Building bridges between dynamic and temporal doxastic logics
Extended Abstract

Johan van Benthem
ILLC Amsterdam
Stanford University

Cédric Dégremont*
ILLC Amsterdam

March 20, 2008

Analyzing the behavior of agents in a dynamic environment requires describing the evolution of their knowledge as they receive new information. Moreover agents entertain beliefs that need to be revised after learning new facts. I might be confident that I will find the shop open, but once I found it closed, I should not crash but rather make a decision on the basis of new consistent beliefs. Such beliefs and information may concern ground-level facts, but also beliefs about other agents. I might be a priori confident that the price of my shares will rise, but if I learn that the market is rather pessimistic (say because the shares fell by 10%), this information should change my higher-order beliefs about what other agents believe.

Tools from modal logic have been successfully applied to analyze knowledge dynamics in multi-agent contexts. Among these, Temporal Epistemic Logic [16], [12]'s Interpreted Systems, and Dynamic Epistemic Logic [2] have been particularly fruitful. A recent line of research [8, 7] compares these alternative frameworks, and [7] presents a representation theorem that shows under which conditions a temporal model can be represented as a dynamic one. Thanks to this link, the two languages also become comparable, and one can merge ideas: for example, a new line of research explores the introduction of protocols into PAL (see [7]).

To the best of our knowledge, there are no similar results yet for multi-agent belief revision. One reason is that dynamic logics of belief revision have only been well-understood recently. But right now, there is work on both dynamic doxastic logics [5, 3] and on temporal frameworks for belief revision with [10] as a recent example. The exact connection between these two frameworks is not quite like the case of epistemic update. In this paper we make things clear, by viewing belief revision as priority update over plausibility *pre-orders*. This correspondence allows for similar language links as in the knowledge case, with similar benefits.

We start in section 1 with some relevant background, while section 2 gives the main new definitions needed in the paper. Section 3 presents the key temporal doxastic properties that we will work with. In section 4 we state and prove our main result linking the temporal and the dynamic frameworks, first for the special case of *total* pre-orders and then in general. We also discuss some variations and extensions. In section 5 we introduce formal languages, providing an axiomatization for our crucial properties, and discussing some related definability issues. We state our conclusions and mention some open problems in the last section. Our key proofs are in the appendix.

*The second author was supported by a GLoRiClass fellowship of the European Commission (Research Training Fellowship MEST-CT-2005-020841).

1 Introduction: epistemic background results

Epistemic temporal trees and dynamic logics with product update are complementary ways of looking at multi-agent information flow. Representation theorems linking both approaches were proposed for the first time in [4]. A nice presentation of these early results can be found in [14, ch5]. We start with one recent version from [7], referring the reader to [7] for a proof, as well as generalizations and variations.

Definition 1.1 (Epistemic and Event Models, Product Update)**.**

- *An* epistemic model $\mathcal{M}$ *is of the form* $\langle W, (\sim_i)_{i\in N}, V\rangle$ *where* $W \neq \emptyset$*, for each* $i \in N$*,* $\sim_i$ *is a relation on* W*, and* $V : Prop \to \wp(H)$*.*
- *An* event model $\epsilon = \langle E, (\sim_i)_{i\in N}, \mathtt{pre}\rangle$ *has* $E \neq \emptyset$*, and for each* $i \in N$*,* $\sim_i$ *is a relation on* W*. Finally, there is a precondition map* $\mathtt{pre} : E \to \mathcal{L}_{EL}$*, where* $\mathcal{L}_{EL}$ *is the usual language of epistemic logic.*
- *The* product update $\mathcal{M} \otimes \epsilon$ *of an epistemic model* $\mathcal{M} = \langle W, (\sim'_i)_{i\in N}, V\rangle$ *with an event model* ϵ *is the model* $\langle E, (\sim_i)_{i\in N}, \mathtt{pre}\rangle$*, whose worlds are pairs* (w, e) *with the world* w *satisfying the precondition of the event* e*, and accessibilities defined as:*

$$(w, e) \sim'_i (w', e') \text{ iff } e \sim_i e', w \sim_i w'$$

Intuitively epistemic models describe what agents currently know while the product update describe the new multi-agent epistemic situation after some epistemic event has taken place. Nice intuitive examples are in [1]. Next we turn to the epistemic temporal models introduced by [16]. In what follows, Σ^* is the set of finite sequences on any set Σ.

Definition 1.2 (Epistemic Temporal Models)**.** *An* epistemic temporal model *(ETL model for short)* $\mathcal{H}$ *is of the form* $\langle \Sigma, H, (\sim_i)_{i\in N}, V\rangle$ *where* Σ *is a finite set of events,* $H \subseteq \Sigma^*$ *and* H *is closed under non-empty prefixes. For each* $i \in N$*,* $\sim_i$ *is a relation on* H*, and* $V : Prop \to \wp H$*.*

The crucial epistemic temporal properties driving [7]'s main theorem are:

Definition 1.3. *Let* $\mathcal{H} = \langle \Sigma, H, (\sim_i)_{i\in N}, V\rangle$ *be an ETL model.* $\mathcal{H}$ *satisfies:*
Let $\sim^*$ *be the reflexive transitive closure of the relation* $\bigcup_{i\in N} \sim_i$*:*

- **Local Bisimulation Invariance** *if, whenever* $h \sim^* h'$ *and* h *and* h' *are epistemically bisimilar*[1]*, we have* $h'e \in H$ *iff* $he \in H$*.*
- **Perfect Recall** *if, whenever* $ha \sim_i h'b$*, we also have* $h \sim_i h'$ *.*
- **Local No Miracles** *if, whenever* $ga \sim g'b$ *and* $g \sim^* h \sim h'$*, then for every* $h'a, hb \in H$*, we also have* $h'a \sim hb$*.*

These properties describe the idealized epistemic agents needed in:

[1]The reader is referred to subsection 3.1 for a definition of bisimulation invariance.

