# Modal Fixed-Point Logic and Changing Models

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This paper is dedicated to Professor Boris Trakhtenbrot, whose work and spirit have inspired us and so many others in our field.

**Abstract.** We show that propositional dynamic logic and the modal $\mu$-calculus are closed under product modalities, as defined in current dynamic-epistemic logics. Our analysis clarifies the latter systems, while also raising some new questions about fixed-point logics.

## 1 Basic Closure Properties of Logics

Standard first-order logic has some simple but important closure properties. First, it is closed under *relativization*: for every formula $\phi$ and unary predicate letter $P$, there is a formula $(\phi)^P$ which says that $\phi$ holds in the sub-model consisting of all objects satisfying $P$. One usually thinks of relativization as a syntactic operation which transforms the given formula by relativizing each quantifier $\exists x$ to $\exists x(Px\wedge$ and each quantifier $\forall x$ to $\forall x(Px \rightarrow$. But one can also think of evaluating the original formula itself, but then in a changed semantic model. The connection between the two viewpoints is stated in

**Fact 1 (Relativization Lemma).**

$$\mathbf{M}, s \models (\phi)^P \iff \mathbf{M}\,|\,P, s \models \phi.$$

where $\mathbf{M} \mid P$ is the restriction of the model $\mathbf{M}$ to its sub-model defined by the predicate (or formula with one free variable) $P$. Relativization is a useful property of abstract logics, and it is used extensively in proofs of Lindström theorems. Also useful is closure under *predicate substitutions* $[\psi/P]\phi$, which may again be read as either a syntactic operation, or as a shift to evaluation in a suitably changed model, via the following well-known

**Fact 2 (Substitution Lemma).**

$$\mathbf{M}, s \models [\psi/P]\phi \iff \mathbf{M}[P := \psi^{\mathbf{M}}], s \models \phi.$$

where $\mathbf{M}[P := \psi^{\mathbf{M}}]$ is the model $\mathbf{M}$ with the denotation of the predicate letter $P$ changed as indicated. Substitutions may be viewed as *translations* of basic predicates into newly defined ones.

Even more ambitious operations on models occur in the theory of *relative interpretation* between theories. E.g., embedding the first-order ordering theory of the rational numbers into that of the integers requires taking rationals as ordered pairs of relatively prime integers (a definable subset of the full Cartesian product $\mathbb{Z} \times \mathbb{Z}$), and redefining their order $<$ accordingly. Thus, we now also have a *product construction* where certain definable tuples become the new objects. As is easy to see, the first-order language is also closed under such product constructions - in a sense which we will not spell out. For our purpose here, we will define a precise sense of 'product closure' in terms of modal logic below, returning to the general situation at the end.

The three mentioned properties also hold of many languages extending first-order logic, such as LFP(FO), first-order logic with added fixed-point operators. But as we just said, our focus in this note will be on *modal languages*, which are rather fragments of a full first-order logic over directed graphs with unary predicates, although we also add fixed-point operators later on. For such modal languages, and especially vividly, in their epistemic interpretation as logics of knowledge and information flow, the above properties acquire special meanings of independent interest.

## 2 Closure Properties of Modal Languages

### 2.1 Epistemic Logic

Take a modal language with proposition letters, Boolean operators, and universal modalities $[i]$ which we read as stating what agent $i$ knows, or maybe better: what is true to the best of $i$'s information. More precisely, in epistemic pointed graph models $\mathbf{M}$ with actual world $s$, representing the information of a group of agents:

$$\mathbf{M}, s \vDash [i]\phi \iff \text{for all } t, \text{if } sR_it, \text{then } \mathbf{M}, t \vDash \phi.$$

### 2.2 Public Announcement and Definable Submodels

In this epistemic setting, taking the relativization of the current model $\mathbf{M}, s$ to its sub-model $\mathbf{M} \mid P, s$ consisting of all points satisfying the formula $P$ is the natural rendering of an informational event $!P$ of *public announcement* that $P$ is currently true. Thus, model change reflects information update. The language of *public announcement logic* PAL extends epistemic logic, making these updates explicit by adding modal operators $[!P]$ for truthful announcement actions:

$$\mathbf{M}, s \vDash [!P]\phi \iff \text{if } \mathbf{M}, s \vDash P, \text{then } \mathbf{M}|P, s \vDash \phi.$$

Here is the relevant completeness result.

**Theorem 1.** *PAL is axiomatized by the minimal modal logic for the new operators* $[i]$ *plus four reduction axioms*:

$$\begin{array}{rcll} [!P]q & \leftrightarrow & P \to q & \textit{for atomic facts } q, \\ [!P]\neg\phi & \leftrightarrow & P \to \neg[!P]\phi, & \\ [!P]\phi \wedge \psi & \leftrightarrow & [!P]\phi \wedge [!P]\psi, & \\ [!P][i]\phi & \leftrightarrow & P \to [i](P \to [!P]\phi). & \end{array}$$

We can read these principles as a complete recursive analysis of what agents know after they have received new information. But as was pointed out in van Benthem 2000 [4], this completeness theorem due to Plaza and Gerbrandy really just states the standard recursive clauses for performing syntactic relativization of modal formulas. Thus the technical question becomes which modal languages are closed under relativization.

This is not always the case. E.g., consider an epistemic language with an operator of common knowledge (everyone knows that everyone knows that, and so on $\cdots$ ), or semantically:

$$\mathbf{M}, s \vDash C_G\phi \iff \text{for all worlds } t \text{ reachable from } s \text{ by some finite sequence of } \sim_i \text{ steps } (i \in G), \mathbf{M}, t \vDash \phi.$$

This amounts to adding an operator of *reflexive-transitive closure* over the union of all individual accessibility relations. This infinitary operation takes us from the basic modal language into a fragment of so-called *propositional dynamic logic* (PDL). It can be shown that this fragment does not have the relativization property: indeed, the formula $[!p]C_Gq$ is not definable without modalities $[!p]$. Van Benthem, van Eijck & Kooi 2006 [5] proved this and go on to propose richer epistemic languages, using richer fragments of PDL which do have relativization closure, using so-called 'conditional common knowledge' $C_G(\phi, \psi)$ which says that $\phi$ is true in every world reachable with steps staying inside the $\psi$-worlds.

*Remark 1.* These observations are reminiscent of the fact that languages with generalized quantifiers may lack relativization closure. An example is first-order logic with the added quantifier "for most objects". To get the closure, one needs to add a truly binary quantifier "Most $\phi$ are $\psi$".

### 2.3 General Observation and Product Update

Public announcement is just one mechanism of information flow. In real-life scenarios, different agents often have different powers of observation. To model this, *dynamic-epistemic logic* (DEL) works with *event models*

$$\mathbf{A} = (E, \{R_i\}_i, \mathrm{PRE}).$$

Here the precondition function maps events $e$ to precondition formulas $\mathrm{PRE}_e$ which must hold in order for the event to occur. Just as worlds in epistemic models, events can be related by accessibility relations $\{R_i\}$ for agents. Now 'product update' turns a current model $\mathbf{M}, s$ into a model $\mathbf{M}\times\mathbf{A}, (s, e)$ recording the information of different agents after some event $e$ has taken place in the epistemic setting represented by $\mathbf{A}$. Product update redefines the universe of relevant possible worlds, and the epistemic accessibility relations between them:

$\mathbf{M}\times\mathbf{A}$ has domain $\{(s, a) \mid s$ a world in $\mathbf{M}, a$ an event in $\mathbf{A}, (\mathbf{M}, s) \vDash \mathrm{PRE}_a\}$.
The new uncertainties satisfy $(s, a)R_i(t, b)$ if both $sR_it$ and $aR_ib$.
The valuation for proposition letters on $(s, e)$ is just as that for $s$ in $\mathbf{M}$.

Here uncertainty among new worlds $(s, a), (t, b)$ can only come from old uncertainty among $s, t$ via indistinguishable events $a, b$. In general, this product construction can blow up the size of the input model $\mathbf{M}$ - it does not just go to a definable sub-model. In what follows, we will assume that the event models are finite, though infinitary versions are possible.

Despite the apparent complexity of this product construction, there is a natural matching dynamic epistemic language DEL with a new modality $[\mathbf{A}, e]$:

$$\mathbf{M}, s \vDash [\mathbf{A}, e]\phi \iff \text{ if } \mathbf{M}, s \vDash \mathrm{PRE}_e, \text{ then } \mathbf{M}\times\mathbf{A}, (s, e) \vDash \phi.$$

**Theorem 2.** *DEL is completely axiomatizable.*

*Proof.* The argument, due to Baltag, Moss & Solecki 1998 [2], is as follows. The atomic and Boolean reduction axioms involved are like the earlier ones for public announcement, but here is the essential clause for the knowledge modality:

$$[\mathbf{A}, e][i]\phi \iff \mathrm{PRE}_e \to \bigwedge_{eR_if \text{ in } \mathbf{A}} [i][\mathbf{A}, f]\phi.$$

By successive application of such principles, all dynamic modalities can be eliminated to obtain a standard epistemic formula. □

We sum this up, somewhat loosely, by stating the following:

**Fact 3.** *Basic epistemic logic is product-closed.*

But again, the situation gets more complicated when we add common knowledge. In this case, no reduction to the language without $[\mathbf{A}, e]$ modalities is possible. Van Benthem, van Eijck & Kooi 2006 [5] solve this problem by moving to the language E-PDL which is just the propositional dynamic logic version of epistemic logic, but now allowing the formation of arbitrary 'complex agents' using the standard PDL program vocabulary:

basic agents $i$, tests $?\phi$ on arbitrary formulas $\phi$ of the language,
unions, compositions, and Kleene iteration.

