THE IMPORTANCE OF BEING DISCRETE

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ABSTRACT. The paper discusses discrete frames as an attractive semantics for modal logic. We study questions of completeness, persistence, duality and definability. Notions of completeness, strong global completeness and complexity of dual varieties coincide for discrete frames; moreover, they are equivalent to conservativity of minimal hybrid extensions. The paper also provides some criteria of di-persistence and a Goldblatt-Thomason theorem for discrete frames.

1. INTRODUCTION

In this paper, we study *discrete general frames*, i.e., general frames in which all singletons are admissible, as a semantics for modal logics. Discrete frames form a natural and interesting generalization of Kripke frames, and they provide a rather well behaved semantics for modal logics. For instance, the usual notions of completeness, strong completeness and strong global completeness coincide for discrete frames (unlike in the case of Kripke frames or neighbourhood frames). From an algebraic perspective, discrete frames correspond to atomic and completely additive algebras ($\mathcal{AV}$-BAOs).

Not every modal logic is determined by a class discrete frames. In fact, completeness for discrete frames ("di-completeness") is a non-trivial property (unlike completeness with respect to arbitrary or descriptive general frames). Clearly, every Kripke frame is in particular a discrete frame, and Kripke completeness implies di-completeness. If a Kripke incomplete modal logic is di-complete, this shows that the logic in question is still reasonably well-behaved. In this paper we characterize the modal logics that are complete for discrete frames.

Besides completeness, we also study persistence with respect to discrete frames ("di-persistence"). Note that these are independent notions: the van Benthem formula $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$ is di-persistent but axiomatizes a di-incomplete logic (Lemma 17). The Church-Rosser formula $\Diamond\Box p \to \Box\Diamond p$, on the other hand, is Kripke complete and hence di-complete, but it is not di-persistent (Example 33). We provide several sufficient conditions for di-persistence.

Finally, we also study the duality theory of discrete frames to obtain an analogue of The Goldblatt-Thomason Theorem (Theorem 32), characterizing the modally definable classes of these frames.

A note on the terminology: according to some, discrete frames should better be referred to as *atomic frames*, and the qualification *discrete* should be reserved for frames whose accessibility relation is a discrete order. We acknowledge this

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unfortunate ambiguity. However, the term *discrete frames* has appeared in the literature already several times and some of its derivatives, such as *di-persistence*, have achieved a certain notoriety. We have therefore decided not to change terminology.

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2. Preliminaries

2.1. **Modal logics, general frames, and completeness notions.** By a *modal logic*, we mean a set of modal formulas Λ containing all propositional tautologies as well as $\Box(p \to q) \to (\Box p \to \Box q)$, that is closed under the rules Modus Ponens (if $\phi \in \Lambda$ and $\phi \to \psi \in \Lambda$ then $\psi \in \Lambda$), Necessitation (if $\phi \in \Lambda$ then $\Box\phi \in \Lambda$) and Substitution (if $\phi \in \Lambda$ then every substitution instance of ϕ is in Λ). Thus, we restrict attention to *normal* modal logics; the adjective *normal* will be dropped in what follows. As usual, we write $\Lambda \vdash \phi$ instead of $\phi \in \Lambda$. A formula ϕ is called Λ-consistent if $\neg\phi \notin \Lambda$. We say that ϕ is *a global* Λ*-consequence of* Γ if it can be derived from Γ by using all formulas in Λ, Modus Ponens and Necessitation—but *not* substitution. If ϕ can be derived without using Necessitation, we say it is *a local* Λ*-consequence of* Γ. Denote $\Box^{\leq n}\phi := \phi \wedge \Box\phi \wedge \cdots \wedge \Box^n\phi$. Thus, ϕ is a global consequence of Λ iff it is a local consequence of $\{\Box^{\leq n}\gamma \mid \gamma \in \Gamma\}$. We say that Γ is Λ*-consistent* if $\bot$ is not a local Λ-consequence of Γ.

A general frame is a structure of the form $\langle W, R, \mathbb{A}\rangle$, where $R \subseteq W \times W$, and $\mathbb{A}$ is a family of subsets of W closed under the Boolean operations and the operator $\Diamond_R X = \{w \in W \mid \exists x \in X(wRx)\}$. If $\mathbb{A} = 2^W$, we say that the general frame is *a Kripke frame* or *a relational structure*. If for every $x \neq y$ there exists a set $X \in \mathbb{A}$ s.t. $x \in X$ and $y \notin X$, we say that the frame is *differentiated*. If for every x and y s.t. y is not an R-successor of x, there exists $X \in \mathbb{A}$ s.t. $x \in \Diamond_R X$ and $y \notin X$, we say the frame is *tight*. Frames which are both differentiated and tight are called *refined*. Finally, if every family $P \subseteq \mathbb{A}$ with the finite intersection property has non-empty intersection, we say the frame is *compact*. Frames which are both refined and compact are called *descriptive*. As is well-known, a finite general frame is a Kripke frame iff it is differentiated iff it is descriptive. It will become clear soon that none of the equivalences holds in in the infinite case.

Given a general frame $\mathfrak{F} = \langle W, R, \mathbb{A}\rangle$, an *admissible valuation* is a function V that assigns to each proposition letter an element of $\mathbb{A}$. *A model* is a pair consisting of a frame and a valuation in it. The satisfaction definition is extended to all formulas in the standard inductive way. A formula is *globally valid* in a model if it is satisfied at every point of the model. A formula ϕ is *valid* on a general frame $\mathfrak{F} = (W, R, \mathbb{A})$, denoted by $\mathfrak{F} \models \phi$, if $\langle W, R, V\rangle, w \models \phi$ for all $w \in W$ and for all admissible valuations V. A formula ϕ is valid on a class of general frames $\mathcal{X}$, denoted by $\mathcal{X} \models \phi$, if it is valid on each general frame in $\mathcal{X}$. We say that a class of frames $K \subseteq K'$ is *a modally definable class of frames* in K' (or *a modal class of frames* in K') if there is a set of formulas Λ s.t. $K = \{\mathfrak{F} \in K' \mid \mathfrak{F} \vDash \Gamma\}$. As for underlying K', we will be interested only in three cases: when K' is the class of all general frames, the class of all Kripke frames or, mostly, the class of all *discrete* frames (to be defined in Section 2.2). We will not mention the underlying class if it is clear from the context.

A logic Λ is said to be *complete* with respect to a class of general frames X iff for each formula φ, $\varphi \in L$ iff $X \models \varphi$. If the "only if" direction holds, we say that Λ is *sound* with respect to X. It is well known that every modal logic is complete with respect to some class of general frames. The Sahlqvist Theorem shows that a large class of modal logics are complete with respect to a class of Kripke frames. Nevertheless, there are uncountably many Kripke incomplete modal logics (i.e, modal logics that are not complete with respect to any class of Kripke frames). In fact, there is a famous theorem known as The Blok Alternative (cf., e.g., [2], [13]), which states that each logic that is not a union-splitting of the lattice of all modal logics defines the same class of Kripke frames as uncountably many other, Kripke incomplete logics. Most well-known logics such as **K4**, **S4** and **T** are not union-splittings.

There are other, stronger forms of completeness. A logic Λ is said to be *strongly (locally) complete* with respect to a class of general frames X if Λ is sound with respect to $\mathcal{X}$ and for every set of formulas Γ that is Λ-consistent is satisfiable at some point of a frame from X. Finally, Λ is *strongly globally complete* with respect to X if Λ is sound with respect to X and for every set of formulas Γ and formula α, if α is not a global Λ-consequence of Γ, then there exists a model $\mathfrak{M}$ based on a frame in X such that $\mathfrak{M}$ globally satisfies Γ but does not globally satisfy α.