Theorem 1.4 (van Benthem et al. [7]). *Let $\mathcal{H}$ be an ETL model, $\mathcal{M}$ an epistemic model, and the 'protocol' P a set of finite sequences of pointed events models closed under prefixes. We write $\otimes$ for* product *update. Let $Forest(M, P) = \bigcup_{\vec{\epsilon} \in P} M \otimes \vec{\epsilon}$ be the 'epistemic forest generated by' $\mathcal{M}$ and sequential application of the events in P.* [2] *The following are equivalent:*

1. *$\mathcal{H}$ is isomorphic to $Forest(M, P)$.*
2. *$\mathcal{H}$ satisfies propositional stability, synchronicity, local bisimulation invariance, Perfect Recall, and Local No Miracles.*

Thus, epistemic temporal conditions characterize just those trees that arise from performing iterated product update governed by some protocol. [7] and [14, ch5] have details. Our paper extends this analysis to the richer case of belief revision, where plausibility orders of agents evolve as they observe possibly surprising events. We prove two main results:

Theorem. *Let $\mathcal{H}$ be a doxastic temporal model, $\mathcal{M}$ a plausibility model, $\vec{\epsilon}$ a sequence of event models, and $\otimes$* priority *update. The following are equivalent (details will follow later):*

1. *$\mathcal{H}$ is isomorphic to the forest generated by $\mathcal{M} \otimes \vec{\epsilon}$*
2. *$\mathcal{H}$ satisfies propositional stability, synchronicity, invariance for bisimulation, as well as principles of Preference Propagation, Preference Revelation and Accommodation.*

Theorem. *Preference Propagation, Preference Revelation and Accommodation are definable in an extended doxastic modal language.*

2 Definitions

We now turn to the definitions needed for the simplest version of our main representation theorem, postponing matching formal languages to Section 5. Let $N = \{1, \ldots, n\}$ be a finite set of agents.

2.1 Plausibility models, event models and priority update

We first introduce static models that encode the current prior (conditional) beliefs of agents. These carry a pre-order $\leq$ between worlds encoding a plausibility relation. Often this relation is taken to be total, but when we think of elicited beliefs as *multi-criteria decisions*, a pre-order allowing for incomparable situations may be all we get [11]. We will therefore assume reflexivity and transitivity, but not totality. We write $a \simeq b$ ('indifference') if $a \leq b$ and $b \leq a$, and $a < b$ if $a \leq b$ and $b \not\leq a$.

Definition 2.1 (Doxastic Plausibility Models). *A* doxastic plausibility model *$\mathcal{M} = \langle W, (\preceq_i)_{i \in N}, V \rangle$ has $W \neq \emptyset$, for each $i \in N$, $\preceq_i$ is a pre-order on W, and $V : Prop \to \wp H$.*

We now consider how such models evolve as agents observe events.

Definition 2.2 (Plausibility Event Model). *A* plausibility event model *(event model, for short) ϵ is a tuple $\langle E, (\preceq_i)_{i \in N}, \texttt{pre} \rangle$ with $E \neq \emptyset$, each $\preceq_i$ a pre-order on E, and $\texttt{pre} : E \to \mathcal{L}$, where $\mathcal{L}$ is a doxastic language.* [3]

[2] For a more precise definition of this notion, see Section 2 below.

[3] This definition is incomplete without specifying the relevant language, but all that follows can be understood by considering the formal language as a 'parameter'.

Definition 2.3 (Priority Update; [3]). *The* priority update *of plausibility model* $\mathcal{M} = \langle W, (\preceq_i)_{i \in N}, V \rangle$ *and event model* $\epsilon = \langle E, (\preceq_i)_{i \in N}, \mathtt{pre} \rangle$ *is the plausibility model* $\mathcal{M} \otimes \epsilon$*:*

- $W' = \{(w, e) \in W \times E \mid \mathcal{M}, w \Vdash \mathtt{pre}(e)\}$
- $(w, e) \preceq'_i (w', e')$ *iff either* $e \prec_i e'$, *or* $e \simeq_i e'$ *and* $w \preceq_i w'$
- $V'((s, e)) = V(s)$

The idea behind priority update is that beliefs about the last event override prior beliefs. If the agent is indifferent, the old plausibility order applies. More motivation can be found in [3, 6].

2.2 Doxastic Temporal Models

Definition 2.4 (Doxastic Temporal Models). *A* doxastic temporal model *(DoTL model for short)* $\mathcal{H}$ *is of the form* $\langle \Sigma, H, (\leq_i)_{i \in N}, V \rangle$, *where* Σ *is a finite set of events,* $H \subseteq \Sigma^*$ *is closed under non-empty prefixes, for each* $i \in N$, $\leq_i$ *is a pre-order on* H, *and* $V : Prop \to \wp H$.

Our task is to identify just when a doxastic temporal model is isomorphic to the 'forest' generated by a sequence of priority updates:

2.3 Dynamic Models Generate Doxastic Temporal Models

Definition 2.5 ($DoTL$ model generated by a sequence of updates). *Each initial plausibility model* $\mathcal{M} = \langle W, (\preceq_i)_{i \in N}, V \rangle$ *and sequence of event models* $\epsilon_j = \langle E_j, (\preceq^j_i)_{i \in N}, \mathtt{pre}_j \rangle$ *yields a* generated $DoTL$ plausibility model $\langle \Sigma, H, (\leq_i)_{i \in N}, \mathbf{V} \rangle$ *as follows:*

- *Let* $\Sigma := \bigcup_{i=1}^{m} e_i$.
- *Let* $H_1 := W$ *and for any* $1 < n \leq m$ *let* $H_{n+1} := \{(we_1 \ldots e_n) | (we_1 \ldots e_{n-1}) \in H_n \text{ and } \mathcal{M} \otimes \epsilon_1 \otimes \ldots \otimes \epsilon_{n-1} \Vdash \mathtt{pre}_n(e_n)\}$. *Finally let* $H = \bigcup_{1 \leq k \leq m} H_k$.
- *If* $h, h' \in H_1$, *then* $h \leq_i h'$ *iff* $h \preceq^{\mathcal{M}}_i h'$.
- *For* $1 < k \leq m$, $he \leq_i h'e'$ *iff 1.* $he, h'e' \in H_k$, *and 2. either* $e \prec^k_i e'$, *or* $e \simeq^k_i e'$ *and* $h \leq_i h'$.
- *Let* $wh \in \mathbf{V}(p)$ *iff* $w \in V(p)$.

Now come the key doxastic temporal properties of our idealized agents.

3 Crucial Frame Properties for Priority Updaters

We first introduce the notion of bisimulation, modulo a choice of language.

