Games, Actions, and Social Software

Jan van Eijck and Rineke Verbrugge (eds.)

September 10, 2010

Contents

CHAPTER 1

In Praise of Strategies

Johan van Benthem[1]

ABSTRACT. This programmatic note high-lights a major theme in my lecture notes "Logic in Games" (van Benthem 1999 – 2002): the need for explicit logics that define agents' strategies, as the drivers of interaction in games. Our text outlines issues, recalls recent results, and raises new open problems. Results are mainly quoted, and the emphasis is on new notions and questions. For more details on the various topics discussed, see the relevant references. [2]

1. Strategies as first-class citizens

Much of game theory is about the question whether strategic equilibria exist. But there are hardly any explicit languages for defining, comparing, or combining strategies as such: the way we have languages for actions and plans, maybe the closest intuitive analogue to strategies. True, there are many current logics for describing game structure – but these tend to have existential quantifiers saying that "players have a strategy" for achieving some purpose, while descriptions of these strategies themselves are not part of the logical language (cf. Parikh & Pauly 2003, van der Hoek, van Otterloo & Wooldridge 2005).[3] In contrast with this, I consider strategies 'the unsung heroes of game theory' – and I want to show how the right kind of logic can bring them to the fore. One guide-line of adequacy for doing so, in the fast-growing rain forest of 'game logics', is the following. We would like to explicitly represent the elementary reasoning about strategies underlying basic game-theoretic results, starting from, say, Zermelo's Theorem or Backeward Induction. Or in more general terms, we want to explicitly represent agents' reasoning about their plans.

---

[1]Amsterdam and Stanford, `http://staff.science.uva.nl/~johan/`

[3]Van Benthem 1999-2003 calls this general tendency of hiding crucial information behind existential quantifiers "∃-sickness", and gives many more instances in logic.

2. Games as models for modal process logics

Logic of strategies must start with logical analysis of games. In particular, modal logic fits naturally with extensive games, viewed as process models from computer science, viz. labeled transition systems with some special annotation for players' activities (cf. van Benthem 2002). Just to be concrete, Figure 1 shows a simple 2-step game between two players i, j, with possible moves a, b, c, d marked at players' turns, while p is an atomic predicate stating some property of outcomes ('i wins' is a possible candidate, but so are others):

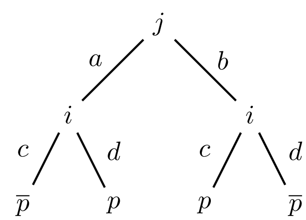


FIGURE 1. Two-step game between players i and j.

Basic modal logic Extensive game trees like these may be viewed as unfoldings, or state spaces, for some multi-agent process. Labeled modalities $\langle a\rangle\phi$ then express properties of nodes, namely, that some move a is available leading to a next node in the game tree satisfying ϕ. In this setting, modal operator combinations describe potential interaction. For instance, in a self-explanatory notation, the formula

$$[\text{move}_j]\langle\text{move}_i\rangle\phi$$

says that, at the current node of evaluation, player i has a strategy for responding to j's initial move which ensures that ϕ results after two steps of play.[4]

Extending this modal analysis to extensive games up to some finite depth k, and using alternations $\Box\Diamond\Box\Diamond\cdots$ of modal operators up to length k to reach the endpoints of the game tree, we can express the existence of winning strategies and the like in fixed finite games. Indeed, given this connection, with games of finite depth, standard logical laws have immediate game-theoretic import. In particular, consider the valid law of Excluded Middle in the following modal form

$$\Box\Diamond\Box\Diamond\cdots\phi \vee \Diamond\Box\Diamond\Box\cdots\neg\phi,$$

where the dots indicate the depth of the tree. This expresses the determinacy of these games: either the second player or the first has a stategy for ensuring ϕ. This

[4]In what follows, like here, we often refer only to the union *move*$_k$ of all available moves for players k. But of course, the game of Figure 1 also contains more specific information about moves, such as the following modal truth at the root: $[a]\langle d\rangle p \wedge [b]\langle c\rangle p$.

is exactly the content of Zermelo's Theorem that all finite zero-sum two-player games of perfect information are determined.

Modal μ-calculus Unfortunately, such modal game-by-game definitions are not 'generic', as they depend on the particular model considered – and Zermelo's inductive argument is rendered much more faithfully by means of just one fixed formula in the modal μ-calculus (Bradfield & Stirling 2006). To make the relevant point more generally, let us first define the following 'forcing modality' in games:

> $M, s \models \{i\}\phi$ iff
> player i has a strategy for the sub-game starting at s which guarantees that only nodes will be visited where ϕ holds, whatever the other player does.

Forcing talk is widespread in games, and it is an obvious target for logical formalization. Note that, for convenience, $\{i\}\phi$ also talks about intermediate nodes, not just end nodes of the game. The existence of a winning strategy for player i can then be expressed as the special case

$$\{i\}(\text{end} \to \text{win}_i).$$

Here, *end* is a proposition letter for end nodes, or a complex modal formula saying that no move is possible, and *win*$_i$ is a proposition letter saying that player i wins at the current node.

Here is an explicit definition for this assertion in the modal μ-calculus, using some further obvious proposition letters and action symbols for indicating players' turns and moves at nodes of the game tree. In this formula, the symbol j is used for the other player in the game:

$$\{i\}\phi = \nu q \cdot (\phi \wedge (\text{turn}_i \wedge \langle \text{move}_i \rangle q) \vee (\text{turn}_j \wedge [\text{move}_j] q)).$$

This definition is faithful to the obvious recursive meaning of having a strategy for player i, regardless of what the others do – and it is also generic, since it works in all games viewed as models M for our language.[5]

Propositional dynamic logic But now to strategies as such! An obvious candidate for defining these is *propositional dynamic logic PDL*, combining propositions about nodes in the game tree with programs defining transition relations between such nodes. [6] Without loss of information, a strategy for player i is a binary relation between nodes. At turns for player i, it picks out one transition, at turns for the others, it allows all available moves. Here is a simple fact for a start:

---

[5]Incidentally, we use a greatest fixed-point operator $\nu q \cdot$ here, rather than a smallest fixed-point operator $\mu q \cdot$, for easier extension to infinite games later on. See Gheerbrant 2010 for many more details on fixed-point logics over finite game trees.

[6]For instance, in Figure 1, strategies may be thought of a sets of move arrows. Player i has two functional strategies, giving a unique move at the root, and player j has four strategies, with two independent choices for going left or right.

FACT 1.1. *General PDL-programs now define strategies σ.*

Note that, in general, these programs may denote arbitrary transition relations, not just uniquely valued functions. Such relations constrain i's moves at her turns, without necessarily narrowing them down to just a single one. This possible non-determinacy makes eminent sense for our plans of action, and by extension, also for a broader notion of strategies in games.