They provide an explicit axiomatization for the dynamic-epistemic version of this with added modalities $[\mathbf{A}, e]\phi$. Thus E-PDL has a completeness theorem like the earlier ones; but cf. Section 4 for remaining desiderata.

For present purposes, however, we summarize the gist of this result as follows: 'E-PDL is closed under the product construction'. In what follows, for convenience, we use obvious existential counterparts to the earlier universal modalities. Here is the central observation of the above paper:

**Theorem 3.** *For all $\phi$ $\in$E-PDL, and all action models* $\mathbf{A}$ *with event* $a$*, the formula* $\langle \mathbf{A}, a\rangle\phi$ *has an equivalent formula in E-PDL.*

Public announcements $!P$ are special action models with just one event with precondition $P$, equally visible to all agents. Thus, the theorem also says that E-PDL, or PDL, is closed under relativization - as observed earlier in van Benthem 2000 [4]. In addition, E-PDL has been shown to be closed under predicate substitutions in Kooi 2007 [11].

The point of the current paper is to analyze this situation more formally, in terms of general closure properties of modal languages, and their fixed-point extensions. In particular, we provide a new proof of Theorem 3 clarifying its background in modal fixed-point logic.

## 3 Closure under Relativization for Modal Standard Languages

It is easy to see that the basic modal language is closed under relativization. The procedure relativizes modalities, just as one does with quantifiers in first-order logic. Likewise, we already mentioned that propositional dynamic logic is closed under relativization. This requires an operation which also transforms program expressions, as follows:

$$([\pi]\phi)^P = [\pi \,|\, P](\phi)^P.$$

Here one must also *relativize programs* $\pi$ to programs $\pi \,|\, P$, as follows:

$$\begin{aligned} i \,|\, P &= ?P; i; ?P \\ ?\phi \,|\, P &= ?(\phi \wedge P) \\ (\pi \cup \theta) \,|\, P &= \pi \,|\, P \cup \theta \,|\, P \\ (\pi; \theta) \,|\, P &= \pi \,|\, P; \theta \,|\, P \\ (\pi^*) \,|\, P &= (\pi \,|\, P)^*. \end{aligned}$$

Finally, consider the most elaborate modal fixed-point language, the so-called *$\mu$-calculus.* Formulas $\phi(q)$ with only positive occurrences of the proposition letter $q$ define a monotonic set transformation in any model $\mathbf{M}$:

$$F_\phi^{\mathbf{M}}(X) = \{s \in \mathbf{M} \mid (\mathbf{M}[q := X], s) \vDash \phi.\}$$

The formula $\mu q \bullet \phi(q)$ defines the smallest fixed point of this transformation, which can be computed in ordinal stages starting from the empty set as a first approximation. Likewise, $\nu q \bullet \phi(q)$ defines the greatest fixed point of $F_\phi^{\mathbf{M}}$, with ordinal stages starting from the whole domain of $\mathbf{M}$ as a first approximation. Both exist for monotone maps, by the Tarski-Knaster theorem (Bradfield and Stirling 2006 [8]). For convenience, we assume that each occurrence of a fixed-point operator binds a unique proposition letter. Here is our first observation.

**Fact 4.** *The modal $\mu$-calculus is closed under relativization.*

*Proof.* We show the universal validity of the following interchange law:

$$\langle !P\rangle \mu q \bullet \phi(q) \leftrightarrow P \wedge \mu q \bullet \langle !P\rangle \phi(q). \tag{1}$$

Here the occurrences of $q$ are still syntactically positive in $\langle !P\rangle \phi(q)$ - in an obvious sense. Now to prove (1), compare the following identities, for all sets $X \subseteq P^{\mathbf{M}}$:

$$\begin{aligned} F^{\mathbf{M}}_{\langle !P\rangle\phi}(X) &= \{s \in \mathbf{M} \mid \mathbf{M}[q := X], s \vDash \langle !P\rangle \phi(q)\} \\ &= \{s \in \mathbf{M}|P \mid (\mathbf{M}|P)[q := X], s \vDash \phi(q)\} \\ &= F^{\mathbf{M}|P}_{\phi}(X). \end{aligned}$$

It should be clear that the approximation maps on both sides now work in exactly the same way. □

Still, there is a difference with standard fixed-point logic. One usually thinks of, e.g., a smallest fixed-point formula $\mu q \bullet \phi(q)$ as defining the limit of a sequence of ordinal approximations starting from the empty set, whose successor stages are computed by substitution of earlier ones:

$$\phi^0 = \bot, \quad \phi^{\alpha+1} = \phi(\phi^\alpha / q).$$

But this analogy breaks down between the two sides of the above equation (2). The approximation sequences defined in a direct manner will diverge. Consider the modal formulas

$$\phi(q) = \Box q, \quad P = \Diamond \top$$

in a model consisting of the numbers $1, 2, 3$ in their natural order. Both sequences in equation (2) start with the empty set, defined by $\bot$, but then they diverge:

| for $\langle !\Diamond\top\rangle \mu q \bullet \Box q$: | for $\Diamond\top \wedge \mu q \bullet \langle !\Diamond\top\rangle \Box q$: |
|---|---|
| $\langle !\Diamond\top\rangle \Box\bot$, only true at 2 | $\Diamond\top \wedge \langle !\Diamond\top\rangle \Box\bot$, only true at 2 |
| $\langle !\Diamond\top\rangle \Box\Box\bot$, true at $1, 2$ | $\Diamond\top \wedge \langle !\Diamond\top\rangle \Box \langle !\Diamond\top\rangle \Box\bot$, only true at 2. |

The reason for the divergence is that the formula on the right-hand side keeps prefixing formulas with dynamic model-changing modalities, so that we are now evaluating in models of the form $(\mathbf{M}|P)|P$, etc.

The general observation explaining this divergence involves another basic closure property of logical languages that we mentioned in Section 1, viz. closure under *substitutions*:

**Fact 5.** *The Substitution Lemma fails even for the basic modal language when announcement modalities $\langle !P\rangle$ are added.*

E.g., consider again our three-point model $\mathbf{M}$, with a proposition letter $p$ true at 2 only, and let $\phi$ be the formula $\langle !\Diamond\top\rangle\Diamond p$. Now consider the substitution $[(\langle !\Diamond\top\rangle\top)/p]$. First consider the model after performing this substitution: it will assign $p$ to $\{1,2\}$. Hence $[p := (\langle !\Diamond\top\rangle\top)^{\mathbf{M}}]\langle !\Diamond\top\rangle\Diamond p$ will be true in 1. Next perform the substitution syntactically to obtain the formula $\langle !\Diamond\top\rangle\Diamond\big(\langle !\Diamond\top\rangle\top\big)$ : this is true nowhere in the model $\mathbf{M}$.

Since the modal language is simply translatable into first-order logic, a similar observation holds for first-order logic with relativization operators $(\phi)^P$ added as part of its syntax. The resulting language does not satisfy the usual Substitution Lemma, since the model-changing operators $()^P$ create new contexts where formulas can change their truth values. So, model-changing operators are nice devices, but they exact a price.

*Remark 2 (Alternative dynamic definitions of substitution).*

Fact 5 holds for the straightforward operational definition of substitutions $[\phi/p]\psi$ as syntactically replacing each occurrence of $p$ in $\psi$ by an occurrence of $\phi$. However, there is an alternative. In line with earlier approaches in 'dynamic semantics' of first-order logic (cf. van Benthem 1996 [3]), Kooi 2007 [11] treats substitutions $[\phi/p]$ as modalities changing the current model in its denotation for $p$. These new modalities satisfy obvious recursive axioms pushing them through Booleans and standard modal operators. To push them also through public announcement modalities, one can first rewrite the latter via their PAL recursion axioms, and only then apply the substitution to the components. Van Eijck 2007 [9] shows how this provides an alternative syntactic operational definition of substitution, working inside out. One first reduces innermost PAL or DEL formulas to their basic modal equivalents, and then performs standard syntactic substitution in these. Though not compositional, this procedure is effective. When applied to the two approximation sequences in our earlier problematic example, these would now come out being the same after all.

Thus, dynamic modal languages are closed under semantic substitutions, but finding the precise corresponding syntactic operation in their static base language requires some care.

## 4 Closure of Dynamic Logic under Products

Theorem 3 said that the language E-PDL is closed under the product operation $\langle \mathbf{A}, e\rangle\phi$. The proof in van Benthem, van Eijck & Kooi 2006 [5] uses special arguments involving Kleene's Theorem for finite automata and program transformations. We provide a new proof which provides further insight by restating the situation within modal fixed-point logic.

