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A NEW PROOF OF THE JAYNE–ROGERS THEOREM

Abstract

We give a new simpler proof of a theorem of Jayne–Rogers.

1 Introduction

In this paper we will give a new proof of a theorem of Jayne and Rogers. First recall from [2] the following definitions:

Definition 1. Let X, Y be metric spaces. A function $f \colon X \to Y$ is said to be $\boldsymbol{\Delta}^0_2$-*function* if $f^{-1}(S) \in \boldsymbol{\Sigma}^0_2$ for every $S \in \boldsymbol{\Sigma}^0_2$ (equivalently, $f^{-1}(U) \in \boldsymbol{\Delta}^0_2$ for every open $U \subseteq Y$). Sometimes these functions are also called *first level Borel functions* (see [2]).

The function f is said to be *piecewise continuous* if X can be expressed as the union of an increasing sequence $X_0, X_1, \dots$ of closed sets such that $f \restriction X_n$ is continuous for every $n \in \omega$.

Observe that f is piecewise continuous if and only if[1] there is a $\boldsymbol{\Delta}^0_2$-partition $\langle D_n \mid n \in \omega \rangle$ of X such that $f \restriction D_n$ is continuous for every $n \in \omega$. For one direction, if f is piecewise continuous then putting $D_0 = X_0$ and $D_{n+1} = X_{n+1} \setminus X_n$ we have the desired partition. Conversely, let $P_{m,n} \in \boldsymbol{\Pi}^0_1$ be such that $D_n = \bigcup_{m \in \omega} P_{m,n}$ and $P_{m,n} \subseteq P_{m',n}$ for every $m \leq m'$ and $n \in \omega$, and let $X_n = \bigcup_{i \leq n} P_{n,i}$. It is easy to check that the X_n are increasing and closed, and that $f \restriction X_n$ is continuous (since $P_{n,i} \cap P_{n,j} = \emptyset$ whenever $i \neq j$). Thus, in the rest of this paper, when we will refer to some piecewise continuous function we will generally have in mind a function with this "partition" property.

Key Words: Baire class 1 functions, piecewise continuous functions, first level Borel functions, $\boldsymbol{\Delta}^0_2$-functions

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[1] A third equivalent definition is that X can be covered by a countable family $P_0, P_1, \dots$ of closed sets such that $f \restriction P_n$ is continuous for every $n \in \omega$.

Definition 2. A set S in a metric space is said to be Souslin-$\mathscr{F}$ set if it belongs to $\mathcal{A}\mathbf{\Pi}^0_1$, where $\mathcal{A}$ is the usual Souslin operation (see Definition 25.4 in [1]).

A metric space X is said to be an *absolute Souslin-$\mathscr{F}$ set* if X is a Souslin-$\mathscr{F}$ set in the completion of X under its metric.

Observe that if X is separable then it is an absolute Souslin-$\mathscr{F}$ set if and only if it is Souslin, that is if and only if it is the continuous image of the Baire space ${}^{\omega}\omega$.

Now we are ready to give the statement of the original Theorem.

Theorem 1.1 (Jayne–Rogers). *If X is an absolute Souslin-$\mathscr{F}$ set, then $f\colon X \to Y$ is a $\mathbf{\Delta}^0_2$-function if and only if it is piecewise continuous.*

According to the authors of [2], their proof "even in the case when X and Y are separable, is complicated". Sixteen years later, Sławomir Solecki provided in [3] a new proof of Theorem 1.1 in the case when X and Y are separable and X is Souslin (in fact he proved a much stronger result which refines Theorem 1.1), but even in that case the proof was quite complicated. Our goal is to provide a simpler proof of Theorem 1.1 that works under the same conditions of Solecki's version of the theorem — see Corollary 2.2.

