THE KUZNETSOV-GERČIU AND RIEGER-NISHIMURA LOGICS: THE BOUNDARIES OF THE FINITE MODEL PROPERTY

GURAM BEZHANISHVILI, NICK BEZHANISHVILI, DICK DE JONGH

To the memory of A.V. Kuznetsov (1926–1984)

Abstract. We give a systematic method of constructing extensions of the Kuznetsov-Gerčiu logic **KG** without the finite model property (fmp for short), and show that there are continuum many such. We also introduce a new technique of gluing of cyclic intuitionistic descriptive frames and give a new simple proof of Gerčiu's result [8, 7] that all extensions of the Rieger-Nishimura logic **RN** have the fmp. Moreover, we show that each extension of **RN** has the poly-size model property, thus improving on [8]. Furthermore, for each function $f : \omega \to \omega$, we construct an extension L_f of **KG** such that L_f has the fmp, but does not have the f-size model property. We also give a new simple proof of another result of Gerčiu [8] characterizing the only extension of **KG** that bounds the fmp for extensions of **KG**. We conclude the paper by proving that $\mathbf{RN.KC} = \mathbf{RN} + (\neg p \vee \neg\neg p)$ is the only pre-locally tabular extension of **KG**, introduce the internal depth of an extension L of **RN**, and show that L is locally tabular if and only if the internal depth of L is finite.

1. Introduction

A.V. Kuznetsov was one of the pioneers in the study of extensions of intuitionistic propositional calculus **IPC**. He coined them as (*propositional*) *superintuitionistic logics* and undertook a systematic study of their structure (see, e.g., the survey articles [14, 15, 17, 16]). Kuznetsov was especially interested whether a logical system is decidable. A theorem by Harrop [10] states that if a propositional logical system is finitely axiomatizable and has the fmp, then it is decidable. This led Kuznetsov to study systematically the fmp and finite axiomatizability of superintuitionistic logics. In collaboration with his student V. Ja. Gerčiu, Kuznetsov introduced a superintuitionistic logic—we call it the *Kuznetsov-Gerčiu logic* and denote it by **KG**—and studied the fmp and finite axiomatizability of extensions of **KG** [13, 9]. Kuznetsov and Gerčiu proved that there exist extensions of **KG** that do not have the fmp and are not finitely axiomatizable.

The logic **KG** is defined as the logic of sums of cyclic Heyting algebras. Dually they correspond to sums of cyclic intuitionistic descriptive frames. It follows that **KG** is contained in the logic of the free cyclic Heyting algebra, known as the *Rieger-Nishimura lattice*. The dual frame of the Rieger-Nishimura lattice is the well-known *Rieger-Nishimura ladder*. We call this logic the *Rieger-Nishimura logic* and denote it by **RN**. It turns out that **RN** is the greatest 1-conservative extension of **IPC**. In this paper we introduce a new technique of gluing of cyclic intuitionistic descriptive frames and give a new simple proof of a result of Gerčiu [8, 7] that all extensions of **RN** have the fmp. We also show that each extension of **RN** has the poly-size model property, thus improving on [8]. On the other hand, for each function $f : \omega \to \omega$, we construct an extension L_f of **KG** such that L_f has the fmp, but does not have the f-size model property. Moreover, we give a systematic method of constructing extensions of **KG** without the fmp, and show that there are continuum many such. We conclude the paper by giving a new simple proof of another result of Gerčiu [8] characterizing the only extension of **KG** that bounds the fmp for extensions of **KG**, show that the logic **RN**.**KC**—which is obtained by adding the law of weak excluded middle to **RN**—is the only pre-locally tabular extension of **KG**, introduce the internal depth of an extension L of **RN**, and prove that L is locally tabular if and only if the internal depth of L is finite.

The paper is organized as follows. Section 2 consists of preliminaries to make the paper as self-contained as possible. In Section 3 we introduce the logics **RN** and **KG**, give a simple finite axiomatization of **KG**, and describe finite and finitely generated rooted descriptive **KG**-frames. We also describe finite rooted **RN**-frames. In Section 4 we introduce our technique of gluing, describe finitely generated rooted descriptive **RN**-frames, and give a simple finite axiomatization of **RN**. In Section 5 we prove that all extensions of **RN** have the fmp, and construct continuum many extensions of **KG** that do not have the fmp. In Section 6 we show that each extension of **RN** has the poly-size model property, and for each function $f : \omega \to \omega$, construct an extension of **KG** with the fmp but without the f-size model property. In Section 7 we describe the extension of **KG** that bounds the fmp in extensions of **KG**. Finally, in Section 8 we show that **RN**.**KC** is the only pre-locally tabular extension of **KG**, define the internal depth of an extension L of **RN**, and prove that L is locally tabular if and only if the internal depth of L is finite.

2. Preliminaries

We assume the reader's familiarity with the intuitionistic propositional calculus **IPC** and its Kripke semantics. For details we refer to [4, 3].

2.1. **Descriptive frames and frame based formulas.** We recall that an *intuitionistic Kripke frame* is a partially ordered set (poset) $\mathfrak{F} = (W, \le)$. For a poset $\mathfrak{F} = (W, \le)$, $w \in W$, and $U \subseteq W$, let $\uparrow w = \{v \in W : w \le v\}$, $\uparrow U = \{w \in W : \exists u \in U \text{ with } u \le w\}$, $\downarrow w = \{v \in W : v \le w\}$, and $\downarrow U = \{w \in W : \exists u \in U \text{ with } w \le u\}$. We also recall that $U \subseteq W$ is an *upset* of W if $u \in U$ and $u \le v$ imply $v \in U$. Let $\mathrm{Up}(\mathfrak{F})$ denote the set of upsets of $\mathfrak{F}$.

Definition 2.1. [4, Section 8.1] *An* intuitionistic general frame *or simply a* general frame *is a triple $\mathfrak{F} = (W, \le, \mathcal{P})$ such that $(W, \le)$ is an intuitionistic Kripke frame and $\mathcal{P}$ is a set of upsets of $\mathfrak{F}$ such that $\emptyset, W \in \mathcal{P}$ and $\mathcal{P}$ is closed under $\cup$, $\cap$, and $\to$, where:*

$$U \to V = \{w \in W : \uparrow w \cap U \subseteq V\} = W - \downarrow(U - V).$$

Definition 2.2. [4, Section 8.4] *Let $\mathfrak{F} = (W, \le, \mathcal{P})$ be a general frame.*

1. *We call $\mathfrak{F}$* refined *if for each $w, v \in W$, from $w \not\le v$ it follows that there is $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$.*
2. *We call $\mathfrak{F}$* compact *if for each $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \{W - U : U \in \mathcal{P}\}$, whenever $\mathcal{X} \cup \mathcal{Y}$ has the* finite intersection property (*that is, finite intersections of elements of $\mathcal{X} \cup \mathcal{Y}$ are nonempty*), then $\bigcap(\mathcal{X} \cup \mathcal{Y}) \ne \emptyset$.
3. We call $\mathfrak{F}$ *descriptive* if $\mathfrak{F}$ is refined and compact.

The elements of $\mathcal{P}$ are called admissible sets*. A* descriptive valuation *is a map ν from the set of propositional letters to $\mathcal{P}$. A pair $(\mathfrak{F}, \nu)$, where $\mathfrak{F}$ is a descriptive frame and ν is a descriptive valuation, is called a* descriptive model.

For the definition of generated subframes and p-morphisms of descriptive frames and models we refer to [4, Section 8.5], and for the definition of subframes we refer to [4, Section 9.1]. An important property of generated subframes and p-morphic images, which we will use frequently, is that they preserve validity of formulas.

Definition 2.3. [3, Definition 2.3.15] *A descriptive frame* $\mathfrak{F} = (W, \leq, \mathcal{P})$ *is called* rooted *if there exists* $w \in W$ *such that* $W = \uparrow w$ *and* $W - \{w\} \in \mathcal{P}$.

It is well known (see, e.g., [3, Section 2.3.2]) that each superintuitionistic logic is complete with respect to the class of its rooted descriptive frames.

Definition 2.4. *Let* $\mathfrak{F} = (W, R, \mathcal{P})$ *be a descriptive frame. We say that* $\mathfrak{F}$ *is* n-generated *if there exist* $G_1, \ldots, G_n \in \mathcal{P}$ *such that each* $E \in \mathcal{P}$ *is a polynomial of* $G_1, \ldots, G_n$ *in the signature* $\wedge, \vee, \to, \bot$. *We say that* $\mathfrak{F}$ *is* finitely generated *if* $\mathfrak{F}$ *is* n*-generated for some* $n \in \omega$.

It is well known that each superintuitionistic logic is complete with respect to its finitely generated rooted descriptive frames [3, Corollary 3.4.3]. For a detailed description of the structure of finitely generated descriptive frames we refer to [4, Section 8.7] and [3, Section 3.2].

Let $\mathfrak{F}$ be a finite rooted frame. We recall that with $\mathfrak{F}$ we can associate the Jankov-de Jongh formula $\chi(\mathfrak{F})$ and the subframe formula $\beta(\mathfrak{F})$ [4, Section 9.4], [3, Section 3.3]. Although the actual shapes of $\chi(\mathfrak{F})$ and $\beta(\mathfrak{F})$ do not really matter, the following theorem is of fundamental importance.

Theorem 2.5.

(1) (*For two different proofs see* [4, Proposition 9.41] *and* [3, Theorem 3.3.3]) *Let* $\mathfrak{F}$ *be a finite rooted frame and let* $\chi(\mathfrak{F})$ *be the Jankov-de Jongh formula of* $\mathfrak{F}$. *Then for each descriptive frame* $\mathfrak{G}$ *we have:*

$$\mathfrak{G} \not\models \chi(\mathfrak{F}) \text{ if and only if } \mathfrak{F} \text{ is a } p\text{-morphic image of a generated subframe of } \mathfrak{G}.$$

(2) (*For two different proofs see* [4, Section 9.4] *and* [3, Theorem 3.3.16]) *Let* $\mathfrak{F}$ *be a finite rooted frame and let* $\beta(\mathfrak{F})$ *be the subframe formula of* $\mathfrak{F}$. *Then for each descriptive frame* $\mathfrak{G}$ *we have:*

$$\mathfrak{G} \not\models \beta(\mathfrak{F}) \text{ if and only if } \mathfrak{F} \text{ is a } p\text{-morphic image of a subframe of } \mathfrak{G}.$$

2.2. **Sums of descriptive frames.**

Definition 2.6. (*see, e.g.,* [5, p. 17 and p. 179]) *Let* $\mathfrak{F}_1 = (W_1, \leq_1)$ *and* $\mathfrak{F}_2 = (W_2, \leq_2)$ *be Kripke frames. The* linear sum *of* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *is the Kripke frame* $\mathfrak{F}_1 \oplus \mathfrak{F}_2 = (W_1 \uplus W_2, \leq)$ *such that* $W_1 \uplus W_2$ *is the disjoint union of* W_1 *and* W_2 *and for each* $w, v \in W_1 \uplus W_2$ *we have:*

$$\begin{array}{lrl} w \leq v & \text{iff} & w, v \in W_1 \text{ and } w \leq_1 v, \\ & \text{or} & w, v \in W_2 \text{ and } w \leq_2 v, \\ & \text{or} & w \in W_2 \text{ and } v \in W_1. \end{array}$$

We extend the definition of linear sum to descriptive frames.

Definition 2.7. [2, Sections 2.3 and 2.4]

(1) *Let* $\mathfrak{F}_1 = (W_1, \leq_1, \mathcal{P}_1)$ *and* $\mathfrak{F}_2 = (W_2, \leq_2, \mathcal{P}_2)$ *be descriptive frames. The* linear sum *of* $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *is the descriptive frame* $\mathfrak{F}_1 \oplus \mathfrak{F}_2 = (W, \leq, \mathcal{P})$ *such that* $(W, \leq)$ *is the linear sum of* $(W_1, \leq_1)$ *and* $(W_2, \leq_2)$, *and* $U \in \mathcal{P}$ *if and only if* $U \in \mathcal{P}_1$ *or* $U = W_1 \cup V$, *where* $V \in \mathcal{P}_2$.

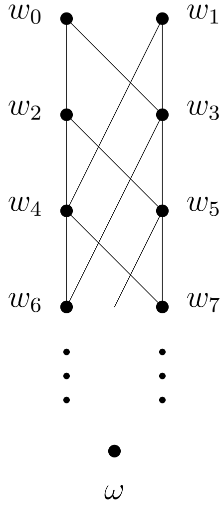

FIGURE 1. The Rieger-Nishimura ladder $\mathfrak{L}$

(2) *Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ be descriptive frames. We define $\bigoplus_{i=1}^{n} \mathfrak{F}_i = (\bigoplus_{i=1}^{n-1} \mathfrak{F}_i) \oplus \mathfrak{F}_n$. If each $\mathfrak{F}_i$ is equal to $\mathfrak{F}$, then we simply write $\bigoplus_n \mathfrak{F}$.*

(3) *Let $\{\mathfrak{F}_i : i \in \omega\}$ be a countable family of descriptive frames, where $\mathfrak{F}_i = (W_i, \leq_i, \mathcal{P}_i)$ for each $i \in \omega$. Let $W = \biguplus_{i\in\omega} W_i \cup \{\infty\}$, where $\infty \notin W_i$ for each $i \in \omega$. The* linear sum *of $\{\mathfrak{F}_i : i \in \omega\}$ is the frame $\bigoplus_{i\in\omega} \mathfrak{F}_i = (W, \leq, \mathcal{P})$ such that for each $w, v \in \biguplus_{i\in\omega} W_i$ we have:*

$$
\begin{array}{lll}
w \leq v \ \ \textit{iff} & & w \in W_i, v \in W_j, \ \textit{and}\ i > j, \\
& \textit{or} & \textit{there is}\ i \in \omega\ \textit{such that}\ w, v \in W_i\ \textit{and}\ w \leq_i v, \\
& \textit{or} & w = \infty,
\end{array}
$$

and $U \in \mathcal{P}$ if and only if U is an upset of W, $U \neq \biguplus_{i\in\omega} W_i$, and $U \cap W_i \in \mathcal{P}_i$ for each $i \in \omega$.

It is obvious that $\oplus$ is an associative operation, and it is easy to verify that the linear sum of a countable family of descriptive frames is again a descriptive frame [2, Section 2.4]. If each $\mathfrak{F}_i$ is equal to $\mathfrak{F}$, then we simply write $\bigoplus_\omega \mathfrak{F}$. Figuratively speaking, the operation $\oplus$ puts $\mathfrak{F}_2$ below $\mathfrak{F}_1$, and the operation $\bigoplus$ forms a tower of $\{\mathfrak{F}_i : i \in \omega\}$ by putting the $\mathfrak{F}_i$ below each other and then adjoining a new root to it. Note that the complement of the new root is not admissible.

