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VISUALIZATION OF ORDINALS*

ABSTRACT

We describe the pictorial representations of infinite ordinals used in teaching set theory, and discuss a possible use in naturalistic foundations of mathematics.

1 INTRODUCTION

Pictures are everywhere in mathematics. A dining group of mathematicians will leave a visual trace of their mathematical thoughts and ideas on the napkins of their chosen restaurant. When the ideas consolidate and reach publishable form, most of the pictures tend to disappear.[1] Our modern publishing routine that requires authors to produce camera-ready illustrations may have something to do with this, as mathematicians are notoriously bad with graphics software. But this pragmatic issue alone cannot account for the surprising lack of illustrations in mathematical papers.

Mathematics underwent a formalization process starting in the nineteenth century. An ideal of formal mathematics was developed and informal mathematics tried to fit into this new paradigm. Pictures and in particular picture proofs did not conform to the views of this formalization period, as can be seen most stringently in the following quote of Moritz Pasch:

> Wir werden nur diejenigen Beweise anerkennen, in denen man Schritt für Schritt sich auf vorhergehende Sätze beruft oder

*The author would like to thank Stefan Bold, Hannes Leitgeb, Paolo Mancosu, Thomas Müller, John Pais, Sara Uckelman, and an anonymous referee for comments and technical assistance. The blackboard photographs have been taken and edited by Sara Uckelman and are included with her permission.

[1] We randomly picked volume 13 (2000) of the **Journal of the American Mathematical Society** as a representative selection of current published high-level research mathematics. The volume had fewer than 50 illustrations on 1009 pages of articles (not counting two-dimensional notation like matrices or commutative diagrams). 73.5% of the articles had no illustration, 88.2% of the articles had fewer than three illustrations, and only one article in the entire volume had more than 5 illustrations. Note that this volume contains papers from research areas like “Convex Geometry”, “Algebraic Geometry” and “Differential Geometry” without any illustrations.

> berufen kann. Wenn zur Auffassung eines Beweises die entsprechende Abbildung unentbehrlich ist, so genügt der Beweis nicht den Anforderungen, die wir an ihn stellen. [...] Bei einem vollkommenen Beweise ist die Abbildung entbehrlich.
>
> (Pas26, p. 45)

Our traditional philosophy of mathematics, rooted in the foundational debates of the early twentieth century, rests on the idea of mathematical activity as an approximation to formal reasoning. The notion of a logical deduction from the axioms to the theorems dominates mainstream discussions about mathematical truth and epistemology.

If pictures enter traditional discussions, it is either as a heuristic instrument or as part of a discussion of Gödelian Platonism and Gödel's ideas of a human faculty of perception of abstract objects. Philosophers found Gödel's faculty of mathematical intuition puzzling[2], and a lot of the philosophical literature on mathematical intuition or the use of pictures struggles with understanding the relationship of visualization to foundational questions of ontology and mathematical truth.[3]

However, no one would dispute that pictures, diagrams, metaphors and the like are epistemically important for the working mathematician. Outside of the mathematical journals, mathematical practice uses a lot of drawings on blackboards and napkins, fingers symbolizing objects, gestures, movements[4], sound, and many other features that traditional views of philosophy of mathematics tend to disregard.[5] Of course, foundationalists would claim that the use of these means by mathematicians tells us nothing about mathematics.

In this paper, we shall take a decidedly anti-foundationalist stance, following Maddy's maxim of *Naturalism in Mathematics* (Mad97). We be-

[2]To give just one example: "Talk of this mysterious faculty of intuition is tainted with an air of occult mysticism." (Che97, p. 121)

[3]*Cf.* (Kva97; Kva00), the discussion between Brown and Folina (Bro97; Fol99), (Rot00), and (Gia94). In his recent project on informal provability, Leitgeb agrees with our point of view: "There is nothing 'mystic' about intuition: whenever mathematicians speak of intuition, they refer to *non-conceptual*/*non-propositional* representations of mathematical structures. (Lei07)".

[4]*Cf.* (Núñ04).

[5]Visualization has always played a major rôle in the didactics of mathematics, *cf.*—as a completely unrepresentative sample—, *e.g.*, (Nel93) and (Pai01). Recently, it has become a major focus in philosophy of mathematics as well, as witnessed by (Man05) and the other papers in the collection (ManJørPed05).