We will assume $\mathsf{ZF} + \mathsf{DC}(\mathbb{R})$ throughout the paper (note that the Jayne–Rogers' and Solecki's proofs are carried out in ZFC, but by a simple absoluteness argument the result must hold also in $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})$). *All spaces considered are metric.* Our notation will be quite standard: the set of the natural numbers will be denoted by ω, while if X is any topological space and A is a subset of X we will denote the closure of A with $\mathrm{Cl}(A)$. The set of all the binary sequences of finite length will be denoted by ${}^{<\omega}2$, and ${}^{\omega}2$ will denote the Cantor space. A function $f\colon X \to Y$ will be said of *Baire class* 1 if it is the pointwise limit of a sequence of continuous functions $f_n\colon X \to Y$. Finally, if (X, d) is any metric space, a set $U \subseteq X$ will be called *basic open* if it is an open ball of X, i.e. if $U = \{x \in X \mid d(x, x_0) < r\}$ where $x_0 \in X$ and $r \in \mathbb{R}^+$. For all the other undefined symbols and notions we refer the reader to the standard monograph [1].

2 The proof of the Jayne–Rogers Theorem

The main result of this paper is the following Theorem, from which the Jayne–Rogers Theorem will follow — see Corollary 2.2.

Theorem 2.1. *Let X and Y be metric spaces with X Polish, and let $f\colon X \to Y$ be of Baire class 1. If f is a $\mathbf{\Delta}^0_2$-function then it is piecewise continuous.*

Recall that if $f\colon X \to Y$ is of Baire class 1 then it is also $\mathbf{\Sigma}^0_2$-*measurable*, i.e. $f^{-1}(U) \in \mathbf{\Sigma}^0_2$ for every open set $U \subseteq Y$, but the converse in general fails. Nevertheless, if we require that X and Y are separable and that X is zero-dimensional then f is of Baire class 1 just in case it is $\mathbf{\Sigma}^0_2$-measurable (see e.g. Theorem 24.10 in [1]).

Corollary 2.2. *Let X and Y be separable metric spaces with X Souslin. Then $f\colon X \to Y$ is a $\mathbf{\Delta}^0_2$-function if and only if it is piecewise continuous.*

Proof. One direction is trivial. For the other direction, assume toward a contradiction that f is a $\mathbf{\Delta}^0_2$-function but *not* piecewise continuous. Let $\mathcal{F}$ be the collection of all the closed sets C of (the completion of) X such that $f \restriction C$ is continuous. By Corollary 1 of [4], we have that either there is a countable family of sets in $\mathcal{F}$ which cover X, or else there is some $Z \subseteq X$ which is homeomorphic to the Baire space ${}^\omega\omega$ and such that Z can not be covered by a countable family of sets from $\mathcal{F}$. Since the first alternative easily implies that f is piecewise continuous, we can assume that the second alternative holds and therefore that $f' = f \restriction Z$ is not piecewise continuous. Note that we can assume also that f' is of Baire class 1 (otherwise, since Z is Polish and zero-dimensional and Y is separable, we would have that f' is not even $\mathbf{\Sigma}^0_2$-measurable and hence not a $\mathbf{\Delta}^0_2$-function), and therefore we can apply Theorem 2.1 to f': this gives the desired contradiction. □

Before proving Theorem 2.1 we need a couple of technical lemmas. For the next few results, X' *will be an arbitrary subset of the Polish space* X. Given $A, B \subseteq Y$ we will say that A and B are *strongly disjoint* if $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \emptyset$. Moreover if $h\colon X' \to Y$ is any function we put $A^h = h^{-1}(Y \setminus \mathrm{Cl}(A))$. Note that for every $A, B \subseteq Y$ one has $(A \cup B)^h = A^h \cap B^h$. If h is $\mathbf{\Sigma}^0_2$-measurable and $U, V \subseteq Y$ are strongly disjoint, then we have that if $h \restriction U^h$ and $h \restriction V^h$ are both piecewise continuous then the whole h is piecewise continuous. In fact, U^h and V^h is a finite $\mathbf{\Sigma}^0_2$-covering of X' (by the strongly disjointness of U and V), which by the reduction property of $\mathbf{\Sigma}^0_2$ can be refined to a $\mathbf{\Delta}^0_2$-partition $\langle D_0, D_1\rangle$ of X' such that $D_0 \subseteq U^h$, $D_1 \subseteq V^h$, and hence both $h \restriction D_0$ and $h \restriction D_1$ are piecewise continuous. But if $h'\colon X' \to Y$ is such that for some $\mathbf{\Delta}^0_2$-partition $\langle D'_n \mid n \in \omega\rangle$ of X' we have that $h' \restriction D'_n$ is piecewise continuous for every n, then h' is piecewise continuous on the whole X': therefore h is piecewise continuous as well.