2.3. **The Rieger-Nishimura ladder.** Rieger [19] and Nishimura [18] described independently the free cyclic (1-generated) Heyting algebra. The corresponding dual descriptive frame is known as the *Rieger-Nishimura ladder* and is shown in Fig. 1. We denote the Rieger-Nishimura ladder by $\mathfrak{L}$. Let $\mathcal{P}_{\mathfrak{L}}$ denote the set of admissible upsets of $\mathfrak{L}$, and let $\mathfrak{L}_0 = \mathfrak{L} - \{\omega\}$. Then $\mathfrak{L}_0$ is the only non-admissible upset of $\mathfrak{L}$. Consequently, $\mathrm{Up}(\mathfrak{L}_0)$ is isomorphic to $\mathcal{P}_{\mathfrak{L}}$, and so one can work with either $\mathfrak{L}$ and the admissible upsets of $\mathfrak{L}$, or equivalently, with $\mathfrak{L}_0$ and all upsets of $\mathfrak{L}_0$. As a result, some authors concentrate mostly on $\mathfrak{L}_0$ (see, e.g., [4, Section 8.7]). Since in this paper we mostly work with descriptive frames, we prefer to work with $\mathfrak{L}$, and call it the Rieger-Nishimura ladder.

Definition 2.8. [18] *The* Nishimura polynomials *are given by the following recursive definition:*

(1) $g_0(p) = p$,
(2) $g_1(p) = \neg\, p$,
(3) $f_1(p) = p \vee \neg p$,
(4) $g_2(p) = \neg\,\neg\, p$,
(5) $g_3(p) = \neg\,\neg\, p \to p$,
(6) $g_{n+4}(p) = g_{n+3}(p) \to (g_n(p) \vee g_{n+1}(p))$,
(7) $f_{n+2}(p) = g_{n+2}(p) \vee g_{n+1}(p)$.

For $k \in \omega$ let $\mathfrak{L}_{g_k} = {\uparrow} w_k$, and for $k \geq 1$ let $\mathfrak{L}_{f_k} = {\uparrow} w_k \cup {\uparrow} w_{k-1}$. Let also $\nu(p) = \{w_0\}$. The next proposition, which is straightforward to verify, shows that $\mathfrak{L}_{g_k}$ and $\mathfrak{L}_{f_k}$ are precisely the generated subframes of $\mathfrak{L}$ satisfying $g_k(p)$ and $f_k(p)$, respectively.

Proposition 2.9.

(1) *For $k \in \omega$ we have ${\uparrow} w_k = \{w \in \mathfrak{L} : w \models g_k(p)\}$.*
(2) *For $k \geq 1$ we have ${\uparrow} w_k \cup {\uparrow} w_{k-1} = \{w \in \mathfrak{L} : w \models f_k(p)\}$.*

We conclude this brief survey of the Rieger-Nishimura ladder by mentioning a rather natural appearance of $\mathfrak{L}_0$ in a different setting. Define $\preccurlyeq$ on ω by

$$n \preccurlyeq m \text{ if and only if } n - m \geq 2.$$

As was observed by Esakia [6], the frame $(\omega, \preccurlyeq)$ is isomorphic to $\mathfrak{L}_0$.

3. Rieger-Nishimura and Kuznetsov-Gerčiu logics

In this section we introduce the Rieger-Nishimura logic **RN** and the Kuznetsov-Gerčiu logic **KG**. We give a finite axiomatization of **KG** and describe finite and finitely generated rooted descriptive **KG**-frames. We also describe finite rooted **RN**-frames.

For a frame $\mathfrak{F}$, let $Log(\mathfrak{F}) = \{\varphi : \mathfrak{F} \models \varphi\}$; that is, $Log(\mathfrak{F})$ is the set of formulas valid in $\mathfrak{F}$. For a class K of frames, let $Log(\mathsf{K}) = \bigcap\{Log(\mathfrak{F}) : \mathfrak{F} \in \mathsf{K}\}$. It is well-known (see, e.g., [4, Theorem 4.3]) that both $Log(\mathfrak{F})$ and $Log(\mathsf{K})$ are superintuitionistic logics. We call $Log(\mathfrak{F})$ the *logic of* $\mathfrak{F}$, and we call $Log(\mathsf{K})$ the *logic of* K.

Definition 3.1. *We set* **RN** $= Log(\mathfrak{L})$*; that is,* **RN** *is the logic of the Rieger-Nishimura ladder.*

A purely syntactic motivation for studying **RN** comes from n-conservative extensions and n-scheme logics. Let L and S be superintuitionistic logics. We recall that S is an *n-conservative extension* of L if $L \subseteq S$ and for each formula $\varphi(p_1, \ldots, p_n)$ in n variables, we have $L \vdash \varphi$ if and only if $S \vdash \varphi$. We also recall that for a superintuitionistic logic L, a set of formulas $L(n)$ is called the *n-scheme logic* of L if for each k and each formula $\psi(p_1, \ldots, p_k)$ in k variables, $\psi(p_1, \ldots, p_k) \in L(n)$ if and only if for all $\chi_1(p_1, \ldots p_n), \ldots, \chi_k(p_1, \ldots, p_n)$, we have $L \vdash \psi(\chi_1, \ldots, \chi_k)$. It is easy to see that $L(n)$ is a superintuitionistic logic for each $n \in \omega$. It follows from [3, Proposition 4.1.9] that for each superintuitionistic logic L, a superintuitionistic logic S is an n-conservative extension of L if and only if $L \subseteq S \subseteq L(n)$, and that $L(n)$ is the greatest n-conservative extension of L. It turns out that **RN** is the 1-scheme logic of **IPC** and the greatest 1-conservative extension of **IPC** [3, Theorem 4.1.10].

We call a descriptive frame $\mathfrak{F}$ *cyclic* if it is isomorphic to $\mathfrak{L}$, $\mathfrak{L}_{g_k}$, or $\mathfrak{L}_{f_k}$ for some $k \in \omega$. Thus, $\mathfrak{F}$ is cyclic if and only if it is a generated subframe of $\mathfrak{L}$, and each cyclic frame is finite

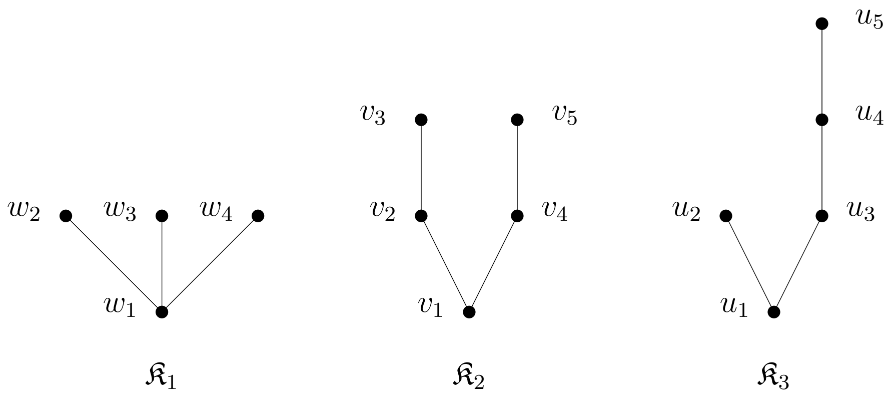

FIGURE 2. The frames $\mathfrak{K}_1, \mathfrak{K}_2$, and $\mathfrak{K}_3$

except $\mathfrak{L}$. Cyclic frames are exactly the duals of cyclic Heyting algebras ([2, Proposition 4], [3, Section 4.1.1]), which is the motivation for the definition. It follows that **RN** is the logic of the cyclic frames. In fact, **RN** is the logic of the finite cyclic frames (see [13, Section 4] and Section 5 below). A natural relative of **RN** is the logic of finite linear sums of cyclic frames.

Definition 3.2. *We set* $\mathbf{KG} = Log(\{\bigoplus_{i=1}^{n} \mathfrak{F}_i : \text{each } \mathfrak{F}_i \text{ is cyclic}\})$*; that is,* **KG** *is the logic of finite linear sums of cyclic frames.*

It follows from the definition that $\mathbf{KG} \subseteq \mathbf{RN}$. In fact, as we will see below, **RN** is a proper extension of **KG**, and there are continuum many logics in the interval $[\mathbf{KG}, \mathbf{RN}]$. The logic **KG** was introduced and studied by Kuznetsov and Gerčiu [13]. They showed that **KG** is finitely axiomatizable. Consider the formula

$$\varphi_{KG} = (p \to q) \vee (q \to r) \vee ((q \to r) \to r) \vee (r \to (p \vee q)).$$

Theorem 3.3. [13, Corollary 4.3.9] $\mathbf{KG} = \mathbf{IPC} + \varphi_{KG}$.

A more convenient axiomatization of **KG** was given in [12, Theorem 16] and [3, Theorem 4.3.4] by means of subframe formulas. Consider the frames $\mathfrak{K}_1$, $\mathfrak{K}_2$, and $\mathfrak{K}_3$ shown in Fig. 2.

Theorem 3.4. $\mathbf{KG} = \mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$.

Proof. It is shown in [12, Theorem 16] that the greatest modal companion of **KG** is axiomatized by adding the subframe formulas of $\mathfrak{K}_1, \mathfrak{K}_2$, and $\mathfrak{K}_3$ to the Grzegorczyk logic **S4.Grz**, which is the greatest modal companion of **IPC**. It follows that $\mathbf{KG} = \mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3)$. A more detailed direct proof can be found in [3, Theorem 4.3.4]. □

Consequently, **KG** is a subframe logic. Finitely generated subdirectly irreducible Heyting algebras that belong to the variety of Heyting algebras corresponding to **KG** were characterized in [13, Lemma 4]. This gives the following characterization of rooted finitely generated descriptive **KG**-frames. For a detailed proof, which is different from that in [13], we refer to [3, Corollary 4.3.9]. A similar characterization was also established in [12, Theorem 16] for the least modal companion of **KG**.

Theorem 3.5. *A rooted descriptive* **KG***-frame* $\mathfrak{F}$ *is finitely generated if and only if* $\mathfrak{F}$ *is isomorphic to* $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$*, where each* $\mathfrak{F}_i$ *is a cyclic frame and* $k \in \omega$.

FIGURE 3. The frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$

This theorem, in particular, implies that each finite rooted **KG**-frame is isomorphic to $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is a finite cyclic frame. Our next task is to single out the class of finite rooted **RN**-frames from the class of finite rooted **KG**-frames. We recall that a descriptive frame $\mathfrak{F}$ is a generated subframe of $\mathfrak{L}$ if and only if $\mathfrak{F}$ is isomorphic to $\mathfrak{L}$, $\mathfrak{L}_{g_k}$, or $\mathfrak{L}_{f_k}$ for some $k \in \omega$, and that each proper generated subframe of $\mathfrak{L}$ is finite ([2, Proposition 4], [3, Theorem 4.2.1]). Next we recall a characterization of the p-morphic images of $\mathfrak{L}$. Up to isomorphism, there are three different types of p-morphic images of $\mathfrak{L}$, which can be described by means of linear sums of descriptive frames. Let $\mathfrak{G}_1$ denote the frame consisting of a single point, and let $\mathfrak{G}_2$ denote the frame consisting of two distinct points that are not related to each other (see Fig.3). The following result was established independently in [12, Section 6] and [2, Proposition 4]. For a purely algebraic proof, we refer to [3, Theorem 4.2.6 and Corollary 4.2.7].

Theorem 3.6. *A descriptive frame $\mathfrak{F}$ is a p-morphic image of $\mathfrak{L}$ if and only if $\mathfrak{F}$ is isomorphic to one of the following frames: $\mathfrak{L}$, $\bigoplus_{i\in\omega} \mathfrak{F}_i$, $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{G}_1$, or $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}$, where each $\mathfrak{F}_i$ is isomorphic to either $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $n \in \omega$.*

Theorem 3.6 enables us to characterize the generated subframes of p-morphic images of $\mathfrak{L}$.

Theorem 3.7.

(1) *An infinite descriptive frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$ if and only if $\mathfrak{F}$ is isomorphic to $\bigoplus_{i\in\omega} \mathfrak{F}_i$ or $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $n \in \omega$.*

(2) *A finite frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$ if and only if $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$ or $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}_{f_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $k, n \in \omega$.*

(3) *A finite rooted frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$ if and only if $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $k, n \in \omega$.*

Proof. (1) The right to left implication follows from Theorem 3.6. Conversely, suppose an infinite descriptive frame $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$. Then there exists an infinite descriptive frame $\mathfrak{G}$ such that $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$ and $\mathfrak{G}$ is a p-morphic image of $\mathfrak{L}$. Then by Theorem 3.6, $\mathfrak{G}$ is isomorphic to $\bigoplus_{i\in\omega} \mathfrak{F}_i$ or $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}$. It is easy to see that neither $\bigoplus_{i\in\omega} \mathfrak{F}_i$ nor $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}$ contains a proper infinite generated subframe. Therefore, $\mathfrak{F}$ is isomorphic to either $\bigoplus_{i\in\omega} \mathfrak{F}_i$ or $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}$.

(2) The right to left implication follows from Theorem 3.6. Conversely, suppose $\mathfrak{G}$ is a p-morphic image of $\mathfrak{L}$ and $\mathfrak{F}$ is a finite generated subframe of $\mathfrak{G}$. Then by Theorem 3.6, $\mathfrak{G}$ is isomorphic to $\mathfrak{L}$, $\bigoplus_{i\in\omega} \mathfrak{F}_i$, $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{G}_1$, or $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}$. In the first case $\mathfrak{F}$ is isomorphic to $\mathfrak{L}_{g_k}$ or $\mathfrak{L}_{f_k}$, in the second and third cases $\mathfrak{F}$ is isomorphic to $\bigoplus_{i=1}^n \mathfrak{F}_i$, and in the fourth case $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$ or $(\bigoplus_{i=1}^n \mathfrak{F}_i) \oplus \mathfrak{L}_{f_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$.

(3) follows from (2) since $\mathfrak{L}_{f_k}$ is not rooted for each $k > 0$. $\square$

Corollary 3.8. *A finite rooted frame $\mathfrak{F}$ is an* **RN***-frame if and only if $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $k,n\in\omega$.*

Proof. It follows from Theorem 3.7 that if a finite rooted frame $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$, then $\mathfrak{F}$ is an **RN**-frame. Conversely, suppose that $\mathfrak{F}$ is a finite rooted **RN**-frame. By Theorem 2.5.1, $\mathfrak{F}$ is a generated subframe of a p-morphic image of $\mathfrak{L}$. Thus, by Theorem 3.7.3, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. $\square$

As an immediate consequence, we obtain that **RN** is a proper extension of **KG**.

Theorem 3.9. $\mathbf{KG}\subsetneqq\mathbf{RN}$.

Proof. That none of $\mathfrak{K}_1,\mathfrak{K}_2,\mathfrak{K}_3$ is a p-morphic image of a subframe of $\mathfrak{L}$ is routine to check. Therefore, by Theorem 2.5.2, $\mathfrak{L}\models\beta(\mathfrak{K}_1),\beta(\mathfrak{K}_2),\beta(\mathfrak{K}_3)$. This, by Theorem 3.4, means that $\mathfrak{L}$ is a **KG**-frame, and so $\mathbf{KG}\subseteq Log(\mathfrak{L})=\mathbf{RN}$. Now we show that $\mathbf{KG}\neq\mathbf{RN}$. Consider the frame $\mathfrak{L}_{g_4}\oplus\mathfrak{G}_1$. By Theorem 3.5, $\mathfrak{L}_{g_4}\oplus\mathfrak{G}_1$ is a rooted **KG**-frame. On the other hand, by Corollary 3.8, $\mathfrak{L}_{g_4}\oplus\mathfrak{G}_1$ is not an **RN**-frame. Thus, by Theorem 2.5.1, $\chi(\mathfrak{L}_{g_4}\oplus\mathfrak{G}_1)\in\mathbf{RN}$ but $\chi(\mathfrak{L}_{g_4}\oplus\mathfrak{G}_1)\notin\mathbf{KG}$, and so $\mathbf{RN}\not\subseteq\mathbf{KG}$. $\square$

Similar to **KG**, we have that **RN** is finitely axiomatizable. This was first observed by Kuznetsov and Gerčiu [13, Theorem 1]. But their axiomatization was rather complicated. In order to give a more convenient axiomatization of **RN**, using a mixture of subframe and Jankov-de Jongh formulas, we need to characterize finitely generated rooted **RN**-frames.