3.1 Bisimulation Invariance

Definition 3.1 ($\leq$-Bisimulation)**.** *Let $\mathcal{H}$ and $\mathcal{H}'$ be two DoTL plausibility models $\langle H, (\leq_1, \ldots, \leq_n), V\rangle$ and $\langle H', (\leq'_1, \ldots, \leq'_n), V'\rangle$. A relation $Z \subseteq H \times H'$ is a $\leq$-Bisimulation if, for all $h \in H$, $h' \in H'$, and all $\leq_i$ in $(\leq_1, \ldots, \leq_n)$,*

(prop) h and h' satisfy the same proposition letters,

(zig) If hZh' and $h \leq_i j$, then there exists $j' \in H'$ such that jZj' and $h' \leq'_i j'$,

(zag) If hZh' and $h' \leq'_i j'$, then there exists $j \in H$ such that jZj' and $h \leq_i j$.

If Z is a $\leq_n$-bisimulation and hZh', we call h and h' are $\leq$-bisimilar.

Definition 3.2 ($\leq$-Bisimulation Invariance)**.** *A DoTL model $\mathcal{H}$ satisfies* $\leq$-bisimulation invariance *if, for all $\leq$-bisimilar histories $h, h' \in H$, and all events e, $h'e \in H$ iff $he \in H$.*

3.2 Agent-Oriented Frame Properties

In the following we drop agent labels and the "for each $i \in N$" for the sake of clarity. Also, when we write ha we will always assume that $ha \in H$. We will make heavy use of the following notion:

Definition 3.3 (Accommodating Events)**.** *Two events $a, b \in \Sigma$ are* accommodating *if, for all $ga, g'b$, $(g \leq g' \leftrightarrow ga \leq g'b)$ and similarly for $\geq$, i.e., a, b preserve and anti-preserve plausibility.*

Definition 3.4. *Let $\mathcal{H} = \langle \Sigma, H, (\leq_i)_{i \in N}, V\rangle$ be a DoTL model. $\mathcal{H}$ satisfies:*

- **Propositional stability** *if, whenever h is a finite prefix of h', then h and h' satisfy the same proposition letters.*
- **Synchronicity** *if, whenever $h \leq h'$, we have $len(h) = len(h')$.*

The following three properties trace the belief revising behavior of agents in doxastic trees.

- **Preference Propagation** *if, whenever $ja \leq j'b$, then $h \leq h'$ implies $ha \leq h'b$.*
- **Preference Revelation** *if, whenever $jb \leq j'a$, then $ha \leq h'b$ implies $h \leq h'$.*
- **Accommodation** *if, a and b are* accommodating *whenever both $ja \leq j'b$ and $ha \not\leq h'b$.*

These properties are somewhat trickier than in the epistemic case, reflecting the peculiarities of priority update in settings where incomparability is allowed. But we do have:

Fact 3.5. *If $\leq$ is a* total *pre-order and $\mathcal{H}$ satisfies Preference Propagation and Preference Revelation, then $\mathcal{H}$ satisfies Accommodation.*

We can also prove a partial converse without assuming totality:

Fact 3.6. *If $\mathcal{H}$ satisfies Accommodation, it satisfies Preference Propagation.*

By contrast *Accommodation* does not imply *Preference Revelation.*

4 The Main Representation Theorem

We start with a warm-up case, with plausibility a *total* pre-order.

4.1 Total pre-orders

Theorem 4.1. *Let $\mathcal{H}$ be a total doxastic-temporal model, $\mathcal{M}$ a total plausibility model, $\vec{\epsilon}$ a sequence of total event models, and let $\otimes$ stand for* priority *update. The following are equivalent:*

- *$\mathcal{H}$ is isomorphic to the forest generated by $\mathcal{M} \otimes \vec{\epsilon}$.*
- *$\mathcal{H}$ satisfies propositional stability, synchronicity, bisimulation invariance, Preference Propagation, and Preference Revelation.*

4.2 The general case: pre-orders allowing incomparability

While the argument went smoothly for *total* pre-orders, it gets somewhat more interesting when incomparability enters the stage. In the case of pre-orders we need the additional axiom of Accommodation as stated below:

Theorem 4.2. *Let $\mathcal{H}$ be a doxastic-temporal model, $\mathcal{M}$ a plausibility model, $\vec{\epsilon}$ be a sequence of event models while $\otimes$ is* priority *update. The following assertions are equivalent:*

- *$\mathcal{H}$ is isomorphic to the forest generated by $\mathcal{M} \otimes \vec{\epsilon}$,*
- *$\mathcal{H}$ satisfies bisimulation invariance, propositional stability, synchronicity, Preference Revelation and Accommodation.*

By Fact 3.6, Accommodation also gives us Preference Propagation.

Given a doxastic temporal model describing the evolution of the beliefs of a group of agents, we have determined whether it could have been generated by successive 'local' priority updates of a plausibility model. Of course, further scenarios are possible, e.g., bringing in knowledge as well.

4.3 Extensions and variations of the theorem

4.3.1 Unified plausibility models

There are two roads to merging epistemic indistinguishability and doxastic plausibility. The first works with a plausibility order *and* an epistemic indistinguishability relation, explaining the notion of *belief* with a mixture of the two. Baltag and Smets [3] apply product update to epistemic indistinguishability and priority update to the plausibility relation. A characterization for the doxastic epistemic temporal models induced in this way follows from van Benthem et al. [7] Theorem 1.4 plus Theorem 4.2 of the previous subsection. All this has the flavor of working with *prior* beliefs and information partitions, taking the *posteriors* to be computed from them. However there are also reasons for working with (*posterior*) beliefs only (see e.g. [15]). Indeed, Baltag and Smets [3] take this second road, using *unified* 'local' plausibility models with just one explicit relation $\trianglelefteq$. We briefly show how our earlier results transform to this setting. Due to space restrictions, we only state the following result which is proved in the extended version of this paper.

