Probabilistic Dependence Logic

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Abstract

Given a finite model $\mathcal{M}$, it is possible to associate to every sentence ϕ of Backslash Logic and Dependence Logic the value of the corresponding imperfect information game $H(\phi)$.
Hodges' compositional semantics can then be adapted to this new logic, and the value of atomic dependence formulas in the resulting framework is seen to correspond to one of Kivinen and Mannila's measures of approximate functional dependency.

Keywords: Semantic games, Imperfect information, Nash equilibria, Probabilistic logic, Dependence logic

1 Preliminaries

Logics of Imperfect Information are extensions of First-Order Logic which allow more general patterns of dependence and independence between quantified variables - for example, in Hodges' Slash Logic [6] the formula

$$\forall x_1 \exists y_1 \forall x_2 (\exists y_2 / \{x_1, y_1\}) \psi(x_1, x_2, y_1, y_2) \tag{1}$$

corresponds to the Skolem normal form

$$\exists f \exists g \forall x_1 \forall x_2 \psi(x_1, x_2, f(x_1), g(x_2))$$

that is, the values of y_1 and y_2 may only depend respectively on the values of x_1 and x_2.

The two forms of Imperfect Information Logic which will be considered in this work are Backslash Logic (called also Dependence-Friendly Logic, or DF-Logic for short: [16], [17]), and Dependence Logic ([16]).
The former uses backslashed quantifiers $(Qx_n \backslash \{x_1 \ldots x_{n-1}\})$, whose intuitive meaning is that the value of x is dependent only on the variables $x_1 \ldots x_{n-1}$;

the latter, instead, introduces *dependence atomic formulas* $=(t_1 \dots t_n)$, which hold if and only if the value of t_n is determined by the values of $t_1 \dots t_{n-1}$.

For example, the formula (1) would be written as

$$\forall x_1 \exists y_1 \forall x_2 (\exists y_2 \backslash x_2) \psi(x_1, x_2, y_1, y_2)$$

in DF-Logic, where $(\exists y_2 \backslash x_2)$ is used as a shorthand for $(\exists y_2 \backslash \{x_2\})$, and

$$\forall x_1 \exists y_1 \forall x_2 \exists y_2 (=(x_2, y_2) \wedge \psi(x_1, x_2, y_1, y_2))$$

in Dependence Logic.

These two logics can be easily translated into one another: indeed, in ([16], §3.6) it is verified that

$$(Qx_n \backslash \{x_1 \dots x_{n-1}\}) \psi(x_1 \dots x_n) \equiv Qx_n (=(x_1 \dots x_n) \wedge \psi(x_1 \dots x_n)) \quad (2)$$

for all $Q \in \{\forall, \exists\}$, and

$$=(t_1 \dots t_n) \equiv \exists y_1 \dots y_{n-1} (\exists y_n \backslash \{y_1 \dots y_{n-1}\}) (\bigwedge_{i=1}^{n} (y_i = t_i)) \quad (3)$$

The game-theoretic semantics for First-Order Logic can be easily adapted to DF-Logic: the interpretation of the first-order connectives does not change, and for the backslashed quantifiers we simply restrict the available information of the corresponding player (Player **I** for $\forall$, Player **II** for $\exists$)) as required.

Thus, for every model $\mathcal{M}$, initial assignment s and DF-Logic formula ϕ it is possible to build a game $H_s^{\mathcal{M}}(\phi)$ such that $\mathcal{M}, s \models \phi$ if and only if Player **II** (also called the Verifier) has a winning strategy for $H_s^{\mathcal{M}}(\phi)$ which is *uniform*, that is, which does not "cheat" by employing unavailable information.

Because of (2), the same also holds for formulas of Dependence Logic. The semantic games $H_s(\phi)$ could also be defined directly for formulas of Dependence Logic, as in ([16], §5.3); however, in this work dependence atomic formulas will be interpreted as shorthands for the corresponding formulas of DF-Logic.

2 Game Values

A *behavioral strategy* for Player **I** [**II**] is defined as follows:

Definition 1 (Behavioral Strategy)
A behavioral strategy β *for Player* **I** [**II**] *in the game* $H_s(\phi)$ *is a family of functions* β_i *from partial plays* $\bar{p} = (p_1 \dots p_i)$*, where each* p_i *is of the form* (ψ, s') *for some subformula instance* ψ *of* ϕ *and Player* **I** [**II**] *has to make a choice in* p_i*, to* probability distributions *of possible successors* p_{i+1}*.*

Then, a behavioral strategy is said to be *uniform* if it respects the information constraints stated in the formula:

Definition 2 (Uniform Behavioral Strategy)
A behavioral strategy γ for Player **II** *is said to be* uniform *if and only if, for every two partial plays $(p_1 \dots p_i)$ and $(p'_1 \dots p'_i)$ in which Player* **II** *follows γ, if*

- *It holds that*
$$p_i = ((\exists x \backslash V)\psi, s)$$
and
$$p'_i = ((\exists x \backslash V)\psi, s')$$
for the same instance of the subformula $(\exists x \backslash V)\psi$;

- *The assignments s and s' coincide over the set of variables V;*

then γ induces the same distribution over x in both plays, that is,

$$\gamma(p_1 \dots p_i)(\psi, s[m/x]) = \gamma(p'_1 \dots p'_i)(\psi, s'[m/x]), \text{ for all } m \in M$$

The definition of behavioral strategy for Player **I** *is analogous, except that now the condition is concerned with backslashed universal quantifiers $(\forall x \backslash V)\psi$.*

As the games $H_s(\phi)$ can be of imperfect recall, Kuhn's Theorem [10] does not hold, and not all uniform mixed strategies (that is, not all probability distributions over uniform pure strategies), correspond to an uniform behavioral strategy.

Example 1
Let $dom(\mathcal{M}) = (\{a, b\})$, let $\phi := \exists x(\exists y \backslash x)(x = y)$, and the two pure strategies τ', τ'' given by

$$\tau'_1(\phi, \emptyset) = ((\exists y \backslash \{\})(x = y), \emptyset[a/x]);$$
$$\tau'_2((\phi, \emptyset)((\exists y \backslash \{\})(x = y), s)) = (x = y, s[a/y]).$$

and

$$\tau''_1(\phi, \emptyset) = ((\exists y \backslash \{\})(x = y), \emptyset[b/x]);$$
$$\tau''_2((\phi, \emptyset)((\exists y \backslash \{\})(x = y), s)) = (x = y, s[b/y]).$$

That is, τ' is "always choose a" and τ'' is 'always choose b"; it is easy to see that both these strategies are winning for Player **II** *in the game.*

Now, let g be the mixed strategy which selects either τ' or τ'' with equal probability: $g(\tau') = g(\tau'') = 1/2$.
Then g does not correspond to any uniform behavioral strategy, as the probability distribution of y according to g is not independent from that of x.

However, every uniform behavioral strategy β induces a probability distribution β^* over uniform pure strategies. Because of this, it is possible to define the *payoff* for Player **II** of the game $H_s(\phi)$, when Player **I** uses β and Player **II** uses γ, as follows:

Definition 3 (Payoffs of Behavioral Strategies)
Let $\mathcal{M}$ be a finite model, let ϕ be any formula, let s be a variable assignment over $FV(\phi)$ and let β, γ be two behavioral strategies for the two players.