4. Gluing and finitely generated rooted RN-frames

In this section we introduce our technique of gluing, characterize finitely generated rooted **RN**-frames, and give a convenient finite axiomatization of **RN**.

Theorem 4.1. *Let $\mathfrak{F}$ be a finitely generated rooted descriptive* **KG***-frame. If $\mathfrak{F}$ is an* **RN***-frame, then there exist $k,n\in\omega$ such that $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}$, $\mathfrak{G}_1$, or $\mathfrak{G}_2$.*

Proof. By Theorem 3.5, $\mathfrak{F}$ is isomorphic to a linear sum $(\bigoplus_{k=1}^{n}\mathfrak{F}_i)\oplus\mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is a cyclic frame and $k\in\omega$. If for each $j\leq n$ we have that $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$, $\mathfrak{G}_1$, or $\mathfrak{G}_2$, then $\mathfrak{F}$ satisfies the condition of the theorem. Suppose that there exists $j\leq n$ such that $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}_{g_m}$ for some $m\geq 4$ or $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}_{f_l}$ for some $l\geq 2$. (For $m<4$ and $l<2$ the frames $\mathfrak{L}_{g_m}$ and $\mathfrak{L}_{f_l}$ are isomorphic to linear sums of $\mathfrak{G}_1$ and $\mathfrak{G}_2$.) Let $j\leq n$ be the the least such j. If $j>1$, then we define $f:\mathfrak{F}\to\mathfrak{G}_1\oplus\mathfrak{F}_j\oplus\mathfrak{G}_1$ by mapping all the points above $\mathfrak{F}_j$ onto the top node of $\mathfrak{G}_1\oplus\mathfrak{F}_j\oplus\mathfrak{G}_1$, all the points below $\mathfrak{F}_j$ onto the bottom node of $\mathfrak{G}_1\oplus\mathfrak{F}_j\oplus\mathfrak{G}_1$, and each point in $\mathfrak{F}_j$ to itself; and if $j=1$, then we define $f:\mathfrak{F}\to\mathfrak{F}_j\oplus\mathfrak{G}_1$ by mapping all the points below $\mathfrak{F}_j$ onto the bottom node of $\mathfrak{F}_j\oplus\mathfrak{G}_1$, and each point in $\mathfrak{F}_j$ to itself. In either case it is easy to verify that f is a p-morphism. Thus, either $\mathfrak{G}_1\oplus\mathfrak{F}_j\oplus\mathfrak{G}_1$ or $\mathfrak{F}_j\oplus\mathfrak{G}_1$ is a finite **RN**-frame, which contradicts Corollary 3.8. The obtained contradiction proves that such a j does not exist. $\square$

To show that the converse of Theorem 4.1 holds, we introduce a new technique of gluing. For a Kripke frame $\mathfrak{F}$ let $\max(\mathfrak{F})$ denote the set of maximal points and $\min(\mathfrak{F})$ denote the set of minimal points of $\mathfrak{F}$.

Definition 4.2.

(1) *Let* $\mathfrak{F}_1 = (W_1, \leq_1)$ *and* $\mathfrak{F}_2 = (W_2, \leq_2)$ *be Kripke frames such that* $\min(\mathfrak{F}_1)$ *and* $\max(\mathfrak{F}_2)$ *are nonempty. Let* $x \in \min(\mathfrak{F}_1)$ *and* $y \in \max(\mathfrak{F}_2)$. *The* gluing sum *of* $(\mathfrak{F}_1, x)$ *and* $(\mathfrak{F}_2, y)$ *is the frame* $(\mathfrak{F}_1, x)\widehat{\oplus}(\mathfrak{F}_2, y) = (W_1 \uplus W_2, \leq)$ *such that* $W_1 \uplus W_2$ *is the disjoint union of* W_1 *and* W_2, *and* $\leq = \leq_1 \cup \leq_2 \cup [(W_2 \times W_1) - \{(y, x)\}]$.

(2) *Let* $\mathfrak{F}_1 = (W_1, \leq_1, \mathcal{P}_1)$ *and* $\mathfrak{F}_2 = (W_2, \leq_2, \mathcal{P}_2)$ *be descriptive frames and let* $x \in \min(\mathfrak{F}_1)$ *and* $y \in \max(\mathfrak{F}_2)$. *The* gluing sum *of* $(\mathfrak{F}_1, x)$ *and* $(\mathfrak{F}_2, y)$ *is the frame* $(\mathfrak{F}_1, x)\widehat{\oplus}(\mathfrak{F}_2, y) = (W_1 \uplus W_2, \leq, \mathcal{P})$, *where* $(W_1 \uplus W_2, \leq)$ *is the gluing sum of* $((W_1, \leq_1), x)$ *and* $((W_2, \leq_2), y)$, *and*
$$\mathcal{P} = \{U \subseteq W_1 \uplus W_2 : U \text{ is a } \leq\text{-upset, } U \cap W_1 \in \mathcal{P}_1, \text{ and } U \cap W_2 \in \mathcal{P}_2\}.$$

Figuratively speaking, we take the linear sum of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ and erase an arrow going from y to x.

Lemma 4.3. *Let* $k, m \in \omega$ *and let* m *be odd.*

(1) $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}, w_0)$ *is isomorphic to* $\mathfrak{L}$.

(2) $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_{g_k}, w_0)$ *is isomorphic to* $\mathfrak{L}_{g_{k+m+1}}$.

Proof. Easy. □

Next we recall the definition of the complexity of a formula.

Definition 4.4. *The* complexity $c(\varphi)$ *of a formula* φ *is defined inductively as follows:*
$$c(p) = 0,$$
$$c(\perp) = 0,$$
$$c(\varphi \wedge \psi) = \max\{c(\varphi), c(\psi)\},$$
$$c(\varphi \vee \psi) = \max\{c(\varphi), c(\psi)\},$$
$$c(\varphi \to \psi) = 1 + \max\{c(\varphi), c(\psi)\}.$$

Now we recall the notion of the depth of a frame.

Definition 4.5. *Let* $\mathfrak{F}$ *be a frame.*

(1) *We say that* $\mathfrak{F}$ *is of* depth $n < \omega$, *and write* $d(\mathfrak{F})$, *if there is a chain of* n *points in* $\mathfrak{F}$ *and no other chain in* $\mathfrak{F}$ *contains more than* n *points.*

(2) *We say that* $\mathfrak{F}$ *is of* infinite depth, *and write* $d(\mathfrak{F}) = \omega$, *if* $\mathfrak{F}$ *contains a chain consisting of* n *points for each* $n \in \omega$.

(3) *We say that* $\mathfrak{F}$ *is of* finite depth *if* $d(\mathfrak{F}) < \omega$.

(4) *The* depth *of a point* w *of* $\mathfrak{F}$ *is the depth of the subframe of* $\mathfrak{F}$ *generated by* w. *We denote the depth of* w *by* $d(w)$.

(5) *For an upset* U *of* $\mathfrak{F}$, *the* depth $d(U)$ *of* U *is defined as* $d(U) = \sup\{d(x) : x \in U\}$.

Definition 4.6. *Let* $p_1, \ldots, p_n$ *be propositional variables and let* ν *be a descriptive valuation of* $p_1, \ldots, p_n$ *on* $\mathfrak{L}$.

(1) *Let* $\mathrm{rank}(\nu) = \max\{d(\nu(p_i)) : \nu(p_i) \subsetneq \mathfrak{L}\}$.

(2) *For each formula* $\varphi(p_1, \ldots, p_n)$, *let* $M_\nu(\varphi) = \mathrm{rank}(\nu) + c(\varphi) + 1$.

Lemma 4.7. *Let* ν *be a descriptive valuation on* $\mathfrak{L}$. *For each formula* $\varphi(p_1, \ldots, p_n)$ *and for each* $x, y \in \mathfrak{L}$ *with* $d(x), d(y) > M_\nu(\varphi)$, *we have:*
$$x \models \varphi \text{ if and only if } y \models \varphi.$$

Proof. By induction on the complexity of φ. If $c(\varphi) = 0$; that is, if φ is either $\bot$ or a propositional letter, then the lemma is obvious. Suppose that $c(\varphi) = k$ and that the lemma holds for each formula ψ such that $c(\psi) < k$. The cases when $\varphi = \psi_1 \wedge \psi_2$ and $\varphi = \psi_1 \vee \psi_2$ are trivial. Suppose that $\varphi = \psi_1 \to \psi_2$. Then $c(\psi_1), c(\psi_2) < k$. Let $x, y \in \mathfrak{L}$ be such that $d(x), d(y) > M_\nu(\varphi)$. Without loss of generality we may assume that $x \not\models \varphi$ and show that $y \not\models \varphi$. From $x \not\models \psi_1 \to \psi_2$ it follows that there exists $z \in \mathfrak{L}$ such that $x \leq z$, $z \models \psi_1$, and $z \not\models \psi_2$. If $d(z) < d(y) - 1$, because of the structure of $\mathfrak{L}$, we have that $y \leq z$, and so $y \not\models \varphi$. If $d(z) \geq d(y) - 1$, then $d(z) > M_\nu(\varphi) - 1 = \operatorname{rank}(\nu) + c(\varphi) \geq \operatorname{rank}(\nu) + c(\psi_i) + 1 = M_\nu(\psi_i)$ for each $i = 1, 2$. Thus, $d(z), d(y) > M_\nu(\psi_i)$, and by the induction hypothesis, $y \models \psi_1$ and $y \not\models \psi_2$, which again implies that $y \not\models \varphi$. □

Lemma 4.8.

1. *If $\mathfrak{L} \oplus \mathfrak{L} \not\models \varphi$, then $\mathfrak{L} \not\models \varphi$.*
2. *If $\mathfrak{L} \oplus \mathfrak{L} \oplus \mathfrak{G} \not\models \varphi$ for some frame $\mathfrak{G}$, then $\mathfrak{L} \oplus \mathfrak{G} \not\models \varphi$.*
3. *If $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{L} \not\models \varphi$ for some frame $\mathfrak{F}$, then $\mathfrak{F} \oplus \mathfrak{L} \not\models \varphi$.*
4. *If $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{L} \oplus \mathfrak{G} \not\models \varphi$ for some frames $\mathfrak{F}$ and $\mathfrak{G}$, then $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{G} \not\models \varphi$.*
5. *If for some $k \in \omega$ we have $\mathfrak{L} \oplus \mathfrak{L}_{g_k} \not\models \varphi$, then $\mathfrak{L}_{g_m} \not\models \varphi$ for some $m \geq k$.*
6. *If for some $k \in \omega$ and some frame $\mathfrak{G}$ we have $\mathfrak{L} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G} \not\models \varphi$, then $\mathfrak{L}_{g_m} \oplus \mathfrak{G} \not\models \varphi$ for some $m \geq k$.*
7. *If for some $k \in \omega$ and some frame $\mathfrak{F}$ we have $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_k} \not\models \varphi$, then $\mathfrak{F} \oplus \mathfrak{L}_{g_m} \not\models \varphi$ for some $m \geq k$.*
8. *If for some $k \in \omega$ and some frames $\mathfrak{G}$ and $\mathfrak{F}$ we have $\mathfrak{F} \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{F} \not\models \varphi$, then $\mathfrak{F} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G} \not\models \varphi$ for some $m \geq k$.*

Proof. (1) Let ν be a descriptive valuation on $\mathfrak{L} \oplus \mathfrak{L}$ such that $(\mathfrak{L} \oplus \mathfrak{L}, \nu) \not\models \varphi$. In order to make a distinction, we denote the copy of $\mathfrak{L}$ on top by $\mathfrak{L}_1$ and the copy underneath by $\mathfrak{L}_2$. Let ν_1 and ν_2 be the restrictions of ν to $\mathfrak{L}_1$ and $\mathfrak{L}_2$, respectively; that is, $\nu_i(p) = \nu(p) \cap \mathfrak{L}_i$ for each $i = 1, 2$. Let $M_1(\varphi) = \operatorname{rank}(\nu_1) + c(\varphi) + 1$ and let $m = 2 \cdot M_1(\varphi) + 1$. Consider the gluing sum $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0)$, and let μ be the restriction of ν to $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0)$. By Lemma 4.3.1, $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0)$ is isomorphic to $\mathfrak{L}$. Thus, to finish the proof we only need to show that $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), \mu) \not\models \varphi$, which we do in the next claim.

Claim 4.9. $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), \mu) \not\models \varphi$.

Proof. By induction on the complexity of φ. The cases when φ is either $\bot$, a propositional letter, a conjunction, or a disjunction of two formulas are simple. Let $\varphi = \psi \to \chi$. Since $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, \nu) \not\models \varphi$, there exists $x \in \mathfrak{L}_1 \oplus \mathfrak{L}_2$ such that $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, \nu), x \models \psi$ and $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, \nu), x \not\models \chi$. If x belongs to $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0)$, then we are done. If x does not belong to $(\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0)$, then we take a point y in $\mathfrak{L}_{f_m}$ of depth $M_1(\varphi)$. Since $c(\psi), c(\chi) < c(\varphi)$, we have $M_1(\psi), M_1(\chi) < M_1(\varphi)$. It follows from Lemma 4.7 that $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, \nu), y \models \psi$ and $(\mathfrak{L}_1 \oplus \mathfrak{L}_2, \nu), y \not\models \chi$. Therefore, $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), \mu), y \models \psi$ and $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), \mu), y \not\models \chi$. Thus, $((\mathfrak{L}_{f_m}, w_m)\widehat{\oplus}(\mathfrak{L}_2, w_0), \mu), y \not\models \varphi$. □

The proof of (2) is similar to that of (1). The proofs of (3) and (4) are similar to those of (1) and (2) with the only difference that in these cases we should consider $\mathfrak{F} \oplus \mathfrak{L}_{f_m}$ instead of $\mathfrak{L}_{f_m}$. The proof of (5) is similar to that of (1): We take the upset $\mathfrak{F}$ consisting of $M_\nu(\varphi)$ layers of $\mathfrak{L}$ and then consider a gluing sum of $\mathfrak{F}$ with $\mathfrak{L}_{g_k}$. The proofs of (6), (7), and (8) are similar to that of (5). □

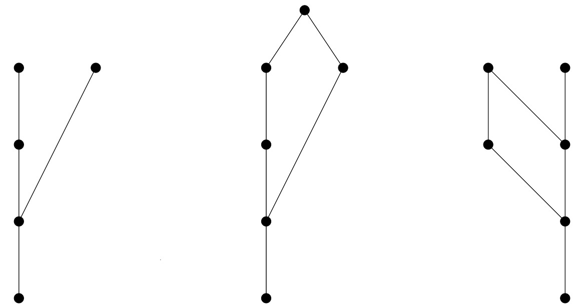

FIGURE 4. The frames $\mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_6$

We point out that a modal analogue of Lemma 4.8.1 can be found in [12, Lemma 17]. We will also need the following auxiliary lemma [3, Lemma 4.2.12], which is an analogue of Theorem 3.6.