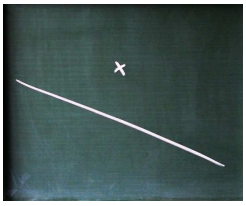
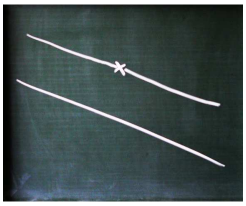
	“Take a line and a point that does not lie on the line . . . ”
	“. . . and draw the unique parallel through the point.”

Figure 1: The annotated kinetigram $\mathfrak{K}_0$.

lieve that a philosophy of mathematics should take mathematical practice and the actions and standards of mathematicians very seriously. A philosophy of mathematics that requires major re-interpretation of statements of mathematicians in order to fit into the philosophical frame is unacceptable from our point of view. Similarly, a philosophy of mathematics that ignores a large part of the actual research practice (and focuses on its last step: the preparation of the published version of the proof) cannot account for all of mathematical activity.

We observe that our naturalistic position has a number of consequences, in particular for the philosophy of set theory:

(1) If our philosophy of mathematics has to pay attention to mathematical practice and attitudes and behaviour of mathematicians, then we need to rest our philosophical decisions on empirical data. Note that this does not mean that philosophy of mathematics becomes sociology of mathematics, or becomes subordinate to sociology of mathematics, but just that the data typically collected by sociologists of mathematics should play a major rôle in the development and the testing of our philosophical hypotheses.

(2) The actual attitudes of set theorists and mathematicians towards axioms are much more subtle than traditional philosophy of mathematics can accommodate. It is conceivable that mathematicians accept

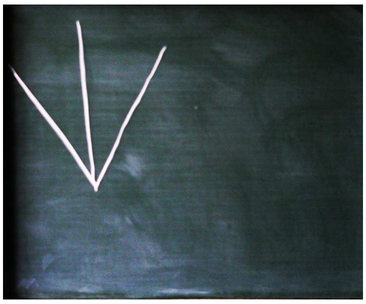
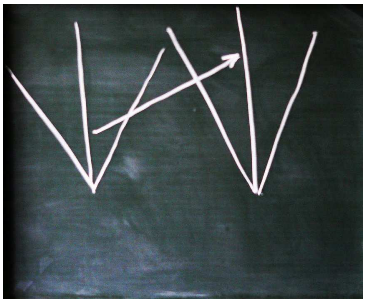
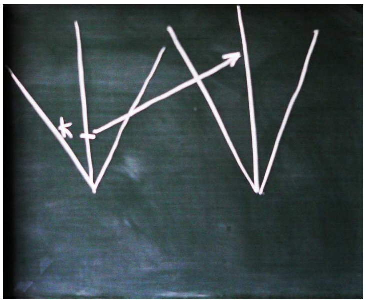
	“Let’s take a model of set theory...”
	“...and an elementary embedding into an inner model...”
	“...and let κ be its critical point.”

Figure 2: The annotated kinetigram $\mathfrak{K}_1$.

certain instances of an axiom for intrinsic reasons, but the most general form of the axiom is only accepted for extrinsic reasons (such as having a coherent and presentable theory). If these differences are relevant for mathematical practice, a philosophy of mathematics following our naturalistic principles should account for this perceived distinction.

(3) If pictures and diagrams play such a big rôle in mathematical practice, then every naturalistic philosophy of mathematics should be able

to explain their impact for mathematical epistemology in particular and for mathematics in general. This calls for a philosophical investigation of visualization and picture proofs.

Observation (1) has been made by Thomas Müller and the present author, and has led to a comprehensive project called "*Empirical Philosophy of Mathematics*". Not only in philosophy of mathematics, but in analytic philosophy in general, there is a tendency to make claims about intuitions and the general perception of communities without any empirical basis. For instance, in philosophy of language, many claims about grammaticality of phrases are made based on the personal intuition of the author as a native speaker. Similarly, in epistemology, philosophers decide whether the sentence "yesterday I knew it was true, but today I know that it is false" can be uttered adequately and if so, what it means, based on their linguistic and conceptual introspection.

In philosophy of mathematics, many claims have been made about mathematicians and their attitudes without realizing that statements like "most mathematicians are Platonists" or "most mathematicians think that the Axiom of Choice is true" are empirical statements that need to be checked.