Lemma 1. *Strong global completeness with respect to X implies strong completeness with respect to X, which in turn implies completeness with respect to X.*

Proof. The second implication is trivial. That strong global completeness implies strong completeness may be easily proven in the same way as implication (2) $\implies$ (1) in Wolter [25] Theorem 1.4.1. We sketch the proof to make the paper more self-contained. Assume the set Γ is Λ-consistent. We claim that the set $\{\Box^{\leq n}(p \to \gamma) \mid \gamma \in \Gamma, n \in \omega\} \cup \{p\}$ for p which does not appear in Γ is Λ-consistent. To see this, observe that every finite subset $\Gamma' \subseteq_{fin} \Gamma$ is satisfiable in a model $\langle \mathfrak{F}, \mathfrak{V} \rangle$, where $\mathfrak{F}$ is a general frame s.t. $\mathfrak{F} \vDash \Lambda$; otherwise Γ wouldn't be Λ-consistent. Because of the assumption that p does not appear in Γ, the valuation $\mathfrak{V}'$ defined in the same way as $\mathfrak{V}$ with the exception of $\mathfrak{V}'(p) := \mathfrak{V}(\bigwedge \Gamma')$ (we may define a valuation this way because Γ' is finite) satisfies $\{\Box^{\leq n}(p \to \gamma) \mid \gamma \in \Gamma', n \in \omega\}$. Now, it is enough to use compactness to prove that $\neg p$ is not a global Λ-consequence of $\{p \to \gamma \mid \gamma \in \Gamma\}$. But then, by strong global completeness, there must be a model based on Λ-frame from $\mathcal{X}$ which verifies all $p \to \gamma$ globally and refutes $\neg p$ at some point. At any such point Γ holds. $\Box$

For classes of general frames closed under ultraproducts, completeness and strong completeness coincide. Recall the definition of ultraproducts of general frames from [8, 21]. A standard compactness argument establishes the following (cf. also Corollary 1.8.6 in Goldblatt [8]):

Lemma 2. *If Λ is complete with respect to a class X of general frames closed under ultraproducts, then Λ is strongly complete with respect to X.*

Moreover, we prove now that for classes of general frames closed under taking generated subframes, strong completeness implies strong global completeness. Recall that a *generated subframe* of a general frame $\langle W, R, \mathbb{A} \rangle$ is any frame $\langle W', R', \mathbb{A}' \rangle$

with $W' \subseteq W$, $R' = R \cap (W' \times W')$, $\mathbb{A}' = \{X \cap W' \mid X \in \mathbb{A}\}$ such that for all $\langle w, v\rangle \in R$, if $w \in W'$ then $v \in W'$. *A point-generated subframe* is a generated subframe of $\langle W, R, \mathbb{A}\rangle$ whose universe consists exactly of those points in W which are accessible in finitely many steps from a chosen $x \in W$. A class of general frames $\mathcal{X}$ is *closed under point-generated subframes* if for every $\mathfrak{F} \in \mathcal{X}$ and every point $x \in \mathfrak{F}$, the subframe of $\mathfrak{F}$ point-generated by x is in $\mathcal{X}$.

Lemma 3. *If Λ is strongly complete with respect to a class X of general frames closed under point-generated subframes, then Λ is strongly globally complete with respect to X.*

Proof. It can again be extracted from proof of Theorem 1.4.1 in [25]. Assume α is not a global Λ-consequence of Γ. It is equivalent to Λ-consistency of $\{\Box^n\gamma | n \in \omega, \gamma \in \Gamma\} \cup \{\neg\alpha\}$. Then by strong completeness, this set must be satisfied in a point x of a frame $\mathfrak{F} \in X$ under a valuation $\mathfrak{V}$. But then the subframe of $\mathfrak{F}$ point-generated by x validates Λ, and the restriction of $\mathfrak{V}$ to this subframe satisfies globally Γ and refutes α at x. □

In general, however, none of the converse implications from Lemma 1 hold. The well-known logic **GL** is Kripke complete without being strongly complete for any class of Kripke frames. The proof of this fact provides us also with an interesting class of general frames which is not closed under point-generated subframes provides us with an example that strong completeness does not necessarily imply strong global completeness. A general frame is *complete* if for every family of admissible sets $\{A_i\}_{i\in I}$ there exists a smallest admissible set containing all A_i's. It follows from results of Shehtman [17] that every Kripke-complete extension of **K4** is strongly complete with respect to a class of complete general frames. In particular, **GL** is strongly complete with respect to complete frames. But it is not very hard to prove

Proposition 4. **GL** *is not strongly globally complete with respect to complete frames. Neither is any logic between* **GL** *and* **GL.3** *(the logic of well-founded strict linear orders).*

Proof. We will show that $\{p_i \to \Diamond p_{i+1} | i \in \omega\}$ globally implies $\neg p_0$ on all complete frames of the logic **GL**, but that $\neg p_0$ is not a global **GL**-consequence of $\{p_i \to \Diamond p_{i+1} | i \in \omega\}$.

Suppose for the sake of contradiction there were a complete frame $\mathfrak{F} = \langle W, R, \mathbb{A}\rangle$ and an admissible valuation V, such that $\mathfrak{F} \models$ **GL**, $\langle \mathfrak{F}, V\rangle$ globally satisfies $\{p_i \to \Diamond p_{i+1} | i \in \omega\}$ and $V(p_0) \neq \emptyset$. Let P be the smallest admissible set of which each $V(p_i)$ is a subset ($i \in \omega$). By construction, P is non-empty and $P \subseteq \Diamond P$ (note that $V(p_i) \subseteq \Diamond P$ for all $i \in \omega$). It follows that we can refute **GL** on $\mathfrak{F}$, which contradicts the assumption that $\mathfrak{F} \models$ **GL**.

In order to see that $\neg p_0$ is not a global **GL.3**-consequence of $\{p_i \to \Diamond p_{i+1} | i \in \omega\}$, let $\mathfrak{F}$ be the (discrete) frame consisting of a copy of natural numbers $\{0, 1, 2, \ldots\}$ with the usual strict order, followed by a copy of natural numbers with reversed order $\{\ldots, 2', 1', 0'\}$, where the admissible sets are the finite and cofinite sets. It is easy to see that $\mathfrak{F} \models$ **GL.3**. Now, let V be a valuation with $V(p_i) = \{i\}$ (i.e., the first copy of i) for all $i \in \omega$. Then $(\mathfrak{F}, V)$ globally satisfies $\{p_i \to \Diamond p_{i+1} | i \in \omega\}$ and $V(p_0) \neq \emptyset$. □

Proposition 4 strengthens the well-known fact that **GL** is not canonical: note that canonicity implies strong (local and global) Kripke completeness. It also indicates that the claim made in [17] that the consequence over transitive complete frames is of essentially first-order nature (as opposed to second-order nature of consequence over full Kripke frames) requires some qualifications. *Local* consequence relation over such frames is indeed well-behaved. But as far as *global* consequence is concerned, lattice-completeness of the frames involved seems to give rise to non-elementary, non-compact behavior. The discrete frames studied in the present paper are different in this respect, as we will see soon.

2.2. **Discrete frames.** A *discrete frame* is a general frame $\langle W, R, \mathbb{A}\rangle$ which satisfies the additional condition that $\{x\} \in \mathbb{A}$ for each $x \in W$. Discrete frames are a natural generalization of Kripke frames. Many Kripke-incomplete logics are still complete with respect to a class of discrete frames. This is nicely exemplified by results of Wolter [26] on the lattice of all tense logics of linear time flows. Within this lattice, Kripke-incompleteness is a common phenomenon, and yet all logics are di-complete (i.e., complete with respect to a class of discrete frames). In general, however, di-completeness is a non-trivial property, as we will see in Section 3.

Note that no infinite general frame can be discrete and descriptive at the same time for a simple reason: every cofinite subset is admissible in a discrete frame, but on infinite frames, the family of all cofinite sets has the finite intersection property but empty intersection. Thus, the intersection of the discrete frames and the descriptive frames is precisely the class of finite Kripke frames. Still, every discrete frame is refined. Refinedness therefore forms a natural common generalization of discreteness and descriptiveness.

Another straightforward observation linking classes of frames introduced before: the class of Kripke frames is simply the intersection of the class of discrete frames and the class of complete frames. Assume X is not admissible. By completeness and discreteness, there is the smallest set X' containing all singletons from X. X must be then a proper subset of X' and there is $y \in X' - X$. But then by discreteness again $X' - \{y\}$ is also an admissible set containing all singletons from X, a contradiction.

As was mentioned, discrete frames appeared several times in the literature, but it would be hard to provide a detailed historical sketch. One of the earliest references is perhaps van Benthem [21], though the name itself was not used there. Van Benthem proved, e.g., that di-persistence, as opposed to d-persistence, does not imply Kripke completeness. Discrete frames play a crucial role in hybrid logic and in difference logic, where they have been used in proofs of some general completeness results (cf. [6, 22]). They were discussed in papers on *modal model theory*; cf. Goranko and Otto [11] for a reference. General frames where all singletons are admissible turn out to be of importance in the study of *atom structures*; cf. Venema [23] or Goldblatt [10] for some interesting results.