Lemma 4.10. *For each $k, n \in \omega$, the frame $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of $\mathfrak{L}_{g_{k+3n}}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$.*

We are now ready to characterize finitely generated rooted descriptive **RN**-frames.

Theorem 4.11. *A finitely generated rooted descriptive* **KG***-frame $\mathfrak{F}$ is an* **RN***-frame if and only if $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}$, $\mathfrak{G}_1$, or $\mathfrak{G}_2$ and $k \in \omega$.*

Proof. The direction from left to right is Theorem 4.1. For the other direction, suppose that $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}$, $\mathfrak{G}_1$, or $\mathfrak{G}_2$. Let $m \in \omega$ be the number of copies of $\mathfrak{L}$ occurring in $\bigoplus_{i=1}^{n} \mathfrak{F}_i$. Then $\mathfrak{F}$ is isomorphic to $[\bigoplus_m ((\bigoplus_{j=1}^{m_i} \mathfrak{H}_j) \oplus \mathfrak{L})] \oplus (\bigoplus_{j=1}^{s} \mathfrak{H}_j) \oplus \mathfrak{L}_{g_k}$ for some $k, m, m_i, s \in \omega$, where each $\mathfrak{H}_j$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. By Theorem 3.6, $(\bigoplus_{j=1}^{m_i} \mathfrak{H}_j) \oplus \mathfrak{L}$ is a p-morphic image of $\mathfrak{L}$. By Lemma 4.10, $(\bigoplus_{j=1}^{s} \mathfrak{H}_j) \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of $\mathfrak{L}_{g_{k+3s}}$. Thus, $\mathfrak{F}$ is a p-morphic image of $(\bigoplus_m \mathfrak{L}) \oplus \mathfrak{L}_{g_{k+3s}}$. We show that $(\bigoplus_m \mathfrak{L}) \oplus \mathfrak{L}_{g_{k+3s}}$ is an **RN**-frame. If not, then there exists a formula $\varphi(p_1, \ldots, p_n)$ such that $\mathbf{RN} \vdash \varphi$ but $(\bigoplus_m \mathfrak{L}) \oplus \mathfrak{L}_{g_{k+3s}} \not\models \varphi$. Applying Lemma 4.8.2 $m-1$ times, we obtain that $\mathfrak{L} \oplus \mathfrak{L}_{g_k} \not\models \varphi$. By Lemma 4.8.5, there is $t \geq k$ such that $\mathfrak{L}_{g_t} \not\models \varphi$. Therefore, we found an **RN**-frame $\mathfrak{H} = \mathfrak{L}_{g_t}$ such that $\mathfrak{H} \not\models \varphi$. This contradicts the fact that $\mathbf{RN} \vdash \varphi$. Thus, such a φ does not exist, and so $(\bigoplus_m \mathfrak{L}) \oplus \mathfrak{L}_{g_{k+3s}}$ is an **RN**-frame. Consequently, so is $\mathfrak{F}$ as a p-morphic image of $(\bigoplus_m \mathfrak{L}) \oplus \mathfrak{L}_{g_{k+3s}}$. $\square$

Next we give yet another characterization of finitely generated rooted descriptive **RN**-frames. Let $\mathfrak{K}_4 = \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$, $\mathfrak{K}_5 = \mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$, and $\mathfrak{K}_6 = \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$. The frames $\mathfrak{K}_4, \mathfrak{K}_5$, and $\mathfrak{K}_6$ are shown in Fig. 4.

Lemma 4.12. *$\mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$.*

Proof. Let $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ and $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$ be labeled as in Fig. 5. Define $f : \mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1 \to \mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ by $f(y_i) = x_i$ for each $i = 1, \ldots, 5$, and $f(y_6) = x_5$. Then it is easy to check that f is an onto p-morphism. $\square$

Theorem 4.13. *A finitely generated rooted descriptive* **KG***-frame $\mathfrak{F}$ is an* **RN***-frame if and only if $\mathfrak{K}_i$ is not a generated subframe of a p-morphic image of $\mathfrak{F}$ for each $i = 4, 5, 6$.*

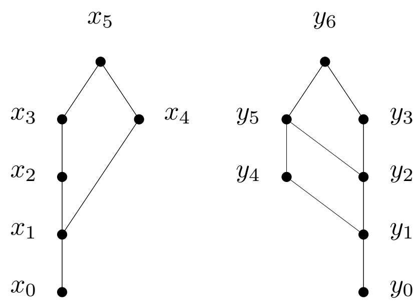

FIGURE 5. The frames $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ and $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$ with the labels

Proof. First suppose that $\mathfrak{F}$ is a finitely generated rooted descriptive **RN**-frame. If there is $i = 4, 5, 6$ such that $\mathfrak{K}_i$ is a generated subframe of a p-morphic image of $\mathfrak{F}$, then the $\mathfrak{K}_i$ is also an **RN**-frame, which contradicts Corollary 3.8. Thus, for no $i = 4, 5, 6$ we have $\mathfrak{K}_i$ is a generated subframe of a p-morphic image of $\mathfrak{F}$. Conversely, suppose that $\mathfrak{F}$ is a finitely generated rooted descriptive **KG**-frame such that for no $i = 4, 5, 6$ we have $\mathfrak{K}_i$ is a generated subframe of a p-morphic image of $\mathfrak{F}$. Since $\mathfrak{F}$ is a **KG**-frame, by Theorem 3.5, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is a cyclic frame. Assume that $\mathfrak{F}$ is not an **RN**-frame. By Theorem 4.11, there exists $i \leq n$ such that $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ or $\mathfrak{L}_{f_l}$ for some $m \geq 4$ and $l \geq 2$. We take the least such i. We consider the case when $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ for some $m \geq 4$. The case when $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{f_l}$ for some $l \geq 2$ is proved similarly. Similar to Theorem 4.1, if $i > 1$, then we define $f : \mathfrak{F} \to \mathfrak{G}_1 \oplus \mathfrak{F}_i \oplus \mathfrak{G}_1$ by mapping all the points above $\mathfrak{F}_i$ onto the top node of $\mathfrak{G}_1 \oplus \mathfrak{F}_i \oplus \mathfrak{G}_1$, all the points below $\mathfrak{F}_i$ onto the bottom node of $\mathfrak{G}_1 \oplus \mathfrak{F}_i \oplus \mathfrak{G}_1$, and each point in $\mathfrak{F}_i$ to itself; and if $i = 1$, then we define $f : \mathfrak{F} \to \mathfrak{F}_i \oplus \mathfrak{G}_1$ by mapping all the points below $\mathfrak{F}_i$ onto the bottom node of $\mathfrak{F}_i \oplus \mathfrak{G}_1$, and each point in $\mathfrak{F}_i$ to itself. In either case it is easy to verify that f is a p-morphism. Looking at the structure of $\mathfrak{L}_{g_m}$ we see that if m is even, then the subframe of $\mathfrak{L}_{g_m}$ consisting of the last three layers of $\mathfrak{L}_{g_m}$ is isomorphic to $\mathfrak{L}_{g_4}$; and if m is odd, then the subframe of $\mathfrak{L}_{g_m}$ consisting of the last three layers of $\mathfrak{L}_{g_m}$ is isomorphic to $\mathfrak{L}_{g_5}$. Therefore, if m is even and $m \geq 4$, then by identifying all but the points of the last three layers of $\mathfrak{L}_{g_m}$ we obtain a p-morphic image of $\mathfrak{L}_{g_m}$ which is isomorphic to $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_4}$ or $\mathfrak{L}_{g_4}$ (depending whether $i > 1$ or $i = 1$); and if m is odd and $m \geq 5$, then by identifying all but the points of the last three layers of $\mathfrak{L}_{g_m}$ we obtain a p-morphic image of $\mathfrak{L}_{g_m}$ which is isomorphic to $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_5}$ or $\mathfrak{L}_{g_5}$ (again depending whether $i > 1$ or $i = 1$). Thus, if $m \geq 4$ and m is even, then $\mathfrak{K}_4 = \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ or $\mathfrak{K}_5 = \mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{F}$; and if $m \geq 5$ and m is odd, then $\mathfrak{K}_6 = \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$ or $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{F}$. Since by Lemma 4.12, $\mathfrak{K}_5$ is a p-morphic image of $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$, we obtain that one of $\mathfrak{K}_4, \mathfrak{K}_5, \mathfrak{K}_6$ is a p-morphic image of $\mathfrak{F}$. The obtained contradiction proves that our assumption was wrong, and that $\mathfrak{F}$ is an **RN**-frame. $\square$

Now we are in a position to give a convenient axiomatization of **RN**.

Theorem 4.14.

(1) $\mathbf{RN} = \mathbf{KG} + \chi(\mathfrak{K}_4) \wedge \chi(\mathfrak{K}_5) \wedge \chi(\mathfrak{K}_6)$.

(2) $\mathbf{RN} = \mathbf{IPC} + \beta(\mathfrak{K}_1) \wedge \beta(\mathfrak{K}_2) \wedge \beta(\mathfrak{K}_3) \wedge \chi(\mathfrak{K}_4) \wedge \chi(\mathfrak{K}_5) \wedge \chi(\mathfrak{K}_6)$.

Proof. (1) It follows from Theorems 2.5.1, 4.11, and 4.13 that **RN** and $\mathbf{KG} + \chi(\mathfrak{K}_4) \wedge \chi(\mathfrak{K}_5) \wedge \chi(\mathfrak{K}_6)$ have the same finitely generated rooted descriptive frames. Now since each

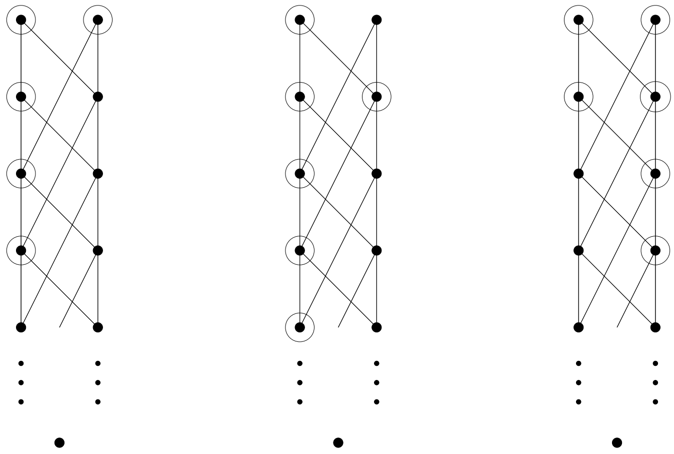

Figure 6. $\mathfrak{K}_4$, $\mathfrak{K}_5$, and $\mathfrak{K}_6$ as subframes of $\mathfrak{L}$

superintuitionistic logic is complete with respect to its finitely generated rooted descriptive frames, we obtain that $\mathbf{RN} = \mathbf{KG} + \chi(\mathfrak{K}_1) \wedge \chi(\mathfrak{K}_2) \wedge \chi(\mathfrak{K}_3)$.

(2) is an immediate consequence of (1) and Theorem 3.4. □

We note that a similar axiomatization of the greatest modal companion of **RN** was claimed in [12, Theorem 18]. However, the argument contained a gap since the formula $\chi(\mathfrak{K}_6)$ was missing from the axiomatization. We conclude this section by showing that unlike **KG**, the logic **RN** is not a subframe logic. For this, by [4, Theorem 11.21], it is sufficient to show that descriptive **RN**-frames are not closed under the operation of taking subframes.

Theorem 4.15. **RN** *is not a subframe logic.*

Proof. By Corollary 3.8, neither of $\mathfrak{K}_4$, $\mathfrak{K}_5$, $\mathfrak{K}_6$ is an **RN**-frame. However, as can be seen in Fig. 6, all three are subframes of $\mathfrak{L}$. Thus, **RN** is not a subframe logic. □

5. Extensions of KG with and without the fmp

In this section we use our gluing technique to give a systematic method of constructing extensions of **KG** with and without the fmp. Our first general theorem states that every extension of **RN** has the fmp. This result was first established by Gerčiu [8] using algebraic technique (the gaps in [8] were corrected in [7]). Kracht [12] claimed that every extension of the greatest modal companion of **KG** has the fmp. This is not true as we will see shortly. In fact, there are continuum many extensions of **KG** that lack the fmp. Nevertheless, Kracht's technique works for all extensions of the greatest modal companion of **RN**.

Theorem 5.1. *Every extension of* **RN** *has the fmp.*

Proof. Let L be an extension of **RN** and let $L \not\vdash \varphi$. Then there exists a finitely generated rooted descriptive L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By Theorem 4.11, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}$, $\mathfrak{G}_1$, or $\mathfrak{G}_2$. If there is no $j \leq n$ such that $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$, then $\mathfrak{F}$ is finite, and so φ is refuted on a finite L-frame. Suppose that $j \leq n$ is the least index for which $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$. Let $\mathfrak{H}$ denote the finite frame $\mathfrak{F}_1 \oplus \cdots \oplus \mathfrak{F}_{j-1}$. Then $\mathfrak{F}$ is isomorphic to $\mathfrak{H} \oplus \mathfrak{F}_j \oplus \cdots \oplus \mathfrak{F}_n \oplus \mathfrak{L}_{g_k}$. It follows from the proof of Theorem 4.11 that there exist $s, m \in \omega$ such that $\mathfrak{F}_j \oplus \cdots \oplus \mathfrak{F}_n \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of

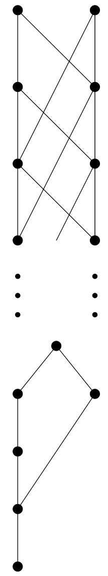

FIGURE 7. The frame $\mathfrak{L} \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$

$\bigoplus_s \mathfrak{L} \oplus \mathfrak{L}_{g_m}$. Therefore, $\mathfrak{F}$ is a p-morphic image of $\mathfrak{G} = \mathfrak{H} \oplus \bigoplus_s \mathfrak{L} \oplus \mathfrak{L}_{g_m}$. Since p-morphisms preserve validity of formulas, $\mathfrak{G} \not\models \varphi$. Applying Lemma 4.8.4 $s-1$ times, we obtain that $\mathfrak{H} \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_m} \not\models \varphi$. By Lemma 4.8.7, there is $t \geq m$ such that $\mathfrak{H} \oplus \mathfrak{L}_{g_t} \not\models \varphi$. As $\mathfrak{H} \oplus \mathfrak{L}_{g_t}$ is a generated subframe of $\mathfrak{H} \oplus \mathfrak{L}$, which is a generated subframe of $\mathfrak{F}$, it follows that $\mathfrak{H} \oplus \mathfrak{L}_{g_t}$ is an L-frame. Thus, φ is refuted on a finite L-frame $\mathfrak{H} \oplus \mathfrak{L}_{g_t}$, so each non-theorem of L is refuted on a finite L-frame, and so L has the fmp. □

Now we show that there exist extensions of **KG** that lack the fmp. Let $\mathfrak{G}$ be a finite rooted **KG**-frame not isomorphic to an **RN**-frame. The simplest such frame is $\mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$. Let $\mathfrak{H} = \mathfrak{L} \oplus \mathfrak{G}$ and let $L = Log(\mathfrak{H})$. The descriptive frame $\mathfrak{L} \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ is shown in Fig. 7.