Now, we can define an explicit version of the earlier forcing modality, indicating the strategy involved – even without the full power of the modal μ–calculus:

FACT 1.2. *For any program expression σ, PDL can define the explicit forcing modality $\{\sigma, i\}\phi$ stating that σ is a strategy for player i forcing the game, against any play of the others, to pass only through states satisfying ϕ.*

The precise definition is an easy exercise in modal logic (cf. van Benthem 2002).

But *PDL* has further uses in this setting. Consider any finite game M with a strategy σ for player i. As a relation, σ is a finite set of ordered pairs (s, t). Thus, it can be defined by enumeration as a program union, if we define these ordered pairs. To do so, assume we have an 'expressive' model M, where states s are definable in our modal language by formulas def_s.[7] Then we define transitions (s, t) by formulas $def_s; a; def_t$, with a the relevant move. This gives:

FACT 1.3. *In expressive finite extensive games, all strategies are PDL-definable.*

Of course, this is a trivial result, but it does suggest that *PDL* is on the right track.

Van Benthem 2002 also discusses further issues about *PDL* as a 'calculus of strategies'. For instance, suppose that player i plays strategy σ, and at the same time, j plays strategy τ. What end nodes are reachable in this way? This calls for an operation on strategies describing the joint strategy of $\{i, j\}$ – which is just the intersection $\sigma \cap \tau$ of the relations σ, τ.[8]

Thus, propositional dynamic logic is a good starting point for logics of strategies, since it can define explicit strategies in simple extensive games. In the next sections, we will extend it to deal with more realistic game structures, such as preferences and imperfect information. But for here, we end with an open problem concerning the purely modal action part of games.

Stronger modal logics of strategies? The modal μ–calculus is a natural strengthening of *PDL* for defining strategic equilibria, but it has no explicit programs or strategies, as its formulas merely define properties of states. Is there a counterpart

[7]This expressive power can be achieved in several ways, e.g., using backward temporal modalities which can describe the total history leading up to s (cf. Rodenhauser 2001).

[8]Adding intersections takes us outside of *PDL* proper, but it still yields a simple modal language.

to the μ–calculus which also extends *PDL* in terms of defining corresponding transition relations? Hollenberg 1996 proposes to use the *PDL*-operations, but these are biased toward terminating programs, whereas the μ–calculus treats terminating and infinite processes on a par. For instance, a natural strategy of 'keep playing a' would guarantee infinite a–branches witnessing true greatest fixed-point formulas of the form $\nu p \cdot \langle a \rangle p$.[9]

PROBLEM 1.1. *Design an explicit action version of the modal μ–calculus.*[10]

3. Preference structure and more realistic games

Real games add an agent dynamics to mere action structure, by encoding preferences for players over outcome states, or utility values beyond 'win' and 'lose'. In this case, defining the so-called Backward Induction procedure for solving extensive games, rather than just Zermelo winning positions, becomes a benchmark for game logics. Many solutions have been proposed for this purpose, mostly involving modal languages for both moves and preference order,[11] with a modality

$$\langle \text{pref}_i \rangle \phi : \text{ player } i \text{ prefers some node where } \phi \text{ holds to the current one.}$$

Definitions for the Backward Induction strategy in modal preference logic have been published by Board, Bonanno, and many others: cf. Harrenstein 2004, De Bruin 2004 and the references therein. For concreteness, we cite just one instance, from van Benthem, van Otterloo & Roy 2006, using the following frame correspondence on finite structures:

FACT 1.4. *The BI strategy is definable as the unique relation σ satisfying the following axiom for all propositions p – viewed as sets of nodes – for all players i:*

$$(\textit{turn}_i \wedge \langle \sigma^* \rangle (\textit{end} \wedge p)) \rightarrow [\textit{move}_i] \langle \sigma^* \rangle (\textit{end} \wedge \langle \textit{pref}_i \rangle p).$$

Properly deciphered, this modal axiom expresses a form of game-theoretic Rationality: "players never play strictly dominated moves, given that they keep playing the strategy at future stages". This frame correspondence is not yet an explicit definition of our earlier sort: the reader might have expected a *PDL*-program defining the above σ with added preference modalities in test conditions. I do not know if the Backward Induction strategy can be defined in such a simple format, but van Benthem & Gheerbrant 2010 transforms the above modal description into explicit definitions in richer fixed-point logics over finite game trees.

---

[9]Even with terminating programs, more strategies may be needed, as in the 'continuous fragment' studied in Fontaine 2010, which properly extends *PDL* to allow any fixed-point recursions that always stop by iteration stage omega.

[10]We discuss an alternative view of defining strategies in fixed-point logics in our Postscript.

[11]Relevant recent work on on modal preference languages that fit with games is found in van Otterloo 2005, Liu 2008, van Benthem, Girard & Roy 2007.

Betterness, preference, and expectation Actually, the situation to be analyzed is subtle conceptually. A game gives players' direct preferences over outcomes. The Backward Induction algorithm then lifts these to a binary order among nodes that does not just represent what players prefer, but what they expect to happen, given rationality assumptions about how the other players will proceed. Thus, the resulting binary order is more like the plausibility relations used to interpret conditional beliefs in doxastic logic, generated from a mixture of preference and assumptions about behaviour of other players. We will return to this issue of strategies mixed with beliefs in the Postscript to this paper.

Action and preference once more As in the preceding section, we conclude with a less standard sort of problem. Modal preference logics work with betterness comparisons between worlds. But in adding preference to *PDL*, our recommended strategy language, we have two levels for putting it: states, but also *transitions between states*. The latter move would make sense, e.g., in analogy with dynamic-epistemic logics where events are treated on a par with states (Baltag, Moss & Solecki 1998).[12] How can one develop such a more radical preferential action logic *PDL*? Working at its two levels also makes sense in philosophy: think of the distinction made in ethics between 'deontology' and 'consequentialism'. The latter compares the worlds resulting from actions when judging our duties, the former qualifies those actions themselves as 'better' or 'obligatory'.

PROBLEM 1.2. *Design a dynamic preference logic on both worlds and actions.*

4. Epistemic logic and extensive games with imperfect information

Next, consider extensive games of imperfect information, which involve 'information sets', or equivalence relations $\sim_i$ between nodes which players i cannot distinguish. These games naturally model a combined epistemic modal language including knowledge operators $K_i\phi$ interpreted in the usual manner as

ϕ is true at all nodes that are $\sim_i$ -related to the current one.

As pointed out in van Benthem 2001, this language can make crucial distinctions such as knowing 'de dicto' that one has a move with effect ϕ, versus having some move of which one knows 'de re' that it yields ϕ:

$$K_i\langle a \cup b\rangle\phi \text{ versus } K_i\langle a\rangle\phi \vee K_i\langle b\rangle\phi.$$

Moreover, as epistemic relations describe what agents can observe in the course of a game, this language can define special properties of agents through modal frame correspondences. An example is the following analysis of the important property of Perfect Recall for a player i in a game model M:

[12] Van der Meyden 1996 goes in this direction, with preferences between transition relations.