Now let $h\colon X' \to Y$ be a $\mathbf{\Sigma}^0_2$-measurable function, $x \in X'$, and A be any subset of Y. We say that x is *h-irreducible outside* A if for every open neighborhood $V \subseteq X'$ of x the function $h \restriction A^h \cap V$ is not piecewise continuous. Otherwise we say that x is *h-reducible outside* A. It is easy to check that if x is h-irreducible outside A and $A' \subseteq A$ then x is also h-irreducible outside

A'. Moreover if $X'' \subseteq X'$ and $x \in X''$ is h'-irreducible outside A (where $h' = h \restriction X''$) then x is also h-irreducible outside A.

Lemma 2.3. *Suppose $h \colon X' \to Y$ is a $\mathbf{\Sigma}^0_2$-measurable function and $U_0, \ldots, U_n \subseteq Y$ are basic open sets of Y such that* $\mathrm{range}(h) \cap \mathrm{Cl}(U_i) = \emptyset$ *for every $i \leq n$. Then h is* not *piecewise continuous if and only if $(*)$ there is an $x \in X'$ and a basic open set $U \subseteq Y$ strongly disjoint from $U_0, \ldots, U_n$ such that $h(x) \in U$ and x is h-irreducible outside U.*

Proof. Put $C = \mathrm{Cl}(U_0) \cup \ldots \cup \mathrm{Cl}(U_n)$. We will prove that h is piecewise continuous if and only if $(*)$ does not hold. If h is piecewise continuous then the same must hold for $h \restriction X''$ where X'' is any subset of X', therefore one direction is trivial. For the other direction, assume toward a contradiction that $(*)$ does not hold, i.e. for every $x \in X'$ and every open set $U \subseteq Y$ strongly disjoint from C such that $h(x) \in U$ we have that x is h-reducible outside U, that is there is some open neighborhood $V \subseteq X'$ of x such that $h \restriction U^h \cap V$ is piecewise continuous. Let Q be the union of all the open sets $W \subseteq X'$ such that $h \restriction W$ is piecewise continuous (this clearly implies that also $h \restriction Q$ is piecewise continuous since X' is assumed to be separable). We claim that $h \restriction X' \setminus Q$ is continuous (from this easily follows that h is piecewise continuous). To see this we fix any $x \in X' \setminus Q$ and let $U \subseteq Y$ be any open set such that $h(x) \in U$. We want to show that there is some open neighborhood V of x such that $h``(V \cap (X' \setminus Q)) \subseteq U$. Let $U' \subseteq Y$ be basic open, strongly disjoint from C, and such that $\mathrm{Cl}(U') \subseteq U$ (U' exists since Y is metric). Let $V \subseteq X'$ be given by the failure of $(*)$ on the inputs x and U', and assume toward a contradiction that $h(x') \notin \mathrm{Cl}(U')$ for some $x' \in V \cap (X' \setminus Q)$. In this case we can clearly find a basic open $U'' \subseteq Y$ strongly disjoint from U' and C, and such that $h(x') \in U''$. Let $V' \subseteq X'$ be the open set given by the failure of $(*)$ on inputs x' and U''. Since V and V' have been chosen in such a way that $h \restriction (U')^h \cap V$ and $h \restriction (U'')^h \cap V'$ are piecewise continuous, and since $\{(U')^h \cap V, (U'')^h \cap V'\}$ is a $\mathbf{\Sigma}^0_2$-covering of $V \cap V'$, by the strong disjointness of U' and U'' we must have that $h \restriction V \cap V'$ is piecewise continuous, and therefore $V \cap V' \subseteq Q$: but this means that $x' \in Q$, a contradiction! $\square$