Then Player **II***'s* payoff *for this pair of strategies is*

$$P_{II}(H_s(\phi);\beta;\gamma) = \sum_{\sigma}\sum_{\tau}\beta^*(\sigma)\gamma^*(\tau)P(H_s(\phi);\sigma;\tau)$$

where $P(H_s(\phi);\sigma;\tau)$ is the payoff, for Player **II***, of the play $(\sigma;\tau)$, that is,*

$$P_{II}(H_s(\phi);\sigma;\tau) = \begin{cases} 1 & \textit{if the play } (\sigma;\tau) \textit{ is winning for Player } \mathbf{II}; \\ 0 & \textit{otherwise.} \end{cases}$$

The value for Player **I** $P_I(H_s(\phi);\beta;\gamma)$ can be defined similarly, and it turns out that

$$P(H_s(\phi);\beta;\gamma) + P_I(H_s(\phi);\beta;\gamma) = 1 \qquad (4)$$

for all β and γ.

When there is no risk of confusion, the subscript II will be omitted in $P_{II}(H(\phi);\beta;\gamma)$.

Now, the *value* of the formula ϕ (in the model M and for the assignment s) is the best expected payoff that Player **II** can guarantee, no matter what Player **I** does:

Definition 4 (Value of a formula)
Let M be a finite model, let ϕ be a formula and let s be an assignment over $FV(\phi)$.
Then

$$V_s(\phi) = \sup_{\gamma}\inf_{\beta} P(H_s(\phi);\beta;\gamma)$$

Van Benthem proposed investigating equilibria of semantic games for IF Logic along similar lines, asking if "could it be that IF games also involve an essential probabilistic feature, which we just have not been able to identify yet" ([2], [18]).
Also, the above definition of value can be seen ate generalization to behavioral strategies of the definition of value considered by Hodges in [7]; moreover, at the very end of [3] an analogous concept is suggested, and Sevenster in [14] formalizes the intuitions of [3] in the context of Branching and Generalized Quantifiers.

It is not difficult to see that it makes no difference if the infimum ranges over pure strategies, that is,

$$V_s(\phi) = \sup_{\gamma} \inf_{\sigma} P(H_s(\phi); \sigma; \gamma)$$

If $\mathcal{M}, s \models \phi$, then Player **II** has an uniform winning strategy for $H_s(\phi)$ and therefore $V_s(\phi) = 1$; analogously, if $\mathcal{M}, s \models \sim\phi$ then Player **I** has an uniform winning strategy for $H_s(\phi)$ and $V_s(\phi) = 0$.

However, undetermined formulas can take values in $(0, 1)$, as the following example demonstrates:

Example 2
Let M be a model with $dom(M) = \{a_1 \ldots a_n\}$, and let

$$\phi := \forall x(\exists y \backslash \{\})(x = y)$$

Then $V_\emptyset(\phi) = 1/n$: indeed, let $\sigma_1 \ldots \sigma_n$ be **I***'s pure strategies, corresponding to the choices "$x = a_1$" ... "$x = a_n$", and analogously let* **II***'s strategies $\tau_1 \ldots \tau_n$ correspond to "$y = a_1$" ... "$y = a_n$".*
Then

$$P(H_\emptyset(\phi); \sigma_i; \tau_j) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

Now, let γ be the strategy for Player **II** *which selects the value for y randomly in $\{a_1 \ldots a_n\}$, that is,*

$$(\gamma_2((\phi, \emptyset), ((\exists y \backslash \{\})(x = y), s)))(x = y, s[a_i/y]) = 1/n, \text{ for all } i \in 1 \ldots n$$

Then γ corresponds to the uniform distribution over the τ_i, since

$$\gamma^*(\tau_i) = 1/n, \text{ for all } i \in 1 \ldots n$$

Then, let β be any uniform behavioral strategy for Player **I***: by definition,*

$$P(H_\emptyset(\phi); \beta; \gamma) = \sum_{i,j} \beta^*(\sigma_i)\gamma^*(\tau_j)P(H_\emptyset(\phi), \sigma_i; \tau_j) = \sum_{i,j} (\beta^*(\sigma_i)\delta_{ij})/n = \sum_i \beta^*(\sigma_i)/n = 1/n$$

Therefore, $V_\emptyset(\phi) \geq 1/n$; on the other hand, if β is given by

$$(\beta_1((\phi, \emptyset)))(\exists y \backslash \{\}(x = y), \emptyset[a_i/x]) = 1/n, \text{ for } i \in 1 \ldots n.$$

then, by the same reasoning used above,

$$P(H_\emptyset(\phi); \beta; \gamma) = 1/n$$

In conclusion, the value $V_\emptyset(\phi)$ is exactly $1/n$, as stated.

3 The Minimax Theorem

The rest of this work will attempt to characterize the logic which associates to every formula ϕ the value $V_s(\phi)$.

An useful tool for this will be the *Minimax Theorem*, in the version found in [15], and adapted to the games $H_s(\psi)$ in ([5], §3.2):

Theorem 1 (The Minimax Theorem for $H_s(\psi)$)
For every game $H_s(\phi)$ in a finite model $\mathcal{M}$, there exist two behavioral strategies β^e and γ^e such that, for all β and γ,

$$P(H_s(\phi); \beta^e; \gamma) \leq P(H_s(\phi); \beta^e; \gamma^e) \leq P(H_s(\phi); \beta; \gamma^e)$$

A pair of strategies (β^e, γ^e) as above is called an equilibrium pair.

This does not contradict the results in [13], which imply that, for games of imperfect recall, the very definitions of behavioral strategy and equilibrium are somewhat problematic: indeed, the games $H_s(\phi)$ do not present *absentmindedness*, to use the terminology of [13] - that is, no information set contains two partial plays $\bar{p}$ and $\bar{p}'$ such that the former is a proper initial segment of the latter - and therefore the so-called "Paradox of the Absentminded Driver" has no analogue in this class of games.

The Minimax Theorem has a well-known, useful corollary:

Corollary 1
For every game $H_s(\psi)$ in a finite model $\mathcal{M}$,

$$\forall\gamma\exists\sigma P_I(H_s(\psi); \sigma; \gamma) \geq r \Leftrightarrow \exists\beta\forall\tau P_I(H_s(\psi); \beta; \tau) \geq r$$

and

$$\forall\beta\exists\tau P_{II}(H_s(\psi); \beta; \tau) \geq r \Leftrightarrow \exists\gamma\forall\sigma P_{II}(H_s(\psi); \sigma; \gamma) \geq r$$

Another consequence of the Minimax Theorem is that, if $V_s^{\mathbf{I}}(\phi)$ is the value for Player **I** of the game $H_s(\phi)$,

$$\begin{aligned} V_s^{\mathbf{I}}(\phi) &= \sup_\beta \inf_\gamma (1 - P_{II}(H_s(\phi); \beta; \gamma)) = 1 - \inf_\beta \sup_\gamma P_{II}(H_s(\phi); \beta; \gamma) = \\ &= 1 - \inf_\beta \sup_\tau P_{II}(H_s(\phi); \beta; \tau) = 1 - \sup_\gamma \inf_\sigma P_{II}(H_s(\phi); \sigma; \gamma) = \\ &= 1 - \sup_\gamma \inf_\beta P_{II}(H_s(\phi); \beta; \gamma) = 1 - V_s(\phi) \end{aligned}$$