FACT 1.5. *The axiom $K_i[a]P \to [a]K_ip$ holds for i w.r.t. any proposition p iff M satisfies Confluence:* $\forall xyz : ((xR_ay \wedge y \sim_i z) \to \exists u : (x \sim_i u \wedge uR_az))$.

Similar analyses work for other powers of memory or observation for agents (cf. Fagin et al. 1995). For instance, as a converse to Perfect Recall, agents satisfy the principle of 'No Miracles' when their epistemic uncertainty can only disappear by observing two subsequent events which they can distinguish.

Now once again for explicit strategies! As before, we can add *PDL*-style programs here to define players' strategies under the new circumstances. But there is a twist. Especially relevant then are the 'knowledge programs' of Fagin et al. 1995, whose only test conditions for actions are knowledge statements for agents. In such programs, the actions prescribed for an agent can only be guarded by conditions which the agent knows to be true or false. It is easy to see that knowledge programs can only define *uniform strategies*, i.e., transition relations where a player always chooses the same move at any two game nodes which she cannot distinguish epistemically. A converse also holds, modulo some assumptions on expressiveness of the game language defining nodes in the game tree (van Benthem 2001):

FACT 1.6. *On expressive finite games of imperfect information, the uniform strategies are precisely those definable by knowledge programs in epistemic PDL.*

But there is still more to games of imperfect information. As with adding preferences, there are two levels for making our base logic *PDL* epistemic. One can connect worlds, as with the above language with standard epistemic modalities K_i. But one can also place epistemic structure on the moves themselves, as happens in dynamic epistemic logic.[13]

Epistemized *PDL* is also a good setting for pursuing a more general aspect of strategies, the famous distinction between 'knowing that' and 'knowing how' (Gochet 2006). Strategies are ways of achieving goals, and hence they represent procedural know-how:

PROBLEM 1.3. *Define what it means to 'know a strategy' in epistemic* PDL*, and then develop a version of DEL with explicit strategies.*

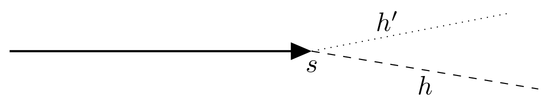
In particular, in an epistemic setting, there is a plausible desideratum that strategies σ should be epistemically transparent, in the sense that when a part of σ has been played, players know at all intermediate stages that playing the rest will achieve the intended result. When players have Perfect Recall in the above sense, this just amounts to knowing the intended effects at the start. But in general, defining epistemic transparency in a generic manner seems non-trivial.

[13] This raises the issue what sort of games correspond to models for dynamic-epistemic logic. This has been determined precisly in van Benthem, Gerbrandy & Pacuit 2007, in terms of Perfect Recall, No Miracles, and Bisimulation Invariance for the domains of events.

5. Ignorance about the future, beliefs and expectations

The phrase 'imperfect information' covers two intuitively different senses of knowledge, which are sometimes confused. One is *observation uncertainty*: players may not have been able to observe all events so far, and so they do not know just where they are in the game tree. This 'past-oriented' knowledge and ignorance is found in *DEL* or epistemic temporal logics (van Benthem & Pacuit 2006). But in action and games of perfect information, there is still 'future-oriented' expectation uncertainty: players may not know where the game is heading since they do not know what others, or they themselves, are going to do. Modeling the latter type of knowledge and ignorance intuitively involves current uncertainty between whole future histories in branching tree of actions, or even between players' strategies (i.e., whole ways in which the game might evolve). Here are a few observations on what this means for strategies.

Branching epistemic temporal models In tree-like models for branching time, 'legal histories' h represent possible evolutions of a given game. At each stage of the game, players are in a node s on some actual history whose past they know, either completely or partially, but whose future is yet to be fully revealed:



This can be described in an action language with knowledge, belief, and added temporal operators. We first consider games of perfect information about the past, with $s\hat{\ }a$ the result of extending the current history up to s with action a:

(a)	$M, h, s \models F_a\phi$	iff	$s\hat{\ }a$ lies on h and $M, h, s\hat{\ }a \models \phi$
(b)	$M, h, s \models P_a\phi$	iff	$s = s'\hat{\ }a$, and $M, h, s' \models \phi$
(c)	$M, h, s \models K_i\phi$	iff	$M, h', s \models \phi$ for all histories h' equal for i to h up to stage s.

Intuitively, along such trees, as moves are played publicly, players receive 'public announcements' of these. This validates logical laws for knowledge:

FACT 1.7. *The following valid principle is the temporal equivalent of the key dynamic-epistemic reduction axiom for public announcement:*[14]

$$F_a\Diamond_i\phi \leftrightarrow (F_a\top \wedge \Diamond_i F_a\phi).$$

As in the earlier modal setting, this implies a form of Perfect Recall: agents' present uncertainties are always inherited from past ones.

[14] For ease of verification, we use a formulation with an existential modality $\Diamond_i = \neg K_i \neg$.

Adding beliefs Next, players i will typically have have beliefs about the further course of the game. To model this, we add binary relations $\leq_i$ of relative plausibility between histories, and we introduce a doxastic modality (which we take to be existential here for convenience):

$$(d) \quad M, h, s \models [B, i]\phi \quad \text{iff} \quad M, h', s \models \phi \text{ for all histories } h' \text{ coinciding with } h \text{ up to stage } s \text{ and most plausible for } i.$$

Beliefs may change gently or drastically over time, and the plausibility relation encodes agents' revision policies. For instance, let B_s^i be the set of most plausible histories for i at h, s, while s' lies on h at some stage later than s. Let some history in B_s agree with h up to s': the actual history is compatible with i's expectations. Then a conservative agent i would let the most plausible histories at s' be the intersection of the most plausible ones in B_s^i with all continuations of s'. But when some unexpected move a is played at s, leading to a history that was not in the most plausible set, a totally new choice needs to be made.

Again, logics will express features of such revision behaviour. Here are two relevant valid axioms for both scenarios (similar principles occur in Bonanno 2007 on temporal AGM theory):[15]

FACT 1.8. *The following laws hold for belief revision along game trees:*

$$\langle B, i\rangle F_a\top \to (F_a\langle B, i\rangle\phi \leftrightarrow (F_a\top \wedge \langle B, i\rangle F_a\phi)).$$

$$F_a\langle B, i\rangle\phi \leftrightarrow (F_a\top \wedge \langle B, i\rangle(F_a\top \wedge F_a\phi)).$$

Temporal models, protocols, and strategies In this setting, there is an issue of how to model the strategies that players follow on branching temporal 'playgrounds'. First of all, a temporal tree model is itself a restricted family of histories, often called a *protocol* (Fagin, Halpern, Moses & Vardi 1995, van Benthem, Gerbrandy, Hoshi & Pacuit 2007). A protocol models a process with certain restrictions on the sequences of events that can occur. But in this sense, a *strategy*, too, is like a protocol, since a players' restriction to certain moves amounts to playing a restricted set of histories of the total game. On the other hand, it also makes sense to distinguish the protocol, as a global constraint defining the total game, from the strategies, further internal choices that players can have in restricting that global protocol. One could then define a strategy as a map from nodes to available histories from that node.[16]

[15]For convenience, we use the existential belief modality in stating the next result.