Theorem 5.2. *Let $\mathfrak{G}$ be a finite rooted* **KG***-frame not isomorphic to an* **RN***-frame, $\mathfrak{H} = \mathfrak{L} \oplus \mathfrak{G}$, and $L = Log(\mathfrak{H})$. Then a finite rooted* **KG***-frame $\mathfrak{F}$ is an L-frame if and only if either of the following two conditions is satisfied.*

(1) *$\mathfrak{F}$ is an* **RN***-frame.*
(2) *$\mathfrak{F}$ is isomorphic to a p-morphic image of a generated subframe of $\bigoplus_{i=1}^n \mathfrak{F}_i \oplus \mathfrak{G}_1 \oplus \mathfrak{G}$, where each $\mathfrak{F}_i$ is either empty or isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$.*

Proof. First we show that if a finite rooted frame satisfies the conditions of the theorem, then it is an L-frame. Since $\mathfrak{L}$ is a generated subframe of $\mathfrak{H}$, we have that each **RN**-frame is an L-frame. By Theorem 3.6, $\bigoplus_{i=1}^n \mathfrak{F}_i \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{L}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. Therefore, $\bigoplus_{i=1}^n \mathfrak{F}_i \oplus \mathfrak{G}_1 \oplus \mathfrak{G}$ is a p-morphic image of $\mathfrak{L} \oplus \mathfrak{G}$. Thus, if $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\bigoplus_{i=1}^n \mathfrak{F}_i \oplus \mathfrak{G}_1 \oplus \mathfrak{G}$, then $\mathfrak{F}$ is an L-frame. Conversely, let $\mathfrak{F}$ be a finite rooted L-frame. By Theorem 2.5.1, $\mathfrak{F}$ is a p-morphic image of a generated subframe $\mathfrak{H}'$ of $\mathfrak{H}$. If $\mathfrak{H}'$ is a generated subframe of $\mathfrak{L}$, then $\mathfrak{F}$ is an **RN**-frame. Suppose that $\mathfrak{H}'$ is isomorphic to $\mathfrak{L} \oplus \mathfrak{H}''$, where $\mathfrak{H}''$ is a generated subframe of $\mathfrak{G}$. By Theorem 3.6, each finite p-morphic image of $\mathfrak{L}$ has the form $\bigoplus_{i=1}^n \mathfrak{F}_i \oplus \mathfrak{G}_1$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. Thus, if $\mathfrak{F}$ is a p-morphic image of $\mathfrak{L} \oplus \mathfrak{H}''$, then $\mathfrak{F}$ is a p-morphic image of $\bigoplus_{i=1}^n \mathfrak{F}_i \oplus \mathfrak{G}_1 \oplus \mathfrak{H}''$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. Finally, since $\mathfrak{H}''$ is a generated subframe of $\mathfrak{G}$, the frame $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathfrak{G}_1 \oplus \mathfrak{H}''$ is a generated subframe of $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathfrak{G}_1 \oplus \mathfrak{G}$. Thus, $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathfrak{G}_1 \oplus \mathfrak{G}$, which concludes the proof. □

Theorem 5.3. *Let $\mathfrak{G}$ be a finite rooted* **KG***-frame not isomorphic to an* **RN***-frame, $\mathfrak{H} = \mathfrak{L} \oplus \mathfrak{G}$, and $L = Log(\mathfrak{H})$. Then L does not have the fmp.*

Proof. Consider the Jankov-de Jongh formulas $\chi_1 = \chi(\mathfrak{G}_1 \oplus \mathfrak{G})$ and $\chi_2 = \chi(\mathfrak{L}_{g_4})$. Without loss of generality we may assume that χ_1 and χ_2 have no variables in common. Let $\varphi = \chi_1 \vee \chi_2$. It is easy to see that $\mathfrak{G}_1 \oplus \mathfrak{G}$ is a p-morphic image of $\mathfrak{H}$ (simply map all the points in $\mathfrak{L}$ to the top node of $\mathfrak{G}_1 \oplus \mathfrak{G}$). This by Theorem 2.5.1 means that $\mathfrak{H} \not\models \chi_1$. Also, $\mathfrak{L}_{g_4}$ is a generated subframe of $\mathfrak{H}$. Applying Theorem 2.5.1 again we obtain that $\mathfrak{H} \not\models \chi_2$. Therefore, $\mathfrak{H} \not\models \varphi$, and so $L \not\vdash \varphi$. Suppose that there is a finite rooted L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Then $\mathfrak{F} \not\models \chi_1$ and $\mathfrak{F} \not\models \chi_2$. By Theorem 2.5.1, $\mathfrak{F} \not\models \chi_1$ implies that $\mathfrak{G}_1 \oplus \mathfrak{G}$ is a p-morphic image of a generated subframe of $\mathfrak{F}$. Thus, if $\mathfrak{F}$ is an **RN**-frame, then $\mathfrak{G}_1 \oplus \mathfrak{G}$ is also an **RN**-frame, which by Corollary 3.8, is a contradiction. Consequently, $\mathfrak{F} \not\models \chi_1$ implies $\mathfrak{F}$ is not an **RN**-frame. By Theorem 5.2.2, this means that $\mathfrak{F}$ is a p-morphic image of some $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{G}_1 \oplus \mathfrak{H}''$, where $\mathfrak{H}''$ is a generated subframe of $\mathfrak{G}$ and each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. Next we show that $\mathfrak{L}_{g_4}$ cannot be a p-morphic image of a generated subframe of $\mathfrak{F}$. Let $\mathfrak{F}'$ be a generated subframe of $\mathfrak{F}$ and let $f : \mathfrak{F}' \to \mathfrak{L}_{g_4}$ be an onto p-morphism. If $|\max(\mathfrak{F}')| = 1$, then clearly $\mathfrak{L}_{g_4}$ cannot be a p-morphic image of $\mathfrak{F}'$. Suppose that $\mathfrak{F}'$ has two maximal points u_1 and u_2. Then $f(u_1) \neq f(u_2)$ and $f(u_1)$ and $f(u_2)$ are the maximal points of $\mathfrak{L}_{g_4}$. Let u be a point of the second layer of $\mathfrak{F}'$. Since the top layers of $\mathfrak{F}'$ are sums of $\mathfrak{G}_1$ and $\mathfrak{G}_2$, we have that $u \leq u_1$ and $u \leq u_2$. Therefore, $f(u) \neq f(u_1)$ and $f(u) \neq f(u_2)$. But then u should be mapped to a point of the second layer of $\mathfrak{L}_{g_4}$, which consists of a single point. This point must see both maximal points of $\mathfrak{L}_{g_4}$, a contradiction. Therefore, no generated subframe of $\mathfrak{F}$ can be p-morphically mapped onto $\mathfrak{L}_{g_4}$, and so $\mathfrak{F} \models \chi_2$, which contradicts our assumption that $\mathfrak{F} \not\models \chi_2$. Thus, there is no finite L-frame that refutes both χ_1 and χ_2. Consequently, φ can not be refuted on a finite rooted L-frame, which means that L does not have the fmp. □

Consequently, there are many extensions of **KG** that lack the fmp. Next we show that there are in fact continuum many such. We use the standard method (introduced by Jankov [11]) of constructing infinite anti-chains of finite rooted **KG**-frames. Let K be the class of non-isomorphic finite rooted **KG**-frames. We define a partial order $\sqsubseteq$ on K as follows. For $\mathfrak{F}, \mathfrak{G} \in \mathsf{K}$ we set:

$$\mathfrak{F} \sqsubseteq \mathfrak{G} \quad \text{if and only if } \mathfrak{F} \text{ is a } p\text{-morphic image of a generated subframe of } \mathfrak{G}.$$

In the next lemma we show how to construct anti-chains of finite rooted **KG**-frames and **RN**-frames. This, using Jankov's technique, will allow us to show that **RN** has continuum many extensions, and that there are continuum many logics in the interval $[\mathbf{KG}, \mathbf{RN}]$.

Lemma 5.4.

(1) *If $k \neq m$, then $\mathfrak{L}_{g_k}$ is not a p-morphic image of $\mathfrak{L}_{g_m}$.*
(2) *The sequence $\Gamma = \{\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1 : k \geq 4\}$ of rooted* **KG***-frames forms an anti-chain in* $(\mathsf{K}, \sqsubseteq)$.
(3) *The sequences $\Delta_1 = \{\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1 : k \geq 4$ and k is even$\}$ and $\Delta_2 = \{\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1 : k \geq 5$ and k is odd$\}$ of rooted* **KG***-frames form anti-chains in* $(\mathsf{K}, \sqsubseteq)$.

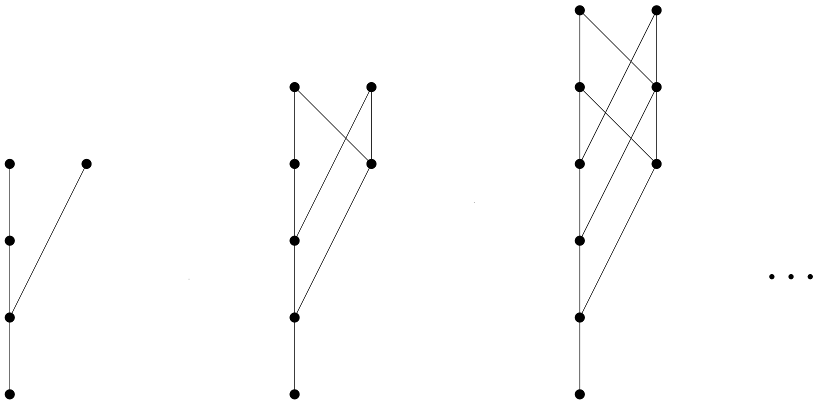

FIGURE 8. The frames $\mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$, $\mathfrak{L}_{g_6} \oplus \mathfrak{G}_1$, and $\mathfrak{L}_{g_8} \oplus \mathfrak{G}_1$

(4) $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1 \not\sqsubseteq (\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$, *where each* $\mathfrak{F}_i$ *is isomorphic to* $\mathfrak{G}_1$ *or* $\mathfrak{G}_2$ *and* $k \neq m$.

(5) *The sequences* $\Upsilon_1 = \{(\bigoplus_{i=1}^{k} \mathfrak{G}_2) \oplus \mathfrak{L}_{g_4} : k \in \omega\}$ *and* $\Upsilon_2 = \{(\bigoplus_{i=1}^{k} \mathfrak{G}_2) \oplus \mathfrak{L}_{g_5} : k \in \omega\}$ *of rooted* **RN***-frames form anti-chains in* $(\mathsf{K}, \sqsubseteq)$.

Proof. (1) is easy; for a short proof see [3, Lemma 4.2.13].

For (2), let $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1, \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1 \in \Gamma$ with $m > k$. Then $|\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1| < |\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1|$, so $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$ cannot be a p-morphic image of a generated subframe of $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$. Suppose that there exists a generated subframe $\mathfrak{H}$ of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$ such that $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{H}$. If $\mathfrak{H}$ is a proper generated subframe of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$, then $\mathfrak{H}$ is an **RN**-frame. By Corollary 3.8, $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is not an **RN**-frame, so cannot be a p-morphic image of $\mathfrak{H}$. Thus, $\mathfrak{H}$ is isomorphic to $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$, and so $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$. Then the least point of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$ is mapped to the least point of $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$. If some other point of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$ were mapped to the least point of $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$, then $\mathfrak{L}_k \oplus \mathfrak{G}_1$ would be a p-morphic image of a generated subframe of $\mathfrak{L}_{g_m}$, so would be an **RN**-frame, a contradiction. Therefore, no other point of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$ is mapped to the least point of $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$. Thus, $\mathfrak{L}_{g_k}$ is a p-morphic image of $\mathfrak{L}_{g_m}$, which contradicts (1). Consequently, Γ forms an anti-chain in $(\mathsf{K}, \sqsubseteq)$.

For (3), suppose that $m > k$ and that $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is a p-morphic image of a generated subframe of $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$. Then there exist a generated subframe $\mathfrak{H}$ of $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$ and an onto p-morphism $f : \mathfrak{H} \to \mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$. Obviously, $\mathfrak{H}$ contains the first three layers of $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$; otherwise, the cardinality of $\mathfrak{H}$ is smaller than that of $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$. First we show that if $x \in \mathfrak{H}$ is such that $d(x) \leq 3$, then $d(f(x)) \leq 3$. If not, then $|{\uparrow} f(x)| > |\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}|$. On the other hand, $|{\uparrow} x| < |\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}|$. So $|{\uparrow} x| < |{\uparrow} f(x)|$, a contradiction. Therefore, the restriction of f to the first three layers of $\mathfrak{H}$ is contained in $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}$. We show that it is exactly $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}$. If not, then it is a proper upset of $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}$. If it is the top node of $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}$, then $\mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is a p-morphic image of a generated subframe of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$, a contradiction. If it contains the top node and at least one other point, then it is easy to see that there exist $z \in \mathfrak{H}$ of depth ≤ 3 and u in $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}$ minus the f image of the first three layers of $\mathfrak{H}$ such that $u \not\leq f(z)$. Since $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{H}$, there exists $x \in \mathfrak{H}$ such that $d(x) > 3$ and $f(x) = u$. But then $x \leq z$ and $f(x) \not\leq f(z)$, a contradiction. Thus, the restriction of f to the first three layers of $\mathfrak{H}$ is equal to $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3}$. By (2), $\mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is not a p-morphic image

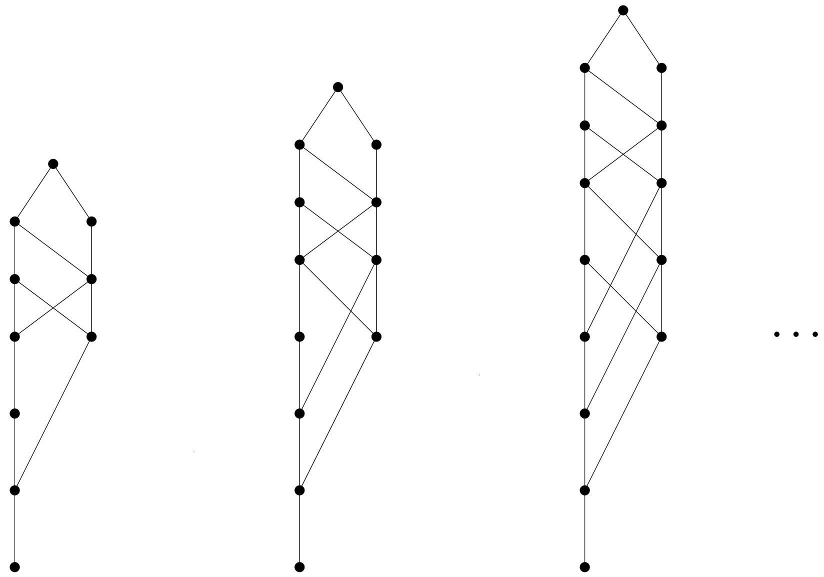

FIGURE 9. The frames in Δ_1

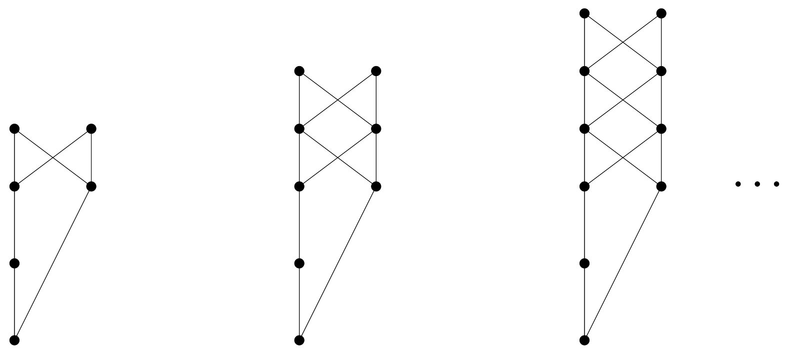

FIGURE 10. The frames in Υ_1

of a generated subframe of $\mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$. Therefore, there is $x \in \mathfrak{H}$ such that $d(x) > 3$ and $d(f(x)) \leq 3$. Let $y \in \mathfrak{H}$ be such that $d(y) \leq 3$. Then $x \leq y$, and so $f(x) \leq f(y)$. This is a contradiction since for each $u \in \mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ of depth ≤ 3, there exists $z \in \mathfrak{H}$ of depth ≤ 3 such that $u \not\leq f(z)$. Thus, there is no generated subframe of $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$ that can be mapped p-morphically onto $\mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$. This proves that both Δ_1 and Δ_2 are anti-chains in $(\mathsf{K}, \sqsubseteq)$.