A preliminary version of this paper was written in 2002, and in spite of the fact that the paper has been completely rewritten, the content of the present paper still reflects its time of origin, which was before Müller and the present author started their project "*Empirical Philosophy of Mathematics*". Most of the claims that we shall make in §3 and §4 follow the criticized tradition of basing empirical claims on personal introspection. We shall, however, take **Observations (2) and (3)** very seriously.

In §2, we discuss the diagrammatic reasoning used in informal mathematics, developing an informal notion of **annotated kinetigram** which will be used throughout this paper. We then apply this notion in §3 to graphical representations of ordinal numbers as many mathematicians use them in introductory set theory courses.

In §4, we shall discuss an interesting phenomenon relating to **Observation (2)**: set theorists observe a difference in the difficulty of working in various axiomatic frameworks. Certain ways of contradicting the axiom of choice are less natural than others. The naturalistic philosopher of mathematics needs to find an explanation of this phenomenon. In §5, we shall tie this phenomenon to our visualizations of ordinals.

As mentioned, this paper does not contain an empirical analysis of the claims made. In our conclusion, §6, we shall discuss what empirical studies will have to be made in order to make the results from §5 acceptable from the point of view of *Empirical Philosophy of Mathematics*.

2 DIAGRAMS, KINETIGRAMS, ANNOTATED KINETIGRAMS

As mentioned, pictures are ubiquitous in mathematical practice, but static diagrams don't suffice to describe how mathematicians discuss mathematics: Check a blackboard after a mathematics professor has finished a class; the blackboard will have lots of pictures and drawings of constructions, but it will be hard if not impossible to recover the constructions from the picture. The development of the pictures through time is needed to understand the full meaning of these pictures. This leads naturally to the notion of a "kinetigram", which was introduced by Pais as "a diagram that changes" (Pai01).

We shall go beyond Pais: the typical blackboard presentation of a mathematical argument or definition requires additional verbal information in order to be properly understood. In this paper we shall therefore be concerned with **annotated kinetigrams**. These are drawings that change over time with natural language utterances at particular points of the drawing process. In this paper, we shall represent annotated kinetigrams by a sequence of snapshots of the drawn picture with the natural language annotations associated with the snapshot that corresponds to the moment of utterance. Note that this is not a mathematically precise definition, but it will be enough for the purposes of this paper.[6]

In a given mathematical context, annotated kinetigrams can **describe** or **represent** a mathematical concept. Again, this is an informal notion that is difficult to make precise, but it is empirically easy to establish whether a

[6]In fact, one of the striking features of the use of pictures in mathematics is their enormous flexibility and reluctance to be put into a formal framework. The same kinetigram can have different meanings in different areas of mathematics because the respective audiences use different rules of vaguely defined pictorial semantics. Annotations can be used to evoke certain connotations that activate a different pictorial semantics. While doing mathematics, you can invent your own diagrammatic language on the fly and be understood by your audience sharing the same background context. Thus, not even the pictorial vocabulary of our kinetigrams is fixed.

In artificial intelligence and computer science (for instance, in automated theorem proving), researchers are interested in approximating the human capacity of pictorial communication and argumentation by formal systems. In this context –in contrast with the described mathematical practice–, the formal vocabulary is fixed. A large body of work exists in this direction, of which we list (Shi94; AllBar$_1$96; Jam01; Bar$_0$+02) as examples. In this paper, however, we are not interested in formalisms of this kind.

given annotated kinetigram describes a particular concept or not by showing them to mathematicians knowledgeable in the relevant area of mathematics.

In some cases, annotated kinetigrams do not *per se* describe a concept, but only under certain assumptions. As an example of this, consider the annotated kinetigram $\mathfrak{K}_0$ in Figure 1. We can imagine a geometer drawing the chalk lines on a blackboard. Even though the blackboard itself is Euclidean, the lines could represent lines in some non-Euclidean geometry. The second line drawn in the second snapshot represents the parallel through the point, but this parallel only exists uniquely if the geometry satisfies the parallel postulate. We shall say that the second line in $\mathfrak{K}_0$ represents the parallel under the assumption that the geometry is Euclidean.