[16]*Technical caveat:* One cannot in general identify a strategy with just a subset of all the total histories, since strategies are 'subforests' rather than 'subtrees', containing instructions for playing 'counterfactually' at later nodes that will not be reached.

Now temporal models are just given: there is no definition of their set of histories inside the above logical languages. But when strategies are considered as further restrictions of the models, we can indeed talk about them explicitly in terms of *definable submodels* of the complete given tree, allowing us to discuss dynamic matters like changing strategies. Technically, this can be done either in terms of sets of nodes, or sets of histories at nodes. Van Benthem & Pacuit 2007 proposes a simple language for this purpose merging ideas from temporal logic with dynamic logics for plans of action for players – but this is only a start (Wang 2010 is a much more elaborate development):

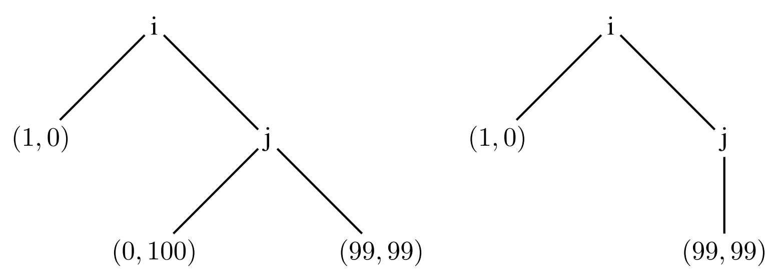
PROBLEM 1.4. *Develop doxastic-epistemic temporal logics of explicit strategies.*

Richer temporal game models Once strategies become first-class citizens again in this manner, further issues of modeling arise. For instance, the natural uncertainty one can have between strategies ("Am I playing a simple automaton, or a sophisticated learner?") cannot be modeled in the above fashion, since it is not about being unsure about histories, but about whole subtrees. This requires full-fledged epistemic game models with worlds including whole strategy profiles. Van Benthem to appearA, Chapter 10, discusses a hierarchy of models for this purpose.

6. Public announcement and changing games

An extensive game, or a branching temporal model, may be viewed as a complete static description of all the histories that a game might go through. Now we shift our logical perspective to dynamic-epistemic logics that *change models*. What happens to strategies and their logics then?

Promises and intentions Here is one concrete illustration of game change affecting strategies. One can break the impasse of a bad Backward Induction solution by making *promises*. E.g., in the following game, the bad equilibrium $(1, 0)$ can be avoided by j's promise that she will not go left – and the new equilibrium $(99, 99)$ results, making both players better off by restricting the freedom of one of them!



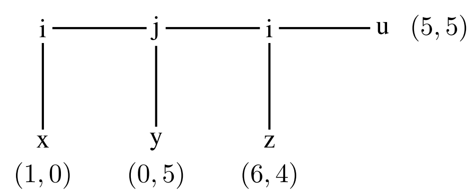
Thus, changing a game tree to a submodel consisting of those nodes satisfying some announced proposition can drastically affect strategic behaviour. Indeed, van Otterloo 2005 has a logic of strategic 'enforceability' plus preference, where models change by reliably announcing players' intentions.[17] [18]

More generally, this perspective highlights an issue that seems somewhat neglected in game theory: what happens to solutions when we make changes in games? In logical model theory, one studies the preservation behaviour of assertions when models change under various constructions (taking sub-models, forming products, etcetera): this perspective also makes eminent sense here.

Rational game dynamics Public announcement and its effects on strategies has many further uses. Van Benthem 2003 uses *iterated public announcement* of 'Rationality' assertions to analyze game solution procedures: the procedure ends when a first fixed-point is reached where the model no longer changes. At that stage, Rationality has become common knowledge, and only strategy profiles remain that fall under the relevant solution concept. Now this operates at the level of strategic games, which we will largely ignore in this paper, even though it raises interesting logical issues of its own.[19] But a similar analysis applies to extensive games. Let us call a move a *dominated* if the player who owns it has another move whose final outcomes would all be better than those reachable by playing on after a.

FACT 1.9. *The Backward Induction solution for extensive games is obtained through repeated announcement of the assertion "no player chooses a strictly dominated move in the current game".*

Here is how this would work out for a Centipede game, with three turns as indicated, branches indicated by name, and pay-offs indicated for i, j, in that order:



Stage 0 of the announcement procedure rules out branch u, Stage 1 then rules out z, while Stage 2 finally rules out y. We suppress some technicalities here:

[17]Other game changes might include *addition* of moves, which seems to call for product-update techniques: cf. van Benthem, van Eijck & Kooi 2006.

[18]Admittedly, one might rephrase this setting in our earlier temporal models by coding up all possible changes beforehand in one 'Super Game', but that would lose the flavour of understanding strategic behaviour in a stepwise manner.

[19]The structure of strategic games calls for epistemic-preferential logics of matrices or *'grid structures'* rather than trees, and matching logical languages may differ in spirit from the above. We leave their study for another occasion.

with a little more care, the announcement procedure really records all Backward Induction moves, also those off the eventually chosen path.[20]

Similar points apply to dynamic scenarios where games change through actions of *belief revision* rather than knowledge update.

Matching languages and logics for strategies The preceding models and scenarios call for dynamic-epistemic languages of both games and strategies. We start with a simple toy example.

If we use *PDL* for strategies and moves in extensive games as before, the simplest combination with dynamics would be the combined modal logic *PDL+PAL* adding public announcements $[!\phi]$. Essentially, the latter modality describes a relativization to the sub-model defined by ϕ, and it is easy to show that *PDL* is closed under relativization in both its propositional and its program parts.[21]

FACT 1.10. *The logic PDL+PAL is axiomatized completely by merging the separate laws of PDL and PAL, while adding the following reduction axiom:*

$$[!\phi]\{\sigma\}\phi \leftrightarrow (\phi \rightarrow \{\sigma|\phi\}[!\phi]\phi).$$

For *PDL* plus full *DEL*-style product update, cf. van Benthem, van Eijck & Kooi 2006, and for μ–calculus versions, van Benthem & Ikegami 2007.

The wrong way around? Maybe the above reduction axiom misses the real issue in strategy analysis. It explains what old plan I should have had in order to make some new plan work in the changed game model. But usually, the issue is the other way around. I already have a plan σ for playing game G to obtain a certain effect ϕ. Now G changes to G': my machine lost some functionality, my game got some extra moves, and the like. How should I then revise that current plan σ to get some intended effect ψ in the new game G'? This is not answered by our system.[22]

However this may be, the above logics are clearly just a start. We need much richer languages that can also deal with preferences and other crucial game structure:

PROBLEM 1.5. *Give a logic that can deal with strategies under real game change.*

---

[20]There are many other options here: instead of the temporally 'forward-looking' assertion of Rationality, van Benthem 2003 also discusses temporally 'backward-looking' announcements where players remind themselves of the *legitimate rights of other players*, and build up trust. In line with this, type space models for games (cf. Brandenburger 2007) allow for much greater freedom in assumptions about other players – turning, one might say, Game Theory into a *Theory of Players*.