The proof of (4) is a routine adaptation of that of (3). The proof of (5) is similar to that of (2), and is based on the fact that for $m \neq n$ there is no p-morphism from $\bigoplus_{i=1}^{n} \mathfrak{G}_2$ onto $\bigoplus_{i=1}^{m} \mathfrak{G}_2$. □

We point out that the anti-chain in Lemma 5.4.5 was first constructed in [12, Lemma 20].

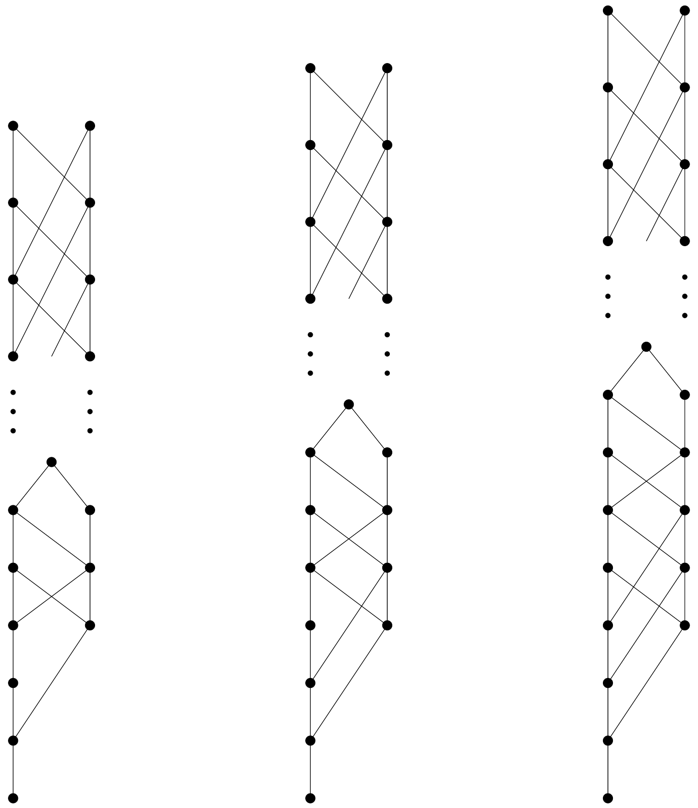

FIGURE 11. The frames $\mathfrak{H}_k$

Theorem 5.5.

(1) *There are continuum many extensions of* **RN**. *Consequently, there are continuum many extensions of* **KG** *with the fmp.*
(2) *There are continuum many extensions of* **KG** *that are not contained in* **RN**.
(3) *There are continuum many logics in the interval* [**KG**, **RN**].

Proof. (1) It follows from Lemma 5.4.5 that Υ_1 (resp. Υ_2) is an infinite anti-chain of finite rooted **RN**-frames. For $\Delta, \Theta \subseteq \Upsilon_1$, if $\Delta \neq \Theta$, then the standard application of the Jankov-de Jongh formulas gives us that $Log(\Delta) \neq Log(\Theta)$ [11]. Since there are continuum many subsets of Υ_1, the result follows.

(2) is similar to (1). We only need to observe that none of the frames in Γ constructed in Lemma 5.4.2 is an **RN**-frame. Therefore, for $\Delta \subseteq \Gamma$, $Log(\Delta)$ is an extension of **KG** not contained in **RN**.

(3) is similar to (1) and (2). For each $\Delta \subseteq \Gamma$, the logic $Log(\{\mathfrak{L}\} \cup \Delta)$ is an extension of **KG** that is properly contained in **RN**. □

Now we show that there are continuum many extensions of **KG** without the fmp. Let $\mathfrak{H}_k = \mathfrak{L} \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$, where $k \geq 4$ is even (see Fig. 11), and let $\Theta = \{\mathfrak{H}_k : k \geq 4$ is even$\}$.

Theorem 5.6.

(1) *For $k \geq 4$ the logic $Log(\mathfrak{H}_k)$ lacks the fmp.*
(2) *For each $\Delta \subseteq \Theta$, the logic $Log(\Delta)$ lacks the fmp.*

(3) *For each $\Delta, \Gamma \subseteq \Theta$, if $\Delta \neq \Gamma$, then $Log(\Delta) \neq Log(\Gamma)$.*

Proof. (1) is a consequence of Theorem 5.3 since $\mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is not an **RN**-frame. For (2), we first show that a finite rooted frame $\mathfrak{F}$ is a $Log(\Delta)$-frame if and only if $\mathfrak{F}$ is a $Log(\mathfrak{H}_k)$-frame for some $\mathfrak{H}_k \in \Delta$. Indeed, it is clear that if $\mathfrak{F}$ is a finite rooted $Log(\mathfrak{H}_k)$-frame for some $\mathfrak{H}_k \in \Delta$, then $\mathfrak{F}$ is a $Log(\Delta)$-frame. Conversely, if $\mathfrak{F}$ is a finite rooted $Log(\Delta)$-frame, then $Log(\mathfrak{F}) \supseteq Log(\Delta) = \bigcap\{Log(\mathfrak{H}_k) : \mathfrak{H}_k \in \Delta\}$. By Theorem 2.5.1, there is $\mathfrak{H}_k \in \Delta$ such that $Log(\mathfrak{F}) \supseteq Log(\mathfrak{H}_k)$. Thus, $\mathfrak{F}$ is a $Log(\mathfrak{H}_k)$-frame. Now the same technique as in the proof of Theorem 5.3 shows that $Log(\Delta)$ lacks the fmp for each $\Delta \subseteq \Theta$. For (3), suppose that $\Delta, \Gamma \subseteq \Theta$ and that $\Delta \neq \Gamma$. Without loss of generality we may assume that there is $\mathfrak{H}_k \in \Delta$ such that $\mathfrak{H}_k \notin \Gamma$. Then it is easy to see that $\mathfrak{G}_k = \mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_k} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{H}_k$, and so $\mathfrak{G}_k$ is a $Log(\Delta)$-frame. Suppose that $\mathfrak{G}_k$ is a $Log(\Gamma)$-frame. Then, as was shown in (2), there exists $\mathfrak{H}_m \in \Gamma$ such that $m \neq k$ and $\mathfrak{G}_k$ is a $Log(\mathfrak{H}_m)$-frame. Similar to Theorem 5.2, we can show that all finite rooted frames of $Log(\mathfrak{H}_m)$ are finite rooted **RN**-frames or p-morphic images of generated subframes of $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. Then $\mathfrak{G}_k$ is a p-morphic image of a generated subframe of $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{f_3} \oplus \mathfrak{L}_{g_m} \oplus \mathfrak{G}_1$, which contradicts Lemma 5.4.3 and 5.4.4. Therefore, $\mathfrak{G}_k$ is not a $Log(\Gamma)$-frame. Then the Jankov-de Jongh formula of $\mathfrak{G}_k$ belongs to $Log(\Gamma)$ but does not belong to $Log(\Delta)$. Thus, $Log(\Delta) \neq Log(\Gamma)$. □

As an immediate consequence, we obtain:

Corollary 5.7. *There are continuum many extensions of* **KG** *without the fmp.*

6. Poly-size model property

In this section we strengthen Theorem 5.1 and show that every extension of **RN** has the poly-size model property. We recall that a logic L has the *poly-size model property* if for each formula φ with $L \nvdash \varphi$, there exists an L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$ and the size of $\mathfrak{F}$ is polynomial in the size of φ.

Theorem 6.1. *Every extension of* **RN** *has the poly-size model property.*

Proof. Let L be an extension of **RN** and let $L \nvdash \varphi$. By Theorem 5.1, there exists a finite rooted L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. Since L is an extension of **RN**, we have that $\mathfrak{F}$ is an **RN**-frame. Therefore, by Corollary 3.8, $\mathfrak{F}$ is isomorphic to $\mathfrak{F}_1 \oplus \mathfrak{F}_2$, where $\mathfrak{F}_2$ is a finite generated subframe of $\mathfrak{L}$ and $\mathfrak{F}_1$ is a finite sum of the frames $\mathfrak{G}_1$ and $\mathfrak{G}_2$. It is our goal to find a finite L-frame $\mathfrak{G}$ such that $\mathfrak{G} \not\models \varphi$ and the size of $\mathfrak{G}$ is polynomial in the size of φ. We split the proof in two parts. First we 'compress' $\mathfrak{F}_1$ into a smaller frame and then we 'cut out' some parts of $\mathfrak{F}_2$ to make $\mathfrak{F}$ even smaller. Let ν be a valuation on $\mathfrak{F}$ such that $(\mathfrak{F}, \nu) \not\models \varphi$ and let $p_1, \ldots, p_n$ be the variables occurring in φ. Define an equivalence relation $\sim$ on $\mathfrak{F}$ by $w \sim v$ if $w \in \nu(p_i)$ if and only if $v \in \nu(p_i)$ for each $i = 1, \ldots, n$. Since each $\nu(p_i)$ is an upset, we have that each equivalence class is convex; that is, from $w \sim v$ and $w \leq u \leq v$, it follows that $u \sim w$. We show that there are at most $(n+1) + 2n$ equivalence classes of $\mathfrak{F}_1$. If there are $w, v \in \mathfrak{F}_1$ such that $d(w) = d(v)$, $w \models p_i$, and $v \not\models p_i$ for some p_i, then for each u with $w, v \leq u$ we have $u \models p_i$, and for each u with $u \leq w, v$ we have $u \not\models p_i$. Looking at the structure of $\mathfrak{F}_1$, we see that for each u different from w, v we have $w, v \leq u$ or $u \leq w, v$. Therefore, for each p_i there is at most one layer of $\mathfrak{F}_1$ with points that have different values of p_i. Since there are n propositional variables, there are at most n non-equivalent layers of $\mathfrak{F}_1$, say $l_1, \ldots, l_n$. Note that the number of equivalence classes of $\mathfrak{F}_1$ is less than or equal to the number of equivalence classes of $\mathfrak{F}_1 - \bigcup_{i=1}^n l_i$ plus the number of equivalence classes of $\bigcup_{i=1}^n l_i$. The cardinality of $\bigcup_{i=1}^n l_i$ is $2n$. Therefore, there are at most $2n$ equivalence classes of $\bigcup_{i=1}^n l_i$. Moreover, for each $i, j \leq n$ we have that $\nu(p_i) \cap (\mathfrak{F}_1 - \bigcup_{i=1}^n l_i) \subseteq \nu(p_j) \cap (\mathfrak{F}_1 - \bigcup_{i=1}^n l_i)$ or $\nu(p_j) \cap (\mathfrak{F}_1 - \bigcup_{i=1}^n l_i) \subseteq \nu(p_i) \cap (\mathfrak{F}_1 - \bigcup_{i=1}^n l_i)$. Thus, there are at most $n+1$ equivalence classes of $\mathfrak{F}_1 - \bigcup_{i=1}^n l_i$. Consequently, there are at most $(n+1)+2n$ equivalence classes of $\mathfrak{F}_1$. We let $\mathfrak{H}_1$ be the frame obtained from $\mathfrak{F}_1$ by replacing each equivalence class C in $\mathfrak{F}_1 - \bigcup_{i=1}^n l_i$ by a single point w_C, and define a map $f : \mathfrak{F} \to \mathfrak{H}_1 \oplus \mathfrak{F}_2$ as follows. Let f be the identity on all the points of $\mathfrak{F}_2 \cup \bigcup_{i=1}^n l_i$, and for each $w \in \mathfrak{F}_1 - \bigcup_{i=1}^n l_i$ let $f(w) = w_C$, where C is the equivalence class containing w. It is easy to check that f is an onto p-morphism. We define a valuation μ on $\mathfrak{H}_1 \oplus \mathfrak{F}_2$ by $\mu(f(x)) = \nu(x)$ for each $x \in \mathfrak{F}$. It follows from the definition of f that μ is well-defined. Therefore, the new model $(\mathfrak{H}_1 \oplus \mathfrak{F}_2, \mu)$ is a p-morphic image of the model $(\mathfrak{F}, \nu)$. Since the truth of a formula is preserved and reflected by p-morphisms between models [4, Theorem 2.15], we have that $(\mathfrak{H}_1 \oplus \mathfrak{F}_2, \mu) \not\models \varphi$ and that $|\mathfrak{H}_1 \oplus \mathfrak{F}_2| \leq |\mathfrak{F}_2| + (n+1) + 2n$.

