathsf{C} := \Big\{ \bigwedge_{0\le i<n} B_i \mid n \in \mathbb{N},\ \text{and either } B_i \equiv \Box D_i \text{ or } B_i \equiv \Sigma_1 D_i \text{ for some } D_i \in \mathcal{L}_{\Sigma\mathrm{L}} \Big\},$$

and $\mathsf{C}_\vee := \{\bigvee_{0\le i<n} C_i \mid n \in \mathbb{N}, C_i \in \mathsf{C}\}$. Observe that not only is $\mathsf{C}_\vee$ closed under disjunctions, but (up to equivalence) also under conjunctions: suppose

$$(A \wedge B) \vee (C \wedge D), (E \wedge F) \vee (G \wedge H) \in \mathsf{C}_\vee,$$

then by distributivity of $\wedge$ over $\vee$ and associativity of $\wedge$, their conjunction is equivalent to

$$(A \wedge B \wedge E \wedge F) \vee (A \wedge B \wedge G \wedge H) \vee (C \wedge D \wedge E \wedge F) \vee (C \wedge D \wedge G \wedge H),$$

which is again a member of $\mathsf{C}_\vee$. Additionally, we have $\top, \bot \in \mathsf{C}_\vee$.

Theorem 2.1. *Let A be a formula of $\mathcal{L}_{\Sigma\mathrm{L}}$. Then A is essentially Σ_1 w.r.t. T iff there exists $\tilde{A} \in \mathsf{C}_\vee$ with $\vdash_{\Sigma\mathrm{L}} A \leftrightarrow \tilde{A}$.*

We will break the proof of this main theorem down into several lemmata.

Lemma 2.2. *If $A \in \mathsf{C}_\vee$, then A is essentially Σ_1 w.r.t. T.*

Proof. Let $*\colon \mathcal{L}_{\Sigma\mathrm{L}} \to \mathcal{L}_{\mathrm{T}}$ be an arithmetical realization. First of all, both $(\Box B)^*$ and $(\Sigma_1 B)^*$ are Σ_1 for any ΣL formula B. Secondly, a conjunction of Σ_1 formulae is again Σ_1, so any element of C is Σ_1. Since a disjunction of Σ_1 formulae is also Σ_1, we conclude that any element of $\mathsf{C}_\vee$ has a Σ_1 interpretation. Since $*$ was arbitrary, any element of $\mathsf{C}_\vee$ must be essentially Σ_1 with respect to T. $\Box$

The right to left direction of Theorem 2.1 follows from Lemma 2.2. We prove the other direction by contraposition. First, we show that if A does not have the desired shape, we can find two ΣILM-maximal consistent sets, one containing A, the other containing $\neg A$, with a $\subseteq_{\Box,\Sigma}$-relation between them (see below). Next, we use these MCSs to create a ΣILM model invalidating a special ΣILM formula (containing A). Finally, we show that if this special formula is not a theorem of ΣILM (which it cannot be, by modal soundness of ΣILM), then A cannot be essentially Σ_1.

The lemma below was originally Lemma 7.9 ('the Σ-Lemma') in [4]; our proof extends that of Goris and Joosten. We use the following notation: if X, Y are sets of formulas we say that $X \subseteq_{\Box,\Sigma_1} Y$ if $\Box B \in X \Rightarrow \Box B \in Y$ and $\Sigma_1 B \in X \Rightarrow \Sigma_1 B \in Y$ for all $B \in \mathcal{L}_{\Sigma\mathrm{L}}$.

Lemma 2.3. *Let A be a ΣL formula such that for no $\tilde{A} \in \mathsf{C}_\vee$ we have $\vdash_{\Sigma\mathrm{ILM}} A \leftrightarrow \tilde{A}$. Then there exist ΣILM-maximal consistent sets Γ_0, Γ_1 such that $A \in \Gamma_0 \subseteq_{\Box,\Sigma_1} \Gamma_1 \ni \neg A$.*

Proof. Assume that A is not equivalent to any member of $\mathsf{C}_\vee$. First, we define

$$\mathcal{C}_{\mathrm{con}} := \{Y \subseteq \mathsf{C}_\vee \mid \{\neg A\} + Y \text{ is } \Sigma\text{ILM-consistent and maximally such}\}.$$