Theorem 4.3. *Let $\mathcal{H}$ be a unified doxastic-temporal model, $\mathcal{M}$ a unified plausibility model, $\vec{\epsilon}$ be a sequence of unified event models, while $\otimes$ is* priority *update. The following assertions are equivalent:*

- *$\mathcal{H}$ is isomorphic to the forest generated by $\mathcal{M} \otimes \vec{\epsilon}$,*
- *$\mathcal{H}$ satisfies bisimulation invariance, propositional stability, synchronicity, $\trianglelefteq$-Perfect Recall, $\trianglelefteq$-Preference Propagation, $\trianglelefteq$-Preference Revelation and $\trianglelefteq$-Accommodation.*

4.3.2 Variations: languages and protocols

One issue left open so far may have bothered some readers. Our definition of event models presupposed a language for the preconditions, and correspondingly, the right notion of bisimulation in our representation results should match the precondition language used. This issue is discussed in the extended version of the paper. Also, so far, the same sequences of events were executable uniformly anywhere in the initial doxastic model, provided the worlds fulfilled the preconditions. This strong assumption might be lifted (cf. [7]) and leads to interesting variations.

5 Dynamic and Temporal Doxastic Languages

To conclude, we turn from structural properties of models to logical logical languages that can express these, and thus the type of doxastic reasoning our agents can be involved with.

5.1 Dynamic doxastic language

We first look at a core language that matches dynamic belief update.

Definition 5.1 (Dynamic Doxastic-Epistemic language)**.** *The language of dynamic doxastic language $DDE\mathcal{L}$ is defined as follows:*

$$\phi := p \mid \neg\phi \mid \phi \vee \phi \mid \langle \leq_i \rangle \phi \mid \langle i \rangle \phi \mid \mathtt{E}\phi \mid \langle \epsilon, \mathbf{e} \rangle \phi$$

where i ranges over over N, p over a countable set of proposition letters $Prop$, and $(\epsilon, \mathbf{e})$ ranges over a suitable set of symbols for event models.

Definition 5.2 (Epistemic Plausibility Models)**.** *Epistemic Plausibility Models are the obvious combination of epistemic models (of section 1) and plausibility models (of section 2), while the* priority update *of an epistemic plausibility model is as expected, i.e. the crucial clauses are:*

Definition 5.3 (Priority update)**.**

- $(w, e) \preceq'_i (w', e')$ *iff* $e {\prec_i} e'$, *or* $e {\simeq_i} e'$ *and* $w \preceq_i w'$
- $(w, e) \sim'_i (w', e')$ *iff* $e \sim_i e'$ *and* $w \sim_i w'$

Here is how we interpret the $DDE(L)$ language. A pointed event model is an event model plus an element of its domain. We write $\mathtt{pre}(e)$ for $\mathtt{pre}_\epsilon(e)$ when it is clear from context.

Definition 5.4 (Truth definition)**.** *Let $K_i[w] = \{v \mid w \sim_i v\}$. Booleans cases are expected.*

$$\begin{array}{lll}
\mathcal{M}, w \Vdash \langle \leq_i \rangle \phi & \text{iff} & \exists v \text{ such that } w \preceq_i v \text{ and } \mathcal{M}, v \Vdash \phi \\
\mathcal{M}, w \Vdash \langle i \rangle \phi & \text{iff} & \exists v \text{ such that } v \in K_i[w] \text{ and } \mathcal{M}, v \Vdash \phi \\
\mathcal{M}, w \Vdash \mathtt{E}\phi & \text{iff} & \exists v \in W \text{ such that } \mathcal{M}, v \Vdash \phi \\
\mathcal{M}, w \Vdash \langle \epsilon, e \rangle \phi & \text{iff} & \mathcal{M}, w \Vdash \mathtt{pre}(e) \text{ and } \mathcal{M} \times \epsilon, (w, e) \Vdash \phi
\end{array}$$

Knowledge K_i and the universal modality $\mathtt{A}$ are defined as usual.

Reduction axioms. The methodology of dynamic epistemic and dynamic doxastic logics revolves around *reduction* axioms. On top of some complete static base logic, these fully describe the dynamic component. Here is well-known *Action − Knowledge* reduction axiom of [2]:

$$[\epsilon, \mathbf{e}]K_i\phi \leftrightarrow (\mathtt{pre}(e) \;\rightarrow\; \bigwedge\{K_i[\epsilon, \mathbf{f}]\phi \;:\; e \sim_i f\}) \tag{1}$$

Similarly here are the key reduction axioms for $\langle \epsilon, \mathbf{e} \rangle \langle \leq_i \rangle$ with priority update:

Proposition 5.5. *The following dynamic-doxastic principle is sound for plausibility change:*

$$\langle \epsilon, \mathbf{e} \rangle \langle \leq_i \rangle \phi \leftrightarrow (\mathtt{pre}(e) \wedge (\langle \leq_i \rangle \bigvee\{\langle f \rangle \phi \;:\; e \simeq_i f\} \vee \mathtt{E} \bigvee\{\langle g \rangle \phi \;:\; e <_i g\})) \tag{2}$$

The crucial feature of such a dynamic 'recursion step' is that the order between *action* and *belief* is reversed. This works because, conceptually, the current beliefs already *pre-encode* the beliefs after some specified event. In the epistemic setting, principles like this reflect agent properties of Perfect Recall and No Miracles [8]. Here, they rather encode radically 'event-oriented' revision policies, and the same point applies to the principles we will find later in a doxastic temporal setting.

Proposition 5.6. *The following axiom is valid for the existential modality:*

$$\langle \epsilon, \mathbf{e} \rangle \mathtt{E}\phi \leftrightarrow (\mathtt{pre}(e) \wedge (\mathtt{E} \bigvee\{\langle f \rangle \phi \;:\; f \in Dom(\epsilon)\})) \tag{3}$$

We do not pursue completeness, since we are just after the model theory.

5.2 Doxastic epistemic temporal language

Our *epistemic-doxastic temporal models* are simply our old doxastic temporal models $\mathcal{H}$ extended with epistemic accessibility relations $\sim_i$. Our main language is as follows:

Definition 5.7 (Doxastic Epistemic Temporal Language)**.** *Here is the $DET\mathcal{L}$ syntax:*

$$\phi := p \mid \neg\phi \mid \phi \vee \phi \mid \langle e \rangle \phi \mid \langle e^{-1} \rangle \phi \mid \langle \leq_i \rangle \phi \mid \langle i \rangle \phi \mid \mathtt{E}\phi$$

where i ranges over N, e over Σ, and p over proposition letters $Prop$.