The assumptions needed such that a given annotated kinetigram represents a concept can be used as a measure of quality of depiction. If the assumptions needed are very strong, then the annotated kinetigram could be seen as a *bad representation* of the concept. Conversely, if an annotated kinetigram is agreed to be (independently) the natural depiction of a concept, but only represents the concept under certain assumptions, then this could be seen as a strong argument to restrict our attention to those cases where the assumptions are satisfied. The case of the Euclidean geometry can be used as an example, and we shall see another example in §5.

It should be stressed that annotations are crucial for the representation of mathematical concepts. Dots and lines can represent many different things in drawings, and can be disambiguated only by the context and natural language explanations. We shall see a relevant example in §3. Also note that annotations are immensely powerful: by our use of natural language we can let simple lines and arrows stand for excruciatingly complicated mathematical concepts, as the standard example $\mathfrak{K}_1$ from set theory (Figure 2) shows.

Of course, for a mathematical concept, there is never a unique annotated kinetigram that represents it. Often, mathematicians have to choose how to represent their concepts pictorially, and some of the representations are better than others, thus leading to measures of quality of representation. We already discussed the assumptions needed for being a representation as one measure of quality, but there are others.

In general, a depiction with fewer or less powerful annotations would be considered better. In particular, it could be considered preferable if the annotations only refer to properties of the diagram as it is drawn. For instance,

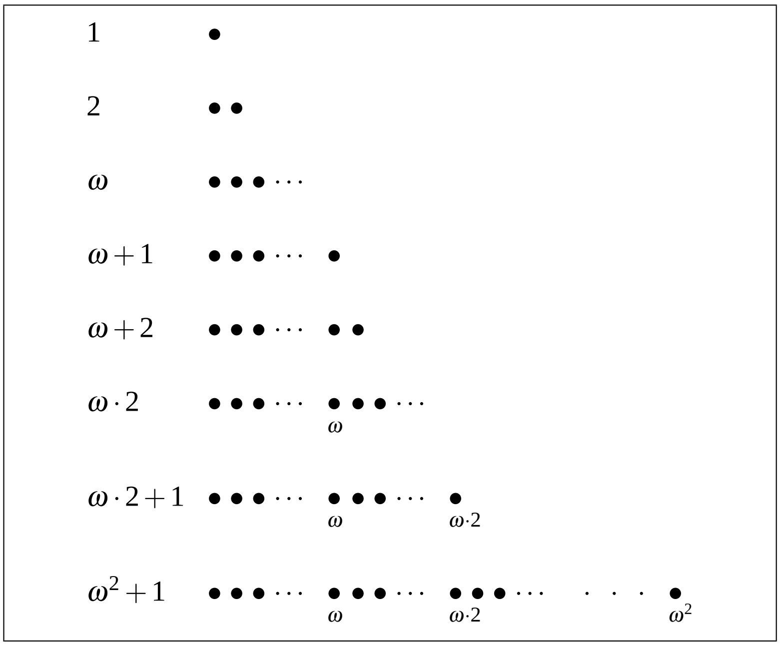


Figure 3: Dot diagrams of the ordinals 1, 2, ω, $\omega+1$, $\omega+2$, $\omega\cdot 2$, $\omega\cdot 2+1$, and ω^2+1.

the annotations in $\mathfrak{K}_0$ refer to geometric properties that can be found in the picture itself, whereas those of $\mathfrak{K}_1$ refer to properties that are not geometrically represented such as the elementarity of the embedding.

Note that the informal notions introduced in this section are not precisely defined in a mathematical sense, but they are concrete enough to serve as empirically testable features of mathematical practice. Whether a given annotated kinetigram represents a concept is something that can be evaluated by polling the appropriate community of mathematicians; similarly, you can ask mathematicians whether one annotated kinetigram represents a concept better than another one. So, in spite of the lack of mathematical precision in these definitions, the approach of the "*Empirical Philosophy of Mathematics*" nevertheless makes our notions meaningful and testable (*cf.* § 6).