(Note that if $\mathcal{C}_{\mathrm{con}}$ is empty, then $\vdash_{\Sigma\mathrm{ILM}} A \leftrightarrow \top$, contradicting our assumption about A.) A useful property of elements Y of $\mathcal{C}_{\mathrm{con}}$ is that

$$B \vee C \in Y \text{ implies } B \in Y \text{ or } C \in Y. \tag{2.1}$$

For if $Y \in \mathcal{C}_{\mathrm{con}}$ and $B \vee C \in Y$, then $B, C \in \mathsf{C}_\vee$. If neither B nor C were consistent with $\{\neg A\} + Y$, then we would have $\neg A + Y \vdash_{\Sigma\mathrm{ILM}} \neg B \wedge \neg C$ and $\{\neg A\} + Y$ would be inconsistent. So either B or C must be consistent with $\{\neg A\} + Y$, whence by Y's maximality, $B \in Y$ or $C \in Y$. We conclude that (2.1) holds.

Claim 1. For some $Y \in \mathcal{C}_{\mathrm{con}}$, the set $\{A\} + \{\neg\sigma \mid \sigma \in \mathsf{C}_\vee \setminus Y\}$ is consistent.

From the assumption that the claim is false we will derive a contradiction to our initial assumption about A. If the claim is false, then for each $Y \in \mathcal{C}_{\mathrm{con}}$ there is some finite $Y^{\mathrm{fin}} \subseteq \mathsf{C}_\vee \setminus Y$ such that $\{A\} + \{\neg\sigma \mid \sigma \in Y^{\mathrm{fin}}\}$ is inconsistent. Therefore,

$$\text{for each } Y \in \mathcal{C}_{\mathrm{con}} \text{ there is } Y^{\mathrm{fin}} \subseteq \mathsf{C}_\vee \setminus Y \text{ s.t. } \vdash_{\Sigma\mathrm{ILM}} A \to \bigvee_{\sigma \in Y^{\mathrm{fin}}} \sigma. \tag{2.2}$$

Next, we will show that

$$\{\neg A\} + \{\bigvee_{\sigma \in Y^{\mathrm{fin}}} \sigma \mid Y \in \mathcal{C}_{\mathrm{con}}\} \text{ is inconsistent.} \tag{2.3}$$

(Note that the 'right half' of the set above is a subset of $\mathsf{C}_\vee$.) For if this set is consistent then that fact must be witnessed by some $S \in \mathcal{C}_{\mathrm{con}}$ such that $\{\bigvee_{\sigma \in Y^{\mathrm{fin}}} \sigma \mid Y \in \mathcal{C}_{\mathrm{con}}\} \subseteq S$. Now we are in a case of fatal self-reference, because this means that in particular $\bigvee_{\sigma \in S^{\mathrm{fin}}} \sigma \in S$, so by (2.1), we have $\sigma \in S$ for some $\sigma \in S^{\mathrm{fin}}$, contradicting the fact that $S^{\mathrm{fin}} \subseteq \mathsf{C}_\vee \setminus S$. We conclude that (2.3) holds.

There must be some finite $\mathcal{C}^{\mathrm{fin}}_{\mathrm{con}} \subseteq \mathcal{C}_{\mathrm{con}}$ witnessing this situation, so we get

$$\vdash_{\Sigma\mathrm{ILM}} \Big(\bigwedge_{Y \in \mathcal{C}^{\mathrm{fin}}_{\mathrm{con}}} \bigvee_{\sigma \in Y^{\mathrm{fin}}} \sigma \Big) \to A.$$

As a consequence of (2.2), we also get

$$\vdash_{\Sigma\mathrm{ILM}} A \to \Big(\bigwedge_{Y \in \mathcal{C}^{\mathrm{fin}}_{\mathrm{con}}} \bigvee_{\sigma \in Y^{\mathrm{fin}}} \sigma \Big).$$

Combining the above two results we get

$$\vdash_{\Sigma\mathrm{ILM}} A \leftrightarrow \Big(\bigwedge_{Y \in \mathcal{C}^{\mathrm{fin}}_{\mathrm{con}}} \bigvee_{\sigma \in Y^{\mathrm{fin}}} \sigma \Big).$$