[21]The key to the syntactic relativization is a recursive operation $\pi|\phi$ for programs π which surrounds every atomic move with tests $?\phi$.

[22]There may be analogies here with open model-theoretic problems concerning *PDL* with model change. For instance, there is no good Łoś-Tarski theorem syntactically characterizing those *PDL* formulas that are preserved under sub-models.

Coda: announcement games Here is another way of thinking. Instead of using announcements over arbitrary games and abstract strategies, one can also view announcements themselves as actions in a 'conversation game'. In that case, strategies rather have to do with what assertions are available, and how these lead to intended effects for the players. Cf. Agotnes & van Ditmarsch 2009 for more on such 'knowledge games'.

7. A more global level: logics of powers

Here is another approach to our topic, moving to a 'coarser' input-output view of social interaction than extensive games by hiding the strategies, so to speak.

Powers over outcomes and dynamic game logic A global view of games only cares about outcomes, not about local moves on the way to the end result. This is the level of powers for players i, being the sets of outcomes X for which i has a strategy forcing the game to end inside X, whatever the other players do. This is the level of the 'forcing modality' defined in Section 1, and it supports its own style of modal logic over modal 'neighbourhood models' (van Benthem & Blackburn 2006), including bisimulations and other basic notions.

Here is also where Parikh's dynamic game logic lives (DGL, Parikh & Pauly 2003). Its models M encode forcing relations $\rho^{G,i}sX$ for players i in atomic games G, played from some initial sate s.[23] This notion is then lifted inductively to the right powers for both players in complex games G formed using further operations of 'dual' ^{d}, 'choice' $\cup$, 'sequential composition' ; , and 'iteration' *. The matching forcing modality is this:

$$M, s \models \{G, i\}\phi \text{ iff } \exists X : \rho^{G,i}sX \wedge \forall x \in X : M, x \models \phi.$$

The laws of this logic reflect basic properties of powers for players i, j:

$C1$	if $\rho^{G,i}sY$ and $Y \subseteq Z,$ then $\rho^{G,i}sZ.$	Monotonicity
$C2$	if $\rho^{G,i}sY$ and $\rho^{G,j}sZ,$ then Y, Z overlap.	Consistency

In the usual version of dynamic game logic, they also satisfied Determinacy:

$C3$ if not $\rho^{G,i}sY,$ then $\rho^{G,j}s(S - Y),$ and the same for j versus i.

Conversely, these three conditions are also all that must hold:

FACT 1.11. *Families F_1, F_2 of subsets of some set S satisfying $C1$, $C2$, $C3$ are the powers for players i, j at the root of some determined two-step game.*

The result also extends to games with imperfect information (van Benthem 2001), using just the first two conditions $C1$, $C2$.

[23] Here it is crucial that the same game can be played from different starting states.

The power analysis fits naturally with the Thompson Transformations in game theory (Osborne & Rubinstein 1994) for games of imperfect information with players having Perfect Recall. It is also the level at which logicians view 'logic games' of evaluation, model comparison, or proof (van Benthem 2007C).

For concreteness, we state some crucial compositional axioms of DGL (cf. Pauly 2001 for details on the model theory and axiomatics):

FACT 1.12. *The following principles are valid in modal forcing semantics:*

$$\begin{array}{ll} \{G_1; G_2\}\phi \leftrightarrow \{G_1\}\{G_2\}\phi & \{G_1 \cup G_2\}\phi \leftrightarrow \{G_1\}\phi \vee \{G_2\}\phi \\ \{G^d\}A \leftrightarrow \neg\{G\}\neg A & \{?P\}\psi \leftrightarrow (P \rightarrow \psi). \end{array}$$

The soundness arguments for these axioms involve some basic reasoning with strategies. But this reasoning is the essence of what makes the logic tick, so let's formalize it explicitly!

'Strategizing' logics of powers

DGL describes the compositional structure of the games under sequential operations, on the model of *PDL* with neighbourhood semantics.[24] But now we want to formalize its reasoning about strategies explicitly, i.e., we need to 'strategize' it:

PROBLEM 1.6. *Add strategies to* DGL*, with modalities $\{\sigma, i, G\}\phi$ for "strategy σ for player i forces outcome ϕ in game G"; and determine the logic.*

There are grounds for optimism. We already know that making witness objects for existential quantifiers explicit can be done with beneficial effects in the study of proof and computation. Artemov 1998's 'logic of proofs' did this for modal provability logic, replacing boxes $\Box$ for 'provability' by boxes $[t]$ containing terms t for proofs or pieces of evidence – and it has been extended to deal with general epistemic evidence by the 'New York School'.[25]

One obvious attempt would 'do a double *PDL*' on Dynamic Game Logic, by also bringing in a description of strategies in our earlier spirit. Here is an illustration. Viewing a strategy as something which we can unpack into a 'head' (the first move to be played) and a 'tail' (the rest), we have that

$$\{\sigma, i, G \cup H\}\phi \leftrightarrow \{\text{tail}(\sigma), i, G\}\phi \vee \{\text{tail}(\sigma), i, H\}\phi.$$

What is the right semantic setting for making precise sense of this?[26] *DGL* expressions are interpreted on abstract board models, while strategies seem to live inside

[24]Dealing with parallel game composition is another story: see below.

[25]Renne 2006 adds explicit strategies linking up with dynamic epistemic logic.

[26]Note the direction: *DGL* analyzes strategies top-down into substrategies in component games. What about bottom-up strategy construction laws like $\{\sigma, i, G\}\phi \rightarrow \{\langle \text{LEFT}; \sigma\rangle, i, G \cup H\}\phi$?

concrete games. Van Benthem 1999-2002 (Chapter 5) proposed using two languages for this purpose – one game-external over 'board models', and one-game internal, referring to turns, moves, and other procedural features.

Parallel operations There are also further natural game constructions affecting strategies. *DGL* is about sequential game operations: what about concurrency, and operations for parallel play – surely, a natural scenario in practice? Van Benthem, Ghosh & Liu 2007 propose a concurrent dynamic game logic *CDGL* that works with non-determined games. Its crucial axiom is a simple reduction for game products $G_1 \times G_2$ where players pay simultaneously without any intermediate communication:

$$\{G_1 \times G_2, i\}\phi \leftrightarrow \{G_1, i\}\phi \wedge \{G_2, i\}\phi.$$

But again, explicit strategies are missing, so we have an obvious extension of our earlier question, viz. to 'strategize' $CDGL$.[27]

Temporal logics once more Finally, opening up existential modalities of abilities for agents also seems natural in temporal logics. Consider the logic of agency *STIT* (Belnap et al. 2001) which describes agents' powers for 'seeing to it that' certain states of the world are realized without mentioning explicit actions. The intuition underlying *STIT* is that of some agents acting simultaneously, knowing only their own action, but not that of the others – something reminiscent of games with a mild form of imperfect information, as well as our epistemic temporal logics, but now enriched with an intersection operation for parallel action.