2.3. **Algebra and duality theory.** As is well-known, with every general frame $\mathfrak{F} = \langle W, R, \mathbb{A}\rangle$, we can associate its dual algebra $\mathfrak{F}^+$ whose universe is $\mathbb{A}$ equipped with standard set-theoretical operations and $\Diamond_R$. This is an example of a BAO — *a boolean algebra with operators.* [1] Instead of speaking of satisfiability (refutability) of formulas in a frame, we may then speak of satisfiability (refutability) in algebra: valuations in algebra are assigning arbitrary elements of algebra to propositional variables. In other words, we identify formulas of modal logics with terms of BAOs. *An algebraic model* is a pair consisting of an algebra and a valuation in it. A formula ϕ is *satisfiable in an algebra* if there is a valuation $\mathfrak{V}$ s.t. $\mathfrak{V}(\phi) \neq \bot$. ϕ is *globally satisfiable* if there is a valuation $\mathfrak{V}$ s.t. $\mathfrak{V}(\phi) = \top$. Finally, ϕ is *validated* in a given algebra if it is globally satisfied by any valuation. The class of all algebras validating Λ is a variety, i.e., equationally definable class of algebras. We denote this variety by V_Λ.

Definitions of strong completeness and strong global completeness with classes respect to classes of general frames can be thus directly translated for corresponding classes of algebras. Lemma 1 holds without any changes. Algebraic perspective, however, provides us with one more notion of completeness, which turns out to be the strongest one. Let K be a class of algebras. We say that a variety V is *K-complex* if every $\mathfrak{A} \in K$ can be isomorphically embedded into an algebra from $V \cap K$. A logic Λ is strongly globally complete with respect to K if the corresponding variety V_Λ is K-complex. This observation hardly merits the name of a lemma: by standard algebraic techniques, every logic Λ is strongly globally complete with respect to V_Λ; i.e., ϕ is not a global Λ-consequence of Γ iff there is $\mathfrak{A} \in V_\Lambda$ and a valuation in $\mathfrak{A}$ which globally satisfies Γ while refuting ϕ. Now embed $\mathfrak{A}$ into an algebra from $V_\Lambda \cap K$.

It is the converse direction which is more interesting. It turns out that a sufficient condition for the converse implication to hold is the closure of K under direct products. The proof can again be adopted from Wolter [25]:

Lemma 5. *If K is closed under direct products and Λ is strongly globally complete with respect to K, then the corresponding variety V_Λ is K-complex.*

Proof. Take any $\mathfrak{A} \in V_\Lambda$. Assign a new variable p_a to every element a of $\mathfrak{A}$; let $\mathfrak{V}_A(p_a) = a$. $Diag_A$ is the set of all formulas which are mapped onto $\top$ by this valuation. Now for any $a \neq \top$, p_a is obviously not a global Λ-consequence of $Diag_A$, so there must be $\mathfrak{A}_a \in K$ and a valuation $\mathfrak{V}_a$ in $\mathfrak{A}_a$ globally satisfying $Diag_A$ and refuting p_a. Now take the product of all $\mathfrak{A}_a$'s and the product valuation $\mathfrak{V}$. It is left for the reader to verify that $f(a) := \mathfrak{V}(p_a)$ is indeed an embedding. $\square$

Thus, Proposition 4 is a thinly disguised proof that the variety corresponding to **GL** (i.e., the variety of *diagonalizable* or *Löb* algebras) is not closed under completions. That is, certain algebras from that variety cannot be embedded into any lattice-complete algebra from the same variety. The class of complete frames is just a counterpart of the class of lattice-complete algebras and the latter are closed under direct products.

Discrete frames have a natural algebraic counterpart, too. From an algebraic point of view, they correspond to atomic and completely additive atomic Boolean algebras with operators ($\mathcal{AV}$-BAO's). Recall that a BAO is atomic if every element is above an atom, and it is completely additive if $\Diamond \bigvee_{s \in S} s = \bigvee_{s \in S} \Diamond s$ whenever $\bigvee_{s \in S} s$ is defined. A simple example of an atomless BAO: the algebra whose carrier is $\{\{s{\cdot}t \mid t \in 2^\omega, s \in S\} \mid S \subseteq^{fin} 2^*\}$ with the usual set-theoretic Boolean operations and a trivial modal operator (i.e., the identity operator), where 2^* (2^ω) is the set of all binary sequences of finite (infinite) length. A simple example of an atomic BAO that is not completely additive: the algebra of finite and cofinite subsets of natural numbers, with $\Diamond S = \emptyset$ for finite S and $\Diamond S = \omega$ for cofinite S. This BAO is not completely additive, as can be seen by comparing $\Diamond \bigvee_{n \in \omega} \{n\}$ to $\bigvee_{n \in \omega} \Diamond \{n\}$.

It is fairly easy to see that the dual algebra of a discrete frame must always be atomic and completely additive. The converse direction, i.e, the question how turn an $\mathcal{AV}$-BAO into a discrete frame, will be postponed until Section 5.

2.4. **First-order logic, second-order logic and many sorted logic.** In this paper, we will use two correspondence languages. One is standard first-order logic with one binary relation constant. Another is a very weak form of (monadic) second-order logic, identified here with many-sorted first-order logic. It does not even contain the full Comprehension Principle for first-order formulas or Universal Instantiation as defined in van Benthem [21]. The axioms governing the behaviour of set variables are simply the same as the axioms governing the behaviour of first-order variables: $\forall X.(\phi \to \psi) \to (\forall X.\phi \to \forall X.\psi)$ and $\phi \to \forall X.\phi$, if X does not appear free in ϕ. Completeness of this language can be established by the standard Henkin construction. Ultraproducts of Henkin structures are defined in the standard way and used to prove compactness. General frames for modal logic are exactly those Henkin structures which satisfy in addition

$$\forall X \exists Y.(\forall y.y \in Y \leftrightarrow y \notin X), \quad \forall X, Y.\exists Z(\forall x.x \in Z \leftrightarrow x \in X \& x \in Y),$$

i.e., closure under boolean connectives and

$$\forall X \exists Y.(\forall y.y \in Y \leftrightarrow \exists x.(yRx \& x \in X)),$$

that is, closure under modal connectives. Hence, these conditions are definable in our many-sorted language and preserved under ultraproducts. And indeed, for those Henkin structures which happen to be general frames, the many-sorted ultraproduct construction coincides with modal ultraproduct of general frames. Discreteness is definable too (and hence preserved under ultraproducts):

$$\forall x \exists Y.(\forall y.y \in Y \leftrightarrow x = y).$$

Thus, elementarity of a class of general frames means something much more trivial than elementarity of a class of Kripke frames. The condition of being closed under ultraproducts of Kripke frames taken as relational structures is far harder to satisfy than being closed under ultraproducts of Kripke frames taken as general frames. Let us say that call the weaker notion of elementarity *two-sorted elementarity* and sum up this observation as

Proposition 6. *Every modally definable class of discrete frames is two-sorted elementary, hence closed under ultraproducts.*

Proof. Use the standard translation, and prefix the resulting formula with universal quantifiers. Recall that discreteness itself is also preserved under ultraproducts. □

2.5. **Hybrid logics.** While this paper deal with modal logics, we make use of a number of results from hybrid logic. For this reason, let us briefly survey hybrid logic and the results that we need. The minimal hybrid language is obtained by extending the basic modal language with an infinite set of nominals $NOM = \{i, j, \ldots\}$. The formulas of this language are generated by the following recursive definition:

$$\phi ::= \top \mid p \mid i \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond\phi$$

where p is a proposition letter and i is a nominal. It is usually assumed that the set of nominals NOM, as well as the set of proposition letters $PROP$, is countable. A formula will be called *pure* if it contains no proposition letters.