Our next task is to make $\mathfrak{F}_2$ smaller. Let $D_1, \ldots, D_s$ be the partition of $\mathfrak{F}_2$ into the equivalence classes of $\sim$. We first show that $s \leq (n+1) + 2(2n)$. The proof is similar to that for $\mathfrak{F}_1$. It follows from the structure of $\mathfrak{F}_2$ that for each propositional variable p_i there are at most two adjacent layers of $\mathfrak{F}_2$ with points that have different values of p_i. Therefore, there are at most $2n$ layers of $\mathfrak{F}_2$ with non-equivalent points. Let these layers be $e_1, \ldots, e_{2n}$. Then, as in the above, we can show that $s \leq (n+1) + 2(2n)$. Therefore, $|\mathfrak{F}_2| \leq \max(\{|D_i| : i = 1, \ldots, n\}) \cdot ((n+1) + 2(2n))$. Next we show that without loss of generality we may assume that $|D_i| \leq 2 \cdot (c(\varphi) + 5)$. If there is i such that D_i has more than $c(\varphi) + 5$ layers, then let $k' = \max\{d(x) : x \in D_i\}$ and let $m' = \min\{d(x) : x \in D_i\}$. We also let $k = k' - 2$ and $m'' = m' + 2$. We add and subtract 2 to m' and k', respectively, to make sure that each layer in between k and m'' is properly contained in D_i. Lastly, let $m = m'' + (c(\varphi) + 1)$. Similar to Lemma 4.7, we can show that if x, y are such that $m \leq d(x), d(y) \leq k$, then for each subformula ψ of φ we have $x \models \psi$ if and only if $y \models \psi$. Now we 'cut out' all the layers in between m and k as follows. Let $\mathfrak{K} = \mathfrak{F}_2 - \mathfrak{L}_{f_t}$, where $t = 2k-1$; that is, $\mathfrak{K}$ is obtained from $\mathfrak{F}_2$ by cutting out the first k layers. Then $\mathfrak{K}$ is isomorphic to $\mathfrak{L}_{g_a}$ for some a. Consider the gluing sum $(\mathfrak{H}_1 \oplus \mathfrak{L}_{f_r}, w_r)\widehat{\oplus}(\mathfrak{K}, w_0)$, where $r = 2m - 1$. By Lemma 4.3.2, $(\mathfrak{H}_1 \oplus \mathfrak{L}_{f_r}, w_r)\widehat{\oplus}(\mathfrak{K}, w_0)$ is isomorphic to $\mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\mathfrak{H}_2$ is isomorphic to $\mathfrak{L}_{g_{r+a+1}}$. On the other hand, $\mathfrak{F}_2$ is isomorphic to $\mathfrak{L}_{g_b}$, where $b = t+a+1 = (r+a+1)+(t-r) = (r+a+1) + ((2k-1) - (2m-1)) = (r+a+1) + 2(k-m)$. Therefore, $\mathfrak{H}_2$ is isomorphic to a generated subframe of $\mathfrak{F}_2$. As in Claim 4.9, we can show that $(\mathfrak{H}_1 \oplus \mathfrak{L}_{f_r}, w_r)\widehat{\oplus}(\mathfrak{K}, w_0) \not\models \varphi$. Continuing this process for each i such that D_i contains more than $c(\varphi)+5$ layers, we obtain a frame $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ such that $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \not\models \varphi$ and $\mathfrak{H}_2$ is isomorphic to a generated subframe of $\mathfrak{F}_2$ of the size at most $2 \cdot (c(\varphi) + 5) \cdot ((n+1) + 2(2n))$. Thus, $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is isomorphic to a generated subframe of a p-morphic image of $\mathfrak{F}_1 \oplus \mathfrak{F}_2$, so $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is an L-frame, and the size of $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is bounded by $((n+1) + 2n) + 2 \cdot (c(\varphi) + 5) \cdot ((n+1) + 2(2n))$. It follows that the size of $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is polynomial in the size of φ. Consequently, every non-theorem of L is refuted in an L-frame whose size is polynomial in the size of φ, and so L has the poly-size model property. $\square$

Next we show that although every extension of **RN** has the poly-size model property, there exist extensions of **KG** that have the fmp, but do not have the poly-size model property. In fact, for each function $f : \omega \to \omega$, we construct a logic $L_f \supset$ **KG** such that L_f has the fmp, but it does not have the f-size model property. We recall that for a given function

$f : \omega \to \omega$, a logic L has the *f-size model property* if for each formula φ with $L \not\vdash \varphi$, there is a finite L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$ and $|\mathfrak{F}| < f(|\varphi|)$. Our construction is similar to that of [4, Theorem 18.20], however our proof is different and uses the Jankov-de Jongh formulas.

Theorem 6.2. *For each function $f : \omega \to \omega$ there is a an extension L_f of* **KG** *such that L_f has the fmp, but L_f does not have the f-size model property.*

Proof. If $f : \omega \to \omega$ is not order-preserving, then we consider an order-preserving function $g : \omega \to \omega$ such that $f(n) < g(n)$ for each $n \in \omega$. If the theorem holds for g, it obviously holds for f as well. Thus, without loss of generality we may assume that $f : \omega \to \omega$ is order-preserving. Let $\mathfrak{G}$ be a finite rooted **KG**-frame which is not an **RN**-frame. For each $k \in \omega$ let $\mathfrak{C}_k$ be the chain of depth k and let $\mathfrak{H}_k = \mathfrak{G}_1 \oplus \mathfrak{G} \oplus \mathfrak{C}_k$. We set $\varphi_k = \chi(\mathfrak{H}_k) \vee \chi(\mathfrak{L}_{g_4})$. Then $|\mathfrak{H}_k| = k + |\mathfrak{G}| + 1$ and $|\varphi_k| = |\chi(\mathfrak{H}_k)| + |\chi(\mathfrak{L}_{g_4})| + 1$. It follows from the syntactic description of Jankov-de Jongh formulas (see, e.g., [3, Section 3.3]) that there is a polynomial P such that $|\chi(\mathfrak{H}_k)| < P(|\mathfrak{H}_k|)$. Therefore, $|\varphi_k| < P(|\mathfrak{H}_k|) + c_1 = P(k + c_2) + c_1$ for some constants c_1 and c_2. Thus, without loss of generality we may assume that there is a polynomial P such that $|\varphi_k| < P(k)$. Since f is order-preserving, $f(|\varphi_k|) < f(P(k))$. Consider $\mathfrak{L}_{g_{f(P(k))}}$ consisting of the first $f(P(k))$ layers of $\mathfrak{L}$. Clearly $\mathfrak{L}_{g_{f(P(k))}}$ is a generated subframe of $\mathfrak{L}$. For each $k \in \omega$ let $\mathfrak{F}_k$ denote the frame $\mathfrak{L}_{g_{f(P(k))}} \oplus \mathfrak{G} \oplus \mathfrak{C}_k$. We let $L_f = Log(\{\mathfrak{F}_k : k \in \omega\})$. It follows from the definition of L_f that L_f has the fmp.

Claim 6.3. *$\mathfrak{F}_k$ is the smallest L_f-frame that refutes φ_k.*

Proof. The proof is similar to that of Theorem 5.3. We will be a bit sketchy here. First note that arguments similar to those in the proof of Theorem 5.2 show that if a finite rooted frame $\mathfrak{F}$ is an L_f-frame, then it is isomorphic to either of the following frames:

1. $\mathfrak{F}_k$ for some $k \in \omega$,
2. Some **RN**-frame,
3. A p-morphic image of a generated subframe of $(\bigoplus_{i=1}^n \mathfrak{K}_i) \oplus \mathfrak{G}_1 \oplus \mathfrak{G} \oplus \mathfrak{C}_k$, where each $\mathfrak{K}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$.

As in the proof of Theorem 5.3, we can show that if $\mathfrak{F}$ is isomorphic to some **RN**-frame, then $\mathfrak{F} \models \chi(\mathfrak{H}_k)$ for each $k \in \omega$, and if $\mathfrak{F}$ is isomorphic to a frame described in (3), then $\mathfrak{F} \models \chi(\mathfrak{L}_{g_4})$. Moreover, it is clear that $\mathfrak{F}_n \models \chi(\mathfrak{H}_k)$ for each $k > n$. Therefore, $\mathfrak{F} \not\models \varphi_k$ only if $\mathfrak{F}$ is isomorphic to $\mathfrak{F}_n$ for $n \geq k$. Obviously the smallest among the $\mathfrak{F}_n$ with $n \geq k$ is the frame $\mathfrak{F}_k$. □

To finish the proof we observe that $|\mathfrak{F}_k| = 2f(P(k)) + |\mathfrak{G}| + k$. Moreover, $|\varphi_k| < P(k)$ and f is order-preserving. Thus, $|\mathfrak{F}_k| > f(|\varphi_k|)$, and so L_f does not have the f-size model property. □

7. Pre-finite model property

In this section we characterize the logic that bounds the fmp in extensions of **KG**. This was first established by Gerčiu [8]. He gave a very sketchy algebraic proof. We give a new full proof of this result by means of descriptive frames.

Definition 7.1. *A logic L is said to have the* pre-finite model property *if L does not have the fmp, but all proper extensions of L have the fmp.*

Let $\mathfrak{T}_1 = \mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{L} \oplus \mathfrak{G}_1$ and $\mathfrak{T}_2 = \mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{L} \oplus \mathfrak{G}_1$. The frames $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are shown in Fig. 12.

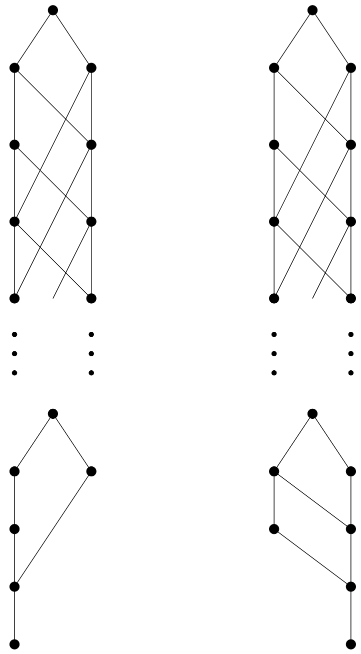

FIGURE 12. The frames $\mathfrak{T}_1$ and $\mathfrak{T}_2$

Lemma 7.2. *$\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{T}_2$.*

Proof. The proof is a simple adaptation of the proof of Lemma 4.12. $\square$

Theorem 7.3. *Let $L \supseteq \mathbf{KG}$.*

1. *If L does not have the fmp, then $L \subseteq Log(\mathfrak{T}_1)$.*
2. *$Log(\mathfrak{T}_1)$ is the only extension of $\mathbf{KG}$ with the pre-finite model property.*

Proof. (1) Suppose that $L \supseteq \mathbf{KG}$ does not have the fmp. Then there is a formula φ such that $L \not\vdash \varphi$ and for each finite L-frame $\mathfrak{G}$ we have $\mathfrak{G} \models \varphi$. Since each superintuitionistic logic is complete with respect to its finitely generated rooted descriptive frames, there is a finitely generated rooted descriptive L-frame $\mathfrak{F}$ such that $\mathfrak{F} \not\models \varphi$. By our assumption, $\mathfrak{F}$ is infinite. This implies that $Log(\mathfrak{F})$ does not have the fmp. Obviously we have that $L \subseteq Log(\mathfrak{F})$. Thus, to prove that $L \subseteq Log(\mathfrak{T}_1)$, it is sufficient to show that $Log(\mathfrak{F}) \subseteq Log(\mathfrak{T}_1)$. We prove this by showing that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{F}$. By Theorem 3.5, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where $k, n \in \omega$ and each $\mathfrak{F}_i$ is a cyclic frame. Since $\mathfrak{F}$ is infinite, there is $j \leq n$ such that $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$. Let j be the least such index. First suppose that $j > 1$. Then $\mathfrak{F}$ is isomorphic to $\mathfrak{G} \oplus \mathfrak{F}_j \oplus \mathfrak{F}_{j-1} \oplus \cdots \oplus \mathfrak{F}_n \oplus \mathfrak{L}_{g_k}$, where $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$ and $\mathfrak{G}$ is finite. If there is no i with $n \geq i \geq j-1$ such that $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ or $\mathfrak{L}_{f_l}$ for some $m \geq 4$ and $l \geq 2$, then the same argument as in the proof of Theorem 5.1 shows that $Log(\mathfrak{F})$ has the fmp, which is a contradiction. Therefore, there is such i and we take the least such i. Then there are two possible cases: (i) $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ for $m \geq 4$, or (ii) $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{f_l}$ for $l \geq 2$. We only consider the case when $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ for $m \geq 4$. The case when $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{f_l}$ for $l \geq 2$ is similar. We define a p-morphism f from $\mathfrak{F}$ to $\mathfrak{G}_1 \oplus \mathfrak{F}_j \oplus \mathfrak{G}_1 \oplus \mathfrak{F}_i \oplus \mathfrak{G}_1$ as follows: We send all the elements of $\mathfrak{G}$ to $\mathfrak{G}_1$, each element of $\mathfrak{F}_j$ to itself, all the elements of $\mathfrak{F}_{j-1} \oplus \cdots \oplus \mathfrak{F}_{i-1}$ to $\mathfrak{G}_1$, each element of $\mathfrak{F}_i$ to itself, and all the elements of $\mathfrak{F}_{i+1} \oplus \cdots \oplus \mathfrak{L}_{g_k}$ to $\mathfrak{G}_1$. It is easy to check that f is an onto p-morphism, and so $\mathfrak{G}_1 \oplus \mathfrak{F}_j \oplus \mathfrak{G}_1 \oplus \mathfrak{F}_i \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{F}$. Moreover, $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$ and $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}_{g_m}$ for $m \geq 4$. Next we apply the same argument as in the proof of Theorem 4.13. If $m > 4$ is even, then $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_4}$ is a p-morphic image of $\mathfrak{L}_{g_m}$; and if $m > 4$ is odd, then $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_5}$ is a p-morphic image of $\mathfrak{L}_{g_m}$. Therefore, if $m > 4$ and m is even, then $\mathfrak{H}_1 = \mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{G}_1 \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{F}$; and if $m > 4$ is odd, then $\mathfrak{H}_2 = \mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{G}_1 \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{F}$. Clearly if $m = 4$, then $\mathfrak{H}'_1 = \mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{F}$; and if $m = 5$, then $\mathfrak{H}'_2 = \mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{G}_1 \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{F}$. It is easy to see that $\mathfrak{H}'_1$ is a p-morphic image of $\mathfrak{H}_1$, and that $\mathfrak{H}'_2$ is a p-morphic image of $\mathfrak{H}_2$. Now by identifying the greatest element of $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ with the least element of $\mathfrak{L} \oplus \mathfrak{G}_1$, we obtain that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{H}'_1$. Exactly the same argument shows that $\mathfrak{T}_2$ is a p-morphic image of $\mathfrak{H}'_2$. Finally, Lemma 7.2.2 ensures that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{T}_2$, which means that $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{F}$. The proof in case $j = 1$ is analogous, with the only difference that we also need to use Theorem 3.6, by which $\mathfrak{G}_1 \oplus \mathfrak{L}$ is a p-morphic image of $\mathfrak{L}$, and so $\mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{L} \oplus \mathfrak{L}_{g_4} \oplus \mathfrak{G}_1$, and $\mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$ is a p-morphic image of $\mathfrak{L} \oplus \mathfrak{L}_{g_5} \oplus \mathfrak{G}_1$. Thus, in either case, $\mathfrak{T}_1$ is a p-morphic image of $\mathfrak{F}$, and so $Log(\mathfrak{T}_1) \supseteq Log(\mathfrak{F})$.

(2) Suppose that L has the pre-finite model property. Then L does not have the fmp, so by (1), $L \subseteq Log(\mathfrak{T}_1)$. Moreover, since $Log(\mathfrak{T}_1)$ does not have the fmp, L can not be properly contained in $Log(\mathfrak{T}_1)$. Thus, $L = Log(\mathfrak{T}_1)$. □

8. Locally tabular extensions of **RN** and **KG**

In this section we show that $\mathbf{RN}.\mathbf{KC} = \mathbf{RN} + (\neg p \vee \neg\neg p)$ is the only pre-locally tabular extension of **KG**. This gives a criterion for an extension of **KG** to be locally tabular. We also introduce the internal depth of a descriptive **RN**-frame and prove that an extension L of **RN** is locally tabular if and only if the internal depth of L is finite. This provides another criterion of local tabularity for extensions of **RN**.

Definition 8.1.