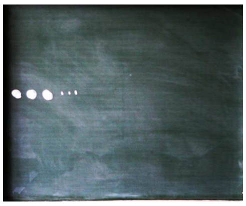
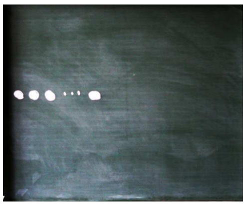
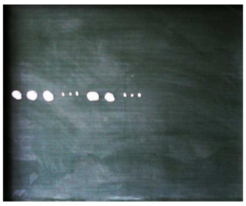
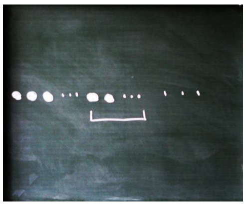
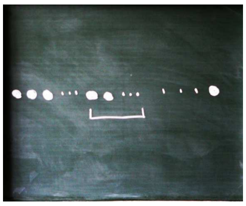
	“You start with infinitely many dots...”
	“...then you add a dot that is bigger than all of these...”
	“...and repeat drawing infinitely many dots...”
	“...and now you repeat this procedure infinitely many times...”
	“...and add a final dot.”

Figure 4: The annotated kinetigram $\mathfrak{K}_2$.

3 Pictures of infinite ordinals

When the present author teaches students how to conceive ordinals, he normally draws them as dot diagrams as exemplified for some small ordinal in the list of diagrams from Figure 3. Most likely, many other set theorists do the same.[7]

While the depictions of 1, 2, ω, $\omega+1$, $\omega+2$, $\omega\cdot 2$, $\omega\cdot 2+1$ can still be understood as diagrams without change and annotation, the ordinal ω^2+1 is more complicated. Note that there are two different kinds of ellipses: one indicating the infinite continuation in an ω-sequence, the other indicating the infinite continuation of the pattern seen before the ellipsis. An annotation will disambiguate these two different kinds of ellipses, as seen in the annotated kinetigram $\mathfrak{K}_2$ in Figure 4.

The dot diagrams stress the order structure of the ordinals, and are particularly good for the visualization of concepts that have to do with this order structure (like, e.g., the notion of *cofinality*; *cf.* § 5).

It is more complicated to use the dot diagrams for the notion of *cardinality* as this cannot preserve the order structure. However, for small infinite ordinals, we can give diagrams to depict their countability by displaying the (definable) bijections to ω as depicted in Figures 5 and 6.

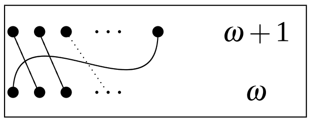


Figure 5: Bijection between $\omega+1$ and ω.

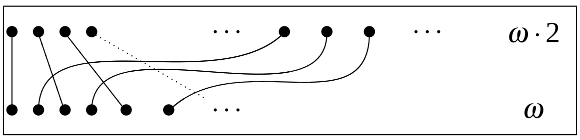


Figure 6: Bijection between $\omega\cdot 2$ and ω.

Already at ω^2 drawing bijections becomes very cumbersome, and in general, we know that there is no general procedure that allows us to present bijections like this for all countable ordinals.[8] So, the notion of *cardinality*

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[7]This visualization of ordinals has already been discussed in a letter (dated 16 August 1926) of Oskar Becker to Hermann Weyl. Becker, however, disagrees with the present author's belief that the visualization is crucial for our understanding of the ordinals (*cf.* (ManRyc02, p. 187)).

[8]This would amount to a function assigning codes of wellorderings of ω to all countable

is not easily represented in our diagrams. In an annotated kinetigram, it can easily (yet trivially) be incorporated by an annotation, as can be seen in the annotated kinetigram $\mathfrak{K}_3$ in Figure 7. The arrow from the part of the diagram representing ω^2 to the part of $\mathfrak{K}_3$ representing ω stands for something that is not easily depicted in the diagram. In §2, we mentioned that the geometric representation of properties in the diagram is a measure of quality for the representation. With this in mind, we should not consider $\mathfrak{K}_3$ as a high-quality pictorial representation of the fact that ω^2 is countable.

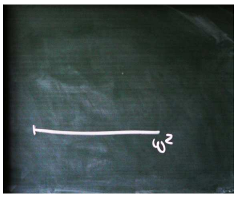
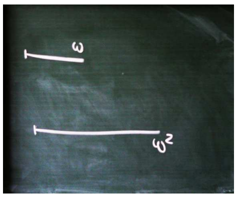
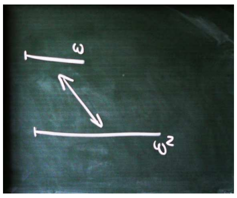
	"Let's take ω^2..."
	"...and ω..."
	"...and a bijection between them."