By a *hybrid logic*, we mean in this paper a set Λ of formulas of the minimal hybrid language which contains all propositional tautologies, the axiom $\Box(p \to q) \to (\Box p \to \Box q)$, and all instances of the axiom scheme $\Box^{\leq m}(i \to \phi) \vee \Box^{\leq m}(i \to \neg\phi)$ (for $i \in NOM$, $m \in \omega$ and ϕ a hybrid formula), and is closed under the rules Modus Ponens, Necessitation, Substitution (i.e., uniformly replacing proposition letters by formulas and nominals by nominals), as well as the following rule, called COV:

$$\text{If } \Lambda \vdash \ell(\neg i) \text{ for all } i \text{ then } \Lambda \vdash \ell(\bot),$$

where ℓ is an arbitrary necessity form. Here, *necessity forms* are defined as follows (cf. Goldblatt [9]): fix an arbitrary symbol # not occurring in the language, and let the set of necessity forms be the smallest set containing # such that for all necessity forms $\ell(\#)$ and hybrid formulas ϕ, $(\phi \to l(\#))$ and $\Box\ell(\#)$ are also necessity forms. Given a necessity form $\ell(\#)$ and a formula ϕ, we will use $\ell(\phi)$ to denote the formula obtained by replacing the unique occurrence of # in ℓ by ϕ.

For a modal logic Λ, we use Λ_H to denote its *minimal hybrid extension*, i.e., the smallest hybrid logic containing it.

Discrete frames form a very natural semantics for hybrid logic. Valuations of nominal variables are required to range over singletons. Notions of satisfiability and completeness can be formulated without any changes and we can prove

Theorem 7 (Di-completeness of hybrid logics)**.** *Every hybrid logic is strongly complete for a class of discrete frames. Moreover, assuming $PROP$ and NOM are countable, the universes of all frames from the class can be assumed to be countable.*

Proof. See, e.g., [7, 6]. □

This implies a Kripke completeness result for hybrid logics axiomatized by pure formulas:

Corollary 8 (Kripke completeness of pure hybrid logics)**.** *Every hybrid logic axiomatized by pure formulas is strongly Kripke complete. Moreover, assuming $PROP$ and NOM are countable, the universes of all frames from the class can be assumed to be countable.*

Definition 9 (Bisimulation systems)**.** *Given a bisimulation Z between frames $\mathfrak{F}$ and $\mathfrak{G}$, and a subset X of the domain of $\mathfrak{G}$, we say that Z* respects *X if the following two conditions hold for all $x \in X$:*

(1) There exists exactly one w such that wZx.
(2) For all w, v, if wZx and wZv then $v = x$.

A bisimulation system from $\mathfrak{F}$ to $\mathfrak{G}$ is a function $\mathcal{Z}$ that assigns to each finite subset $X \subseteq \mathfrak{G}$ a total bisimulation $\mathcal{Z}(X) \subseteq \mathfrak{F} \times \mathfrak{G}$ respecting X. $\mathfrak{G}$ is called the image of bisimulation system *$\mathcal{Z}$.*

Definition of a total bisimulation is standard and can be found in, e.g., [18]. That work gives several analogues of the Goldblatt-Thomason theorem for languages with nominals. For present purposes, the following result will be important, concerning definability by means of pure hybrid formulas.

Theorem 10 (Frame classes definable by pure hybrid formulas [18]). *Every class of frames defined by a set of pure formulas is Δ-elementary. Moreover, a class K of Kripke frames is defined by a pure formula iff the following hold.*

(1) K is elementary,
(2) K is closed under images of bisimulation systems,
(3) For all frames $\mathfrak{F}$, if every point-generated subframe of $\mathfrak{F}$ is a generated subframe of a frame in K, then $\mathfrak{F} \in K$.

Note that every modal frame class satisfies the last condition. A typical example of a modally definable frame class that is not closed under images of bisimulation systems is the class of confluent frames, defined by $\Diamond\Box p \to \Box\Diamond p$.

Call a formula *di-persistent* if its validity is preserved under passage from a discrete frame to the underlying Kripke frame. We will study di-persistence for modal formulas in detail in Section 4. In doing so, we will make use of the following elegant characterization of the di-persistent hybrid formulas.

Theorem 11 (Gargov and Goranko [12]). *Every pure hybrid formula is di-persistent. Conversely, every di-persistent hybrid formula defines the same class of discrete general frames as a pure hybrid formula.*

Proof. The first part of the result is clear. Next, suppose ϕ is a di-persistent hybrid formula, and let Σ be the set consisting of all pure instantiations of ϕ, i.e., $\Sigma = \{\phi^\sigma \mid \sigma$ is a substitution that maps every proposition letter to a pure formula$\}$. We will show that Σ defines the same class of discrete general frames as ϕ. It follows then by compactness that ϕ is equivalent on discrete general frames to a finite conjunction of elements of Σ (here is where we use two-sorted elementarity of the class of discrete general frames).

Let $\mathfrak{F}$ be any discrete frame. If $\mathfrak{F} \models \phi$, then clearly, $\mathfrak{F} \models \Sigma$. Conversely, suppose $\mathfrak{F} \models \Sigma$. Let $\mathfrak{G}$ be the smallest discrete frame based on the underlying Kripke frame of $\mathfrak{F}$. More precisely, let V be any valuation for $\mathfrak{F}$ under which every point in $\mathfrak{F}$ is named by a nominal, and let $\mathfrak{G}$ be the discrete general frame in which the admissible subsets are precisely those definable under V by means of pure hybrid formulas. Clearly, $\mathfrak{G} \models \phi$. By di-persistence, we obtain that ϕ is valid on the underlying Kripke frame of $\mathfrak{G}$ (which is also the underlying Kripke frame of $\mathfrak{F}$), and hence, $\mathfrak{F} \models \phi$. □

Call a class K of frames *singleton-persistent* if whenever a discrete frame belongs to K, all discrete frames built on the same Kripke frame also belong to K. A similar notion has been introduced by Goldblatt [10] in an algebraic setting. Then, the following analogue of Theorem 11 holds for classes of discrete frames and *sets* of hybrid formulas.

Theorem 12. *Let K be any class of discrete frames defined by a set of hybrid formulas. Then K is singleton-persistent iff K is defined by a set of pure formulas.*

Proof. A straightforward adaptation of the proof of Theorem 11. The only difference in the proof is that compactness is no longer used, since we do not insist on definability by means of a single formula. □

3. Completeness for discrete frames

Recall that we call a modal logic *di-complete* if it is complete for some class of discrete frames. In this section, we study di-(in)completeness in detail. The following central result characterizes di-completeness in terms of conservativity of the minimal hybrid extension.

Theorem 13 (Di-completeness)**.** *The following are equivalent for any modal logic* Λ*.*

(1) Λ *is di-complete.*
(2) Λ *is strongly globally complete for a class of discrete frames. If* $PROP$ *is countable, then all frames in the class may furthermore be assumed to be countable.*
(3) The minimal hybrid extension Λ_H *is conservative over* Λ*.*

Proof. [$1 \Rightarrow 2$] If Λ is complete with respect to some class of discrete frames, then in particular, Λ is complete with respect to the class of discrete frames it defines. By Proposition 6 and Lemma 2, it follows that Λ is strongly complete for the same class of discrete frames. Finally, by Lemma 3 Λ is in fact strongly globally complete with respect to this class.

[$2 \Rightarrow 3$] Suppose Λ is strongly complete for a class K of discrete frames. Since the additional axioms of Λ_H are valid on all discrete frames and the inference rules of Λ_H preserve validity on all classes of discrete frames, Λ_H is sound with respect to K, which implies that Λ_H is conservative over Λ.

[$3 \Rightarrow 1$] Conversely, suppose Λ_H is conservative over Λ. By Theorem 7, Λ_H is complete with respect to some class of discrete frames. Hence, so is Λ.

□

This theorem has surprising algebraic implications. Completeness of Λ with respect to discrete frames means that the corresponding variety is HSP-generated from its $\mathcal{AV}$-algebras. The class of $\mathcal{AV}$-algebras is closed under direct products. Hence, by Lemma 5, we obtain the following

Corollary 14. *For arbitrary variety* V *of BAOs, t.f.a.e.*
(1) V *is* HSP*-generated from its* $\mathcal{AV}$*-algebras.*
(2) V *is* $\mathcal{AV}$*-complex, i.e., every algebra from* V *can be isomorphically embedded into an* $\mathcal{AV}$*-algebra from* V*.*

In other words, given the class of all $\mathcal{AV}$-algebras from the variety corresponding to a di-complete logic, it is enough to consider *only* their subalgebras to obtain all algebras from this variety, without any help of homomorphic images and products. As the example of **GL** above shows, nothing like this holds for dual algebras of Kripke frames: Kripke completeness does not imply Kripke complexity.