First, consider the obvious inductive proof of Theorem 3, the 'Main Reduction'. Its steps follow the construction of the formula $\phi$. The atomic case, Booleans $\neg, \vee$, and basic epistemic modalities $\langle i \rangle$ are taken care of by the standard DEL reduction axioms. The remaining case is that of formulas $\langle \mathbf{A}, a\rangle\langle \pi\rangle\psi$ with an E-PDL modality involving a complex epistemic program $\pi$. To proceed, we need a deeper analysis of program structure. The following result can be proved together with Theorem 3 by a simultaneous induction:

**Theorem 4.** *For all $\mathbf{A}, a$, and programs $\pi' \in$ E-PDL, there exist E-PDL programs $T^{\pi'}_{a,b}$ (for each $b \in A$) such that, for all E-PDL formulas $\psi$,*

$$\mathbf{M}, s \models \langle \mathbf{A}, a\rangle\langle \pi'\rangle\psi \iff \mathbf{M}, s \models \bigvee_{b\in\mathbf{A}} \langle T^{\pi'}_{a,b}\rangle\langle \mathbf{A}, b\rangle\psi.$$

*Proof.* We use induction on the construction of the program $\pi'$.

Case 1: $\pi' = i$.

$$\langle \mathbf{A}, a\rangle\langle i\rangle\psi \iff \text{PRE}_a \text{ and } \bigvee_{aR_ib \text{ in } \mathbf{A}} \langle i\rangle\langle \mathbf{A}, b\rangle\psi.$$

This can be brought into our special form by setting

$$T^i_{a,b} = ?\text{PRE}_a; i \text{ if } aR_ib \text{ in } A, \text{ and } T^i_{a,b} = \bot, \text{ otherwise.}$$

Case 2: $\pi' = ?\alpha$ for some formula $\alpha$.

$$\langle \mathbf{A}, a\rangle\langle ?\alpha\rangle\psi \iff \langle \mathbf{A}, a\rangle(\alpha \wedge \psi) \iff$$
$$\langle \mathbf{A}, a\rangle\alpha \wedge \langle \mathbf{A}, a\rangle\psi \iff \langle ?(\langle \mathbf{A}, a\rangle\alpha)\rangle\langle \mathbf{A}, a\rangle\psi.$$

Here the less complex formula $\langle \mathbf{A}, a\rangle\alpha$ can be taken to be in the language of E-PDL already, by the simultaneous induction proving Theorem 3. It is easy to then define the correct transition predicates $T^{?\alpha}_{a,b}$ for all events $b \in \mathbf{A}$.

Case 3: $\pi' = \alpha \cup \beta$ for some formulas $\alpha$, $\beta$.

$$\langle \mathbf{A}, a\rangle\langle \alpha \cup \beta\rangle\psi \iff \langle \mathbf{A}, a\rangle(\langle \alpha\rangle\psi \vee \langle \beta\rangle\psi) \iff$$
$$\langle \mathbf{A}, a\rangle\langle \alpha\rangle\psi \vee \langle \mathbf{A}, a\rangle\langle \beta\rangle\psi \overset{\text{ind.hyp.}}{\iff} \bigvee_{b\in\mathbf{A}} \langle T^{\alpha}_{a,b}\rangle\langle \mathbf{A}, b\rangle\psi \vee \bigvee_{b\in\mathbf{A}} \langle T^{\beta}_{a,b}\rangle\langle \mathbf{A}, b\rangle\psi$$

and, by recombining parts of this disjunction, using the valid PDL-equivalence $\langle \alpha\rangle\psi \vee \langle \beta\rangle\psi \leftrightarrow \langle \alpha \cup \beta\rangle\psi$, we get the required normal form.

Case 4: $\pi' = \alpha; \beta$ for some formulas $\alpha$, $\beta$.

$$\langle \mathbf{A}, a\rangle\langle \alpha; \beta\rangle\psi \iff \langle \mathbf{A}, a\rangle\langle \alpha\rangle\langle \beta\rangle\psi \overset{\text{ind. hyp.1}}{\iff}$$
$$\bigvee_{b\in\mathbf{A}} \langle T^{\alpha}_{a,b}\rangle\langle \mathbf{A}, b\rangle\langle \beta\rangle\psi \overset{\text{ind. hyp.2}}{\iff} \bigvee_{b\in\mathbf{A}} (\langle T^{\alpha}_{a,b}\rangle \bigvee_{c\in\mathbf{A}} \langle T^{\beta}_{b,c}\rangle\langle \mathbf{A}, c\rangle\psi)$$

and here, using the minimal logic of PDL again, substituting one special form in another once more yields a special form. E.g., we have the equivalence $\langle \alpha\rangle(\langle \beta\rangle p \vee \langle \gamma\rangle q) \iff \langle \alpha; \beta\rangle p \vee \langle \alpha; \gamma\rangle q$.

Case 5: $\pi' = \pi^*$ for some program $\pi$.

The crux lies in this final case: combinations with Kleene iterations $\langle \mathbf{A}, a\rangle\langle \pi^*\rangle\psi$ do not reduce as before. But even so, we can analyze them in the same style, using a *simultaneous fixed-point operator* $\mu \mathbf{q}_b \bullet$ defining the propositions $\langle \mathbf{A}, b\rangle\langle \pi^*\rangle\psi$ for all events $b \in \mathbf{A}$ in one fell swoop. The need for this simultaneous recursion explains earlier difficulties in the literature with reduction axioms for common knowledge with product update. To find the right schema, first recall the PDL fixed-point equation for Kleene iteration:

$$\langle \mathbf{A}, a\rangle\langle \pi^*\rangle\psi \iff \langle \mathbf{A}, a\rangle(\psi \vee \langle \pi\rangle\langle \pi^*\rangle\psi) \iff$$
$$\langle \mathbf{A}, a\rangle\psi \vee \langle \mathbf{A}, a\rangle\langle \pi\rangle\langle \pi^*\rangle\psi \overset{\text{ind. hyp.}}{\iff} \langle \mathbf{A}, a\rangle\psi \vee \bigvee_{b\in\mathbf{A}} \langle T^{\pi}_{a,b}\rangle\langle \mathbf{A}, b\rangle\langle \pi^*\rangle\psi.$$

Here, again because of the simultaneous inductive proof with Theorem 3, we can think of the first disjunct as being some formula $\alpha_a$ of E-PDL. The result of this unpacking are simultaneous equivalences of the form (with propositional variables $q_a$ for each $a \in \mathbf{A}$):

$$q_a \leftrightarrow \alpha_a \vee \bigvee_{b\in\mathbf{A}} \langle T^{\pi}_{a,b}\rangle q_b. \qquad (*)$$

**Lemma 1.** *The denotations of the modal formulas $\langle \mathbf{A}, a\rangle\langle \pi^*\rangle\psi$ in a model $\mathbf{M}$ are precisely the $a$-projections of the smallest fixed-point solution to the simultaneous equations $(*)$.*

*Proof (Lemma 1).* Here, smallest fixed-points for simultaneous equations in the $\mu-$calculus are computed just as those for single fixed-point equations. lemma 1 follows by a simple induction, showing that the standard meanings of the modal formulas $\langle \mathbf{A}, a\rangle\langle \pi^*\rangle\psi$ in a model $\mathbf{M}$ are contained in any solution for the simultaneous fixed-point equation.

We calculate the meaning of the least fixed-point of $(*)$ through the approximation procedure and show it is equal to that of $\langle \mathbf{A}, a\rangle\langle \pi^*\rangle\psi$ $(a \in \mathbf{A})$.

From now on, we identify formulas by their truth sets in $\mathbf{M}$, reading $\phi$ as $\{m \in \mathbf{M} \mid (\mathbf{M}, m) \vDash \phi\}$. For simplicity, we rewrite $(*)$ as follows:

$$q_i = \alpha_i \vee \bigvee_{1\le j\le n} \langle T^{\pi}_{i,j}\rangle q_j \quad (1 \le i \le n)$$

Let $F$ be the monotone operator from $\mathcal{P}(\mathbf{M})^n$ to itself induced by the right hand side of $(*)$, where $n$ is the number of elements in $\mathbf{A}$. More precisely, for $\boldsymbol{X} = (X_1, \cdots, X_n) \in \mathcal{P}(\mathbf{M})^n$, $F(\boldsymbol{X}) = (Y_1, \cdots, Y_n)$ where for each $1 \le i \le n$,

$$Y_i = \big\{m \in \mathbf{M} \mid (\mathbf{M}[\{q_j := X_j\}_{j=1,\cdots,n}], m) \vDash \alpha_i \vee \bigvee_{1\le j\le n} \langle T^{\pi}_{i,j}\rangle q_j\big\}.$$

Next, for $\boldsymbol{X} \in \mathcal{P}(\mathbf{M})^n$, define $\langle F^\xi(\boldsymbol{X}) \mid \xi \in \mathrm{On}\rangle$ as follows:

$$
\begin{aligned}
F^0(\boldsymbol{X}) &= \boldsymbol{X} \\
F^{\xi+1}(\boldsymbol{X}) &= F(F^\xi(\boldsymbol{X})) \\
F^\xi(\boldsymbol{X}) &= \bigcup_{\eta<\xi} F^\eta(\boldsymbol{X}) \text{ if } \xi \text{ is a limit ordinal.}
\end{aligned}
$$