The language $DET\mathcal{L}$ is interpreted over nodes h in our trees (cf. [8]):

Definition 5.8 (Truth definition)**.** *Let $K_i[h] = \{h' \mid h \sim_i h'\}$. Booleans cases are expected.*

$$\begin{array}{lll}
\mathcal{H}, h \Vdash \langle e \rangle \phi & \text{iff} & \exists h' \in H \text{ such that } h' = he \text{ and } \mathcal{H}, h' \Vdash \phi \\
\mathcal{H}, h \Vdash \langle e^{-1} \rangle \phi & \text{iff} & \exists h' \in H \text{ such that } h'e = h \text{ and } \mathcal{H}, h' \Vdash \phi \\
\mathcal{H}, h \Vdash \langle \leq_i \rangle \phi & \text{iff} & \exists h' \text{ such that } h \leq_i h' \text{ and } \mathcal{H}, h' \Vdash \phi \\
\mathcal{H}, h \Vdash \langle i \rangle \phi & \text{iff} & \exists h' \text{ such that } h' \in K_i[h] \text{ and } \mathcal{H}, h' \Vdash \phi \\
\mathcal{H}, h \Vdash \mathtt{E}\phi & \text{iff} & \exists h' \in H \text{ such that } \mathcal{H}, h' \Vdash \phi
\end{array}$$

5.3 Defining the frame conditions

We will prove semantic *correspondence results* (cf. [9]) for our crucial properties, using somewhat technical axioms that simplify the argument. Afterwards, we discuss their intuitive meaning.

5.3.1 The key correspondence result

Theorem 5.9 (Definability). *Preference Propagation, Preference Revelation and Accommodation are definable in the doxastic-epistemic temporal language $DET\mathcal{L}$.*

- *$\mathcal{H}$ satisfies Preference Propagation iff the following axiom is valid:*

$$\mathtt{E}\langle a\rangle\langle\leq_i\rangle\langle b^{-1}\rangle\top \rightarrow (((\langle\leq_i\rangle\langle b\rangle p \wedge \langle a\rangle q) \rightarrow \langle a\rangle(q \wedge \langle\leq_i\rangle p) \tag{PP}$$

- *$\mathcal{H}$ satisfies Preference Revelation iff the following axiom is valid:*

$$\mathtt{E}\langle b\rangle\langle\leq_i\rangle\langle a^{-1}\rangle\top \rightarrow ((\langle a\rangle\langle\leq_i\rangle(p \wedge \langle b^{-1}\rangle\top) \rightarrow \langle\leq_i\rangle\langle b\rangle p) \tag{PR}$$

- *$\mathcal{H}$ satisfies Accommodation iff the following axiom is valid:*

$$\begin{aligned} &\mathtt{E}\langle a\rangle\langle\leq_i\rangle\langle b^{-1}\rangle\top \\ &\qquad \wedge\ \mathtt{E}\,[\langle a\rangle\ (p_1\ \wedge\ \mathtt{E}\,(p_2 \wedge \langle b^{-1}\rangle\top)\ )\ \wedge\ [a]\ (p_1\ \rightarrow [\leq_i]\neg p_2)] \\ &\qquad \rightarrow\ (\ (\langle\leq_i\rangle\langle b\rangle q \rightarrow [a]\langle\leq_i\rangle q) \\ &\qquad \wedge\ (\langle a\rangle\langle\leq_i\rangle(r \wedge \langle b^{-1}\rangle\top) \rightarrow \langle\leq_i\rangle\langle b\rangle r) \end{aligned} \tag{AC}$$

The preceding correspondence arguments are *Sahlqvist* substitution cases (cf. [9]). We do not prove completeness, but here is a nice syntactic counterpart (Fact 5.11) to our earlier Fact 3.5:

Fact 5.10. *On total doxastic temporal models the following axiom is valid:*

$$\langle a\rangle(\psi\ \wedge\ \mathtt{E}\,(\phi \wedge \langle b^{-1}\rangle\top)) \quad\rightarrow\quad (\ \langle a\rangle(\psi\ \wedge\ \langle\leq_i\rangle\phi)\ \vee\ \mathtt{E}\langle b\rangle(\phi \wedge\ \langle\leq_i\rangle(\psi \wedge \langle a^{-1}\rangle\top)) \tag{Tot}$$

Fact 5.11.

$$\vdash ((PP) \wedge (PR) \wedge (Tot)) \rightarrow (\mathrm{AC}) \tag{4}$$

5.3.2 Two intuitive explanations

Reformulation with safe belief. An intermediate notion of knowledge first considered by [17] has been argued for doxastically as *safe belief* by [3] as describing those beliefs we do not give up under true new information. The safe belief modality $\Box^{\geq}$ is just the universal dual of $\langle\geq\rangle$. Here is an example of how we can rephrase our earlier axioms:

- $\mathcal{H}$ satisfies Preference Propagation iff the following axiom is valid on $\mathcal{H}$:

$$\mathtt{E}\langle a\rangle\langle\geq\rangle\langle b^{-1}\rangle\top \rightarrow (\langle a\rangle\Box^{\geq_i} p \rightarrow \Box^{\geq_i}[b]p) \tag{PP'}$$

Such principles reverse action modalities and safe belief much like the better-known Knowledge-Action interchange laws in the epistemic-temporal case.

Analogies with reduction axioms Another way to understand the above axioms with existential modalities is their clear analogy with the reduction axiom for priority update.

$$\langle \epsilon, \mathbf{e} \rangle \langle \leq_i \rangle p \leftrightarrow (\texttt{pre}(e) \wedge (\langle \leq_i \rangle \bigvee \{ \langle f \rangle p \; : \; e \simeq_i f \} \vee \texttt{E} \bigvee \{ \langle g \rangle p \; : \; e <_i g \})) \tag{2}$$

$$\texttt{E}\langle a \rangle \langle \leq_i \rangle \langle b^{-1} \rangle \top \rightarrow (\langle \leq_{\texttt{i}} \rangle \langle \texttt{b} \rangle p \rightarrow [\texttt{a}] \langle \leq_{\texttt{i}} \rangle p) \tag{PP}$$

$$\texttt{E}\langle b \rangle \langle \leq_i \rangle \langle a^{-1} \rangle \top \rightarrow (\langle \texttt{a} \rangle \langle \leq_{\texttt{i}} \rangle (p \wedge \langle b^{-1} \rangle \top) \rightarrow \langle \leq_{\texttt{i}} \rangle \langle \texttt{b} \rangle p) \tag{PR}$$

The family resemblance is obvious, and indeed, PP and PR may be viewed as the two halves of the reduction axiom, transposed to the more general setting of arbitrary doxastic-temporal models.