There is an interesting open problem here. Recall that the hybrid logic Λ_H extends the modal logic Λ with an axiom scheme and rule COV forcing di-completeness.

Question 1. *Can we find a non-standard rule that would force di-completeness in the basic modal language?*

Despite several promising attempts, we have not been able to find such a rule. Another open question is the following.

Question 2. *Is there a di-incomplete modal logic extending* **K4***?*

Having considered the issue of completeness, we should mention a corollary of a recent result of Venema concerning a strong version of di-incompleteness.

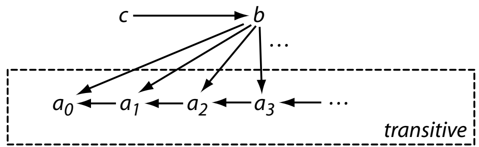

FIGURE 1. The Van Benthem frame $\mathfrak{F}$.

Theorem 15 (Venema [24]). *There is a consistent bi-modal logic* Λ *that has no discrete frames.*

We will give a easy proof of this result below (cf. Theorem 19). Note that this radical form of incompleteness can only occur among multi-modal logics. For uni-modal frames, the situation is a little better.

Theorem 16 (Makinson [16]). *Every consistent uni-modal logic has at least one discrete frame.*

Actually, it can be shown that the so-called Blok Alternative holds for discrete frames, too. We are not going to pursue this issue here; it is going to be discussed in [15].

In the remainder of this section, we discuss applications of the above results. A famous example of a Kripke incomplete logic is the logic **vB** [20, 5], which is obtained by extending the basic modal logic **K** with the axiom $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$; if no confusion arises, **vB** is used to denote this axiom itself. Litak [14] observed that **vB** is in fact incomplete with respect to discrete frames:

Lemma 17. *The formula* $\Diamond\Box\bot \vee \Box\bot$ *is valid on every discrete frame of* **vB**, *even though it does not belong to the logic.*

Proof. Assume $\Diamond\Box\bot \vee \Box\bot$ fails in a discrete frame $\mathfrak{F} = (W, R, \mathbb{A})$. Then there is an $x \in \mathfrak{F}$ s.t. $\mathfrak{F}, x \vDash \Diamond\top \wedge \Box\Diamond\top$. In other words, x has at least one R-successor y and every R-successor of x (in particular y) has an R-successor.

Consider any admissible valuation V such that $V(p) := W \setminus \{y\}$ (note that, since $\mathfrak{F}$ is discrete, $W \setminus \{y\} \in \mathbb{A}$). Then $\mathfrak{F}, y, V \vDash \Box(\Box p \to p)$. For, consider any R-successor z of y, and suppose $\mathfrak{F}, z, V \models \Box p$. Then z must be distinct from y, for otherwise it would follow that $\mathfrak{F}, V, y \models p$. Hence, by the definition of V, $\mathfrak{F}, z, V \models p$.

Since $\mathfrak{F}, y, V \not\models p$, this shows that $\mathfrak{F}, x, V \not\models$ **vB**, and hence $\mathfrak{F} \not\models$ **vB**.

The second part of the statement was proved by van Benthem [20], by means of the general frame depicted in Figure 1, where the admissible sets are all finite sets not containing b, and their complements. This general frame (which is not discrete) is known as *the van Benthem frame.* □

As an application of our results, we now show that the minimal hybrid extension of **vB** *is* Kripke complete and has the finite model property.

Proposition 18. $\mathbf{vB}_H$ *is Kripke complete and has the finite model property.*

Proof. By Lemma 17 and Theorem 7, $\Diamond\Box\bot\vee\Box\bot$ belongs to $\mathbf{vB}_H$. It is also possible to derive this formula directly; an interested reader may try it as an exercise. Clearly, $\Diamond\Box\bot \vee \Box\bot$ implies $\Box\Diamond\top \to \Box(\Box(\Box p \to p) \to p)$. It follows that $\mathbf{vB}_H$ coincides with the minimal hybrid extension of **K** plus $\Diamond\Box\bot \vee \Box\bot$. The latter is strongly Kripke complete by Corollary 8 (note that $\Diamond\Box\bot \vee \Box\bot$ is pure) and has finite model property by a straightforward filtration argument. $\Box$

By the same construction, we may prove stronger results concerning nonconservativity of minimal hybrid extensions. For example, adding an arrow from a_1 to c and deleting the arrow from b to a_0 in the van Benthem frame allows one to obtain a Kripke incomplete logic whose minimal hybrid extension is determined by a single finite frame. Similarly, by expanding the Van Benthem frame with an extra relation we obtain a simple proof of Theorem 15:

Theorem 19. *The bi-modal logic extending* **vB** *with the axiom* $\Diamond'(\Box\Diamond\top \wedge \Diamond\top)$ *is consistent, but has no discrete frames, and hence has an inconsistent minimal hybrid extension.*

Proof. Consider the Van Benthem frame expanded with the total accessibility relation $W \times W$. This general frame (which is not discrete) validates the logic under consideration, thus showing consistency. That the logic has no discrete frames follows immediately from Lemma 17. $\Box$

Incidentally, Proposition 18 contradicts Theorem 6.1 in [6], which states that the minimal hybrid extension of a uni-modal logic is always conservative. We believe there is a mistake in the proof of Theorem 6.1 (more precisely in the last sentence of the proof).

4. Persistence for discrete frames

One of the most fruitful notions in the study of modal logics has been *persistence with respect to descriptive frames* (*d-persistence*), also known as *canonicity.* Since we claim that discrete frames provide a natural semantics for modal logics, it seems sensible to investigate the discrete analogue of canonicity, i.e., di-persistence. In fact, Conradie et al. [4] argue that, in the context of hybrid logic, di-persistence is the appropriate notion of canonicity. This seems reasonable in the light of Corollary 8 and Theorem 11. But Theorem 21 below may suggest that the hybrid notion of canonicity should combine *both* kinds of persistence.

A first observation on di-persistence is the following.

Theorem 20. *Every di-persistent modal formula defines an elementary class of Kripke frames that is closed under images of bisimulation systems.*

Proof. Follows from Theorem 11 and Theorem 10. $\Box$

It follows that non-elementary formulas such as $\Box\Diamond p \to \Diamond\Box p$ are not di-persistent, and similarly for formulas that are not preserved by bisimulation systems, such as $\Diamond\Box p \to \Box\Diamond p$. We do not known, though, whether the converse of Theorem 20 holds:

Question 3. *Is every modally definable elementary class of Kripke frames that is closed under images of bisimulation systems definable by means of di-persistent modal formulas?*

Without non-standard rules di-persistence does not imply Kripke completeness or even di-completeness. The logic used in Theorem 19 is axiomatized by di-persistent formulas (the conjunction of its two axioms defines the empty class of discrete frames, hence is trivially di-persistent) and yet the logic is di-incomplete.

This is a significant difference with persistence for refined frames or d-persistence. It is natural then to ask what di-persistent formulas are Kripke-complete. The answer is provided by the following

Theorem 21. *For every modal logic Λ axiomatized by di-persistent formulas, the following are equivalent:*

(1) Λ is complete with respect to a Δ-elementary class of frames.
(2) Λ is canonical, i.e., persistent for descriptive frames.
(3) Λ is Kripke-complete.
(4) The minimal hybrid extension Λ_H is conservative over Λ.

Proof. [1 $\Rightarrow$ 2] This is the Fine-van Benthem theorem; cf., e.g., [2, Theorem 10.19].

[2 $\Rightarrow$ 3] Trivial.

[3 $\Rightarrow$ 4] Kripke-completeness implies di-completeness, which implies conservativity by Theorem 13.

[4 $\Rightarrow$ 1] Suppose Λ_H is conservative over Λ. Λ is axiomatized by a set of di-persistent formulas Γ. By Theorem 11, each $\gamma \in \Gamma$ is equivalent over the class of discrete frames to a pure formula γ'. Then Theorem 7 yields Λ_H is axiomatized by a set of pure formulas $\Gamma' := \{\gamma' \mid \gamma \in \Gamma\}$. Hence, by Corollary 8, Λ is Kripke-complete. By Theorem 10, the class of frames for Λ is Δ-elementary (i.e., it is defined by a set of first-order sentences). $\square$

This is a telling result. It implies that di-persistence, as an intermediate step for proving Kripke completeness, does not offer any more generality than canonicity. For, if a logic can be proven to be Kripke complete via di-persistence, then it could also have been proven Kripke complete via d-persistence.