Then, for the game-theoretic negation $\sim$ and for all formulas ϕ,

$$V_s(\sim\phi) = V_s^{\mathbf{I}}(\phi) = 1 - V_s(\phi) \tag{5}$$

It is worth observing that if the model $\mathcal{M}$ is not finite, the Minimax Theorem does not hold:

Example 3
Let $\mathcal{M} = (\mathbb{N}, <)$ *and*

$$\phi := \forall x(\exists y\backslash\{\})(y > x)$$

Then $V_\emptyset(\phi) = V_\emptyset(\sim\phi) = 0$*, contradicting the corollary of the Minimax Theorem.*

This can be verified as follows:

- *Let* γ *be any behavioral strategy for Player* **II** *in* $H_\emptyset(\phi)$*.*
  As γ *must choose the value of* y *independently from the value of* x*, it induces a probability distribution over* $\mathbb{N}$*: for any assignment* s *with* $dom(s) = \{x\}$*,*

  $$Prob(y = m) = \gamma_2((\phi, \emptyset), (\exists y\backslash\{\}(y > x), s))(y > x, s[m/x])$$

  Then, for any $r > 0$ *there exists a* $n_r \in \mathbb{N}$ *such that* $Prob(y > r) \leq r$*; thus, if* σ^r *is the pure strategy for Player* **I** *which chooses* n_r *for* x*, that is,*

  $$\sigma_1^r((\phi, \emptyset)) = (\exists y\backslash\{\}(y > x), \emptyset[n_r/x])$$

  it turns out that

  $$P_{II}(H_\emptyset(\phi); \sigma^r; \gamma) = \sum_{n > n_0} Prob(y = n) = Prob(y > n_r) \leq r$$

  Hence,

  $$\inf_\sigma P_{II}(H_\emptyset(\phi); \sigma; \gamma) = 0$$

  and, as this holds for all γ*,*

  $$V_\emptyset(\phi) = \sup_\gamma \inf_\sigma P_{II}(H_\emptyset(\phi); \sigma; \gamma) = 0$$

- *Let* β *be any behavioral strategy for Player* **I** *in* $H_\emptyset(\phi)$*: as above, it induces a probability distribution of* x *over* $\mathbb{N}$*, and for all* $r > 0$ *it is possible to find a* m_r *such that*

  $$Prob(x > m_r) \leq r$$

  Thus, if τ^r *is the pure strategy for Player* **II** *which chooses* m_r *for* y*,*

  $$P_{II}(H_\emptyset(\phi); \beta; \tau^r) \leq r$$

  And, since such a τ^r *can be found for all* β *and for all* $r > 0$*,*

  $$V_\emptyset(\sim\phi) = 0$$

4 Probabilistic teams and trumps

In [6], Hodges described a semantics for a game of imperfect information in terms of sets of assignments, which, following [16], will be called *teams*.
In particular, the first-order semantics notion of *assignment satisfying a formula* was substituted in Hodges' semantics with the notion of *team satisfying a formula*, called *trump*.
In the light of [7] and [16], it seems clear that this shift from assignments to sets of assignments is caused by the fact that logics of imperfect information can express statements about *functional dependencies*, and such a concept cannot be meaningfully applied to single assignments.

Some variations of the definitions found in [6] and in the subsequent work will now be presented.

Definition 5 (Probabilistic Team)
A probabilistic team μ *with domain* $dom(\mu) = \{x_1 \dots x_n\}$ *is a probability function over the set of all assignments on* $\{x_1 \dots x_n\}$*, that is, a function*

$$\mu : \{s : dom(s) = \{x_1 \dots x_n\}\} \to [0, 1]$$

such that

$$\sum_{dom(s)=dom(\mu)} \mu(s) = 1$$

Then, the game $H_\mu^{\mathcal{M}}$ is defined as follows:

Definition 6 ($H_\mu^{\mathcal{M}}(\phi)$)
Let ϕ *be a formula,* $\mathcal{M}$ *a model, and let* μ *be a probabilistic team.*
The game $H_\mu^{\mathcal{M}}(\phi)$ *is then played as follows:*

1. *First, an assignment* s *is selected randomly, according to the distribution* μ*;*
2. *Then, the game* $H_s^{\mathcal{M}}(\phi)$ *is played.*

The definitions of strategy, uniform strategy, behavioral strategy and uniform behavioral strategy are as usual; however, this time a play *will be determined by a triple* (s, σ, τ)*, where* s *is the initial assignment (chosen according to* μ*) and* σ*,* τ *are pure strategies.*
Thus,

$$P(H_\mu(\phi); \beta; \gamma) = \sum_{dom(s)=dom(\mu)} \mu(s) P(H_s(\phi); \beta; \gamma)$$

It can be easily verified that

$$P(H_s(\phi); \beta; \gamma) = P(H_{\eta_s}; \beta; \gamma)$$

where η_s is the probabilistic team which chooses s with certainty, that is,

$$\eta_s(s') = \begin{cases} 1 & \text{if } s' = s; \\ 0 & \text{otherwise.} \end{cases}$$

A *trump*, in the original Hodges semantics, is a team X such that Player **II** had an uniform winning strategy for the game $H_X(\phi)$, in which a $s \in X$ is extracted and then $H_s(\phi)$ is played; analogously,

Definition 7 (r-trumps and $\mathcal{T}$)
*A probabilistic team μ is a r-*trump *of a formula ϕ if and only if*

$$\exists\gamma\forall\sigma P(H_\mu(\phi);\sigma;\gamma) \geq r$$

where, as usual, it makes no difference whether Player **I** *can use behavioral strategies β or if he is limited to pure strategies σ.*

Then $\mathcal{T}$ is defined as

$$\mathcal{T} = \{(\phi,\mu,r) : \mu \textit{ is a } r\textit{-trump of } \phi\}$$

The following operations will be useful to characterize $\mathcal{T}$:

Definition 8 (Linear combination)
If μ_1, μ_2, and μ' are probabilistic teams with $dom(\mu_1) = dom(\mu_2) = dom(\mu')$ and $p \in [0,1]$, it holds that

$$\mu' = p\mu_1 + (1-p)\mu_2$$

if and only if

$$\mu'(s) = p\mu_1(s) + (1-p)\mu_2(s), \textit{ for all } s \textit{ with } dom(s) = dom(\mu')$$

It is easy to verify that μ' is still a team:

$$\sum_s \mu'(s) = p\sum_s \mu_1(s) + (1-p)\sum_s \mu_2(s) = p + (1-p) = 1$$

Definition 9 (Supplementation)
If μ is a probabilistic team, F is a function from $\{s : dom(s) = dom(\mu)\}$ to probability distributions over M, that is, a mapping

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

where

$$\mathcal{D}(M) = \{f : M \to [0,1], \sum_{m\in M} f(m) = 1\}$$

and $y \notin dom(\mu)$, then $\mu[F/y]$ is defined as the probabilistic team such that

$$\mu[F/y](s[m/y]) = \mu(s) \cdot F(s)(m)$$

for all s such that $dom(s) = dom(\mu)$ and for all $m \in M$.