5.4 Variations and extensions of the language

The above doxastic-temporal language is by no means the only reasonable one. Weaker forward-looking modal fragments also make sense, dropping both converse and the existential modality. But they do not suffice for the purpose of our correspondence.

Proposition 5.12 (Undefinability)**.** *Preference propagation, Preference Revelation and Accommodation are* not *definable in the forward looking fragment of* $\mathbf{DET}\mathcal{L}$

Richer doxastic temporal languages are studied in the extended version of the paper: including common belief, mixtures of knowledge and beliefs, and unrestricted past and future operators.

6 Conclusion

Agents that update their knowledge and revise their beliefs can behave very differently over time. We have determined the special constraints that capture agents operating with the 'local updates' of dynamic doxastic logic. This took the form of some representation theorems that state just when a general doxastic temporal model is equivalent to the forest model generated by successive priority updates of an initial doxastic model by a protocol sequence of event models. We have also shown how these conditions can be defined in an appropriate extended modal language, making it possible to reason formally about agents engaged in such updates and revisions. Our methods are like those of existing epistemic work, but the doxastic case came with some interesting new notions.

But from where we are standing now, we see several larger directions to pursue:

- A systematic analysis of axiomatic completeness for constrained revision processes, analogous to the purely epistemic theory of observation and conversation protocols initiated in [7],
- A comparison of our 'constructive' DDL-inspired approach to DTL universes with the more abstract AGM-style postulational approach of [10],
- A theory of variation for agents with different abilities and tendencies (cf. [14]),
- Connections with formal learning theory over epistemic-doxastic temporal universes (cf. [13]).

References

[1] A. Baltag and L.S. Moss. Logics for Epistemic Programs. *Synthese*, 139(2):165–224, 2004.

[2] A. Baltag, L.S. Moss, and S. Solecki. The logic of public announcements, common knowledge, and private suspicions. In *TARK '98: Theoretical aspects of rationality and knowledge*, pages 43–56. Morgan Kaufmann Publishers Inc., 1998.

[3] A. Baltag and S. Smets. Dynamic belief revision over multi-agent plausibility models. In G. Bonanno, W. van der Hoek, and M. Wooldridge, editors, *Logic and the Foundations of Game and Decision Theory: Proceedings of LOFT'06*, Texts in Logic and Games, pages 11–24. Amsterdam University Press, 2006.

[4] J. van Benthem. Games in Dynamic Epistemic Logic. *Bul. Econ. Res.*, 53(4):219–248, 2001.

[5] J. van Benthem. Dynamic logics for belief change. *JANCL*, 17(2):129–155, 2007.

[6] J. van Benthem. Priority product update as social choice. *ILLC PP Series*, PP-2008-09, 2008.

[7] J. van Benthem, J. Gerbrandy, and E. Pacuit. Merging frameworks for interaction: DEL and ETL. In Dov Samet, editor, *Theoretical Aspects of Rationality and Knowledge (TARK 2007)*, 2007.

[8] J. van Benthem and E. Pacuit. The Tree of Knowledge in Action: Towards a Common Perspective. In I. Hodkinson G. Governatori and Y. Venema, editors, *Advances in Modal Logic*, volume 6. College Publications, 2006.

[9] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. CUP, 2001.

[10] G. Bonanno. Belief revision in a temporal framework. In G. Bonanno, W. van der Hoek, and M. Wooldridge, editors, *Logic and the Foundations of Game and Decision Theory: Proceedings of LOFT'06*, Texts in Logic and Games, pages 43–50. Amsterdam University Press, 2006.

[11] K. Eliaz and E.A. Ok. Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences. *Games and Economic Behavior*, 56(1):61–86, July 2006.

[12] R. Fagin, J.Y. Halpern, Y. Moses, and M.Y. Vardi. *Reasoning About Knowledge*. MIT Press, Cambridge, 1995.

[13] K. Kelly. Ockham's razor, truth, and information. In J. van Behthem and P. Adriaans, editors, *Handbook of the Philosophy of Information*. 2008.

[14] F. Liu. *Changing for the Better*. Phd dissertation, ILLC Amsterdam, 2008.

[15] S. Morris. The common prior assumption in economic theory. *Econ. & Phil.*, 11:227–253, 1995.

[16] R. Parikh and R. Ramanujam. A knowledge based semantics of messages. *JoLLI*, 12(4):453–467, 2003.

[17] R. Stalnaker. A theory of conditionals. In R. Stalnaker, W. L. Harper, and G. Pearce, editors, *Ifs: Conditionals, Belief, Decision, Chance and Time*. D. Reidel, Dordrecht, 1981.

7 Appendix

Theorem. *Let $\mathcal{H}$ be a total doxastic-temporal model, $\mathcal{M}$ a total plausibility model, $\vec{\epsilon}$ a sequence of total event models, and let $\otimes$ stand for* priority *update. The following are equivalent:*

- *$\mathcal{H}$ is isomorphic to the forest generated by $\mathcal{M} \otimes \vec{\epsilon}$.*
- *$\mathcal{H}$ satisfies propositional stability, synchronicity, bisimulation invariance, Preference Propagation, and Preference Revelation.*

Proof. **Necessity.** We first show that the given conditions are indeed satisfied by any $DoTL$ model generated through successive priority updates along some given protocol sequence. Here, Propositional stability and Synchronicity are straightforward from the definition of generated forests.

Preference Propagation Assume that $ja \leq j'b$ (1). It follows from (1) plus the definition of priority update that $a \leq b$ (2). Now assume that $h \leq h'$ (3). It follows from (2), (3) and priority update that $ha \leq h'b$.

Preference Revelation Assume that $jb \leq j'a$ (1). It follows from (1) and the definition of priority update that $b \leq a$ (2). Now assume $ha \leq h'b$ (3). By the definition of priority update, (3) can happen in two ways. Case 1: $a < b$ (4). It follows from (4) by the definition of $<$ that $b \not\leq a$ (5). But (5) contradicts (2). We are therefore in Case 2: $a \simeq b$ (6) and $h \leq h'$ (7). But (7) is precisely what we wanted to show.

Note that we did not make use of totality here.

Sufficiency Given a $DoTL$ model $\mathcal{M}$, we first show how to construct a DDL model, i.e., a plausibility model and a sequence of event models.