Figure 7: The annotated kinetigram $\mathfrak{K}_3$.

4 Fragments of the Axiom of Choice

In axiomatic set theory, mathematicians observe that working in models without the axiom of choice is more difficult than if you are allowed to use the axiom of choice: proofs are necessarily more subtle, and it is easier to make mistakes.

ordinals. This is not possible without the use of the Axiom of Choice, thus there can't be a presentable way to do it (uniformly).

But there are many ways in which the axiom of choice can fail, and from personal introspection and experience with colleagues and students, we can say that there are differences among the possible violations of the Axiom of Choice that are not necessarily captured by logical strength.

Whereas set theorists (even those of platonistic provenance strongly believing in the truth of the Axiom of Choice) seem to have no problem working in some models of $\mathsf{ZF}+\neg\mathsf{AC}$, *e.g.*, in models of $\mathsf{ZF}+\mathsf{AD}$,[9] and need only a little change of perspective to work in models like this,[10] they have more serious problems with models of severe violations of the Axiom of Choice.

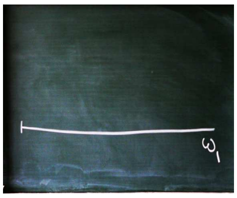
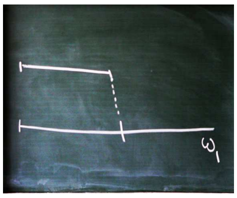
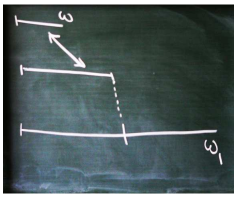
	"The ordinal ω_1 is uncountable, but..."
	"...for each initial segment..."
	"...there is a bijection between the initial segment and ω."

Figure 8: The annotated kinetigram $\mathfrak{K}_4$.

[9]The axiom AD is the *Axiom of Determinacy* stating that every for each infinite two-player perfect information game with countably many moves, one of the two players has a winning strategy. This axiom contradicts AC. More details can be found in (Kan94, §§27–32).

[10]A platonist believing in the Axiom of Choice would think of working in an inner model of $\mathsf{ZF}+\mathsf{AD}$ where mathematicians of different philosophical conviction would think of the universe being a model of $\mathsf{ZF}+\mathsf{AD}$; while of philosophical importance (*cf.* the discussion of Kleinberg's use of language in (Ste97)), this distinction is of little consequence for their mathematical practice.

Let us consider the fragment R stating "ω_1 is regular".[11] The theory $\mathsf{ZF}+\neg\mathsf{R}$ is consistent (if ZF is): the Feferman-Lévy model (Jec03, Example 15.57) is a model of this theory, but the present author has seen that arguing in models of $\mathsf{ZF}+\neg\mathsf{R}$ is much harder for mathematicians and students alike and that people are much more prone to making mistakes when arguing in $\mathsf{ZF}+\neg\mathsf{R}$ than in $\mathsf{ZF}+\mathsf{R}$. This statement has not been empirically tested but is based on anecdotal personal evidence (see §6).

Let us assume that there is indeed a measurable difference that makes $\neg\mathsf{R}$ a particularly bad violation of the axiom of choice. How can we explain this?

Of course, we have the extrinsic argument invoking the degree of familiarity: most working set theorists work in ZFC and for any consequence C of the Axiom of Choice, the more often C is used in our everyday set-theoretic arguments, the more alien the theory $\mathsf{ZF}+\neg\mathsf{C}$ appears to us. Since the regularity of ω_1 is crucially important for many parts of set theory (*e.g.*, the theory of Borel sets), $\mathsf{ZF}+\neg\mathsf{R}$ looks remarkably odd to us. This is a purely extrinsic (and less than satisfactory) reason for accepting R.[12] In §5, we shall give an intrinsic argument for R based on the pictorial representations of ordinals from §3.

5 An argument based on visualization

The usual set-theoretic definition of ω_1 is "the least uncountable ordinal". This definition can be represented with the annotated kinetigram $\mathfrak{K}_4$ in Figure 8. Clearly, this annotated kinetigram represents ω_1. However, as discussed in §3, it is not the most natural kinetigram: it uses an annotation to tell the viewer that the arrow stands for something that is not properly representable in the diagram (see Footnote 8).