It can be verified that $\mu[F/y]$ is a team over $dom(\mu) \cup \{y\}$:

$$\sum_{dom(s')=dom(\mu)\cup\{y\}} \mu[F/y](s') =$$
$$= \sum_{dom(s)=dom(\mu)} \sum_{m \in M} \mu[F/y](s[m/y]) =$$
$$= \sum_{dom(s)=dom(\mu)} \sum_{m \in M} \mu(s) \cdot F(s)(m) =$$
$$= \sum_{dom(s)=dom(\mu)} \mu(s) \sum_{m \in M} F(s)(m) =$$
$$= \sum_{dom(s)=dom(\mu)} \mu(s) = 1$$

5 A compositional semantics

It is now possible to compositionally characterize the set $\mathcal{T}$:

Theorem 2
If $\mathcal{M}$ is a finite *model and ϕ is a formula, the following results hold*[1]*:*

1. *If ϕ is a literal, then $(\phi, \mu, r) \in \mathcal{T}$ if and only if*
$$\sum_{s \models_{FO} \phi} \mu(s) \geq r$$
where the expression $s \models_{FO} \phi$ means that, in the current model, the first-order formula ϕ is satisfied by the assignment s.

2. *$(\psi \vee \theta, \mu, r) \in \mathcal{T}$ if and only if μ can be written as a linear combination of probabilistic teams*
$$\mu = p\xi_1 + (1-p)\xi_2$$
such that, for some r_1 and r_2, the following conditions hold:
$$(\psi, \xi_1, r_1) \in \mathcal{T};$$
$$(\theta, \xi_2, r_2) \in \mathcal{T};$$
$$pr_1 + (1-p)r_2 \geq r$$

3. *$(\psi \wedge \theta, \mu, r) \in \mathcal{T}$ if and only if for all ξ_1, ξ_2, p such that*
$$\mu = p\xi_1 + (1-p)\xi_2$$
there exist r_1, r_2 such that
$$(\psi, \xi_1, r_1), (\theta, \xi_2, r_2) \in \mathcal{T}$$
and
$$pr_1 + (1-p)r_2 \geq r$$

[1] A. Mann has informed us that he has proved a similar result.

4. $(\exists x\psi, \mu, r) \in \mathcal{T}$ *if and only if there exists a*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

such that

$$(\psi, \mu[F/x], r) \in \mathcal{T}$$

5. $(\exists x\backslash\{x_1, \ldots, x_k\}\psi, \mu, r) \in \mathcal{T}$ *if and only if the conditions for the above case hold, and moreover*

$$s(x_1) = s'(x_1), \ldots, s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')$$

for any two s, s' *with* $dom(s) = dom(s') = dom(\mu)$.

6. $(\forall x\psi, \mu, r) \in \mathcal{T}$ *if and only if for all*

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

it holds that

$$(\psi, \mu[F/x], r) \in \mathcal{T}$$

7. $(\forall x\backslash\{x_1 \ldots x_k)\psi, \mu, r) \in \mathcal{T}$ *if and only if the conditions of the previous case hold for all* F *such that*

$$s(x_1) = s'(x_1), \ldots, s(x_k) = s'(x_k) \Rightarrow F(s) = F(s')$$

for every two s, s' *with the same domain of* μ.

8. $(\sim\phi, \mu, r) \in \mathcal{T}$ *if and only if*

$$(\phi, \mu, r') \in \mathcal{T} \Rightarrow r' \leq 1 - r$$

for all $r \in [0, 1]$.

Proof:

1. If ϕ is a literal, there are no strategies available except the trivial ones, and therefore

$$(\phi, \mu, r) \in \mathcal{T} \text{ iff } P(H_\mu(\phi); \emptyset; \emptyset) \geq r \text{ iff } \sum_{s \models_{FO} \phi} \mu(s) \geq r$$

2. Suppose that $(\psi \vee \theta, \mu, r) \in \mathcal{T}$: then there exists an uniform behavioral strategy γ such that, for all σ,

$$P(H_\mu(\psi \vee \theta); \sigma; \gamma) \geq r$$

Now, for every assignment s, let λ_s be the probability, according to γ, that Player **II** chooses the left disjunct ψ when the initial assignment s is extracted - that is,

$$\lambda_s = (\gamma_1(\psi \vee \theta, s))(\psi, s)$$

Then, the total probability that the left disjunct is selected is

$$p = \sum_s \mu(s)\lambda_s$$

As a consequence, the conditional probability distribution

$$Prob(s \text{ is selected in } H_\mu(\psi \vee \theta) \mid \text{the next position is } (\psi, s))$$

is given by

$$\xi_1(s) = \frac{\mu(s)\lambda_s}{p} = \frac{\mu(s)\lambda_s}{\sum_s \mu(s)\lambda_s}$$

And, analogously,

$$Prob(s \text{ is selected in } H_\mu(\psi \vee \theta) \mid \text{the next position is } (\theta, s))$$

is

$$\xi_2(s) = \frac{\mu(s)(1-\lambda_s)}{1-p} = \frac{\mu(s)(1-\lambda_s)}{\sum_s \mu(s)(1-\lambda_s)}$$

Clearly,

$$\mu = p\xi_1 + (1-p)\xi_2$$

Moreover, let γ^L and γ^R be two behavioral strategies for $H(\psi)$ and $H(\theta)$ such that

$$\gamma_i^L((\psi, s)\dots p_i) = \gamma_{i+1}((\psi \vee \theta, s)(\psi, s)\dots p_i);$$
$$\gamma_i^R((\theta, s)\dots p_i) = \gamma_{i+1}((\psi \vee \theta, s)(\theta, s)\dots p_i).$$

Analogously, for each pure strategy σ for Player **I** let us define σ^L and σ^R such that

$$\sigma_i^L((\psi, s)\dots p_i) = \sigma_{i+1}((\psi \vee \theta, s)(\psi, s)\dots p_i);$$
$$\sigma_i^R((\theta, s)\dots p_i) = \sigma_{i+1}((\psi \vee \theta, s)(\theta, s)\dots p_i).$$

Then

$$P(H_\mu(\psi \vee \theta); \sigma; \gamma) = \sum_s \mu(s) P(H_s(\psi \vee \theta); \sigma; \gamma) =$$
$$= \sum_s \mu(s)\lambda_s P(H_s(\psi); \sigma^L; \gamma^L) + \sum_s \mu(s)(1-\lambda_s) P(H_s(\theta); \sigma^R; \gamma^R) =$$
$$= p \sum_s \xi_1(s) P(H_s(\psi); \sigma^L; \gamma^L) + (1-p) \sum_s \xi_2(s) P(H_s(\theta); \sigma^R; \gamma^R) =$$
$$= pP(H_{\xi_1}(\psi); \sigma^L; \gamma^L) + (1-p) P(H_{\xi_2}(\theta); \sigma^R; \gamma^R)$$

Now, by hypothesis $P(H_\mu(\psi \vee \theta); \sigma; \gamma) \geq r$; therefore, there exist r_1 and r_2 such that

$$\begin{aligned}
&P(H_{\xi_1}(\psi); \sigma; \gamma^L) \geq r_1 \text{ for all } \sigma;\\
&P(H_{\xi_2}(\theta); \sigma; \gamma^R) \geq r_2 \text{ for all } \sigma;\\
&pr_1 + (1-p)r_2 \geq r.
\end{aligned}$$

as required.