This does not mean that di-persistence is a useless notion. It becomes useful in the context of axiomatizations of extended modal languages that come with a general completeness result for di-persistent formulas (in particular, this holds for hybrid logics).

Let us note here that if Theorem 11 is replaced by Theorem 12 in the proof of Theorem 21, we obtain the following characterization of logics determined by singleton-persistent classes of discrete frames:

Corollary 22. *Every modal logic Λ determined by a singleton-persistent class of discrete frames is determined by a Δ-elementary class of Kripke frames, hence canonical.*

Venema [23] and Goldblatt [10] offer alternative proofs that the class of Kripke frames for Λ satisfying the assumptions of the above corollary must be Δ-elementary.

There is a natural common generalization of d-persistence and di-persistence, namely *persistence for refined frames* (*r-persistence*). Since discrete frames and descriptive frames are both refined, r-persistence implies both di-persistence and canonicity. Note that the converse does not hold: the density formula $\Diamond p \to \Diamond\Diamond p$ is canonical and di-persistent but not r-persistent [1, Example 5.87].

Let us consider some syntactic criteria for di-persistent and r-persistence. The first one was found by Venema [22]. Let a *very simple Sahlqvist formula* be a formula of the form $\phi \to \psi$, where ψ is positive (i.e., every proposition letter occurs under an even number of negation symbols) and ϕ is built up from proposition letters using conjunction and $\Diamond$'s.

Theorem 23 (Venema [22])**.** *Every very simple Sahlqvist formula is di-persistent (and canonical).*

Venema also showed that, in the absence of tense operators, not every Sahlqvist formula is di-persistent. It will also follow from our Example 33.

A reasonably large class of r-persistent formulas can be obtained by restricting the modal depth at which proposition letters may occur inside a formula. Call a formula *shallow* if every occurrence of a proposition letter is under the scope of at most one modal operator. Then we have the following.

Theorem 24. *Every shallow formula is r-persistent (hence canonical and di-persistent).*

Proof. The proof proceeds by contraposition. Let $\mathfrak{F}$ be a refined general frame, $\mathfrak{G}$ its underlying Kripke frame and suppose $\mathfrak{G}, V, w \not\models \phi$, where ϕ is a shallow modal formula, V a valuation in $\mathfrak{G}$ not necessarily admissible in $\mathfrak{F}$ and and w a world. We will construct an admissible valuation V' such that $\mathfrak{F}, V', w \not\models \phi$, thus showing that $\mathfrak{F} \not\models \phi$.

Let $\chi_1, \dots, \chi_n$ be the variable-free subformulas of ϕ and let $p_1, \dots, p_m$ be the proposition letters occurring in ϕ. In what follows, σ will be always a metavariable ranging over all complete elementary conjunctions of $\chi_1, \dots, \chi_n$, i.e., types of the form $(\neg)\chi_1 \wedge \dots \wedge (\neg)\chi_n$, and τ will be always a metavariable ranging over all complete elementary conjunctions of $p_1, \dots, p_m$. We may in fact assume that ϕ is a Boolean combination of formulas of the form $\sigma \wedge \tau$ or $\Diamond(\sigma \wedge \tau)$. Here is exactly where we used the assumption of shallowness. Let W_σ, W_τ and $W_{\sigma\tau}$ denote the subsets of the domain of $\mathfrak{F}$ defined by σ, τ and $\sigma \wedge \tau$, respectively, under the valuation V.

Fix any σ, and consider the set W_σ. Since σ is a variable-free formula, W_σ is admissible. The proposition letters $p_1, \dots, p_m$ partition W_σ into 2^m disjoint (possibly empty and not necessarily admissible) subsets $W_{\sigma\tau}$ (recall that τ ranges over complete elementary conjunctions of $p_1, \dots, p_m$). We will construct admissible sets $W'_{\sigma\tau}$ (for all τ) that form a partition of W_σ, such that the following requirements hold for all τ:

(1.) $W'_{\sigma\tau}$ contains w iff $W_{\sigma\tau}$ does

(2.) $W'_{\sigma\tau}$ contains a successor of w iff $W_{\sigma\tau}$ does.

Using these new partitions for all the σ's, one can then define an admissible valuation V': for each proposition letter p_k ($k \leq m$), $V'(p_k)$ is the union of all $W'_{\sigma\tau}$ with $\tau \models p_k$. By construction, V' is an admissible valuation, and $\mathfrak{F}, V, w$ and $\mathfrak{F}, V', w$ agree on ϕ. It follows that $\mathfrak{F}, V', w \not\models \phi$, and hence $\mathfrak{F} \not\models \phi$.

In the remainder of the proof, we will show how to construct these sets $W'_{\sigma\tau}$. Fix any σ.

Step 1: Choosing witnesses.

For each τ there are four possibilities:

a. $W_{\sigma\tau}$ contains w and also a successor of w In this case, pick two witnesses: w and a successor of w in this set.
b. $W_{\sigma\tau}$ a successor of w but not w itself. In this case, pick only one witness, namely a successor of w
c. $W_{\sigma\tau}$ contains w but no successor of w. In this case, pick only one witness, namely w.
d. $W_{\sigma\tau}$ does not contain w nor a successor of w. In this case, pick no witness.

Step 2: Separating the witnesses.

By repeated application of differentiatedness, we can now find a partition P of the space W_σ into admissible sets so that (i) any two witnesses belong to a different component of P, and (ii) every component of P contains a witness. Moreover, if no successor of w satisfies exactly the same proposition letters as w does, then by tightness we can ensure that the component of the partition containing w does not contain any successor of w (to see this, note that w is irreflexive in this case).

Step 3: Defining the sets $W'_{\sigma\tau}$

We now define each $W'_{\sigma\tau}$, for each τ, as the union of the components of the partition P that contain a witness belonging to $W_{\sigma\tau}$. By construction, the requirements (1.) and (2.) are met. Note that the left-to-right direction of (2.) follows from the last sentence of Step 2 (the application of tightness). □

Note that this proof generalizes to multi-modal languages, but not to languages with k-ary modalities for $k \geq 2$. Indeed, Goranko and Vakarelov [12] show by means of a formula $\Box(p,p) \to \nabla(p,p)$ that Theorem 24 fails for such languages.

Incidentally, the converse of Theorem 24 does not hold: the formula $\Diamond\Diamond p \to \Box\Box p$ is easily seen to be r-persistent, but there is no shallow formula that defines the same frame property (i.e., $\forall xyzuv(Rxy \wedge Ryz \wedge Rxu \wedge Ruv \to z = v)$). This can be easily shown using two finite frames of depth two; details are left to the reader. Nevertheless, it is not easy to generalize Theorem 24 even within the class of very simple Sahlqvist formulas, as witnessed by the density formula mentioned above.

5. Duality and definability

This section studies the duality between discrete frames and $\mathcal{AV}$-algebras from category-theoretical point of view. Thomason [19] studied the relationship between Kripke frames and *complete* $\mathcal{AV}$-BAOs in the same way; our input is essentially to show that the assumption of completeness is not needed in his proofs. We need these results to obtain Theorem 32, which is an interesting variant of the Goldblatt-Thomason Theorem.

First, let us briefly recall some well known validity preserving operations on general frames. The *disjoint union* of family of general frames $\{\langle W_i, R_i, \mathbb{A}_i\rangle\}_{i\in I}$ is the general frame $\langle W, R, \mathbb{A}\rangle$, where $\langle W, R\rangle$ is the disjoint union of the frames $\langle W_i, R_i\rangle$ $(i \in I)$ and $\mathbb{A} = \{X \mid (X \cap W_i) \in \mathbb{A}_i$ for all $i \in I\}$. Generated subframes were already introduced in Section 2.1. The notion of bounded morphism is the same as in case of Kripke frames with additional requirement that the inverse image of an admissible set is admissible.

We have already seen that the dual $\mathfrak{F}^+$ of a discrete frame is an atomic and completely additive algebra, i.e., a $\mathcal{AV}$-BAO. Now, let $\mathfrak{A}$ be a $\mathcal{AV}$-BAO and let $At\mathfrak{A}$ be the set of its atoms. $\mathfrak{A}_+$ is the frame whose universe is $At\mathfrak{A}$, the accessibility relation $R_\Diamond$ is defined as $aR_\Diamond b$ if $a \leq \Diamond b$ and the admissible subsets are those of the form $\{a \in At\mathfrak{A} \mid a \leq b\}$ for $b \in \mathfrak{A}$. This construction is known as *the atom structure* of $\mathfrak{A}$; it should not be confused with its canonical extension.