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[11] The **cofinality** of a limit ordinal ξ, denoted by $\mathrm{cf}(\xi)$, is the smallest α such that there is a function $f:\alpha\to\xi$ such that $\bigcup\mathrm{ran}(f)=\xi$. Obviously, $\mathrm{cf}(\xi)\leq\xi$. A limit ordinal ξ is called **regular** if $\mathrm{cf}(\xi)=\xi$. The fragment R is an easy consequence of the Axiom of Choice, *cf.* (Jec03, Corollary 5.3).

[12] We are not claiming that intrinsic arguments for axioms are in general better than extrinsic arguments. For a discussion of intrinsic and extrinsic arguments in the history of the axiom of choice, we refer the reader to (Mad88a, p. 487–489); in (Mad88b, §VII), Maddy gives a general discussion of these two types of arguments for foundational axioms in mathematics.

It is, however, part of our empirical assumption that the (extrinsic) argument based on exposure and familiarity is not good enough to explain the "measurable difference" that mathematicians experience when dealing with various systems with violations of the axiom of choice.

Now consider the annotated kinetigram $\mathfrak{K}_5$ in Figure 9. This annotated kinetigram represents some ordinal without using the annotation to introduce notions that are foreign to the diagrammatic representation. As such, it is preferable to $\mathfrak{K}_4$. In ZF, $\mathfrak{K}_5$ represents the smallest ordinal ξ such that $\mathrm{cf}(\xi) > \omega$. The statement "$\xi = \omega_1$" is equivalent to R. Consequently, $\mathfrak{K}_5$ does not represent ω_1 in $\mathsf{ZF} + \neg\mathsf{R}$ and therefore not in ZF, but it does in $\mathsf{ZF} + \mathsf{R}$.

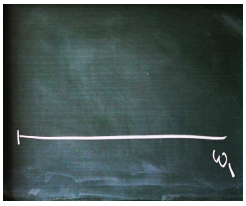
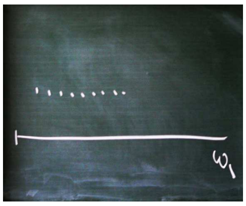
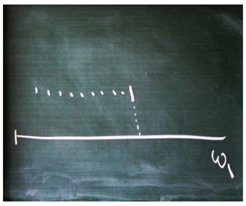
	"The ordinal ω_1 is the smallest ordinal such that..."
	"...whenever you have an increasing ω-sequence..."
	"...it ends before reaching the top."

Figure 9: The annotated kinetigram $\mathfrak{K}_5$.

As discussed in §2, having a natural annotated kinetigram for a mathematical concept under some assumption A might be construed as an argument for restricting our attention to models that satisfy A. If the reader accepts our belief that $\mathfrak{K}_5$ is a more natural representation of ω_1 than $\mathfrak{K}_4$, this could count as an intrinsic explanation for the fact that mathematicians prefer R over $\neg$R.

6 FUTURE WORK

In this paper, we observed differences in the attitude of mathematicians towards different negations of the axiom of choice, identifying $\mathsf{ZF}+\neg\mathsf{R}$ as a particularly bad theory.

In our attempt to explain why mathematicians might view $\mathsf{ZF}+\neg\mathsf{R}$ as bad (or, contrapositively, why they think that R is an important axiom), we gave two different representations of ω_1 by annotated kinetigrams. The annotated kinetigram $\mathfrak{K}_4$ used more powerful annotations and was less natural, whereas $\mathfrak{K}_5$ was more natural, but only represented ω_1 under the assumption of R. This was construed as an intrinsic argument for R.

However, our analysis rests on many empirical claims that we did not even attempt to justify. Is is really true that mathematicians find it more difficult to work in the Feferman-Lévy model than in a model of, say, $\mathsf{ZF}+\mathsf{AD}$? Is it really true that mathematicians prefer annotated kinetigrams with less powerful annotations? In particular, do set theorists prefer $\mathfrak{K}_5$ over $\mathfrak{K}_4$?[13]

In the spirit of "*Empirical Philosophy of Mathematics*", these questions will have to be investigated before the argument of this paper can be accepted as a philosophical analysis of the set-theoretical process of justifying axioms.

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[13]One of the readers of this paper disagrees with this claim. This emphasizes the need of an empirical investigation.

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