For any $m \in \omega$, we can prove the following equation by induction on $m$.

$$
\begin{aligned}
F^m(\bot) = \{ \bigvee_{1\le j_1,j_2,\cdots,j_{m-1}\le n} & [\alpha_i \vee \langle T^\pi_{i,j_1}\rangle \alpha_{j_1} \vee \langle T^\pi_{i,j_1}\rangle\langle T^\pi_{j_1,j_2}\rangle \alpha_{j_2} \\
& \vee \cdots \vee \langle T^\pi_{i,j_1}\rangle\langle T^\pi_{j_1,j_2}\rangle \cdots \langle T^\pi_{j_{m-2},j_{m-1}}\rangle \alpha_{j_{m-1}}]\}_{1\le i\le n}.
\end{aligned}
$$

Hence

$$
F^\omega(\bot) = \Big\{ (\exists m < \omega)\, (\exists j_1, \cdots, j_{m-1})\ \langle T^\pi_{i,j_1}\rangle \cdots \langle T^\pi_{j_{m-2},j_{m-1}}\rangle \alpha_{j_{m-1}} \Big\}_{1\le i\le n},
$$

which implies $F^\omega(\bot) = F^{\omega+1}(\bot)$: the least fixed-point is reached in $\omega$ steps.

Therefore we only have to show that $\{\langle \mathbf{A}, a_i\rangle\langle\pi^*\rangle\psi\}_{1\le i\le n} = F^\omega(\bot)$.

Recall that, by the defining property of $T^\pi_{i,j}$, for any E-PDL formula $\psi'$, any $1 \le i \le n$ and any state $s$ in $\mathbf{M}$,

$$
\mathbf{M}, s \vDash \langle \mathbf{A}, a_i\rangle\langle\pi\rangle\psi' \iff \mathbf{M}, s \vDash \bigvee_{1\le j\le n} \langle T^\pi_{i,j}\rangle\langle \mathbf{A}, a_j\rangle\psi'.
$$

By using this condition repeatedly, we get the following equivalence: for any $n$-tuple $\boldsymbol{s}$ of elements in $M$ and any $i$ with $1 \le i \le n$,

$$
\begin{aligned}
& \mathbf{M}, s_i \vDash \langle \mathbf{A}, a_i\rangle\langle\pi^*\rangle\psi \\
\iff & (\exists m \in \omega)\ \mathbf{M}, s_i \vDash \langle \mathbf{A}, a_i\rangle\langle\pi\rangle^m\psi \\
\iff & (\exists m \in \omega)\, (\exists j_1)\ \mathbf{M}, s_i \vDash \langle T^\pi_{i,j_1}\rangle\langle \mathbf{A}, a_{j_1}\rangle\langle\pi\rangle^{m-1}\psi \\
\iff & (\exists m \in \omega)\, (\exists j_1, j_2)\ \mathbf{M}, s_i \vDash \langle T^\pi_{i,j_1}\rangle\langle T^\pi_{j_1,j_2}\rangle\langle \mathbf{A}, a_{j_2}\rangle\langle\pi\rangle^{m-2}\psi \\
\iff & \cdots \\
\iff & (\exists m \in \omega)\, (\exists j_1, \cdots, j_m)\ \mathbf{M}, s_i \vDash \langle T^\pi_{i,j_1}\rangle\langle T^\pi_{j_1,j_2}\rangle \cdots \langle T^\pi_{j_{m-1},j_m}\rangle\langle \mathbf{A}, a_{j_m}\rangle\psi \\
\iff & (\exists m \in \omega)\, (\exists j_1, \cdots, j_m)\ \mathbf{M}, s_i \vDash \langle T^\pi_{i,j_1}\rangle\langle T^\pi_{j_1,j_2}\rangle \cdots \langle T^\pi_{j_{m-1},j_m}\rangle \alpha_{j_m} \\
\iff & s_i \in \big(F^\omega(\bot)\big)_i
\end{aligned}
$$

where $\big(F^\omega(\bot)\big)_i$ is the $i$-th coordinate of $F^\omega(\bot)$. Hence

$$
(\forall i)\, \big(\mathbf{M}, s_i \vDash \langle \mathbf{A}, a_i\rangle\langle\pi^*\rangle\psi\big) \iff \boldsymbol{s} \in F^\omega(\bot),
$$

which is what we desired. □

What really happens here is this. Computing the explicit solutions for the predicates $q_i$ after $\omega$ steps, one gets the countable disjunction over all finite 'path formulas' of the form $\langle T^{\pi}_{i,j_i}; T^{\pi}_{j_1,j_2}; \cdots ; T^{\pi}_{j_n,k}\rangle\alpha_k$. And the latter are exactly the meanings of the original propositions $\langle \mathbf{A}, a\rangle\langle\pi^*\rangle\psi$.

But we are not done yet. What we need to show next is that the solutions obtained in this way are actually in the language E-PDL! The following lemma tells us the relevant fact about the $\mu$-calculus. Simultaneous fixed-point equations of the above special disjunctive shape $(*)$ can be solved one by one, and the solutions lie inside dynamic logic.

**Lemma 2.** *Any system of simultaneous fixed-point equations of $(*)$ has an explicit minimal solution for each $q_a$ in E-PDL. Moreover, the solutions retain the special disjunctive form described in Theorem 4.*

*Proof (Lemma 2).* The inductive procedure producing explicit E-PDL solutions works line by line - like Gaussian Elimination in a system of linear equations.

- Case 1. There is only one $q$-variable, as with public announcements.
  The line reads $q_1 \leftrightarrow \alpha_1 \vee \langle\beta_{1,1}\rangle q_1$. The explicit solution works just as in standard dynamic logic, in the

$$q_1 = \langle\beta^*_{1,1}\rangle\alpha_1.$$

- Case 2. There are $n$ lines in the recursion schema, with $n > 1$.
  We first solve for the variable $q_1$ as in Case 1 - obtaining an explicit E-PDL formula $\sigma_1(q_2, \cdots, q_n)$ in the other recursion variables. We then substitute this solution in the remaining $n-1$ equations, and solve these inductively. Finally, the solutions thus obtained for the $q_2, \cdots, q_n$ are substituted in $\sigma_1(q_2, \cdots, q_n)$ to also solve for $q_1$.

Some syntactic checking will show that these solutions remain in the syntactic format described in Theorem 4. But of course, we also need to show that this is really a solution for the above fixed-point equations $(*)$, and indeed the smallest one. To prove that, we formulate the algorithm more formally in the following way (cf. Arnold & Niwinski [1] for a more extensive treatment).

For any monotone operator $G$, let $G_*$ denote the least fixed point of $G$. Let $F\colon \mathcal{P}(\mathbf{M})^n \to \mathcal{P}(\mathbf{M})^n$ be the monotone operator induced by the $n$ equations in $(*)$. Now take any $X_2, \cdots, X_n \in \mathcal{P}(\mathbf{M})$ and fix them. Next, define $F_{X_2,\cdots,X_n} : \mathcal{P}(\mathbf{M}) \to \mathcal{P}(\mathbf{M})$ as follows:

$$F_{X_2,\cdots,X_n}(X_1) = \big(F(X_1, \cdots, X_n)\big)_1,$$

where $(\boldsymbol{X})_i$ is the $i$-th coordinate of $\boldsymbol{X}$. Since $F$ is monotone, $F_{X_2,\cdots,X_n}$ is also monotone. Then define $F_{X_3,\cdots,X_n} : \mathcal{P}(\mathbf{M}) \to \mathcal{P}(\mathbf{M})$:

$$F_{X_3,\cdots,X_n}(X_2) = \big(F((F_{X_2,\cdots,X_n})_*, X_2, \cdots, X_n)\big)_2.$$

This is also monotone because $F$ and the function $(X_2, \cdots, X_n) \mapsto (F_{X_2,\cdots,X_n})_*$ are both monotone. Continue this process until we define $F_\emptyset$. Then the solution of the earlier 'Gaussian' algorithm is the unique $F'_*$ such that

$$(F'_*)_i = \left(F_{(F'_*)_{i+1},\cdots,(F'_*)_n}\right)_* \quad (1 \leq i \leq n).$$

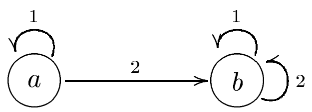
Note how we compute the rightmost fixed-point first here, and then substitute leftward. Hence all we have to show is the following:

**Claim 1.** $F_* = F'_*$.

The proof is in Arnold & Niwinski [1] (see Section 1.4. in this book). To make our paper self-contained, we will put a proof in an Appendix below. □

This concludes the proofs of Theorems 3 and 4. □

**Illustration 1.** *We compute the solutions for the update model* $\mathbf{A} =$




$$PRE_a = p, \qquad PRE_b = \top$$

*This describes a security scenario where agent* 1 *correctly observes that event* $a$ *is taking place, while agent* 2 *mistakenly believes that* $b$ *occurs. Here is a description of the non-trivial common knowledge for* 1, 2 *arising from this scenario, by writing out the fixed point equation for* $\langle \mathbf{A}, a\rangle \langle (1 \cup 2)^* \rangle r$.