Proposition 25. *(1) $\mathfrak{A}_+$ is a discrete frame.*

(2) $(\mathfrak{A}_+)^+$ is isomorphic to $\mathfrak{A}$; the isomorphism is defined as $\phi(b) = \{a \in At\mathfrak{A} \mid a \leq b\}$.

(3) $(\mathfrak{F}^+)_+$ is isomorphic to $\mathfrak{F}$; the isomorphism is defined as $\psi(x) = \{x\}$.

Speaking in category-theoretical terms: this takes care of *objects*, but how about morphisms? The standard notion for morphism for frames is *bounded morphism* and the standard notion of morphism for algebras is *homomorphism*. Alas, as we shall see soon, in this particular case the two notions do not exactly match. In order to find a good counterpart of bounded morphism, we need a stronger algebraic notion: *a complete homomorphism*, that is, a homomorphism preserving all existing joins. Given a complete homomorphism $f : \mathfrak{A} \mapsto \mathfrak{B}$, define its dual $f_+ : \mathfrak{B}_+ \mapsto \mathfrak{A}_+$ as

$$f_+(b) := \text{The single } a \in At\mathfrak{A} \text{ s.t. } b \leq f(a).$$

As f is a homomorphism, there cannot exist two distinct atoms a and a' satisfying this condition. But as f is a complete homomorphism, there must exist at least one a with such a property (otherwise $f(\top) \leq \neg b$) and hence the definition is correct.

Proposition 26. *(1) A discrete frame $\mathfrak{G}$ is a bounded morphic image of a discrete frame $\mathfrak{F}$ iff the dual of $\mathfrak{G}$ is completely embeddable in $\mathfrak{F}^+$*

(2) A $\mathcal{AV}$-BAO $\mathfrak{A}$ is completely embeddable into $\mathcal{AV}$-BAO $\mathfrak{B}$ iff the dual of $\mathfrak{A}$ is a bounded morphic image of $\mathfrak{B}_+$

(3) A discrete frame $\mathfrak{G}$ is (isomorphic to) a generated subframe of a discrete frame $\mathfrak{F}$ iff the dual of $\mathfrak{G}$ is a complete homomorphic image of $\mathfrak{F}^+$

(4) A $\mathcal{AV}$-BAO $\mathfrak{A}$ is a complete homomorphic image of $\mathcal{AV}$-BAO $\mathfrak{B}$ iff the dual of $\mathfrak{A}$ is (isomorphic to) a generated subframe of $\mathfrak{B}_+$.

(5) $\mathfrak{A}$ is isomorphic to the direct product of $\{\mathfrak{A}_i\}_{i\in I}$ iff $\mathfrak{A}_+$ is isomorphic to the disjoint union of $\{\mathfrak{A}_{i+}\}_{i\in I}$.

(6) $\mathfrak{F}$ is isomorphic to the disjoint union of $\{\mathfrak{F}_i\}_{i\in I}$ iff $\mathfrak{F}^+$ is isomorphic to the direct product of $\{\mathfrak{F}_{i+}\}_{i\in I}$.

Those with interest in category theory may prove a far stronger

Proposition 27. *Categories of $\mathcal{AV}$-BAOs with complete homomorphisms and of discrete frames with bounded morphisms are dually equivalent in the sense of Davey and Clark* [3] *by the dual representation $\langle (\,)_+, (\,)^+, \phi, \psi\rangle$.*

The proofs of both facts are analogous to those in Thomason [19].

So what is the proper notion of morphism for discrete frames which corresponds to *arbitrary* morphisms of $\mathcal{AV}$-BAOs? The answer given by Thomason [19] for Kripke frames can be easily adapted to more general case. We have opted for a purely relational definition, formulating di-morphisms as relations between points and admissible sets, rather than as functions from ultrafilters of dual algebras to points as Thomason did.

Definition 28 (Di-morphisms)**.** *A* di-morphism *from a discrete frame $\mathfrak{F} := \langle W, R, \mathbb{A}\rangle$ to a discrete frame $\mathfrak{G} := \langle U, S, \mathbb{B}\rangle$ is a binary relation $F \subseteq W \times \mathbb{B}$ s.t. the following conditions are satisfied for all $w \in W$ and $X, Y \in \mathbb{B}$:*

conjunction: *$wF(X \cap Y)$ iff wFX and wFY,*
negation: *$wF(U \setminus X)$ iff not wFX,*
reverse image: *$\{x \in W \mid xFX\} \in \mathbb{A}$,*
back-and-forth: *$wF\{u \in U \mid \exists u' \in X.uSu'\}$ iff $\exists w' \in W$ such that wRw' and $w'FX$.*

$\mathfrak{F}$ is the source *of F and $\mathfrak{G}$ is its* target*. If for every $u \in U$, there exists $w \in W$ s.t. $wF\{u\}$, we say $\langle U, S, \mathbb{B}\rangle$ is a* di-morphic image *of $\langle W, R, \mathbb{A}\rangle$.*

Definition 29 (Duals)**.** *For every $\mathcal{AV}$-BAO $\mathfrak{A}$, define $\mathfrak{A}_* := \mathfrak{A}_+$. For every discrete frame $\mathfrak{F}$, define $\mathfrak{F}^* := \mathfrak{A}^+$. For every di-morphism F from $\mathfrak{F} := \langle W, R, \mathbb{A}\rangle$ to $\mathfrak{G} := \langle U, S, \mathbb{B}\rangle$, define $F^* : \mathbb{B} \mapsto \mathbb{A}$ as $F^*(B) := \{w \in W \mid wFB\}$. For every* *homomorphism* $f : \mathfrak{A} \mapsto \mathfrak{B}$ *of* $\mathcal{AV}$*-BAOs, define* $f_* \subseteq At(\mathfrak{B}) \times \mathfrak{A}$ *as* bf_*A *if* $b \leq f(A)$.

Proposition 30. *Let* f *be a homomorphism of BAOs and* F *be a di-morphism of discrete frames.*

(1) f_* *is a di-morphism*
(2) F^* *is a homomorphism of BAOs.*
(3) The target of f_* *is a di-morphic image of the source iff* f *is an embedding.*
(4) F^* *is an embedding iff the target of* F *is di-morphic image of the source.*

Proof is the same as in Thomason [19]. That paper also shows how to define notions of *identity di-morphisms* and *composition of di-morphisms* and prove an analogue of Proposition 27.

Proposition 31. *Every modally definable class of discrete frames is closed under point-generated subframes, disjoint unions, ultraproducts (of general frames) and di-morphic images.*

Proof. Follows from Propositions 26, 30 and standard algebraic results (Birkhoff et al.). □

Together, the above results not only provide a necessary, but in fact a sufficient condition for modal definability. Thus, we obtain a complete characterization of the modally definable classes of discrete frame, in the spirit of Goldblatt and Thomason.

Theorem 32 (Modal definability on discrete frames)**.** *A class of discrete frames is modally definable iff it is two-sorted elementary and closed under point-generated subframes, disjoint union and di-morphic images.*

Proof. The "only if" direction has been already proved. For the converse, assume that K is a class of discrete frames closed under the four above mentioned constructions, Λ is the logic determined by K (or, as some would say, the modal theory of K) and $\mathfrak{F} = \langle W, R, P \rangle$ is a discrete frames s.t. $\mathfrak{F} \models \Lambda$. Observe that, as K is closed under ultraproducts and point-generated subframes, Λ is strongly globally complete with respect to K by Lemma 2 and 3.