Lemma 2.4. *Let $h \colon X' \to Y$ be a $\mathbf{\Sigma}^0_2$-measurable function, $x \in X'$, $A \subseteq Y$, and $U_0, \ldots, U_n$ be a sequence of strongly disjoint open subsets of Y. If x is h-irreducible outside A then there is at most one $i \leq n$ such that x is h-*reducible *outside $A \cup U_i$.*

Proof. Assume that $i \leq n$ is such that x is h-reducible outside $A \cup U_i$, i.e. that there is an open neighborhood $V \subseteq X'$ of x such that $h \restriction (A \cup U_i)^h \cap V$ is piecewise continuous. If there were some $j \neq i$ with the same property, then there must be some open neighborhood $V' \subseteq X'$ of x such that $h \restriction$

$(A \cup U_j)^h \cap V'$ is piecewise continuous. But since U_i and U_j are strongly disjoint, this would imply that $h \restriction A^h \cap V \cap V'$ is piecewise continuous as well, and thus $V \cap V'$ would contradict the fact that x is h-irreducible outside A. □

Finally observe that if $f \colon X \to Y$ is the pointwise limit of a sequence of functions $\langle f_m \colon X \to Y \mid m \in \omega \rangle$, then we have the following property: if $x \in X$ and $U_0, U_1, \dots$ are pairwise disjoint open sets such that for infinitely many n's there is an m for which $f_m(x) \in U_n$, then $f(x) \notin U_n$ for each n (otherwise, $f_m(x) \in U_n$ for all but finitely many m's contradicting our hypothesis).

Now we are ready to prove Theorem 2.1. The proof essentially uses recursively Lemma 2.3 applied to smaller and smaller subspaces of X to construct some sequences, and Lemma 2.4 will guarantee that at each stage the construction can be carried out. This is the reason for which we have proved both the Lemmas for arbitrary functions h with domain a generic subset X' of the Polish space X: in fact we will generally apply them to the restriction of the original function f to some subset of X, that is with $h = f \restriction X'$.

Proof of Theorem 2.1. Assume that $f \colon X \to Y$ is of Baire class 1 (hence also $\mathbf{\Sigma}^0_2$-measurable) but not piecewise continuous, and let $\langle f_n \mid n \in \omega \rangle$ be a sequence of continuous functions which converges pointwise to f. We will construct an open set $\hat{U} \subseteq Y$ such that $f^{-1}(\hat{U})$ is a complete $\mathbf{\Sigma}^0_2$-set, and this will imply that f is not a $\mathbf{\Delta}^0_2$-function. To be more specific, we will construct (together with $\hat{U}$) a continuous reduction from the $\mathbf{\Sigma}^0_2$-complete set

$$S = \{z \in {}^\omega 2 \mid \exists i \forall j \geq i (z(j) = 0)\}$$

to $f^{-1}(\hat{U})$, i.e. a function $g \colon {}^\omega 2 \to X$ such that

$$z \in S \iff g(z) \in f^{-1}(\hat{U}).$$

The function g will be defined using a *weak* Cantor scheme $\langle V_s \mid s \in {}^{<\omega}2 \rangle$ (that is a classical Cantor scheme in which we drop the condition $V_{s^\frown 0} \cap V_{s^\frown 1} = \emptyset$) such that for every $s, t \in {}^{<\omega}2$ we have:

1) V_s is an open subset of X;

2) if $s \subsetneq t$ then $\mathrm{Cl}(V_t) \subseteq V_s$;

3) $\mathrm{diam}(V_s) \leq 2^{-\mathrm{length}(s)}$.

It is straightforward to check that, given such a scheme, the function $g \colon {}^\omega 2 \to X \colon z \mapsto \bigcap_{n \in \omega} V_{z \restriction n}$ is well-defined (by the completeness of X) and continuous (in fact it is Lipschitz with constant 1).