(1) *A logic L is called* locally tabular *if for each $n \in \omega$ there are only finitely many pairwise non-L-equivalent formulas in n variables.*
(2) *A logic L is called* pre-locally tabular *if L is not locally tabular but every proper extension of L is locally tabular.*

Let $\mathfrak{K} = \mathfrak{G}_1 \oplus \mathfrak{L}$, which is shown in Fig. 13. It is easy to see that $\mathfrak{K}$ is obtained from $\mathfrak{L}$ by identifying the two maximal nodes of $\mathfrak{L}$.

Theorem 8.2. *$Log(\mathfrak{K})$ is complete with respect to $\{\mathfrak{G}_1 \oplus \mathfrak{L}_{g_k} : k \in \omega\}$.*

Proof. Suppose that $\mathfrak{K} \not\models \varphi$ for some formula φ. Then there exists a descriptive valuation ν and a point x of $\mathfrak{K}$ of finite depth such that $(\mathfrak{K}, \nu), x \not\models \varphi$. We consider the generated subframe $\mathfrak{F}$ of $\mathfrak{K}$ generated by x. It is easy to see that $\mathfrak{F}$ is isomorphic to $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_k}$ for some $k \in \omega$ and that $\mathfrak{F} \not\models \varphi$. Therefore, $Log(\mathfrak{K})$ is complete with respect to $\{\mathfrak{G}_1 \oplus \mathfrak{L}_{g_k} : k \in \omega\}$. □

Definition 8.3. *Let* $\mathbf{RN}.\mathbf{KC} = \mathbf{RN} + (\neg p \vee \neg\neg p)$.

Theorem 8.4. *$Log(\mathfrak{K}) = \mathbf{RN}.\mathbf{KC}$.*

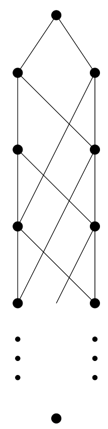

FIGURE 13. The frame $\mathfrak{K}$

Proof. Since $\mathfrak{K}$ is a p-morphic image of $\mathfrak{L}$, it is an **RN**-frame. As $\mathfrak{K}$ has a greatest element, it follows from [4, Proposition 2.37] that $\mathfrak{K}$ validates $\neg p \vee \neg\neg p$, and so $\mathfrak{K}$ is an **RN.KC**-frame. Thus, $Log(\mathfrak{K}) \supseteq$ **RN.KC**. Conversely, **RN.KC** is an extension of **RN**. By Theorem 5.1, **RN.KC** has the fmp. Finite rooted **RN.KC**-frames are finite rooted **RN**-frames with a greatest element. An argument similar to that in the proof of Theorem 3.7 shows that each finite rooted **RN.KC**-frame is a p-morphic image of a generated subframe of $\mathfrak{K}$. Thus, **RN.KC** $\supseteq Log(\mathfrak{K})$. □

To prove a criterion of local tabularity for extensions of **KG**, we reformulate the criterion for a variety of algebras to be locally finite established in [1] for extensions of **KG**.

Theorem 8.5. *An extension L of* **KG** *is locally tabular if and only if the class of finitely generated rooted descriptive L-frames is* uniformly locally tabular*; that is, for each $n \in \omega$ there is $M(n) \in \omega$ such that for each n-generated rooted descriptive L-frame $\mathfrak{F}$ we have $|\mathfrak{F}| \leq M(n)$.*

In proving our criterion, we will use the following auxiliary lemma. For a proof we refer to [3, Lemma 4.1.23].

Lemma 8.6. *Suppose $\mathfrak{F}$ is an n-generated descriptive frame isomorphic to $\bigoplus_{i=1}^{s} \mathfrak{F}_i$. Then $s \leq 2n$.*

Theorem 8.7. *An extension L of* **KG** *is not locally tabular if and only if $L \subseteq Log(\mathfrak{K})$.*

Proof. We first show that $Log(\mathfrak{K})$ is not locally tabular. Observe that for each point x of $\mathfrak{K}$ of finite depth, the point-generated subframe $\mathfrak{F}_x$ of $\mathfrak{F}$ is finite rooted 2-generated and $\sup(\{|\mathfrak{F}_x| : x \text{ is a point of } \mathfrak{F} \text{ of finite depth}\}) = \omega$. Thus, by Theorem 8.5, $Log(\mathfrak{K})$ is not locally tabular. It follows that if $L \subseteq Log(\mathfrak{K})$, then L is not locally tabular. Now suppose that L is not locally tabular. We show that $L \subseteq Log(\mathfrak{K})$. By Theorem 8.5, there are two possible cases:

Case 1: There exists $n \in \omega$ such that there is an n-generated infinite rooted descriptive L-frame $\mathfrak{F}$. By Theorem 3.5, $\mathfrak{F}$ is isomorphic to $\bigoplus_{i=1}^{m} \mathfrak{F}_i$, where each $\mathfrak{F}_i$ is a cyclic frame. Since $\mathfrak{F}$ is infinite, there is $j \leq m$ such that $\mathfrak{F}_j$ is isomorphic to $\mathfrak{L}$. We have that $j > 1$ or $j = 1$.

Case 1.1: If $j > 1$, then we define a p-morphism f from $\mathfrak{F}$ onto $\mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{G}_1$ as follows. We send all the points of $\mathfrak{F}_{j+1} \oplus \cdots \oplus \mathfrak{F}_n$ to $\mathfrak{G}_1$, each point of $\mathfrak{F}_j$ to itself, and all the points of $\mathfrak{F}_1 \oplus \cdots \oplus \mathfrak{F}_{j-1}$ to $\mathfrak{G}_1$. It is easy to check that f is a p-morphism. Finally, by identifying the least point of $\mathfrak{L}$ with the point of $\mathfrak{G}_1$, we obtain a p-morphic image of $\mathfrak{G}_1 \oplus \mathfrak{L} \oplus \mathfrak{G}_1$ isomorphic to $\mathfrak{K}$. Thus, $\mathfrak{K}$ is a p-morphic image of $\mathfrak{F}$, and so $L \subseteq Log(\mathfrak{K})$.

Case 1.2: If $j = 1$, then a similar argument to that in Case 1.1 gives us that $\mathfrak{L}$ is a p-morphic image of $\mathfrak{F}$. But $\mathfrak{K}$ is a p-morphic image of $\mathfrak{L}$. Thus, in this case too, we obtain that $\mathfrak{K}$ is a p-morphic image of $\mathfrak{F}$, and so $L \subseteq Log(\mathfrak{K})$.

Case 2: There exists $n \in \omega$ such that $\sup(\{|\mathfrak{H}| : \mathfrak{H}$ is an n-generated finite rooted L-frame$\}) = \omega$. This means that for each $m \in \omega$ there is a finite rooted n-generated frame $\mathfrak{H}$ such that $|\mathfrak{H}| > m$. Since each $\mathfrak{H}$ is a **KG**-frame, each $\mathfrak{H}$ is isomorphic to $\bigoplus_{i=1}^{s} \mathfrak{H}_i$, where each $\mathfrak{H}_i$ is finite and cyclic. Then we have two possible cases.

Case 2.1: For each $m \in \omega$ there exists an n-generated finite rooted L-frame $\mathfrak{H} = \bigoplus_{i=1}^{s} \mathfrak{H}_i$ such that $|\mathfrak{H}_i| > m$ for some $i \leq s$. Then the same argument as in Case 1 shows that for each $k \in \omega$ the frame $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_k}$ is an L-frame. By Theorem 8.2, this implies that $L \subseteq Log(\mathfrak{K})$.

Case 2.2: There is $m \in \omega$ such that for each n-generated finite rooted L-frame $\mathfrak{H} = \bigoplus_{i=1}^{s} \mathfrak{H}_i$, we have $|\mathfrak{H}_i| \leq m$ for $i = 1, \ldots, s$. By Lemma 8.6, $s \leq 2n$. Therefore, $|\mathfrak{H}| \leq m \cdot 2n$, and by Theorem 8.5, L is locally tabular, which contradicts our assumption.

Consequently, we obtain that L is not locally tabular if and only if $L \subseteq Log(\mathfrak{K})$. □

Corollary 8.8.

(1) *An extension L of* **KG** *is locally tabular if and only if $L \not\subseteq$* **RN.KC***.*

(2) *If an extension L of* **KG** *is finitely axiomatizable, then it is decidable whether L is locally tabular.*

Proof. (1) is an immediate consequence of Theorems 8.4 and 8.7. For (2), first note that since **RN.KC** is finitely axiomatizable and has the fmp, it is decidable. Let $\mathsf{Ax}(L)$ be the finite axiomatization of L. Then L is not locally tabular if and only if **RN.KC** $\vdash \varphi$ for each $\varphi \in \mathsf{Ax}(L)$. This problem is clearly decidable since **RN.KC** is decidable. □

We conclude the paper by giving another criterion of local tabularity for extensions of **RN**. By Corollary 3.8, each finite rooted L-frame is isomorphic to $\bigoplus_{i=1}^{n} \mathfrak{F}_i \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $k, n \in \omega$.

Definition 8.9.

(1) *Let $\mathfrak{F} = (\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $k, n \in \omega$. The* initial segment *of $\mathfrak{F}$ is the frame $\mathfrak{L}_{g_k}$.*

(2) *The* internal depth *of a finite rooted* **RN***-frame $\mathfrak{F}$ is the depth of its initial segment. Let $d_I(\mathfrak{F})$ denote the internal depth of $\mathfrak{F}$.*

(3) *The* internal depth *of a logic $L \supseteq$* **RN** *is $\sup\{d_I(\mathfrak{F}) : \mathfrak{F}$ is a finite rooted L-frame$\}$. Let $d_I(L)$ denote the internal depth of L.*

Theorem 8.10. *A logic $L \supseteq$* **RN** *is locally tabular if and only if $d_I(L) < \omega$.*

Proof. First suppose that $d_I(L) = \omega$. Then for each $m \in \omega$ there exists $k > m$ such that $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$ is an L-frame, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$ and $k, n \in \omega$. By mapping all the points of $\bigoplus_{i=1}^{n} \mathfrak{F}_i$ to $\mathfrak{G}_1$, we obtain that $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_k}$ is a p-morphic image of $(\bigoplus_{i=1}^{n} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$. Therefore, each $\mathfrak{G}_1 \oplus \mathfrak{L}_{g_k}$ is an L-frame, and so $L \subseteq Log(\mathfrak{K})$, by Theorem

8.2. Now apply Theorem 8.7 to obtain that L is not locally tabular. For the converse, suppose that $d_I(L) = m < \omega$. Let $\mathfrak{F}$ be an n-generated rooted descriptive L-frame. By Theorem 4.11, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{s} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}$, $\mathfrak{G}_1$, or $\mathfrak{G}_2$. We show that no $\mathfrak{F}_i$ can be isomorphic to $\mathfrak{L}$. If there is i such that $\mathfrak{F}_i$ is isomorphic to $\mathfrak{L}$, then we consider the least such i. For each $x \in \mathfrak{F}_i$ of finite depth, the generated subframe of $\mathfrak{F}$ generated by x is a finite rooted L-frame. But the internal depth of such frames is unbounded, contradicting the fact that $d_I(L) < \omega$. Therefore, no such $\mathfrak{F}_i$ exists. Thus, $\mathfrak{F}$ is isomorphic to $(\bigoplus_{i=1}^{s} \mathfrak{F}_i) \oplus \mathfrak{L}_{g_k}$, where each $\mathfrak{F}_i$ is isomorphic to $\mathfrak{G}_1$ or $\mathfrak{G}_2$. Since $d_I(L) = m$, we have $|\mathfrak{L}_{g_k}| \leq 2m$. By Lemma 8.6, $s \leq 2n$. Therefore, $|\bigoplus_{i=1}^{s} \mathfrak{F}_i| \leq 2 \cdot (2n) = 4n$. It follows that $|\mathfrak{F}| \leq 4n + 2m$. Thus, the cardinality of each n-generated rooted L-frame is bounded by $4n + 2m$. This, by Theorem 8.5, implies that L is locally tabular. $\square$

References

[1] G. Bezhanishvili. Locally finite varieties. *Algebra Universalis*, 46:531–548, 2001.

[2] G. Bezhanishvili and R. Grigolia. Subalgebras and homomorphic images of the Rieger-Nishimura lattice. In *Proceedings of the Institute of Cybernetics*, volume 1, pages 9–16. Georgian Academy of Sciences, Tbilisi, 2000.

[3] N. Bezhanishvili. *Lattices of Intermediate and Cylindric Modal Logics*. PhD thesis, ILLC, University of Amsterdam, 2006.

[4] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.

[5] B. Davey and H. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.

[6] L. Esakia. Gödel-Löb modal system – addendum. In *Proceedings of the third International Conference, Smirnov's Readings*, pages 77–79, Moscow, 2001. Russian Academy of Sciences.

[7] V. Ja. Gerčiu. Correction to the article "The finite approximability of superintuitionistic logics" (Mat. Issled **7** (1972), no. 1(23), 186–192). *Mat. Issled.*, 7(3(25)):278, 1972. (Russian).

[8] V. Ja. Gerčiu. The finite approximability of superintuitionistic logics. *Mat. Issled.*, 7(1(23)):186–192, 1972. (Russian).

[9] V. Ja. Gerčiu and A. V. Kuznetsov. The finitely axiomatizable superintuitionistic logics. *Soviet Mathematics Doklady*, 11:1654–1658, 1970.

[10] R. Harrop. On the existence of finite models and decision procedures for propositional calculi. *Proc. Cambridge Philos. Soc.*, 54:1–13, 1958.

[11] V.A. Jankov. The construction of a sequence of strongly independent superintuitionistic propositional calculi. *Soviet Mathematics Doklady*, 9:806–807, 1968.

[12] M. Kracht. Prefinitely axiomatizable modal and intermediate logics. *Mathematical Logic Quarterly*, 39:301–322, 1993.

[13] A. Kuznetsov and V. Gerciu. Superintuitionistic logics and finite approximability. *Soviet Mathematics Doklady*, 11(6):1614–1619, 1970.

[14] A. V. Kuznetsov. Superintuitionistic logics. *Mat. Issled.*, 10(2 (36)):150–158, 284–285, 1975. (Russian).

[15] A. V. Kuznetsov. Means for detection of nondeducibility and inexpressibility. In *Logical inference (Moscow, 1974)*, pages 5–33. "Nauka", Moscow, 1979. (Russian).

[16] A. V. Kuznetsov. Algorithms, algebras and intuitionistic logic. *Mat. Issled.*, (98, Neklass. Logiki):10–14, 152, 1987. (Russian).

[17] A. V. Kuznetsov and A. Yu. Muravitsky. On superintuitionistic logics as fragments of proof logic extensions. *Studia Logica*, 45(1):77–99, 1986.

[18] I. Nishimura. On formulas of one variable in intuitionistic propositional calculus. *Journal of Symbolic Logic*, 25:327–331, 1960.

[19] L. Rieger. On the lattice theory of Brouwerian propositional logic. *Acta fac. rerum nat. Univ. Car.*, 189:1–40, 1949.

Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003-8001, USA. Email: gbezhani@nmsu.edu

Department of Computer Science, University of Leicester, University Road, Leicester LE1 7RH, UK. Email: nick@mcs.le.ac.uk

Institute for Logic, Language and Computation, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands. Email: dickdj@science.uva.nl