*By step 1 in the proof,*

$$\begin{array}{ll} T^1_{a,a} = ?PRE_a; 1 = ?p; 1, & T^1_{a,b} = \bot \\ T^1_{b,a} = \bot, & T^1_{b,b} = ?PRE_b; 1 = 1 \\ T^2_{a,a} = \bot, & T^2_{a,b} = ?PRE_a; 2 = ?p; 2 \\ T^2_{b,a} = \bot, & T^2_{b,b} = ?PRE_b; 2 = 2. \end{array}$$

*Then by step 3,*

$$\begin{array}{ll} T^{1\cup 2}_{a,a} = T^1_{a,a} \cup T^2_{a,a} = ?p; 1, & T^{1\cup 2}_{a,b} = T^1_{a,b} \cup T^2_{a,b} = ?p; 2 \\ T^{1\cup 2}_{b,a} = T^1_{b,a} \cup T^2_{b,a} = \bot, & T^{1\cup 2}_{b,b} = T^1_{b,b} \cup T^2_{b,b} = 1 \cup 2. \end{array}$$

*Now put*

$$q_a = \langle \mathbf{A}, a\rangle \langle (1 \cup 2)^* \rangle r, \quad q_b = \langle \mathbf{A}, b\rangle \langle (1 \cup 2)^* \rangle r.$$

*Then by* $(*)$,

$$\begin{aligned} q_a &= \langle \mathbf{A}, a\rangle r \vee \langle T^{1\cup 2}_{a,a}\rangle q_a \vee \langle T^{1\cup 2}_{a,b}\rangle q_b \\ &= (PRE_a \wedge r) \vee \langle ?p;2\rangle q_b \vee \langle ?p;1\rangle q_a \\ &= ((p \wedge r) \vee \langle ?p;2\rangle q_b) \vee \langle ?p;1\rangle q_a \end{aligned}$$

*and*

$$\begin{aligned} q_b &= \langle \mathbf{A}, b\rangle r \vee \langle T^{1\cup 2}_{b,a}\rangle q_a \vee \langle T^{1\cup 2}_{b,b}\rangle q_b \\ &= (PRE_b \wedge r) \vee \langle 1 \cup 2\rangle q_b \\ &= r \vee \langle 1 \cup 2\rangle q_b. \end{aligned}$$

*Since the order of eliminating variables does not influence the solutions, we first solve* $q_b$ *as follows*:

$$q_b = \langle (1 \cup 2)^*\rangle r.$$

*By substituting this solution in the above equation for* $q_a$,

$$\begin{aligned} q_a &= \big((p \wedge r) \vee \langle ?p;2\rangle \, \langle (1 \cup 2)^*\rangle r\big) \vee \langle ?p;1\rangle q_a \\ &= \big((p \wedge r) \vee \langle ?p;2;(1 \cup 2)^*\rangle r\big) \vee \langle ?p;1\rangle q_a. \end{aligned}$$

*Hence*

$$\begin{aligned} q_a &= \langle (?p;1)^*\rangle \big((p \wedge r) \vee \langle ?p;2;(1 \cup 2)^*\rangle r\big) \\ &= \langle (?p;1)^*\rangle (p \wedge r) \vee \langle (?p;1)^*;?p;2;(1 \cup 2)^*\rangle r \end{aligned}$$

*We can easily check that these* $q_a, q_b$ *satisfy the equations we gave by an independent semantic argument.*

*Remark 3.* The calculation in this example is really just the following well-known fact about the modal $\mu$-calculus:

Let $\phi(q_1, q_2)$, $\psi(q_1, q_2)$ be positive formulas in the modal $\mu$-calculus. Then the simultaneous least fixed points of these formulas is

$$\big(\mu q_1.\phi(q_1, \mu q_2.\psi(q_1, q_2)), \mu q_2.\psi(\mu q_1.\phi(q_1, q_2), q_2)\big).$$

In the proof of Claim 1 (cf. the Appendix), we only use the condition that $F$ is monotone. This means we can generalize the result as follows:

**Corollary 1.** *The modal* $\mu$*-calculus is closed under the formation of simultaneous fixed-point operators.*

## 5 Closure of the $\mu$-calculus under Products

Finally, we show how the preceding analysis also extends to the $\mu$-calculus itself, where it even becomes simpler.

**Theorem 5.** *The $\mu$-calculus is closed under product operators.*

*Proof.* We prove the statement by induction on the complexity of formulas. We only consider the fixed point case, as the others go like before.

Our main task is to analyze fixed-point computations in product models $\mathbf{M} \times \mathbf{A}$ in terms of similar computations in the original model $\mathbf{M}$. The following idea turns out to work here. Let $X$ be a subset of $\mathbf{M} \times \mathbf{A}$. Modulo the event preconditions possibly ruling out some pairs, we can describe $X$, without loss of information, in terms of the sequence of its projections to the events in $\mathbf{A}$, viewed as a finite set of indices. Thus, we can describe the computation in $\mathbf{M} \times \mathbf{A}$ by means of a finite set of computations in $M$. The following set of definitions and observations makes this precise.

Take any Kripke model $\mathbf{M}$ and any event model $\mathbf{A}$. Let $n$ be the number of elements of $\mathbf{A}$ and let $\mathbf{A} = \{a_j\}_{1 \leq j \leq n}$. There are canonical mappings $\pi \colon \mathcal{P}(\mathbf{M})^n \to \mathcal{P}(\mathbf{M} \times \mathbf{A})$ and $\iota \colon \mathcal{P}(\mathbf{M} \times \mathbf{A}) \to \mathcal{P}(\mathbf{M})^n$ with $\pi \circ \iota = \mathrm{id}$:

$$\begin{aligned} \pi(\boldsymbol{X}) = & \bigcup_{1 \leq j \leq n} (X_j \times \{a_j\}) \cap (\mathbf{M} \times \mathbf{A}), \\ \iota(Y) = & \{Y_j\}_{1 \leq j \leq n}, \end{aligned}$$

where $Y_j = \{x \in \mathbf{M} \mid (x, a_j) \in Y\}$.

Given a positive formula $\phi(q)$ in the modal $\mu$-calculus, let $F_\phi^{\mathbf{M} \times \mathbf{A}} \colon \mathcal{P}(\mathbf{M} \times \mathbf{A}) \to \mathcal{P}(\mathbf{M} \times \mathbf{A})$ be the monotone function induced by $\phi(q)$. Define $F^{\phi(q)} \colon \mathcal{P}(\mathbf{M})^n \to \mathcal{P}(\mathbf{M})^n$ as follows:

$$F^{\phi(q)} = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}} \circ \pi.$$

We claim that $F_\phi^{\mathbf{M} \times \mathbf{A}}$ is monotone if and only if $F^{\phi(q)}$ is monotone. Suppose $F_\phi^{\mathbf{M} \times \mathbf{A}}$ is monotone. Since $\pi, \iota$ are monotone and compositions of monotone functions are monotone, $F^{\phi(q)}$ is also monotone. To prove the converse, suppose $F^{\phi(q)}$ is monotone. Pick any $X, Y \in \mathcal{P}(\mathbf{M} \times \mathbf{A})$ with $X \subseteq Y$. First note that $F_\phi^{\mathbf{M} \times \mathbf{A}}(X) \subseteq F_\phi^{\mathbf{M} \times \mathbf{A}}(Y)$ holds if and only if $\iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}(X) \subseteq \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}(Y)$ holds. Hence all we have to check is $\iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}(X) \subseteq \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}(Y)$. But

$$\begin{aligned} \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}(X) &= \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}\big(\pi \circ \iota(X)\big) = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}} \circ \pi\big(\iota(X)\big) \\ &= F^{\phi(q)}\big(\iota(X)\big) \subseteq F^{\phi(q)}\big(\iota(Y)\big) = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}} \circ \pi\big(\iota(Y)\big) \\ &= \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}\big(\pi \circ \iota(Y)\big) = \iota \circ F_\phi^{\mathbf{M} \times \mathbf{A}}(Y), \end{aligned}$$

where the above inclusion follows from the monotonicity of $F^{\phi(q)}$ and $\iota$.

Moreover, there is a further canonical correspondence: if $\boldsymbol{X}$ is an $F^{\phi(q)}$-fixed point, then $\pi(\boldsymbol{X})$ is an $F_\phi^{\mathbf{M} \times \mathbf{A}}$-fixed-point, and if $Y$ is an $F_\phi^{\mathbf{M} \times \mathbf{A}}$-fixed-point, then $\iota(Y)$ is an $F^{\phi(q)}$-fixed-point. Hence the least $F^{\phi(q)}$-fixed-point corresponds to the least $F_\phi^{\mathbf{M} \times \mathbf{A}}$-fixed-point.