PROBLEM 1.7. *Develop an explicit action/strategy-based version of STIT, analyzing its time steps as parallel actions for a bunch of players.*

I conclude with a few brief illustrations of two further directions. Our main interest has been finite extensive games with individual players. Let us now look at larger groups, and after that, at wider temporal horizons.

8. Coalition logics and group strategies

Most game logics deal with two individual players, though game theory also deals with coalitions of players. Now consider the logic of agents' powers in Pauly 2001. Here $\{G\}\phi$ means that the agents in the group G can achieve a set of outcomes all of which satisfy the proposition ϕ. In such a language we can talk about group powers, and interesting new principles emerge. For instance, if the groups G, H are disjoint, the following principle describes their cooperation:

$$(\{G\}\phi \wedge \{G\}\psi) \rightarrow \{G \cup H\}(\phi \wedge \psi).$$

[27] For other relevant game logics of concurrency, see Section 9.

But as usual in logics of powers, no explicit actions or strategies are provided. But it is clear what we really want to say here:

FACT 1.13. *The following principle is valid:*

$$(\{G, \sigma\}\phi \wedge \{H, \tau\}\psi) \rightarrow \{G \cup H, \sigma\#\tau\}(\phi \wedge \psi)$$

with $\sigma\#\tau$ for intersection of strategies (viewed as relations).

There are many further interesting laws at this explicit strategy level that describe, e.g., how powers of a coalition combine powers for subgroups.

PROBLEM 1.8. *Give an explicit action/strategy version of Coalition Logic.*

Coalition logic becomes more powerful when explicit preferences of the players are added in the form of suitable modal operators. Van der Hoek 2007 shows what significant collective scenarios can be analyzed then.

9. Infinite games and linear logics

From finite to infinite games Infinite never-ending processes are as fundamental as finite terminating ones, and modern logics of computation favour neither perspective over the other. Likewise, infinite games are crucial, for instance, in evolutionary game theory. Such games, too, can be analyzed with explicit logics of strategies, taking whole histories as outcomes.

This suggests returning to our earlier temporal logics of Section 5, and one good starting point is again our idea that a minimal logic of explicit strategies should at least be able to represent basic game-theoretic arguments.

Here is an illustration. One obvious candidate beyond Zermelo's Theorem in Section 2 is the *Gale-Stewart Theorem.* It says that all infinite two-player games of perfect information, viewed as topological trees, where the winning condition for one of the players is an open set, are determined. The key point in the proof of this result is a universally valid principle of 'Weak Determinacy':

> "A player i either has a winning strategy, or the other player j has a strategy making sure that i never reaches a stage in the game where she has a winning strategy."

This calls for a temporal logic describing these powers. Here is a natural candidate, using our earlier forcing modalities plus a temporal modality G ('always in the future'). Weak Determinacy then becomes a formula of the system:

FACT 1.14. $\{i\}\phi \vee \{j\}G\neg\{i\}\phi$ *is a valid law for infinite games.*

PROBLEM 1.9. *Find a complete temporal logic of players' powers over infinite games. And 'strategize' it using some suitable logic of protocols.*

Game semantics for linear logic Finally, here is a related area of logic where strategies are crucial, viz. the game semantics of Abramsky 1997 for linear logic and concurrency in general. Reasoning about strategies is crucial to understanding the validity of the axioms of linear logic. The protagonist in their verification is an evergreen from game theory, viz. the 'Copy Cat' strategy (or 'Tit for Tat') copying moves from one game played in parallel with another. But again, these strategies are not explicitly represented in the language of linear logic. A basic linear axiom like $A \oplus A^d$, with $\oplus$ for 'parallel disjunction' of games, is interpreted as follows:

> In each concrete game of the form $A \oplus A^d$ (i.e., A played in parallel with its role-switched version), the distinguished player P has a winning strategy.[28]

This is even less informative than the global power formulas of Parikh's *DGL*, which talk about different propositions true after the game. And so we ask:

PROBLEM 1.10. *Develop a logic of strategies which formalizes the soundness arguments for linear logic, and make further statements about the course of games.*

This would again call for a merge of logics of strategies like we had before with a language that can describe complex game structure.

Strategies and proofs The case of linear logic also raises a more general issue. Linear logic arose originally from mathematical proof theory. Now it has been known ever since Lorenzen's dialogue games (cf. van Benthem 2007C) that a proof is like a strategy for winning an interactive debate, cast as a 'logic game'.[29] Thus, proof theories or type theories with explicit proof terms are an explicit calculus of strategies! But in this paper, we have mainly followed a model-theoretic take on games and strategies, leading to modal logics of actions and processes, and strategies in the style of dynamic logic. How the two broad styles of thinking about strategies, proof-theoretic and model-theoretic, are related in general remains a deep issue beyond the scope of this paper.

10. Conclusion

Explicit logics of strategies are desirable, feasible, and most of all: fun to explore!

---

[28] Other parallel operations in logic occur with *Ehrenfeucht games* comparing different models.

[29] We will not go further into the general issue of 'game logics' versus 'logic games' – though this entanglement, too, seems relevant to our view of strategies. For instance, as we have seen, strategies in games can be made explicit using formulas in logical languages. But logical formulas can be interpreted in evaluation games over suitable models. But then, their truth amounts to stating that there exists a winning strategy for the Verifier in this game. The spiral goes on and on.

Acknowledgment I thank the students of the 2007 ILLC seminar in Logic of Games for their responses, and Jan van Eijck and Rineke Verbrugge for her editorial comments. Two anonymous referees also provided valuable feedback.

11. Postscript

In the years since this paper was written, a number of relevant things have happened. The following is a brief list, without any attempt at completeness.

Fixed-point logics for game solution Van Benthem & Gheerbrant 2010 is an exploration of Backward Induction in fixed-point logics on trees, showing how three approaches coincide: unique recursive definitions in terms of Rationality, limits of iterated announcements of Rationality, and limits of upgrade procedures producing common belief in Rationality. In particular, following an idea from Baltag, Smets & Zvesper 2009, it is shown how strategies in extensive games can also be seen equivalently as plausibility orders among leaves which encode conditional beliefs, linking up between action views and epistemic-doxastic views of games. Much more mathematical background on relevant fixed-point logics is found in Gheerbrant 2010, Fontaine 2010. From the perspective of explicit logics of strategies, fixed-point logics are interesting, since their static existence statements also carry procedural information about how to reach solutions via the standard built-in fixed-point approximation procedures.