Conversely, suppose that

$$\mu = p\xi_1 + (1-p)\xi_2$$

with

$$\begin{aligned}
&(\psi, \xi_1, r_1) \in \mathcal{T};\\
&(\theta, \xi_2, r_2) \in \mathcal{T};\\
&pr_1 + (1-p)r_2 \geq r
\end{aligned}$$

Then, by the definition of $\mathcal{T}$, there exist behavioral strategies γ^L, γ^R such that

$$\begin{aligned}
&P(H_{\xi_1}(\psi); \sigma^L; \gamma^L) \geq r_1 \text{ for all } \sigma^L;\\
&P(H_{\xi_2}(\theta); \sigma^R; \gamma^R) \geq r_2 \text{ for all } \sigma^R;
\end{aligned}$$

Then consider the following behavioral strategy γ for Player **II** in $H_\mu(\psi \vee \theta)$: if the assignment s is selected, choose the left disjunct ψ with probability

$$\lambda_s = \frac{p\xi_1(s)}{\mu(s)}$$

that is, let

$$(\gamma_1(\psi \vee \theta, s))(\psi, s) = \lambda_s$$

Then, for the successive moves, let

$$\begin{aligned}
&\gamma_{i+1}((\psi \vee \theta, s)(\psi, s), \ldots) = \gamma_i^L((\psi, s), \ldots);\\
&\gamma_{i+1}((\psi \vee \theta, s)(\theta, s), \ldots) = \gamma_i^R((\theta, s), \ldots).
\end{aligned}$$

Then, for all strategies σ,

$$\begin{aligned}
&P(H_\mu(\psi \vee \theta); \sigma; \gamma) = \sum_s \mu(s) P(H_s(\psi \vee \theta); \sigma; \gamma) =\\
&= \sum_s \mu(s)\lambda_s P(H_s(\psi); \sigma^L; \gamma^L) + \sum_s \mu(s)(1-\lambda_s) P(H_s(\theta); \sigma^R; \gamma^R) =\\
&= p\sum_s \xi_1(s) P(H_s(\psi); \sigma^L; \gamma^L) + (1-p)\sum_s \xi_2(s) P(H_s(\theta); \sigma^R; \gamma^R) =\\
&= pP(H_{\xi_1}(\psi); \sigma^L; \gamma^L) + (1-p)P(H_s(\theta); \sigma^R; \gamma^R) \geq pr_1 + (1-p)r_2 \geq r
\end{aligned}$$

where σ^L, σ^R are defined as above and the fact that

$$1-\lambda_s = \frac{\mu(s) - p\xi_1(s)}{\mu(s)} = \frac{(1-p)\xi_2(s)}{\mu(s)}$$

has been used.

3. Suppose that $(\psi \wedge \theta, \mu, r) \in \mathcal{T}$: then, there exists a behavioral strategy γ for Player **II** such that, no matter which behavioral strategy β Player **I** uses to select ϕ or ψ, the payoff $P(H_\mu(\psi \wedge \theta); \beta; \gamma)$ is greater or equal to r.

   Now, suppose that $\mu = p\xi_1 + (1-p)\xi_2$, and let β^L, β^R be any two behavioral strategies for Player **I** for the games $H(\psi)$ and $H(\theta)$; then, let β be defined as

$$\begin{aligned}
&(\beta_1(\psi \wedge \theta, s))(\psi, s) = p\xi_1(s)/\mu(s);\\
&(\beta_1(\psi \wedge \theta, s))(\theta, s) = (1-p)\xi_2(s)/\mu(s);\\
&\beta_{i+1}((\psi \wedge \theta, s)(\psi, s), \ldots) = \beta_i^L((\psi, s), \ldots);\\
&\beta_{i+1}((\psi \wedge \theta, s)(\theta, s), \ldots) = \beta_i^R((\theta, s), \ldots);
\end{aligned}$$

   Then, for γ^L and γ^R defined as in the previous case,

$$\begin{aligned}
&r \geq P(H_\mu(\psi \wedge \theta; \beta; \gamma) = \sum_s \mu(s) P(H_s(\psi \vee \theta; \beta; \gamma) =\\
&= p \sum_s \xi_1(s) P(H_s(\psi); \beta^L; \gamma^L) + (1-p) \sum_s \xi_2(s) P(H_s(\theta); \beta^R; \gamma^R) =\\
&= pP(H_{\xi_1}(\psi); \beta^L; \gamma^L) + (1-p) P(H_{\xi_2}(\theta); \beta^R; \gamma^R)
\end{aligned}$$

   and, since this holds for every β^L and β^R, there exist r_1, r_2 such that

$$\begin{aligned}
&(\psi, \xi_1, r_1) \in \mathcal{T};\\
&(\theta, \xi_2, r_2) \in \mathcal{T};\\
&pr_1 + (1-p)r_2 \geq r
\end{aligned}$$

   as required.

   Conversely, suppose that whenever

$$\mu = p\xi_1 + (1-p)\xi_2$$

   there exist r_1, r_2 such that

$$\begin{aligned}
&(\psi, \xi_1, r_1) \in \mathcal{T};\\
&(\theta, \xi_2, r_2) \in \mathcal{T};\\
&pr_1 + (1-p)r_2 \geq r
\end{aligned}$$

Then, let β be any behavioral strategy for Player **I**, and, as usual, let

$$\lambda_s = (\beta_1(\psi \wedge \theta, s))(\psi, s)$$

and let ξ_1 and ξ_2 be the conditional assignment distributions when Player **I** chooses ψ or θ, that is,

$$\xi_1(s_i) = \frac{\mu(s_i)\lambda_i}{\sum_i \mu(s_i)\lambda_i}$$

and

$$\xi_2(s_i) = \frac{\mu(s_i)(1-\lambda_i)}{\sum_i \mu(s_i)(1-\lambda_i)}$$

Then, for $p = \sum_i \mu(s_i)\lambda_i$ it holds that

$$\mu = p\xi_1 + (1-p)\xi_2$$

Now, by hypothesis, there exist r_1, r_2 such that $(\psi, \xi_1, r_1), (\theta, \xi_2, r_2) \in \mathcal{T}$ and $pr_1 + (1-p)r_2 \geq r$, and thus it is possible to find two behavioral strategies γ^L and γ^R for Player **II** such that

$$P(H_{\xi_1}(\psi); \beta'; \gamma^L) \geq r_1, \text{ for all } \beta';$$
$$P(H_{\xi_2}(\theta); \beta''; \gamma^R) \geq r_2, \text{ for all } \beta''.$$

Now, let the strategy γ for Player **II** in $H(\psi \wedge \theta)$ be defined as

$$\gamma_{i+1}((\psi \wedge \theta, s), (\psi, s), \ldots) = \gamma_i^L((\psi, s), \ldots);$$
$$\gamma_{i+1}((\psi \wedge \theta, s), (\theta, s), \ldots) = \gamma_i^R((\theta, s), \ldots).$$

Then

$$P(H_\mu(\psi \wedge \theta); \beta; \gamma) = \sum_s \mu(s) P(H_s(\psi \wedge \theta); \beta; \gamma) =$$
$$= p \sum_s \xi_1(s) P(H_s(\psi); \beta^L; \gamma^L) + (1-p) \sum_s \xi_2(s) P(H_s(\theta); \beta^R; \gamma^R) \geq$$
$$\geq pr_1 + (1-p)r_2 \geq r$$

Thus,

$$\forall \beta \exists \gamma P(H_\mu(\psi \wedge \theta); \beta; \gamma) \geq r$$

But then, by the minimax theorem and its corollary,

$$\exists \gamma \forall \beta P(H_\mu(\psi \wedge \theta); \beta; \gamma) \geq r$$

and, in conclusion, $(\psi \wedge \theta, \mu, r) \in \mathcal{T}$.