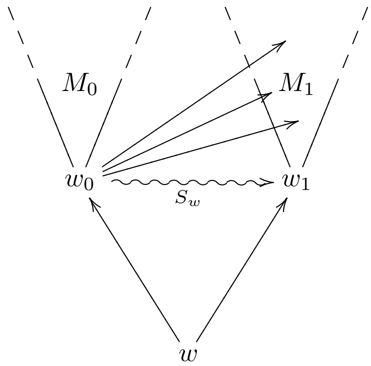

FIGURE 1. Our new model M

Because $\mathsf{C}_\vee$ is closed under disjunctions, for every $Y \in \mathcal{C}^{\text{fin}}_{\text{con}}$ we have $\bigvee_{\sigma \in Y^{\text{fin}}} \sigma \in \mathsf{C}_\vee$. Because $\mathsf{C}_\vee$ is also closed under conjunctions, we conclude that A is equivalent to a member of $\mathsf{C}_\vee$, contradicting the initial assumption of the lemma. Therefore, Claim 1 must be true.

If we take Y to be a set witnessing the truth of the claim above, we find that both $\{A\} + \{\neg\sigma \mid \sigma \in \mathsf{C}_\vee \setminus Y\}$ and $\{\neg A\} + Y$ are consistent, so by Lindenbaum's Lemma, there exist MCS's Γ_0 and Γ_1 extending $\{A\} + \{\neg\sigma \mid \sigma \in \mathsf{C}_\vee \setminus Y\}$ and $\{\neg A\} + Y$, respectively. Also, if $B \equiv \Box C$ or $B \equiv \Sigma_1 C$ for some $C \in \mathcal{L}_{\Sigma\mathrm{L}}$ and $B \notin \Gamma_1$, then $B \in \mathsf{C}_\vee \setminus Y$, whence $\neg B \in \Gamma_0$, so by consistency of Γ_0, $B \notin \Gamma_0$. It follows that $\Gamma_0 \subseteq_{\Box,\Sigma_1} \Gamma_1$. □

Before we proceed, we would like to point out a few subtleties. First of all, because ΣILM is conservative over ΣL (Theorem 4.11 in [3]), it follows that if there is no $\tilde{A} \in \mathsf{C}_\vee$ such that $\vdash_{\Sigma\mathrm{L}} A \leftrightarrow \tilde{A}$, then there can be no $\tilde{A} \in \mathsf{C}_\vee$ for which $\vdash_{\Sigma\mathrm{ILM}} A \leftrightarrow \tilde{A}$. Secondly, although we have two ΣILM-MCS's Γ_0 and Γ_1 that are $\subseteq_{\Box,\Sigma_1}$-related, this relation itself only pertains to ΣL formulae. We continue with the main argument.

Let $\mathrm{Sub}(A)$ denote the (finite) set of all subformulas of A, and define $\neg X := \{B, \neg B \mid B \in X\}$. We now want to constuct models based on Γ_0 and Γ_1. Because of the compactness failure of GL, we will reduce Γ_i to $\Gamma_i^A := \Gamma_i \cap \neg\,\mathrm{Sub}(A)$ for $i = 0, 1$. If it were the case that $\vdash_{\Sigma\mathrm{ILM}} \neg \bigwedge \Gamma_i^A$, then Γ_i^A would be inconsistent which is impossible, since $\Gamma_i^A \subseteq \Gamma_i$ which is an MCS. Therefore we conclude that $\nvdash_{\Sigma\mathrm{ILM}} \neg \bigwedge \Gamma_i^A$, so that by completeness of ΣILM (Theorem 3.13 of [3]), there exists a model M_i with root w_i such that $M_i, w_i \Vdash \bigwedge \Gamma_i^A$. Since $A \in \Gamma_0^A$ and $\neg A \in \Gamma_1^A$, this gives us $M_0, w_0 \Vdash A$ and $M_1, w_1 \Vdash \neg A$. Additionally we may require that for certain fresh variables p and q, we have $[\![\, p \,]\!]^{M_0} = \{w_0\}$, $[\![\, q \,]\!]^{M_1} = \{w_1\}$ and $[\![\, p \,]\!]^{M_1} = [\![\, q \,]\!]^{M_0} = \emptyset$ (p and q mark w_0 and w_1 respectively, so to say).

Claim 2. $\nvdash_{\Sigma\mathrm{ILM}} p \rhd q \to p \wedge A \rhd q \wedge A$.