The construction will be carried out by recursion on the rank of $s \in {}^{<\omega}2$ with respect to the order $\preceq$ defined by

$$s \preceq t \iff \mathrm{length}(s) < \mathrm{length}(t) \vee (\mathrm{length}(s) = \mathrm{length}(t) \wedge s \leq_{\mathrm{lex}} t),$$

where $\leq_{\mathrm{lex}}$ is the usual lexicographical order on ${}^{<\omega}2$ (the strict part of $\preceq$ will be denoted by $\prec$). In fact we will define, together with a scheme $\langle V_s \mid s \in {}^{<\omega}2\rangle$ with the properties above, a sequence $\langle x_s \mid s \in {}^{<\omega}2\rangle$ of points of X and a sequence $\langle U_s \mid s \in {}^{<\omega}2\rangle$ of subsets of Y such that for every $s \in {}^{<\omega}2$:

i) $x_s \in V_s$;

ii) $f(x_s) \in U_s$;

iii) U_s is basic open and for every $t \in {}^{<\omega}2$ we have that U_s and U_t are either equal or strongly disjoint;

iv) there is some $m \in \omega$ such that $f_m``V_s \subseteq U_s$;

v) x_t is f-irreducible outside A for every $t \preceq s$, where $A = \bigcup_{t' \preceq s} U_{t'}$;

vi) if $s = s'{}^\frown 1$ then $U_s \neq U_t$ for every[2] $t \preceq s'{}^\frown 0$ (and therefore, in particular, for every $t \subseteq s'$).

As already noted, to construct these sequences we will recursively apply Lemma 2.3 to the restriction of f to smaller and smaller pieces.

At the first stage, let x and U be given as in Lemma 2.3 applied to the whole f, and let $V = f_m^{-1}(U)$ where $m \in \omega$ is such that $f_m(x) \in U$ (such an m must exists by the fact that f is the limit of the f_n's). Then put $V_\emptyset = V$, $x_\emptyset = x$ and $U_\emptyset = U$. Now let $s \neq \emptyset$ and suppose we have defined V_t, x_t and U_t for $t \prec s$. If the last digit of s is a 0, that is $s = s'{}^\frown 0$, then simply put $V_s = W$, $x_s = x_{s'}$ and $U_s = U_{s'}$, where W is any open set such that $\mathrm{Cl}(W) \subseteq V_{s'}$, $x_s \in W$ and $\mathrm{diam}(W) \leq 2^{-\mathrm{length}(s)}$. Otherwise $s = s'{}^\frown 1$: by the inductive hypothesis, condition v) implies that $h_0 = f \restriction A^f \cap V_{s'}$, where $A = \bigcup_{t \preceq s'{}^\frown 0} U_t$, is not piecewise continuous (otherwise, since $x_{s'{}^\frown 0} \in V_{s'{}^\frown 0} \subseteq V_{s'}$, $x_{s'{}^\frown 0}$ should be f-reducible outside A).

Claim. There are $x_s \in V_{s'}$ and $U_s \subseteq Y$ such that $f(x_s) \in U_s$, U_s is basic open and strongly disjoint from A (which in particular implies $U_s \neq U_t$ for every $t \preceq s'{}^\frown 0$), and x_t is f-irreducible outside $A \cup U_s$ for every $t \preceq s$.

Proof of the Claim. By Lemma 2.3 applied to h_0 there must be an $x_0 \in V_{s'}$ and a basic open set U_0 strongly disjoint from A such that $f(x_0) = h_0(x_0) \in U_0$

[2]Note that $s'{}^\frown 0$ is the immediate predecessor of s with respect to $\preceq$.