Construction Here is the initial plausibility model $\mathcal{M} = \langle W, (\preceq_i)_{i \in N}, \hat{V} \rangle$:

- $W := \{h \in H \mid len(h) = 1\}$.
- Set $h \preceq_i h'$ iff $\leq_i$.
- For every $p \in Prop$, $\hat{V}(p) = V(p) \cap W$.

Now we construct the j-th event model $\epsilon_j = \langle E_j, (\preceq^j_i)_{i \in N}, \mathtt{pre}_j \rangle$:

- $E_j := \{e \in \Sigma \mid$ there is a history$he \in H$ with $len(h) = j\}$
- For each $i \in N$, set $a \preceq^j_i b$ iff there are $ha, h'b \in H$ such that $len(h) = len(h) = j$ and $ha \leq_i h'b$.
- For each $e \in E_j$, let $\mathtt{pre}_j(e)$ be the formula that characterizes the set $\{h \mid he \in H$ and $len(h) = j\}$. By general modal logic, *bisimulation invariance* guarantees that there is such a formula, though it may be an infinitary one in general.

Now we show that the construction is correct in the following sense:

Claim 7.1 (Correctness)**.** *Let $\leq$ be the plausibility relation in the given doxastic temporal model. Let $\preccurlyeq^F_{DDL}$ be the plausibility relation in the forest induced by priority update over the just constructed plausibility model and matching sequence of event models. We have:*

$$h \leq h' \text{ iff } h \preccurlyeq^F_{DDL} h'.$$

Proof of the claim The proof is by induction on the length of histories. The base case is obvious from the construction of our intitial model $\mathcal{M}$. Now for the induction step. To simplify notation we will write $a \leq b$ tfor $a \preceq_i^n b$ with n the length for which the claim has been proved, and i an agent.

From $DoTL$ **to** $Forest(DDL)$ Assume that $h_1 a \leq h_2 b$ (1). It follows that in the constructed event model $a \leq b$ (2). Case 1: $a < b$. By priority update we have $h_1 a \preccurlyeq_{DDL}^F h_2 b$. Case 2: $b \leq a$ (3). This means that there are $h_3 b, h_4 a$ such that $h_3 b \leq h_4 a$. But then by Preference Revelation and (1) we have $h_1 \leq h_2$ (in the doxastic temporal model). It follows by the inductive hypothesis that $h_1 \preccurlyeq_{DDL}^F h_2$. But then by priority update, since by (2) and (3) a and b are indifferent, we have $h_1 a \preccurlyeq_{DDL}^F h_2 b$.

From $Forest(DDL)$ **to** $DoTL$ Next let $h_1 a \preccurlyeq_{DDL}^F h_2 b$. The definition of priority update has two clauses. Case 1: $a < b$. By definition, this implies that $b \not\leq a$. But then by the above construction, for all histories $h_3, h_4 \in H$ we have $h_3 b \not\leq h_4 a$. In particular we have $h_2 b \not\leq h_1 a$. But then by *totality*[4], $h_1 a \leq h_2 b$. Case 2: $a \simeq b$ (4) and $h_1 \preccurlyeq_{DDL}^F h_2$. For a start, by the inductive hypothesis, $h_1 \leq h_2$ (5). By (4) and our construction, there are $h_3 a, h_4 b$ with $h_3 a \leq h_4 b$ (6). But then by Preference Propagation, (5) and (6) imply that we have $h_1 a \leq h_2 b$. □

Theorem. *Let $\mathcal{H}$ be a doxastic-temporal model, $\mathcal{M}$ a plausibility model, $\vec{\epsilon}$ be a sequence of event models while $\otimes$ is* priority *update. The following assertions are equivalent:*

- *$\mathcal{H}$ is isomorphic to the forest generated by $\mathcal{M} \otimes \vec{\epsilon}$,*
- *$\mathcal{H}$ satisfies bisimulation invariance, propositional stability, synchronicity, Preference Revelation and Accommodation.*

By Fact 3.6, Accommodation also gives us Preference Propagation.

Proof. **Necessity of the conditions**

The verification of the conditions in the preceding subsection did not use totality. So we concentrate on the new condition:

Accommodation Assume that $ja \leq j'b$ (1). It follows by the definition of priority update that $a \leq b$ (2). Now let $ha \not\leq h'b$ (3). This implies by priority update that $a \not< b$ (4). By definition, (2) and (4) means that $a \simeq b$ (5). Now assume that $g \leq g'$ (6). It follows from (5), (6) and priority update that $ga \leq g'b$. For the other direction of the consequent assume instead that $g \not\leq g'$ (7). It follows from (5), (7) and priority update that $ga \not\leq g'b$.

Sufficiency of the conditions Given a $DoTL$ model, we again construct a DDL plausibility model plus sequence of event models:

[4]Note that this is the only place in which we make use of totality.

Construction The plausibility model $\mathcal{M} = \langle W, (\preceq_i)_{i\in N}, \hat{V}\rangle$ is as follows:

- $W := \{h \in H \mid len(h) = 1\}$,
- Set $h \preceq_i h'$ whenever $\leq_i$,
- For every $p \in Prop$, $\hat{V}(p) = V(p) \cap W$.

We construct the j-th event model $\epsilon_j = \langle E_j, (\preceq_i^j)_{i\in N}, \mathtt{pre}_j\rangle$ as follows:

- $E_j := \{e \in \Sigma \mid$ there is a history of the form $he \in H$ with $len(h) = j\}$
- For each $i \in N$, $a \preceq_i^j b$ iff either (a) there are $ha, h'b \in H$ such that $len(h) = len(h) = j$ and $ha \leq_i h'b$, or (b) [a new case] a and b are accommodating, and we put $a \simeq b$ (i.e. $a \leq b$ and $b \leq a$).
- For each $e \in E_j$, let $\mathtt{pre}_j(e)$ be the formula that characterizes the set $\{h \mid he \in H$ and $len(h) = j\}$. *Bisimulation invariance* guarantees that there is always such a formula (maybe involving an infinitary syntax).