We need to ensure our language is rich enough. For every $A \in P$ choose a distinct propositional variable p_A and define canonical valuation $V_{\mathfrak{F}}(p_A) = A$. Let Γ be the global theory of $\langle \mathfrak{F}, V_{\mathfrak{F}} \rangle$, i.e, the set of those formulas which are true at every point of $\mathfrak{F}$ under the canonical valuation. Γ is obviously closed not only under Modus Ponens, but also under Necessitation. For any $x \in W$, $\neg p_{\{x\}} \notin \Gamma$. By strong global completeness, for every $x \in W$ there is $\mathfrak{G}_x \in K$ and a valuation V_x on $\mathfrak{G}_x$ s.t. $\mathfrak{G}_x, V_x \models \Gamma$ and $\mathfrak{G}_x, V_x \not\models \neg p_{\{x\}}$. As K is closed under disjoint unions, the disjoint union of all models $\langle \mathfrak{G}_x, V_x \rangle$ is a model based on a frame from K; denote this model as $\langle \mathfrak{G}, V \rangle$. Define a relation F between elements of the universe of $\mathfrak{G}$ and elements of P by yFA iff $\mathfrak{G}, y, V \models p_A$. It is now enough to prove that $\mathfrak{F}$ is a di-morphic image of $\mathfrak{G}$ by F. The conjunction and negation properties follow by the fact that for every $A, B \in P$, $p_{A \cap B} \leftrightarrow p_A \wedge p_B \in \Gamma$ and $p_{W \setminus A} \leftrightarrow \neg p_A \in \Gamma$, respectively. The reverse image property follows from the fact that V is an admissible valuation in $\mathfrak{G}$. The back-and-forth property follows from the fact that $p_{\Diamond A} \leftrightarrow \Diamond p_A \in \Gamma$. Finally, for every $x \in W$, the existence of y s.t. $yF\{x\}$ follows from non-emptiness of $V(p_{\{x\}})$. □

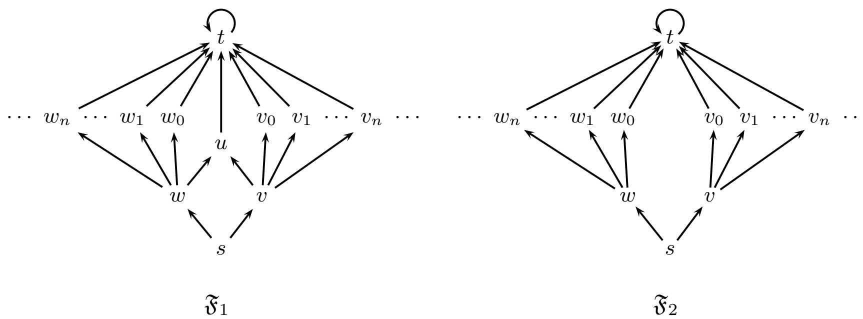

FIGURE 2. Confluence is not preserved under taking di-morphic images

This result read algebraically means that a class of $\mathcal{AV}$-BAOs is the class of all $\mathcal{AV}$-BAOs from some variety V iff it is closed under products, ultraproducts, subalgebras and *complete*-homomorphic images. In other words, all $\mathcal{AV}$-BAOs which are homomorphic images of a class of $\mathcal{AV}$-BAOs can be obtained by the use of these four constructions.

To demonstrate the use of di-morphisms let us consider the following

Example 33. *Call a relation R* confluent *if it satisfies $\forall xyz.(Rxy \wedge Rxz \rightarrow \exists u.(Ryu \wedge Rzu))$. The class of confluent Kripke frames is defined by the elementary and canonical modal formula $\Diamond\Box p \rightarrow \Box\Diamond p$. The class of confluent* discrete *frames, however, is not modally definable, as we will now show by means of a di-morphism. Consider discrete frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ (cf. Venema* [22]*) whose underlying structures are as depicted in Figure 2, and where the admissible sets are the finite and cofinite ones. Define a di-morphism F from $\mathfrak{F}_1$ onto $\mathfrak{F}_2$ by letting xFA iff either $x \in A$ or ($x = u$ and A is cofinite). The only non-trivial property of F that needs to be shown is the back-and-forth property. For this, it is enough to show that $\Diamond A$ is cofinite in $\mathfrak{F}_2$ iff there is a successor x of u s.t. xFA. But $\Diamond A$ is cofinite iff $t \in A$, and t is the only successor of u in $\mathfrak{F}_1$.*

From algebraic point of view, it means that the dual algebra of $\mathfrak{F}_2$ is embeddable into $\mathfrak{F}_1$, but that embedding is not an embedding preserving arbitrary existing joins.

6. Conclusion

Discrete frames provide a natural semantics for modal logics. They are well behaved in many respects, and they offer a natural alternative when Kripke completeness is not obtainable. In this paper, we have addressed a number of basic questions concerning completeness, persistence, duality and definability with respect to discrete frames. There are a few remaining open problems, such as Questions 1, 2 and 3.

References

[1] Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal logic.* Cambridge University Press, Cambridge, UK, 2001.
[2] Alexander Chagrov and Michael Zakharyaschev. *Modal logic.* Oxford University Press, 1997.
[3] David Clark and Brian Davey. *Natural Dualities for a Working Algebraist.* Cambridge Studies in Advanced Mathematics. Cambridge Univerity Press, 1998.

[4] Willem Conradie, Valentin Goranko, and Dimiter Vakarelov. Elementary canonical formulae: a survey on syntactic, algorithmic, and model-theoretic aspects. In *Advances in Modal Logic 5*. King's College Publications, 2005.

[5] M.J. Creswell. An incomplete decidable modal logic. *Journal of Symbolic Logic*, 49:520–527, 1984.

[6] George Gargov and Valentin Goranko. Modal logic with names. *Journal of Philosophical Logic*, 22:607–636, 1993.

[7] George Gargov, Solomon Passy, and Tinko Tinchev. Modal environment for Boolean speculations. In D. Skordev, editor, *Mathematical Logic and its Applications. Proceedings of the Summer School and Conference dedicated to the 80th Anniversary of Kurt Gödel (Druzhba, 1986)*, pages 253–263. Plenum Press, 1987.

[8] Robert Goldblatt. Metamathematics of modal logic I. *Reports on Mathematical Logic*, 6:41–78, 1976.

[9] Robert Goldblatt. *Axiomatizing the logic of computer programming*, volume 130 of *Lecture Notes in Computer Science*. Springer-Verlag, 1982.

[10] Robert Goldblatt. Persistence and atomic generation for varieties of boolean algebras with operators. *Studia Logica*, 68(2):155–171, 2001.

[11] V. Goranko and M. Otto. Modal model theory. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*. Elsevier North-Holland, to appear.

[12] Valentin Goranko and Dimiter Vakarelov. Elementary canonical formulae I: Extending Sahlqvist theorem. *Annals of Pure and Applied Logic*, 2006. In press, doi:10.1016/j.apal.2005.10.005.

[13] Marcus Kracht. *Tools and Techniques in Modal Logic*. Number 142 in Studies in Logic. Elsevier, Amsterdam, 1999.

[14] Tadeusz Litak. On notions of completeness weaker than Kripke completeness. In *Advances in Modal Logic 5*. King's College Publications, 2005.

[15] Tadeusz Litak. Stability of The Blok Theorem. *Algebra Universalis*, to appear.

[16] David Makinson. Some embedding theorems for modal logic. *Notre Dame Journal of Formal Logic*, 12:252–254, 1971.

[17] Valentin Shehtman. On strong neighbourhood completeness of modal and intermediate propositional logic. In J. Gerbrandy, M. Maarx, M. de Rijke, and Y. Venema, editors, *JFAK. Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday*. Vossiuspers AUP, 1999. CD.

[18] Balder ten Cate. *Model theory for extended modal languages*. PhD thesis, University of Amsterdam, 2005. ILLC Dissertation Series DS-2005-01.

[19] S.K. Thomason. Categories of frames for modal logic. *Journal of Symbolic Logic*, 40:439–442, 1975.

[20] J.F.A.K. van Benthem. Syntactic aspects of modal incompleteness theorems. *Theoria*, 45:63–77, 1979.

[21] J.F.A.K. van Benthem. *Modal Logic and Classical Logic*. Bibliopolis, 1983.

[22] Yde Venema. Derivation rules as anti-axioms in modal logic. *Journal of Symbolic Logic*, 58:1003–1034, 1993.

[23] Yde Venema. Atom structures. In *Advances in Modal Logic*, volume I, pages 63–72. CSLI, California, 1998.

[24] Yde Venema. Atomless varieties. *Journal of Symbolic Logic*, 68(2):607–614, 2003.

[25] Frank Wolter. *Lattices of Modal Logic*. PhD thesis, Fachbereich Mathematik, Freien Universität Berlin, 1993.

[26] Frank Wolter. Tense logic without tense operators. *Mathematical Logic Quarterly*, 42:145–171, 1996.

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