Logics of protocols Hoshi 2009 is an extensive study of protocols in dynamic-epistemic logic, including new complete logics, and applications to epistemology. Protocols are represented only partly in these languages, however, through local assertions about the availability of announcements, or of general events that carry ‘procedural information’. The thesis Wang 2010 is a systematic development of a fully explicit view of protocols using *PDL* techniques, and so far, the most extensive representative of a larger CWI project on logics of communication protocols that has also produced papers like van Eijck, Kuppusamy & Wang 2009, van Eijck, Sietsma & Wang 2009.

Logics of strategies and automata perspectives A bunch of papers emananating from Indian researchers has developed various logics of strategies taking ideas from automata theory and dynamic logic, Cf. Ramanujam & Simon 2008A, Ramanujam & Simon 2008B, Ramanujam & Simon 2009, Pacuit & Simon 2010. Ghosh 2008 is also relevant as a congenial ’strategization’ of dynamic logic.

Logics for epistemic foundations of game theory Strategies have also received renewed logical attention in current logical studies of the epistemic foundations of game-theoretic solution procedures. The dissertations Dégrémont 2010 and Zvesper 2010 contain a wide spectrum of issues, from strategic games to extensive games. In particular, Dégrémont and Roy 2009 and Baltag & Smets 2009 study effects of update procedures in terms of Agreement or Rationality that gradually eliminate strategy profiles. Apt & Zvesper 2010 links logical procedures with computational perspectives on game solution algorithms.

Logics of strategies and agency In the agency tradition, various authors have proposed extensions of current framworks with explicit actions and strategies. Van der Hoek, Walther & Wooldridge 2007 adds strategies to Alternating Temporal Logic (a rich logic for players' power in games allowing simultaneous action), Chatterjee, Henzinger & Piterman 2007, too, propose explicit logics of strategies, while Xu 2010 adds actions to the *STIT* logic of agency from the philosophical tradition.

Connections with learning theory Strategies in long-term informational processes are related to learning mechanisms. Dégrémont & Gierasimczuk 2009 show how Learning Theory can be linked with dynamic-epistemic-temporal logics, and Gierasimczuk 2010 is a full-fledged study, including a logical study of learning strategies. Agotnes & van Ditmarsch 2009 studies a special case in more depth: 'knowledge games' over epistemic models.

Finally, two recent books are relevant: van Benthem toappear A on information-driven logical dynamics of agency, and van Benthem to appearB on logic in games. A source of recent information on current developments are the "Yearbooks on Logics for Dynamics of Information and Preferences", edited by D, Grossi, L. Kurzen & F. Velazquez Quesada (ILLC Amsterdam), as well as the LORI webpage on Logics of Rationality and Intelligent Interaction `www.lori.org`.

ADDITIONAL REFERENCES

Th. Agotnes & H. van Ditmarsch, 2009, What Will They Say? Public Announcement Games, University of Bergen & University of Seville. Presented at: Logic, Game Theory and Social Choice 6, Tsukuba, Japan.

K. Apt & J. Zvesper, 2010, 'Proof-Theoretic Analysis of Rationality for Strategic Games with Arbitrary Strategy Sets. CLIMA XI 2010: 186-199.

J. van Benthem & A. Gheerbrant, 2010, 'Game Solution, Epistemic Dynamics and Fixed-Point Logics', Fundamenta Informaticae 100 (1-4): 19-41.

A. Baltag & S. Smets, 2009A, Group Belief Dynamics under Iterated Revision: Fixed Points and Cycles of Joint Upgrades, Proceedings TARK XII, Stanford, 4150.

A. Baltag, S. Smets & J. Zvesper, 2009, Keep Hoping for Rationality: a Solution to the Backward Induction Paradox, Synthese 169, 2009, 301333.

J. van Benthem, to appearA, Logical Dynamics of Information and Interaction, Cambridge University Press, Cambridge.

J. van Benthem, to appearB, Logic in Games, ILLC, University of Amsterdam and Texts in Logic and Games, Springer, Heidelberg.

K. Chatterjee, T. Henzinger & N. Piterman, 2007, 'Strategy Logic', Proceedings 18th International Conference on Concurrency Theory, Springer Lecture Notes in Computer Science 4703, 59-73.

C. Dégrémont, 2010, The Temporal Mind: Observations on Belief Change in Temporal Systems, Dissertation, ILLC, University of Amsterdam.

C. Dégrémont & N. Gierasimczuk, 2009, 'Can Doxastic Agents Learn? On the Temporal Structure of Learning', Proceedings of the 2nd International conference on Logic, Rationality and Interaction, Chongqing China, J. Horty & E. Pacuit eds., Springer Verlag, 90-104.

C. Dégrémont & O. Roy, 2009, Agreement Theorems in Dynamic Epistemic Logic, In A. Heifetz, ed., Proceedings TARK 2009, Stanford, 9198.

J. van Eijck, F. Sietsma & Y. Wang, 2009, Logic of Information Flow on Communication Channels, CWI, Amsterdam, posted at www.lori.org.

J. van Eijck, L. Kuppusamy & Y. Wang, 2009, 'Verifying Epistemic protocols under Common Knowledge', A. Heifetz, ed., Proceedings TARK 2009, 257-266.

R. Fagin, J. Halpern, Y. Moses & M. Vardi, 1997, 'Knowledge-Based Programs', Distributed Computing 10, 199-225.

R. Fagin, J. Halpern, Y. Moses & M. Vardi, 1995, Reasoning about Knowledge, The MIT Press, Cambridge (Mass.).

G. Fontaine, 2010, Modal Fixpoint Logic: some model theoretic questions, Dissertation, ILLC, University of Amsterdam.

A. Gheerbrant, 2010, Fixed-Point Logics on Trees, Dissertation, ILLC, University of Amsterdam.

N. Gierasimczuk, 2010, In the Limits of Knowledge: Logical Analysis of Inductive Inference, Dissertation, ILLC, University of Amsterdam.

P. Girard, 2008, Modal Logic for Belief and Preference Change, Ph.D. thesis DS-2008-04, Institute for Logic, Language and Computation, University of Amsterdam.

S. Ghosh 2008, 'Strategies Made Explicit in Dynamic Game Logic', Proceedings Workshop on Logic and Intelligent Interaction, ESSLLI 2008 Hamburg, 74-81.

W. van der Hoek, D. Walther & M. Wooldridge, 2007, 'Alternating-Time Temporal Logic with Explicit Strategies', Proceedings TARK 11, 269-278.

M. Hollenberg, 1996, Logic and Bisimulation, Ph.D. Thesis, Zeno Institute for Philosophy, University of Utrecht.

T. Hoshi, 2009, Epistemic Dynamics and Protocol Information, Ph.D. thesis, Department of Philosophy, Stanford University (ILLC-DS-2009-08).

F. Liu, 2008, Changing for the Better: Preference Dynamics and Agent Diversity, Ph.D. thesis DS-2008-02, Institute for Logic, Language and Computation, University of Amsterdam.

E. Pacuit & S. Simon, 2010, 'Reasoning with Protocols under Imperfect Information', University of Tilburg & ILLC Amsterdam (Erasmus Mundus project).