and x_0 is h_0-irreducible outside U_0 (hence also h_0-irreducible outside $A \cup U_0$, since $\mathrm{range}(h_0) \cap \mathrm{Cl}(A) = \emptyset$, and thus f-irreducible outside $A \cup U_0$). If there is some $t \preceq s'^\frown 0$ such that x_t is f-reducible outside $A \cup U_0$, we can again apply Lemma 2.3 to $h_1 = h_0 \restriction (A \cup U_0)^f$ (h_1 is not piecewise continuous because x_0 is h_0-irreducible outside $A \cup U_0$) to find $x_1 \in V_{s'}$ and a basic open U_1 such that $f(x_1) \in U_1$, U_1 is strongly disjoint from $A \cup U_0$, and x_1 is h_1-irreducible outside $A \cup U_0 \cup U_1$ (hence also f-irreducible outside $A \cup U_1$). Moreover, by Lemma 2.4 it must be the case that also x_t is f-irreducible outside $A \cup U_1$ (in fact x_t must be f-irreducible outside $A \cup U$ for every open set U which is strongly disjoint from $A \cup U_0$). Now, it could be the case that there is another $t' \preceq s'^\frown 0$ such that $x_{t'}$ is f-reducible outside $A \cup U_1$: if this is the case, apply Lemma 2.3 to $h_2 = h_1 \restriction (A \cup U_0 \cup U_1)^f$ to get x_2 and U_2 such that $f(x_2) \in U_2$, U_2 is basic open and strongly disjoint from $A \cup U_0 \cup U_1$, and x_2 is h_2-irreducible outside $A \cup U_0 \cup U_1 \cup U_2$ (hence, in particular, x_2 is f-irreducible outside $A \cup U_2$). By Lemma 2.4 again, we must have that both x_t and $x_{t'}$ are f-irreducible outside $A \cup U_2$. Arguing inductively in this way, after (at most) $k = |\{t \in {}^{<\omega}2 \mid t \preceq s'^\frown 0\}| + 1$-stages we will have found some $x_k = x_s \in V_{s'}$ and $U_k = U_s$ such that $f(x_s) \in U_s$, U_s is basic open and strongly disjoint from A, x_s is f-irreducible outside $A \cup U_s$, and x_t is f-irreducible outside $A \cup U_s$ for every $t \preceq s'^\frown 0$ as well. □ *Claim*

Let $W \subseteq X$ be an open neighborhood of x_s such that $\mathrm{diam}(W) \leq 2^{-\mathrm{length}(s)}$, $\mathrm{Cl}(W) \subseteq V_{s'}$ and $f_m``W \subseteq U_s$ for some m, and define $V_s = W$. This completes the recursive definition of the sequences required.

It is easy to check that the scheme $\langle V_s \mid s \in {}^{<\omega}2\rangle$ and the sequences $\langle x_s \mid s \in {}^{<\omega}2\rangle$ and $\langle U_s \mid s \in {}^{<\omega}2\rangle$ constructed in this way are as required, i.e. that they satisfy 1)–3) and i)–vi). Now put $\hat{U} = \bigcup_{s \in {}^{<\omega}2} U_s$, and let $g\colon {}^{\omega}2 \to X$ be obtained from $\langle V_s \mid s \in {}^{<\omega}2\rangle$ as described above. We have only to check that g is a reduction of S to $f^{-1}(\hat{U})$. Let $\langle U_k \mid k \in \omega\rangle$ be an enumeration without repetitions of $\langle U_s \mid s \in {}^{<\omega}2\rangle$, so that by condition iii) the U_k's are pairwise disjoint and $\hat{U} = \bigcup_{k \in \omega} U_k$. If $z \in S$, then for some $\bar{n} \in \omega$ we will have that $x_{z \restriction m} = x_{z \restriction \bar{n}} = \bar{x}$ for every $m \geq \bar{n}$, therefore $g(z) = \bar{x}$ and $f(g(z)) = f(\bar{x}) \in U_{z \restriction \bar{n}} \subseteq \hat{U}$. Assume now $z \notin S$: by conditions vi) and iv), for infinitely many k's there is some $m \in \omega$ such that $f_m(g(z)) \in U_k$ (since $g(z) \in V_{z \restriction n}$ for every $n \in \omega$), and therefore $f(g(z)) \notin \hat{U}$ by the observation preceding this proof. □

References

[1] A. S. Kechris, *Classical descriptive set theory*, Graduate Text in Mathematics, no. 156, Springer-Verlag, Heidelberg, New York, 1995.

[2] J. E. Jayne and C. A. Rogers, *First level Borel functions and isomorphisms*, J. Math. pures et appl. **61** (1982), 177–205.

[3] S. Solecki, *Decomposing Borel sets and functions and the structure of Baire class 1 functions*, J. Amer. Math. Soc. **11** (1998), no. 3, 521–550.

[4] S. Solecki, *Covering analytic sets by families of closed sets*, J. Symb. Log. **59** (1994), no. 3, 521–550.