The idea of our proof is to glue $M_0 = \langle W_0, R_0, S_0 \rangle$ and $M_1 = \langle W_1, R_1, S_1 \rangle$ together into a ΣILM model $M = \langle W, R, S \rangle$ with root w, so that $M, w \nVdash p \rhd q \to p \wedge A \rhd q \wedge A$, which proves the lemma by the soundness of ΣILM.

M will be the disjoint union of M_0 and M_1 with a new root w below w_0 and w_1. To that end, we extend the relation R with wRw_0 and wRw_1 to attach the new root. To accomodate an S_w-arrow we will add in a minute, we also add $w_0 R s$ for all $s \in W_1 \setminus \{w_1\}$ (see Figure 1). With these new links in place, we add the R-links required to keep R transitive. This ensures that $\langle W, R \rangle$ is a GL-frame.

Now as promised, we add the new connection $w_0 S_w w_1$. To turn our contraption into a ΣILM-frame, we must also add $sS_w t$ for all s, t such that $wRsRt$ and $uS_{w_0}v$ for all u, v such that $uS_{w_1}v$. After all this, we close S off under reflexive steps. This completes the construction of our new ΣILM model M. It is important to remember that not only have we added R, S_w and S_{w_0}-links, but indirectly we have also added $(R \cup S)^*$-links (which are not directly visible, since this structure resides not in the frame but in the forcing relation). So, if we look closely, we see that no new $(R \cup S)^*$-links have been added in the part of M that is a copy of M_1: this we will need below.

We now want to show that $M, w_0 \Vdash A$ and $M, w_1 \Vdash \neg A$. Since w_1 does not see any new worlds (not even via the new $(R \cup S)^*$-relation, as discussed above), we must have that for all $B \in \neg\,\mathrm{Sub}(A)$, $M_1, w_1 \Vdash B$ implies $M, w_1 \Vdash B$ (so in particular, $M, w_1 \Vdash \neg A$). For w_0, we will show by induction on formula construction that for all $B \in \neg\,\mathrm{Sub}(A)$ and all $s \in W_0$, we have $M, s \Vdash B$ iff $M_0, s \Vdash B$ (it will then follow that $M, w_0 \Vdash A$). The case for propositional variables and Boolean connectives is immediate, so we will turn to the case that $B \equiv \Box C$. If $s \neq w_0$, then s sees the same worlds in both M and M_0, and it follows that $M, s \Vdash \Box C$ iff $M_0, s \Vdash \Box C$. Now suppose that $s = w_0$ and $M_0, w_0 \Vdash \Box C$. Then if $w_0 R t$ (in M), either $t \in W_0$ or $t \in W_1$. In the former case, we know by induction hypothesis that $M, s \Vdash C$. In the latter case, we use the fact that $M_1, w_1 \Vdash \Box C$ (remember the last condition of the lemma) whence $M, w_1 \Vdash \Box C$, so since it must be that $w_1 R t$, we also get $M, t \Vdash C$, so since t was arbitrary, $M, w_0 \Vdash \Box C$. Conversely if $M, w_0 \Vdash \Box C$, then for all t s.t. $w_0 R t$ we have $M, t \Vdash C$, so in particular for all $t \in W_0 \setminus \{w_0\}$, whence $M_0, w_0 \Vdash \Box C$. Finally if $B \equiv \Sigma_1 C$ and $s \neq w_0$, we again immediately get $M, s \Vdash \Sigma_1 C$ iff $M_0, s \Vdash \Sigma_1 C$. Now suppose that $s = w_0$, and assume that $M_0, w_0 \Vdash \Sigma_1 C$, $w_0 (R \cup S)^* w'$ and and $uS_{w'}v$ with $M, u \Vdash C$. If $w' \in W_0 \setminus \{w_0\}$, then $u, v \in W_0$. Since $M_0, v \Vdash C$, by induction hypothesis we get $M, v \Vdash C$. If $w' = w_0$, then either $u, v \in W_0$ or $u, v \in W_1$. In the former case it again follows from $M_0, w_0 \Vdash \Sigma_1 C$ that $M, v \Vdash C$. In the latter case, we need the last condition of the lemma again, which gives us $M_1, w_1 \Vdash \Sigma_1 C$, so also $M, w_1 \Vdash \Sigma_1 C$. Since the connection $uS_{w_0}v$ can only be in place because $uS_{w_1}v$, $M, v \Vdash C$ follows from $M, w_1 \Vdash \Sigma_1 C$. If $w' \in W_1$, then $w_1 (R \cup S)^* w'$, so it again follows from $M, w_1 \Vdash \Sigma_1 C$ that $M, v \Vdash C$. Conversely, if $M, w_0 \Vdash \Sigma_1 C$ then it follows that $M_0, w_0 \Vdash \Sigma_1 C$ as above with $\Box C$. This completes our induction, and as a consequence we get $M, w_0 \Vdash A$.