4. Suppose that $(\exists x\psi, \mu, r) \in \mathcal{T}$: then, there is a behavioral strategy γ such that, for all σ,

$$P(H_\mu(\exists x\psi); \sigma; \gamma) \geq r$$

Then, for all assignments s, let $F(s)$ be defined as

$$F(s)(m) = (\gamma_1(\exists x\psi, s))(\psi, s[m/x]), \text{ for all } m \in \mathbb{M}$$

and, moreover, let

$$\gamma'_i((\psi, s[m/x]) \ldots) = \gamma_{i+1}((\exists x\psi, s)(\psi, s[m/x]) \ldots)$$

Now, let σ' be any strategy for Player **I** in $H_{\mu[F/x]}(\psi)$, and let σ be a strategy for $H_\mu(\psi)$ such that

$$(F(s))(m) > 0 \Rightarrow \sigma((\exists x\psi, s), (\psi, s[m/x]), \ldots) = \sigma'((\psi, s[m/x]), \ldots)$$

Then,

$$\begin{aligned} r \leq P(H_\mu(\exists x\psi); \sigma; \gamma) &= \sum_s \mu(s) P(H_s(\exists x\psi); \sigma; \gamma) = \\ &= \sum_s \mu(s) \sum_m (F(s))(m) P(H_{s[m/x]}(\psi); \sigma'; \gamma') = \\ &= P(H_{\mu[F/x]}(\psi); \sigma'; \gamma') \end{aligned}$$

Since this holds for all σ', one can conclude that

$$(\psi, \mu[F/x], r) \in \mathcal{T}$$

as required.

Conversely, suppose that there exists a behavioral strategy γ' such that

$$P(H_{\mu[F/x]}(\psi); \sigma'; \gamma') \geq r \text{ for all } \sigma'$$

Then, let the strategy γ for $H_\mu(\exists x\psi)$ be as follows:

$$\begin{aligned} &(\gamma_1(\exists x\psi, s))(\psi, s[m/x]) = F(s)(m); \\ &\gamma_{i+1}((\exists x\psi, s), (\psi, s[m/x]), \ldots) = \gamma'_i((\psi, s[m/x]), \ldots) \end{aligned}$$

Now, let σ be any strategy for Player **I** in $H_\mu(\exists x\psi)$, and let σ' be such that

$$\sigma'_i((\psi, s[m/x]) \ldots) = \sigma_{i+1}((\exists x\psi, s)(\psi, s[m/x]), \ldots)$$

Then,

$$\begin{aligned} P(H_\mu(\exists x\psi); \sigma; \gamma) &= \sum_s \mu(s) P(H_s(\exists x\psi); \sigma; \gamma) = \\ &= \sum_s \sum_m \mu(s)(F(s)(m)) P(H_{s[m/x]}(\psi); \sigma'; \gamma') = \\ &= P(H_{\mu[F/x]}(\psi); \sigma'; \gamma') \geq r \end{aligned}$$

as required.

5. Suppose that $(\exists x \backslash V \psi, \mu, r) \in \mathcal{T}$, where V is a set of variables, and let γ be the corresponding uniform behavioral strategy for Player **II** in $H_\mu(\exists x \backslash V \psi)$.
Then, as in the previous case, let F be such that

$$(F(s))(m) = (\gamma_1(\exists x \backslash V \psi, s))(\psi, s[m/x])$$

Then it is possible to verify, using exactly the same argument of the previous case, that

$$(\psi, \mu[F/x], r) \in \mathcal{T};$$

Moreover, since γ is uniform

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

as required.

Conversely, suppose that there exists a γ' such that, for all σ',

$$P(H_{\mu[F/x]}(\psi), \sigma'; \gamma') \geq r$$

where F is such that

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

Then, as in the previous case, let the behavioral strategy γ for $H_\mu(\exists x \backslash V \psi)$ be defined as follows:

$$(\gamma_1(\exists x \backslash V \psi, s))(\psi, s[m/x]) = F(s)(m);$$
$$\gamma_{i+1}((\exists x \backslash V \psi, s), (\psi, s[m/x]), \ldots) = \gamma'_i((\psi, s[m/x]), \ldots)$$

For the same argument used for the non-backslashed existential quantifier. Then,

$$P(H_\mu(\exists x \backslash V \psi); \sigma; \gamma) \geq r$$

and it only remains to verify that γ is uniform.

Indeed, let $(p_1 \ldots p_i)$ and $(p'_1 \ldots p'_i)$ be two partial plays of $H_\mu(\exists x \backslash V \psi)$ in which Player **II** follows γ, p_i and p'_i are of the form $(\exists z \backslash V' \theta, s)$ and $(\exists z \backslash V' \theta, s)$ for the same instance of this subformula, and

$$s(x_i) = s'(x_i) \text{ for all } x \in V'$$

Then

$$\gamma(p_1 \ldots p_i) = \gamma(p'_1 \ldots p'_i)$$

Indeed,

- If $i = 1$, the current subformula is $\exists x \backslash V \psi$, and

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V$$

  Then $F(s) = F(s')$, and therefore

$$(\gamma_1(\exists x \backslash V \psi, s))(\psi, s[m/x]) = (\gamma_1(\exists x \backslash V \psi, s'))(\psi, s[m/x])$$

  for all $m \in M$, as required.

- If $i > 1$ and $(p_1 \dots p_i)$, $(p'_1 \dots p'_i)$ are as above, then $(p_2 \dots p_i)$ and $(p'_2 \dots p'_i)$ are plays of $H_{\mu[F/x]}(\psi)$ in which Player **II** follows γ', and since γ' is uniform the desired result holds.

6. Suppose that there exists a behavioral strategy γ for Player **II** such that, for all behavioral strategies β for Player **I**,

$$P(H_\mu(\forall x \psi); \beta; \gamma) \geq r$$

Now, let F be any function

$$F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$$

and let β' be any strategy of Player **I** for $H_{\mu[F/x]}(\psi)$.

Then, let the strategy β for $H_\mu(\forall x \psi)$ be defined as

$$(\beta_1(\forall x \psi, s))(\psi, s[m/x]) = (F(s))(m);$$
$$\beta_{i+1}((\forall x \psi, s)(\psi, s[m/x]), \dots) = \beta'_i((\psi, s[m/x]) \dots).$$

By hypothesis,

$$P(H_\mu(\forall x \psi); \beta; \gamma) \geq r;$$

Therefore, if one defines the strategy γ' for Player **II** in $H_{\mu[F/x]}(\psi)$ as

$$\gamma'_i((\psi, s[m/x]) \dots) = \gamma_{i+1}((\forall x \psi, s)(\psi, s[m/x]) \dots)$$

then

$$r \leq P(H_\mu(\forall x \psi); \beta; \gamma) = \sum_s \mu(s) P(H_s(\forall x \psi); \beta; \gamma) =$$
$$= \sum_s \mu(s) \sum_m (F(s))(m) P(H_{s[m/x]}(\psi); \beta'; \gamma') = P(H_{\mu[F/x]}(\psi); \beta'; \gamma')$$

and therefore $(\psi, \mu[F/x], r) \in \mathcal{T}$, as required.