*Remark 4 (Relating fixed-point computations in different models).* The argument above may be seen as a special case of the "Transfer Lemma" (Theorem 1.2.15) in Arnold & Niwinski [1]. This lemma only uses our $\iota$ function, while we added the function $\pi$ for clarity, to restrict an input to the inverse image of $\iota$ – which is why the equation $\pi \circ \iota = \mathrm{id}$ holds. For further background to this kind of argument, cf. Bloom and Ésik [7].

So far, we have seen that the least $F_{\phi}^{\mathbf{M}\times\mathbf{A}}$-fixed-point can be correlated with the least $F^{\phi(q)}$-fixed-point in a natural way. Our next task is to show that $\langle \mathbf{A}, a\rangle \mu q.\phi(q)$ is actually definable in the modal $\mu$-calculus. For that purpose, first note that $\langle \mathbf{A}, a_j\rangle\ \mu q \bullet \phi(q)$ defines the $j$-th coordinate of the least $F_{\phi}^{\mathbf{M}\times\mathbf{A}}$-fixed-point. By the definition of $\iota$, it is also the $j$-th coordinate of the least $F^{\phi(q)}$-fixed-point. Now, since the modal $\mu$-calculus is closed under simultaneous fixed-point operators by Corollary 1, if we can express $F^{\phi(q)}$ by a formula of the modal $\mu$-calculus with positive variables, we are done.

To prove this, we generalize the syntactic analysis employed in Section 4 to formulas with many variables $\boldsymbol{q} = q_1, \cdots, q_m$. For any formula $\phi(\boldsymbol{q})$ in the modal $\mu$-calculus, define $F_{\phi(\boldsymbol{q})}^{\mathbf{M}\times\mathbf{A}} : \mathcal{P}(\mathbf{M}\times\mathbf{A})^m \to \mathcal{P}(\mathbf{M}\times\mathbf{A})$ as follows:

$$F_{\phi(\boldsymbol{q})}^{\mathbf{M}\times\mathbf{A}}(\boldsymbol{Y}) = \{(s,a) \mid \big((\mathbf{M}\times\mathbf{A})[q_k := Y_k], (s,a)\big) \vDash \phi(\boldsymbol{q})\},$$

where $\boldsymbol{Y} \in (\mathbf{M}\times\mathbf{A})^m$.

**Claim 2.** *For any formula $\phi(\boldsymbol{q})$ in the modal $\mu$-calculus, there are formulas $\boldsymbol{\psi}_\phi$ such that $F^{\phi(\boldsymbol{q})} = F^{\mathbf{M}}_{\boldsymbol{\psi}_\phi}$ where $F^{\phi(\boldsymbol{q})} : \mathcal{P}(\mathbf{M})^{m\cdot n} \to \mathcal{P}(\mathbf{M})^n$ and*

*(∗) For any $1 \le k \le m$, if all the occurrences of $q_k$ in $\phi$ are positive (negative resp.), then for each $1 \le j, j' \le n$, all the occurrences of $p_{k,j}$ in $(\psi_\phi)_{j'}$ are positive (negative resp.),*

*Proof (Claim 2).* In the following definitions, we only display the essential argument variables needed to understand the function values. We prove the statement by induction on the complexity of $\phi$. As in the proof of Lemma 1, we identify formulas with their truth sets. Also, if $\boldsymbol{\psi}$ is a sequence of formulas, $\psi_j$ is the $j$-th coordinate of $\boldsymbol{\psi}$.

- Case 1: $\phi = p$ ($p$ is not in $\boldsymbol{q}$).

  $$F^{\phi(\boldsymbol{q})} = \big(p \wedge \mathrm{PRE}_{a_1}, \cdots, p \wedge \mathrm{PRE}_{a_n}\big).$$

  Hence $(\psi_{\phi(\boldsymbol{q})})_j = p \wedge \mathrm{PRE}_{a_j}$. It is easy to check (∗).
- Case 2: $\phi = q_k$ ($q_k$ is the $k$-th coordinate of $\boldsymbol{q}$).

  $$F^{\phi(\boldsymbol{q})}(\boldsymbol{X}) = \{X_{k,j} \wedge \mathrm{PRE}_{a_j}\}_{1\le j\le n}.$$

  Hence $(\psi_{\phi(\boldsymbol{q})})_j = p_{k,j} \wedge \mathrm{PRE}_{a_j}$, where $p_{k,j}$ is the $j$-th variable in the $k$-th block corresponding to $q_k$. It is also easy to check (∗).

– Case 3: $\phi = \phi_1 \wedge \phi_2$.

$$F^{\phi(\boldsymbol{q})} = \boldsymbol{\psi}_{\phi_1} \wedge \boldsymbol{\psi}_{\phi_2}.$$

Hence $\boldsymbol{\psi}_{\phi(\boldsymbol{q})} = \boldsymbol{\psi}_{\phi_1} \wedge \boldsymbol{\psi}_{\phi_2}$. It is easy to check $(*)$.

– Case 4: $\phi = \neg\phi'$.

$$F^{\phi(\boldsymbol{q})} = \{\neg(\psi_{\phi'})_j \wedge \mathrm{PRE}_{a_j}\}_{1 \leq j \leq n}.$$

Hence $(\psi_{\phi(\boldsymbol{q})})_j = \neg(\psi_{\phi'})_j \wedge \mathrm{PRE}_{a_j}$. It is easy to check $(*)$ by our inductive hypothesis, and the simultaneous definition for positive and negative occurrences.

– Case 5: $\phi = \langle i \rangle \phi'$.

For any $1 \leq j \leq n$ and $x \in M$,

$$x \in \left(F^{\phi(\boldsymbol{q})}(\boldsymbol{X})\right)_j \iff$$
$$(1 \leq \exists j' \leq n)\ (\exists y \in \mathbf{M})\ \left(xR_i y \wedge a_j R_i a_{j'} \wedge y \in \left(F^{\phi'(\boldsymbol{q})}(\boldsymbol{X})\right)_{j'}\right).$$

To see that this is true, observe that the condition $y \in \left(F^{\phi'(\boldsymbol{q})}(\boldsymbol{X})\right)_{j'}$ implies $(y, a_{j'}) \in \mathbf{M} \times \mathbf{A}$. Therefore, we can put

$$\left(\psi_{\phi(\boldsymbol{q})}\right)_j = \bigvee_{a_j R_i a_{j'}} \langle i \rangle \left(\psi_{\phi'(\boldsymbol{q})}\right)_{j'}.$$

– Case 6: $\phi = \mu q' \bullet \phi'$, where all the occurrences of $q'$ are positive in $\phi'$.

$$\begin{aligned} F^{\phi(\boldsymbol{q})}(\boldsymbol{X}) =& \left\{ \left(F^{\mathbf{M}\times\mathbf{A}}_{\mu q' \bullet \phi'(q', \boldsymbol{q})}(\pi(\boldsymbol{X}))\right)_j \right\}_{1 \leq j \leq n} \\ =& \left\{ \left((F^{\mathbf{M}\times\mathbf{A}}_{\phi'(q', \boldsymbol{q})}(\pi(\boldsymbol{X})))_*\right)_j \right\}_{1 \leq j \leq n} \\ =& \left(\boldsymbol{X'} \mapsto F^{\mathbf{M}}_{\boldsymbol{\psi}_{\phi'}}(\boldsymbol{X'}, \boldsymbol{X})\right)_*, \end{aligned}$$

where $(F(\cdot))_*$ is the least $F$-fixed-point. By induction hypothesis, all the occurrences of $p'_j$ are positive in $(\psi_{\boldsymbol{\phi'}})_{j'}$ for any $1 \leq j, j' \leq n$, where $\boldsymbol{p'}$ corresponds to $q'$. Since the modal $\mu$-calculus is closed under simultaneous fixed-point operators, we can put $\boldsymbol{\psi}_{\phi(\boldsymbol{q})} = \mu \boldsymbol{p'} \bullet \boldsymbol{\psi}_{\phi'}(\boldsymbol{q})$, that are also in the modal $\mu$-calculus. Since $\mu$-operators do not change the positivity (negativity) of variables not bounded by them, $(*)$ also holds in this case.

□

The proof of the last case explains why we needed to 'blow-up' in the number of variables in Claim 2. Also, we proved the claim for arbitrary formulas (not only for positive ones) because otherwise we cannot use the induction hypothesis in Case 4 (if $\phi$ is positive, then $\phi'$ must be negative). □

*Remark 5 (Effective reduction axioms).*

As in Fact 4, we could also an explicit reduction axiom for $\langle \mathbf{A}, a_j \rangle \mu q.\phi(q)$ by taking the $j$-th coordinate of the simultaneous fixed-point expression $\boldsymbol{\mu q}.\boldsymbol{\psi}_{\phi(\boldsymbol{q})}$. Since our proof is effective, we can effectively compute the shape of the axiom.

The common point of the proofs of Theorems 3,4 and Theorem 5 is that both E-PDL and the modal $\mu$-calculus are closed under simultaneous fixed point operators (in the case of E-PDL, such operators have the special form of $(*)$). The proof of that fact is essentially the same (it is that of Claim 1) but the case of the full $\mu$-calculus is easier because we have arbitrary $\mu$-operators, while in E-PDL, we have to check if the solution is also in E-PDL.