Again we show that the construction is correct in the following sense:

Claim 7.2 (Correctness)**.** *Let $\leq$ be the plausibility relation in the doxastic temporal model. Let $\preccurlyeq_{DDL}^F$ be the plausibility relation in the forest induced by successive priority updates of the plausibility model by the sequence of event models we constructed. We have:*

$$h \leq h' \text{ iff } h \preccurlyeq_{DDL}^F h'.$$

Proof of the claim We proceed by induction on the length of histories. The base case is clear from our construction of the initial model $\mathcal{M}$.

Now for the induction step, with the same simplified notation as earlier.

From $DoTL$ to $Forest(DEL)$ We distinguish two cases.

Case 1. $ha \leq h'b, h \leq h'$**.** By the inductive hypothesis, $h \leq h'$ implies $h \preccurlyeq_{DDL}^F h'$ (1). Since $ha \leq h'b$, it follows by construction that $a \leq b$ (2). It follows from (1) and (2) that by priority update $ha \preccurlyeq_{DDL}^F h'b$.

Case 2. $ha \leq h'b, h \not\leq h'$**.** Clearly, then, a and b are not *accommodating* and thus the special clause has not been used to build the event model, though we do have $a \leq b$ (1). By the contrapositive of Preference Revelation, we also conclude that for all $ja, j'b \in H$, we have $j'b \not\leq ja$ (2). Therefore, our construction gives $b \not\leq a$ (3), and we conclude that $a < b$ (4). But then by priority update, we get $ha \preccurlyeq_{DDL}^F h'b$.

From $Forest(DEL)$ to $DoTL$ We distinguish again two cases.

Case 1. $ha \preccurlyeq^F_{DDL} h'b, h \preccurlyeq^F_{DDL} h'$**.** By definition of priority update, $ha \preccurlyeq^F_{DDL} h'b$ implies that $a \leq b$ (1). There are two possibilities. Case 1: The special clause of the construction has been used, and a, b are accommodating (2). By the inductive hypothesis, $h \preccurlyeq^F_{DDL} h'$ implies $h \leq h'$ (3). But (2) and (3) imply that $ha \leq h'b$. Case 2: Clause (1) holds because for some $ja, j'b \in H$, in the $DoTL$ model, $ja \leq j'b$ (4). By the inductive hypothesis, $h \preccurlyeq^F_{DDL} h'$ implies $h \leq h'$ (5). Now, it follows from (4), (5) and Preference Propagation that $ha \leq h'b$.

Case 2. $ha \preccurlyeq^F_{DDL} h'b, h \not\preccurlyeq^F_{DDL} h'$**.** Here is where we put our new accommodation clause to work. Let us label our assertions: $h \not\preccurlyeq^F_{DDL} h'$ (1) and $ha \preccurlyeq^F_{DDL} h'b$ (2). It follows from (1) and (2) by the definition of priority update that $a < b$ (3), and hence, by definition $b \not\leq a$ (4). Clearly, a and b are not accommodating (5): for otherwise, we would have had $a \simeq b$, and hence $b \leq a$, contradicting (4). Therefore, (3) implies that there are $ja, j'b \in H$ with $ja \leq j'b$ (6). Now assume for *contradictio* that (in the $DoTL$ model) $ha \not\leq h'b$ (7). It follows from (6) and (7) by Accommodation that a and b *are* accommodating, contradicting (5). Thus we have $ha \leq h'b$. $\square$

Theorem (Definability)**.** *Preference Propagation, Preference Revelation and Accommodation are definable in the doxastic-epistemic temporal language $DET\mathcal{L}$.*

Proof. We only prove the case of Preference Propagation, the other two are in the extended version of the paper. We drop agent labels for convenience.

(PP) **characterizes Preference Propagation** We first show that (PP) is valid on all models $\mathcal{H}$ based on preference-propagating frames. Assume that $\mathcal{H}, h \Vdash \mathtt{E}\langle a\rangle\langle\leq_i\rangle\langle b^{-1}\rangle\top$ (1). Then there are $ja, j'b \in H$ such that $ja \leq j'b$ (2). Now let $\mathcal{H}, h \Vdash (\langle\leq\rangle\langle b\rangle p \wedge \langle a\rangle q)$ (3). Then there is $h' \in H$ such that $h \leq h'$ (4) and $\mathcal{H}, h' \Vdash \langle b\rangle p$ (5), while also $\mathcal{H}, ha \Vdash q$ (6). We must show that $\mathcal{H}, h \vDash \langle a\rangle(q \wedge \langle\leq_i\rangle p)$ (7). But, from (2),(4),(6) and Preference Propagation, we get $ha \leq h'b$, and the conclusion follows by the truth definition.

Next, we assume that axiom (PP) is valid on a doxastic temporal frame, that is, true under any interpretation of its proposition letters. So, assume that $ja \leq j'b$ (1), and also $h \leq h'$ (2). Moreover, let $ha, h'b \in H$ (3). First note that (1) automatically verifies the antecedent of (PP) in any node of the tree. Next, we make the antecedent of the second implication in (PP) true at h by interpreting the proposition letter p as just the singleton set of nodes $h'b$, and q as just ha (4). Since (PP) is valid, its consequent will also hold under this particular valuation V. Explicitly we have $\mathcal{H}, V, h \Vdash \langle a\rangle(q \wedge \langle\leq_i\rangle p$. But spelling out what p, q mean there, we get just the desired conclusion that $ha \leq h'b$. $\square$

Proposition (Undefinability)**.** *Preference Propagation, Preference Revelation and Accommodation are* not *definable in the forward looking fragment of* **DET$\mathcal{L}$**

Proof. The reason is the same in all cases: we show that these properties are not preserved under taking *bounded p-morphic images.* Figure 1 gives an indication how this works concretely.

$\square$

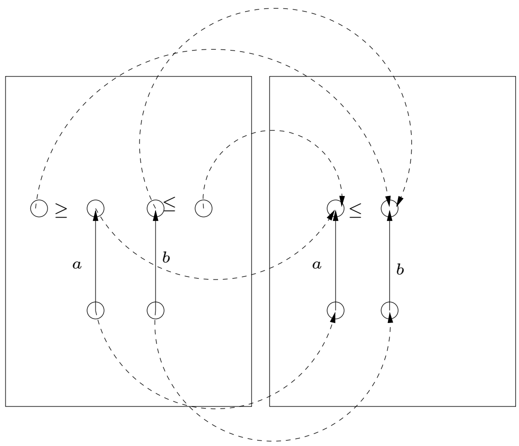

Figure 1: Propagation is not preserved under p-morphic images