Now we have $M, w \Vdash p \rhd q$ (since p is only true at w_0 and $w_0 S_w w_1 \Vdash q$). However, $M, w \nVdash p \wedge A \rhd q \wedge A$, which is witnessed by w_0: we have wRw_0 and $M, w_0 \Vdash p \wedge A$. Towards a contradiction, suppose that there is t such that $w_0 S_w t$ and $M, t \Vdash q \wedge A$. Since $[\![\, q \,]\!]^M = \{w_1\}$, it follows that $t = w_1$, but $M, w_1 \Vdash \neg A$. We conclude that for no t, $w_0 S_w t \Vdash q \wedge A$, whence $M, w \nVdash p \wedge A \rhd q \wedge A$,whence $M, w \nVdash p \rhd q \to p \wedge A \rhd q \wedge A$. It follows that Claim 2 holds.

Lemma 2.4. *Let A be a ΣL formula. If $\nvdash_{\Sigma\mathrm{ILM}} p \rhd q \to p \wedge A \rhd q \wedge A$ for certain propositional letters p, q, then A is not essentially Σ_1 w.r.t. T.*

Proof. We reason by contraposition. Assume that A is essentially Σ_1 with respect to T. Since T extends PA, we know that ΣILM is the logic of Π_1-conservativity over T (by Theorem 4.3 of [3]). Let $*$ be an arbitrary arithmetical realization. Reasoning in T, we will argue that $(p \rhd q)^*$ implies $(p \wedge A \rhd q \wedge A)^*$. Assume $(p \rhd q)^*$, i.e. that $T + q^*$ is Π_1-conservative over $T + p^*$. We want to show that $T + q^* + A^*$ is Π_1-conservative over $T + p^* + A^*$. Let π be an arbitrary Π_1 formula of T, then from $T + p^* + A^* \vdash \pi$ it follows that $T + p^* \vdash A^* \to \pi$. Because A is essentially Σ_1 by assumption, $A^* \to \pi$ is Π_1, whence by Π_1-conservativity of $T + q^*$ over $T + p^*$, $T + q^* \vdash A^* \to \pi$, so $T + q^* + A^* \vdash \pi$. We conclude that $(q \wedge A)^*$ is Π_1-conservative over $(p \wedge A)^*$. Since we reasoned in T, we have $T + (p \rhd q)^* \vdash (p \wedge A \rhd q \wedge A)^*$, and hence $T \vdash (p \rhd q \to p \wedge A \rhd q \wedge A)^*$. Now because $*$ was arbitrary, by arithmetical completeness we may conclude that $\vdash_{\Sigma\mathrm{ILM}} p \rhd q \to p \wedge A \rhd q \wedge A$, concluding our proof. □

This concludes the proof of (the left to right direction of, and hence also the full) Theorem 2.1.

3. Notes and acknowledgements

3.1. As mentioned in §1.2, the first classification of the essentially Σ_1 formulae of GL is found in [8]. De Jongh and Pianigiani ([6]) classified them for GL and R (R is GL extended with a binary modality for witness comparison). Goris and Joosten ([4]) later classified the essentially Σ_1 formulae of ILM. Our proof of Theorem 2.1 is structured like that in [6]. The difference is that presently the two countermodels are constructed out of MCS's obtained by the Σ-Lemma of [4]. We believe that the condition that T contains PA may be weakened; the only thing we need (for Lemma 2.4) is that ΣILM is the logic of Π_1-conservativity over T. In [3] this is proved for superarithmetical[1] T, so presently we have also stuck to T extending PA, even though $\mathrm{I}\Sigma_1$ will probably do, or maybe even less. Additionally, we are optimistic about the possibility to extend the method Goris and Joosten use to classify the essentially Σ_1 formulae of ILM to a method classifying those of ΣILM.

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[1] Strictly speaking, Goris assumes $T = \mathrm{PA}$. However, his results still hold for r.e. superarithmetical T.