## 6 Conclusions and Further Directions

*T* he preceding results place current modal logics of information update in a more general light, relating their 'reduction axiom' approach for obtaining conservative dynamic extensions of existing static logics to abstract closure properties of fixed-point logics. Our observations also suggest a number of more general issues, of which we mention a few.

**Fine-structure of the $\mu$-calculus** Our results show that product closure holds for basic modal logic, propositional dynamic logic PDL, and the $\mu$-calculus itself. We think that there are further natural fragments with this property, including the $\mu$-$\omega$-calculus, which only allows fixed-points whose computations stop uniformly by stage $\omega$. Another case to look for product closure is the hierarchy of nested fixed-point alternation. Our proof removes modal product operators by means of simultaneous fixed-points, which can then be removed by nested single ones, but we have not yet analyzed its precise syntactic details.

On another matter, our proof method in Section 4 suggests that PDL is distinguished inside the $\mu$-calculus as the smallest fragment closed under some very simple 'additive' fixed-point equations. This seems related to the fact that the semantics of dynamic logic only describes linear computation traces, and no more complex constructs, such as arbitrary finite trees. Can this equational observation be turned into a characterization of PDL?

**Connections with automata theory** The first proof of product closure for PDL in van Benthem & Kooi 2004 [6] used finite automata to serve as 'controllers' restricting state sequences in product models $\mathbf{M}\times\mathbf{A}$. The second, different proof in Van Benthem, van Eijck & Kooi 2006 [5] involved a non-trivial use of Kleene's Theorem for regular languages, and hence again a connection with finite automata. What is the exact connection of this proof with our special unwinding of simultaneous 'disjunctive' fixed-point equations inside PDL? Can Kleene's Theorem be interpreted as a normal-form result in fixed-point logic?

There may also be a more general automata-based take on our arguments, given the strong connection between automata theory and $\mu$-calculus.[3]

[3] Added in print. Martin Otto (p.c.) has proposed using the product closure of MSOL and the bisimulation invariance of the mu-calculus with added product modalities for an alternative proof of our Theorem 5, by an appeal to the Janin-Walukiewiz Theorem.

**Logical languages and general product closure** Finally, we know now that many modal languages are product-closed. What about logical systems in general? We would like to have an abstract formulation which applies to a wider class of logical systems, such as first-order logic and its extensions in abstract model theory. We feel that product closure is a natural requirement on expressive power, especially given its earlier motivation in terms of relative interpretability. But the correct formulation may have to be stronger than our notion in this paper. Even in the modal case, our proofs would also go through if we allowed, say, definable substitutions for atomic proposition letters in product models. Also, one might also try to split our modal notion into full product closure plus predicate substitutions, treating our use of preconditions as a case of definable domain relativization.

There may also be a connection here with the Feferman-Vaught Theorem, and product constructions reducing truth in the product to truth of related statements in the component models. After all, our proof of uniform definability of dynamic modal operators $\langle \mathbf{A}, a\rangle\phi$ induces an obvious translation relating truth of $\phi$ in a product model $\mathbf{M}\times\mathbf{A}$ to that of some effective translation of $\phi$ in the component model $\mathbf{M}$.

Finally, one way of seeing how strong product closure really is would be to ask a converse question. For instance, assume that a fragment of the $\mu$-calculus is product-closed. Does it follow that it is closed under simultaneous fixed-points?

In all, our results, though somewhat technical and limited in scope, seem to provide a vantage point for raising many interesting new questions.

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# 7 Appendix

## 7.1 Proof of Claim 1

*Proof (Claim 1).* By the property of $F'_*$, it suffices to show the following:

$$(F_*)_i = \left(F_{(F_*)_{i+1},\cdots,(F_*)_n}\right)_* (1 \leq i \leq n).$$

We prove that by induction on $i$.

– Case 1: $i = 1$.
Since

$$F_{(F_*)_2,\cdots,(F_*)_n}((F_*)_1) = (F(F_*))_1 = (F_*)_1,$$

$(F_*)_1$ is a fixed point of $F_{(F_*)_2,\cdots,(F_*)_n}$. Since $\left(F_{(F_*)_2,\cdots,(F_*)_n}\right)_*$ is the least fixed point of $F_{(F_*)_2,\cdots,(F_*)_n}$, $\left(F_{(F_*)_2,\cdots,(F_*)_n}\right)_* \subseteq (F_*)_1$.
Since $\left(F_{(F_*)_2,\cdots,(F_*)_n}\right)_* \subseteq (F_*)_1$ and $F$ is monotone,

$$\begin{aligned} F\big((F_{(F_*)_2,\cdots,(F_*)_n})_*, (F_*)_2, \cdots, (F_*)_n\big) &\subseteq F((F_*)_1, (F_*)_2, \cdots, (F_*)_n) \\ &= F_*. \end{aligned}$$

Hence

$$\Big(F\big((F_{(F_*)_2,\cdots,(F_*)_n})_*, (F_*)_2, \cdots, (F_*)_n\big)\Big)_j \subseteq (F_*)_j \quad (2 \leq j \leq n).$$

Combining this with

$$\begin{aligned} &\Big(F\big((F_{(F_*)_2,\cdots,(F_*)_n})_*, (F_*)_2, \cdots, (F_*)_n\big)\Big)_1 \\ &= F_{(F_*)_2,\cdots,(F_*)_n}\big((F_{(F_*)_2,\cdots,(F_*)_n})_*\big) = (F_{(F_*)_2,\cdots,(F_*)_n})_*, \end{aligned}$$

we get

$$\begin{aligned} &\Big(F\big((F_{(F_*)_2,\cdots,(F_*)_n})_*, (F_*)_2, \cdots, (F_*)_n\big)\Big)_j \\ &\subseteq \Big(\big((F_{(F_*)_2,\cdots,(F_*)_n}\big)_*, (F_*)_2, \cdots, (F_*)_n\big)\Big)_j \end{aligned}$$

for any $1 \leq j \leq n$, which means that $\big((F_{(F_*)_2,\cdots,(F_*)_n})_*, (F_*)_2, \cdots, (F_*)_n\big)$ is an $F$-prefixed point.
Since $F_*$ is the least $F$-prefixed point, $F_*$ is a subset of $\big((F_{(F_*)_2,\cdots,(F_*)_n})_*, (F_*)_2, \cdots, (F_*)_n\big)$, which implies $(F_*)_1 \subseteq (F_{(F_*)_2,\cdots,(F_*)_n})_*$.

– Case 2: $i > 1$.
  By the induction hypothesis,

$$F_{(F_*)_{i+1},\cdots,(F_*)_n}((F_*)_i) = (F(F_*))_i = (F_*)_i.$$

  Therefore, $(F_*)_i$ is a fixed point of $F_{(F_*)_{i+1},\cdots,(F_*)_n}$. Since $\big(F_{(F_*)_{i+1},\cdots,(F_*)_n}\big)_*$ is the least fixed point of $F_{(F_*)_{i+1},\cdots,(F_*)_n}$, $\big(F_{(F_*)_{i+1},\cdots,(F_*)_n}\big)_* \subseteq (F_*)_i$.
  Let $F_j$ $(1 \le j \le i)$ be the ones uniquely determined by the following equations:

$$\begin{aligned} F_j &= \big(F_{F_{j+1},\cdots,F_i,(F_*)_{i+1},\cdots,(F_*)_n}\big)_*, \quad (1 \le j \le i-1) \\ F_i &= \big(F_{(F_*)_{i+1},\cdots,(F_*)_n}\big)_*. \end{aligned}$$

  By the same argument as before, we can prove

$$\begin{aligned} \big(F(F_1,\cdots,F_i,(F_*)_{i+1},\cdots,(F_*)_n)\big)_j \subseteq F_j \quad (1 \le j \le i), \\ F_j \subseteq (F_*)_j \quad (1 \le j \le i). \end{aligned}$$

  Hence

$$\begin{aligned} &F\big(F_1,\cdots,F_{i-1},(F_{(F_*)_{i+1},\cdots,(F_*)_n})_*,(F_*)_{i+1},\cdots,(F_*)_n\big) \\ &\subseteq \big(F_1,\cdots,F_{i-1},(F_{(F_*)_{i+1},\cdots,(F_*)_n})_*,(F_*)_{i+1},\cdots,(F_*)_n\big), \end{aligned}$$

  which means $\big(F_1,\cdots,F_{i-1},(F_{(F_*)_{i+1},\cdots,(F_*)_n})_*,(F_*)_{i+1},\cdots,(F_*)_n\big)$ is an $F$-prefixed point. Since $F_*$ is the least $F$-prefixed point,
  $F_* \subseteq \big(F_1,\cdots,F_{i-1},(F_{(F_*)_{i+1},\cdots,(F_*)_n})_*,(F_*)_{i+1},\cdots,(F_*)_n\big)$, which implies $(F_*)_i \subseteq \big(F_{(F_*)_{i+1},\cdots,(F_*)_n}\big)_*$. □

### 7.2 Product closure of CF($P$)

In this subsection, we show how our methods apply to the so-called 'continuous fragment' of the modal mu-calculus, where the operators corresponding to formulas are Scott continuous. (Hence, in particular, all fixed-points are reached uniformly in all models by stage omega.) This fragment was recently characterized syntactically in Fontaine [10].