R. Ramanujam & S. Simon, 2008A, 'A Logical Structure for Strategies', Proceedings LOFT 7, Amsterdam University Press, Texts in Logic and Games 3, 183-208.

R. Ramanujam & S. Simon, 2008B, 'Dynamic Logic on Games with Structured Strategies', Proceedings KR 11, AAAI Press, 49-58.

R. Ramanujam & S. Simon, 2009, 'Dynamic Logic of Tree Composition', Perspectives in Concurrency Theory, CRC Press, 408-430.

M. Xu, 2010, 'Combinations of Stit and Actions', in Journal of Logic, Language and Information, on-line first, 13 January 2010.

J. Zvesper, 2010, Playing with Information, Dissertation, ILLC, University of Amsterdam.

UPDATES OLD REFERENCES

S. Abramsky, 1997, 'Semantics of Interaction: an Introduction to Game Semantics', in Proceedings of the 1996 CLiCS Summer School, Isaac Newton Institute,

P. Dybjer and A. Pitts, eds. (Cambridge University Press) 1997, 1 - 31.

S. Artemov. 1998, 'Logic of Proofs: a unified semantics for modality and lambdaterms,' Technical Report CFIS 98-06, Cornell University.

N. Belnap, M. Perloff,& M. Xu, 2001, Facing the Future: Agents and Choices in Our Indeterminist World, Oxford University Press, Oxford.

J. van Benthem, 1999 2003, Logic in Games, lecture notes, Institute for Logic, Language and Computation, University of Amsterdam.

2001, 'Games in Dynamic Epistemic Logic', Bulletin of Economic Research 53:4, 219-248.

2002, 'Extensive Games as Process Models', Journal of Logic, Language and Information 11, 289-313.

2007, 'Rational Dynamics', International Game Theory Review 9:1, 2007, 13-45. Erratum reprint, Volume 9:2, 377-409.

2005, 'Open Problems in Logic and Games', in S. Artemov, H. Barringer, A. d'Avila Garcez, L. Lamb& J. Woods, eds., Essays in Honour of Dov Gabbay, King's College Publications, London, 229-264.

2007A, 'Dynamic Logic of Belief Revision', Journal of Applied Non-Classical Logics 17, 129155.

2007B, Interview, Five Questions in Game Theory, V. Hendricks, ed., Automated Press, 9-19.

2007C, 'Logic Games: from tools to models of interaction', in A. Gupta, R. Parikh & J. van Benthem, eds., Logic at the Crossroads, Allied Publishers, Mumbai, 283317.

J. van Benthem & P. Blackburn, 2006, 'Modal logic, a Semantic Perspective', in P. Blackburn, J. van Benthem & F. Wolter, eds., Handbook of Modal Logic, Elsevier Science Publishers, Amsterdam, 1-84.

J. van Benthem, J. van Eijck & B. Kooi, 2006, 'Logics of Communication and Change'. Information and Computation 204(11), 1620-1662

J. van Benthem, J. Gerbrandy & E. Pacuit, 2007, 'Merging Frameworks for Interaction: DEL and ETL', Proceedings TARK 2007, Namur.

J. van Benthem, P. Girard & O. Roy, 2009, Everything Else Being Equal. A Modal Logic Approach to Ceteris Paribus Preferences, Journal of Philosophical Logic 38, 83125.

J. van Benthem & D. Ikegami, 2008, Modal Fixed-Point Logic and Changing Models, in A. Avron, N. Dershowitz & A. Rabinovich, eds., Pillars of Computer Science: Essays Dedicated to Boris (Boaz) Trakhtenbrot on the Occasion of his 85th Birthday, Springer, Berlin, 146165.

J. van Benthem & F. Liu, 2007, Dynamic Logic of Preference Upgrade, Journal of Applied Non-Classical Logics 17, 157182.

J. van Benthem, S. Ghosh and F. Liu, 2008, 'Modelling Simultaneous Games with Dynamic Logic', Knowledge, Rationality and Action, 165(2): 247 - 268.

J. van Benthem, S. van Otterloo & O. Roy, 2006, 'Preference Logic, Conditionals, and Solution Concepts in Games', in H. Lagerlund, S. Lindstrom & R. Sliwinski, eds., Modality Matters, University of Uppsala, 61 - 76.

J. van Benthem & E. Pacuit, 2006, 'The Tree of Knowledge in Action', Proceedings AiML IV, Melbourne, 87-106.

J. van Benthem & E. Pacuit, 2007, 'Modeling Protocols and Transforming Temporal Models', manuscript, ILLC, University of Amsterdam,

G. Bonanno, 2007, 'Axiomatic Characterization of the AGM Theory of Belief Revision in a Temporal Logic', Artificial Intelligence 171: 2-3, 144-160.

J. Bradfield & C. Stirling, 2006, 'Modal μ-Calculi', in P. Blackburn, J. van Benthem & F. Wolter, eds., Handbook of Modal Logic, Elsevier Science Publishers, Amsterdam, 721-756.

A. Brandenburger, 2007, Tutorial on Game Theory, First Amsterdam-Lausanne-London Graduate Workshop, London School of Economics.

B. de Bruin, 2004, Explaining Games, The Epistemic Program in Game Theory, Dissertation 2004, ILLC, University of Amsterdam. Extended version in Synthese Library, Springer, 2010.

P. Gochet, 2006, La Formalisation du Savoir-Faire, Lecture at Pierre Duhem Colloquium IPHRST Paris, Philosophical Institute, Universit de Liege.

P. Harrenstein, 2004, Logic in Conflict, Dissertation, Department of Computer Science, Utrecht University.

W. van der Hoek, 2007, 'Extended Modal Logic for Social Software', invited lecture, LORI Workshop, Beijing.

W. van der Hoek, S. van Otterloo & M. Wooldridge, 2005, 'A Logic for Strategic Reasoning', Proceedings of the fourth international joint conference on Autonomous Agents and Multiagent Systems AAMAS, Utrecht, 157 164.

R. van der Meyden, 1996, 'The Dynamic Logic of Permission', Journal of Logic and Computation 6:3, 465 - 479.

M. Osborne & A. Rubinstein, 1994, A Course in Game Theory, The MIT Press, Cambridge, Mass.

S. van Otterloo, 2005, A Security Analysis of Multi-Agent Protocols, Dissertation, ILLC, University of Amsterdam, and Department of Computing, University of Liverpool.

R. Parikh & M. Pauly, 2003, 'Game Logic - An Overview', Studia Logica 75(2): 165-182.

M. Pauly, 2001, Logic for Social Software, Dissertation, ILLC, University of Amsterdam.

B. Renne, 2006, 'Propositional Games with Explicit Strategies', in G. Mints and R. de Queiroz (Eds.), Proceedings of the 13th Workshop on Logic, Language, and(WoLLIC), Stanford, CA.

B. Rodenhauser, 2001, Updating Epistemic Uncertainty, Master Thesis MoL-2001-07, ILLC, University of Amsterdam.