Conversely, suppose that for all $F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$ as above there exists a strategy γ^F such that

$$P(H_{\mu[F/x]}(\psi); \beta^F; \gamma^F) \geq r$$

for all behavioral strategies β^F of Player **I**.

Then, let β be any behavioral strategy of Player **I** in $H_\mu(\forall x\psi)$, and let F be defined by

$$(F(s))(m) = (\beta_1(\forall x\psi, s))(\psi, s[m/x])$$

Moreover, let β^F be the strategy such that

$$\beta_i^F((\psi, s[m/x])\ldots) = \beta_{i+1}((\forall x\psi, s)(\psi, s[m/x])\ldots)$$

And let γ be defined by

$$\gamma_{i+1}((\forall x\psi, s)(\psi, s[m/x])\ldots) = \gamma_i^F((\psi, s[m/x])\ldots)$$

Then,

$$P(H_\mu(\forall x\psi); \beta; \gamma) = \sum_s \mu(s) P(H_s(\forall x\psi); \beta; \gamma) =$$

$$= \sum_s \mu(s) \sum_m (F(s))(m) P(H_{s[m/x]}(\psi); \beta^F; \gamma^F) = P(H_{\mu[F/x]}(\psi); \beta^F; \gamma^F) \geq r$$

In conclusion,

$$\forall\beta\exists\gamma P(H_\mu(\forall x\psi); \beta; \gamma) \geq r$$

and, by the corollary of the Minimax Theorem, this implies that $(\forall x\psi; \mu; r) \in \mathcal{T}$, as required.

7. Let γ be such that, for all uniform behavioral strategies β for Player **I**,

$$P(H_\mu(\forall x \backslash V\psi); \beta; \gamma) \geq r$$

and let $F : \{s : dom(s) = dom(\mu)\} \to \mathcal{D}(M)$ be such that

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s')$$

Then, for every uniform behavioral strategy β' for Player **I** in $H_{\mu[F/x]}(\psi)$, let β be defined as

$$(\beta_1(\forall x \backslash V\psi, s))(\psi, s[m/x]) = (F(s))(m);$$
$$\beta_{i+1}((\forall x \backslash V\psi, s)(\psi, s[m/x])\ldots) = \beta'((\psi, s[m/x])\ldots)$$

This β is uniform, since β' is uniform and since

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow F(s) = F(s') \Rightarrow \beta_1(\forall x \backslash V\psi, s) = \beta_1(\forall x \backslash V\psi, s')$$

Therefore, $P(H_\mu(\forall x \backslash V\psi); \beta; \gamma) \geq r$; but then, the γ' defined by

$$\gamma_i'((\psi, s[m/x])\ldots) = \gamma_{i+1}((\forall x \backslash V\psi, s)(\psi, s[m/x])\ldots)$$

is such that

$$\forall \beta', P(H_{\mu[F/x]}(\psi); \beta'; \gamma') \geq r$$

as required.

Conversely, suppose that for all F which satisfy the dependence condition there exists a γ^F such that

$$\forall \beta', P(H_{\mu[F/x]}(\psi); \beta'; \gamma') \geq r$$

Then, let β be any uniform behavioural strategy for $H_\mu(\forall x \backslash V \psi)$, and as usual let F be given by

$$(F(s))(m) = (\beta_1(\forall x \backslash V \psi, s))(\psi, s[m/x])$$

Since β must be uniform,

$$s(x_i) = s'(x_i) \text{ for all } x_i \in V \Rightarrow \beta_1(\forall x \backslash V \psi, s) = \beta_1(\forall x \backslash V \psi, s') \Rightarrow F(s) = F(s')$$

Therefore, F satisfies the dependence requirement, and if β^F is the restriction of β to the subgame $H(\psi)$, as in the case of the non-backslashed universal quantifier,

$$P(H_{\mu[F/x]}(\psi); \beta^F; \gamma^F) \geq r$$

But then, for the γ defined by

$$\gamma_{i+1}((\forall x \backslash V \psi, s)(\psi, s[m/x]) \ldots) = \gamma_i^F((\psi, s[m/x]) \ldots)$$

it holds that

$$P(H_\mu(\forall x \backslash V \psi); \beta; \gamma) \geq r$$

Thus,

$$\forall \beta \exists \gamma P(H_\mu(\forall x \backslash V \psi); \beta; \gamma) \geq r$$

and therefore, by the Minimax Theorem,

$$\exists \gamma \forall \beta P(H_\mu(\forall x \backslash V \psi); \beta; \gamma) \geq r$$

as required.

8. By the Minimax Theorem's corollary,

$$\begin{aligned}
&(\sim\phi, \mu, r) \in \mathcal{T} \text{ iff} \\
&\text{iff } \exists \gamma \forall \sigma P_{II}(H_\mu(\sim\phi); \sigma; \gamma) \geq r, \text{ iff} \\
&\text{iff } \exists \beta \forall \tau P_I(H_\mu(\phi); \beta; \tau) \geq r, \text{ iff} \\
&\text{iff } \exists \beta \forall \tau P_{II}(H_\mu(\phi); \beta; \tau) \leq 1 - r, \text{ iff} \\
&\text{iff } \forall \gamma \exists \sigma P_{II}(H_\mu(\phi); \sigma; \gamma) \leq 1 - r, \text{ iff} \\
&\text{iff } \neg \exists \gamma \forall \beta P_{II}(H_\mu(\phi); \sigma; \gamma) > 1 - r, \text{ iff} \\
&\text{iff } ((\phi, \mu, r') \in \mathcal{T} \Rightarrow r' \leq 1 - r)
\end{aligned}$$

as required.

□

Given the definition of *value*, it is easy to see that

$$V_\mu(\phi) = \sup\{r : (\phi, \mu, r) \in \mathcal{T}\}$$

Thus, by the about results about $\mathcal{T}$, V satisfies the following properties:

Corollary 2

1. *If ϕ is a literal,*
$$V_\mu(\phi) = \sum_{s \models_{FO} \phi} \mu(s);$$

2. *If $\phi = \psi \vee \theta$,*
$$V_\mu(\psi \vee \theta) = \sup\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\};$$

3. *If $\phi = \psi \wedge \theta$,*
$$V_\mu(\psi \wedge \theta) = \inf\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\};$$

4. *If $\phi = \exists x\psi$,*
$$V_\mu(\exists x\psi) = \sup_F V_{\mu[F/x]}(\psi);$$

5. *If $\phi = \exists x \backslash \{x_1 \ldots x_k\}\psi$,*
$$V_\mu(\exists x \backslash \{x_1 \ldots x_k\}\psi) = \sup\{V_{\mu[F/x]}(\psi) : F \textit{ depends only on } x_1, \ldots x_k\};$$

6. *If $\phi = \forall x\psi$,*
$$V_\mu(\forall x\psi) = \inf_F V_{\mu[F/x]}(\psi);$$

7. *If $\phi = \forall x \backslash \{x_1 \ldots x_k\}\psi$,*
$$V_\mu(\forall x \backslash \{x_1 \ldots x_k\}\psi) = \inf\{V_{\mu[F/x]}(\psi) : F \textit{ depends only on } x_1, \ldots x_k\}.$$

8. $V_\mu(\sim\phi) = 1 - V_\mu(\phi)$.