**Definition 1.** *Let PROP be the set of all proposition letters, and $P$ any subset of PROP. Let $I$ be the set of all agents. We define the continuous fragments CF($P$) by induction by induction on the complexity of formulas in the modal $\mu$-calculus as follows:*

$$CF(P)\colon \phi ::= p \in P \mid \psi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \langle i \rangle \phi \mid \mu x \bullet \rho(x)$$

*where $\psi$ is any formula in the modal $\mu$-calculus without any free variable in $P$, $i$ is an agent in $I$, and $\rho(x)$ is a formula in CF($P \cup \{x\}$) and $x$ is not in $P$.*

The following is easy to check:

*Remark 6.* $\{\mathrm{CF}(P) \mid P \subseteq \mathrm{PROP}\}$ is closed under simultaneous fixed points in the following sense: Let $P$, $\{x_1, \cdots, x_n\}$ be sets of propositional letters which are disjoint. Then if $\phi_1(x_1, \cdots, x_n), \cdots, \phi_n(x_1, \cdots, x_n)$ are in $\mathrm{CF}(P \cup \{x_1, \cdots, x_n\})$, then following formula is in $\mathrm{CF}(P)$:

$$\mu \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \bullet \begin{pmatrix} \phi_1(x_1, \cdots, x_n) \\ \phi_2(x_1, \cdots, x_n) \\ \vdots \\ \phi_n(x_1, \cdots, x_n) \end{pmatrix}$$

**Proposition 1.** *For any $P \subseteq PROP$, $CF(P)$ is product closed.*

*Proof.* We prove the statement by induction on the complexity of formulas. We only consider the fixed point case, as other cases go in a standard way. We will use the same notations as in the proof of Theorem 5.

The proof is almost all the same as the case for the modal $\mu$-calculus. The difference is that $\mathrm{CF}(P)$ is not closed under fixed points. But by using the above Remark, we can deal with this problem.

By the same argument in Theorem 5, if we can express $F^{\phi(q)}$ by a formula in $\mathrm{CF}(P \cup \{x_1, \cdots, x_n\})$ for some fresh variables $x_1, \cdots, x_n$, we are done.

To prove this, we need the following Claim:

**Claim 3.** *For any set of propositional letters $Q$, the following is true:*

*Let $\phi(\boldsymbol{q})$ be a formula in $CF(Q)$ where $\boldsymbol{q}$ is a sequence of free variables (possibly not in $\phi$) with length $m$. Take free variables $x_{k,j} (1 \leq k \leq m, 1 \leq j \leq n)$ so that they do not appear in any precondition formulas in $\mathbf{A}$ or in $\boldsymbol{q}$ or in $Q$. We may assume this situation in any subformula of $\phi(\boldsymbol{q})$ by choosing fresh variables properly. Then there is a sequence of formulas $\boldsymbol{\psi}_{\phi(\boldsymbol{q})}$ in $CF(Q \cup \{x_{k,j}\})$ with length $n$ such that $F^{\phi(\boldsymbol{q})} = F^M_{\boldsymbol{\psi}_{\phi(\boldsymbol{q})}}$.*

*Proof.* In the following definitions, we only display the essential argument variables needed to understand the function values. We prove the statement by induction on the complexity of $\phi$. We identify formulas with their truth sets. Also, if $\boldsymbol{\psi}$ is a sequence of formulas, $\psi_j$ is the $j$-th coordinate of $\boldsymbol{\psi}$.

- Case 1: $\phi = p$ ($p$ is not in $\boldsymbol{q}$).

  $$F^{\phi(\boldsymbol{q})} = \left(p \wedge \mathrm{PRE}_{a_1}, \cdots, p \wedge \mathrm{PRE}_{a_n}\right).$$

  Hence $(\psi_{\phi(\boldsymbol{q})})_j = p \wedge \mathrm{PRE}_{a_j}$ and this is in $\mathrm{CF}(Q)$. Since each $x_{k,j}$ does not appear in any precondition formulas in $\mathbf{A}$, $(\psi_{\phi(\boldsymbol{q})})_j$ is also in $\mathrm{CF}(Q \cup \{x_{k,j}\})$.
- Case 2: $\phi = q_{k'}$ ($q_{k'}$ is the $k'$-th coordinate of $\boldsymbol{q}$).

  $$F^{\phi(\boldsymbol{q})}(\boldsymbol{X}) = \{X_{k',j} \wedge \mathrm{PRE}_{a_j}\}_{1 \leq j \leq n}.$$

  Hence $(\psi_{\phi(\boldsymbol{q})})_j = x_{k',j} \wedge \mathrm{PRE}_{a_j}$. By the same reasoning as in Case 1, $\mathrm{PRE}_{a_j}$ is in $\mathrm{CF}(Q \cup \{x_{k,j}\})$ and hence $x_{k',j} \wedge \mathrm{PRE}_{a_j}$ is also in $\mathrm{CF}(Q \cup \{x_{k,j}\})$.

– Case 3: $\phi = \phi_1 \vee \phi_2$.

$$F^{\phi(\boldsymbol{q})} = \boldsymbol{\psi}_{\phi_1} \vee \boldsymbol{\psi}_{\phi_2}.$$

Hence $\boldsymbol{\psi}_{\phi(\boldsymbol{q})} = \boldsymbol{\psi}_{\phi_1} \vee \boldsymbol{\psi}_{\phi_2}$ is in $\mathrm{CF}(Q \cup \{x_{k,j}\})$ by induction hypothesis.

– Case 4: $\phi = \phi_1 \wedge \phi_2$.

$$F^{\phi(\boldsymbol{q})} = \boldsymbol{\psi}_{\phi_1} \wedge \boldsymbol{\psi}_{\phi_2}.$$

Hence $\boldsymbol{\psi}_{\phi(\boldsymbol{q})} = \boldsymbol{\psi}_{\phi_1} \wedge \boldsymbol{\psi}_{\phi_2}$ is in $\mathrm{CF}(Q \cup \{x_{k,j}\})$ by induction hypothesis.

– Case 5: $\phi = \langle i \rangle \phi'$.
For any $1 \leq j \leq n$ and $x \in M$,

$$\begin{aligned} x \in \big(F^{\phi(\boldsymbol{q})}(\boldsymbol{X})\big)_j \iff& \\ &(1 \leq \exists j' \leq n)\ (\exists y \in \mathbf{M})\ \Big(xR_i y \wedge a_j R_i a_{j'} \wedge y \in \big(F^{\phi'(\boldsymbol{q})}(\boldsymbol{X})\big)_{j'}\Big). \end{aligned}$$

To see that this is true, observe that the condition $y \in \big(F^{\phi'(\boldsymbol{q})}(\boldsymbol{X})\big)_{j'}$ implies $(y, a_{j'}) \in \mathbf{M} \times \mathbf{A}$. Therefore, we can put

$$\big(\psi_{\phi(\boldsymbol{q})}\big)_j = \bigvee_{a_j R_i a_{j'}} \langle i \rangle \big(\psi_{\phi'(\boldsymbol{q})}\big)_{j'},$$

which is in $\mathrm{CF}(Q \cup \{x_{k,j}\})$.

– Case 6: $\phi = \mu q' \bullet \phi'$, where $\phi'$ is in $\mathrm{CF}(Q \cup \{q'\})$.

$$\begin{aligned} F^{\phi(\boldsymbol{q})}(\boldsymbol{X}) &= \Big\{ \big(F^{\mathbf{M}\times\mathbf{A}}_{\mu q' \bullet \phi'(q', \boldsymbol{q})}(\pi(\boldsymbol{X}))\big)_j \Big\}_{1 \leq j \leq n} \\ &= \Big\{ \big((F^{\mathbf{M}\times\mathbf{A}}_{\phi'(q', \boldsymbol{q})}(\pi(\boldsymbol{X})))_*\big)_j \Big\}_{1 \leq j \leq n} \\ &= \big(\boldsymbol{X'} \mapsto F^{\mathbf{M}}_{\boldsymbol{\psi}_{\phi'}}(\boldsymbol{X'}, \boldsymbol{X})\big)_*, \end{aligned}$$

where $(F(\cdot))_*$ is the least $F$-fixed-point. By induction hypothesis, $\boldsymbol{\psi}_{\phi'}$ are in $\mathrm{CF}(Q \cup \{x_{k,j}\} \cup \{y_j\})$ where $\{y_j\}$ corresponds to $\boldsymbol{X'}$ in the above formula and satisfies the condition required in the Claim. Then by Remark, we can put $\boldsymbol{\psi}_{\phi(\boldsymbol{q})} = \mu \boldsymbol{y_j} \bullet \boldsymbol{\psi}_{\phi'}$, that are in $\mathrm{CF}(Q \cup \{x_{k,j}\})$. □

This completes the proof of Proposition 1. □