Proof:

1. Obvious.

2. We have that

$$\begin{aligned}
&\sup\{r : (\psi \vee \theta, \mu, r) \in \mathcal{T}\} = \\
&= \sup\{pr_1 + (1-p)r_2 : \mu = p\xi_1 + (1-p)\xi_2, (\psi, \xi_1, r_1) \in \mathcal{T}, (\theta, \xi_2, r_2) \in \mathcal{T}\} = \\
&= \sup\{pr_1 + (1-p)r_2 : \mu = p\xi_1 + (1-p)\xi_2, r_1 < V_{\xi_1}(\psi), r_2 < V_{\xi_2}(\theta)\} = \\
&= \sup\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\}
\end{aligned}$$

3. Similarly to the previous case,

$$
\begin{aligned}
&\sup\{r : (\psi \wedge \theta, \mu, r) \in \mathcal{T}\} = \\
&= \sup\{r : \mu = p\xi_1 + (1-p)\xi_2 \Rightarrow \exists r_1 r_2 \text{ s.t. } (\psi, \xi_1, r_1) \in \mathcal{T}, \\
&\quad (\theta, \xi_2, r_2) \in \mathcal{T}, r \leq pr_1 + (1-p)r_2\} = \\
&= \sup\{r : \mu = p\xi_1 + (1-p)\xi_2 \Rightarrow r < pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta)\} = \\
&= \sup\{r : r < \inf\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\}\} = \\
&= \inf\{pV_{\xi_1}(r_1) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2) = \mu\}
\end{aligned}
$$

4. For the existential quantifier,

$$
\begin{aligned}
&\sup\{r : (\exists x\psi, \mu, r) \in \mathcal{T}\} = \\
&= \sup\{r : \exists F \text{ s.t. } (\psi, \mu[F/x], r) \in \mathcal{T}\} = \\
&= \sup\{r : \exists F \text{ s.t. } r < V_{\mu[F/x]}(\psi)\} = \\
&= \sup_F V_{\mu[F/x]}(\psi)
\end{aligned}
$$

5. The case for the backslashed existential quantifier is similar to that for the non-backslashed one, except that now F must satisfy a dependence condition.

6. For the universal quantifier,

$$
\begin{aligned}
&\sup\{r : (\forall x\psi, \mu, r) \in \mathcal{T}\} = \\
&= \sup\{r : \forall F, (\psi, \mu[F/x], r) \in \mathcal{T}\} = \\
&= \sup\{r : \forall F, r < V_{\mu[F/x]}(\psi)\} = \\
&= \inf_F V_{\mu[F/x]}(\psi)
\end{aligned}
$$

7. The case for the backslashed universal quantifier is exactly as that for the non-backslashed one, except that now F must satisfy a dependence condition.

8. Already proved.

□

These results provide a compositional semantics for Probabilistic Dependence Logic.

6 The range of the value function

Any finite model $\mathcal{M}$ induces now a function $\phi \mapsto V(\phi)$.
In this section, it will be attempted to obtain some results about the range of this mapping.
The following theorem was proved independently in Sevenster and Sandu (to appear).

Theorem 3 *If the domain of $\mathcal{M}$ is finite and contains at least two elements,*

$$\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} = \mathbb{Q} \cap [0,1]$$

Proof:

- $\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} \supseteq \mathbb{Q} \cap [0,1]$:
  Let $r = p/q$, where $p < q$, and let $s = \lceil \log_2(q) \rceil$.
  Then, let ϕ be the following sentence:

$$\begin{aligned}
&\phi \equiv \exists x_0 \exists x_1 ((x_0 \neq x_1) \wedge \\
&\quad \wedge (\exists y_{1,1} \exists y_{1,2} \ldots \exists y_{1,s})(\exists y_{2,1} \exists y_{2,2} \ldots \exists y_{2,s}) \ldots (\exists y_{q,1} \exists y_{q,2} \ldots \exists y_{q,s}) \\
&\quad \left(\bigwedge_{i=1}^{q} \bigwedge_{k=1}^{s} y_{i,k} = x_0 \vee y_{i,k} = x_1 \right) \wedge \left(\bigwedge_{i=1}^{q} \bigwedge_{j=i+1}^{q} \bigvee_{k=1}^{s} y_{i,k} \neq y_{j,k} \right) \wedge \\
&\quad \forall z_1 \forall z_2 \ldots \forall z_s \left(\bigwedge_{i=1}^{q} \bigvee_{k=1}^{s} (z_k \neq y_{i,k}) \vee \right. \\
&\quad \vee (\exists w_{1,1}/\{z_1 \ldots z_s\}) \ldots (\exists w_{1,s}/\{z_1 \ldots z_s\}) \\
&\quad (\exists w_{2,1}/\{z_1 \ldots z_s\}) \ldots (\exists w_{2,s}/\{z_1 \ldots z_s\}) \\
&\quad \ldots \\
&\quad (\exists w_{p,1}/\{z_1 \ldots z_s\}) \ldots (\exists w_{p,s}/\{z_1 \ldots z_s\}) \\
&\quad \left. \left(\bigvee_{i=1}^{p} \bigvee_{j=1}^{q} \bigwedge_{k=1}^{s} w_{i,k} = z_{j,k} \right) \right) \Bigg)
\end{aligned}$$

  where $\exists w/\{z_1 \ldots z_s\}$ is the *slashed quantifier* of IF-logic, which requires the choice of w to be *independent* from the choice of $z_1 \ldots z_s$, and can be easily translated in terms of the usual backslashed quantifier.

  Then $V(\phi) = p/q$: indeed, the game $H(\phi)$ can be described as follows:

  1. Player **II** selects two distinct elements $x_0, x_1 \in M$;
  2. Player **II** selects q distinct strings $y_1, \ldots y_q$ in $\{x_0, x_1\}^s$;
  3. Player **I** selects a string $z \in \{y_1, \ldots y_q\}$;
  4. Player **II** selects p strings $w_1 \ldots w_p$, without knowing z, and wins if and only if $w_i = z$ for some $i = 1 \ldots p$.

  Now, let γ be the following strategy for Player **II**:

  1. Select two fixed distinct elements x_0 and x_1.
  2. Select q fixed distinct strings $y_1, \ldots, y_q \in \{x_0, x_1\}^s$;

3. Extract p strings $w_1, \dots w_p$ from $\{y_1 \dots y_q\}$, with uniform probability and without repetition - that is, w_1 can be each y_i with probability $1/q$, w_2 can be each remaining element with probability $1/(q-1)$, and so on.

Now, consider any strategy σ for Player **I**: by definition, σ selects an element $z \in \{y_1 \dots y_q\}$, and Player **II** wins if it is one of $\{w_1 \dots w_p\}$.
When **II** uses the behavioral strategy γ,

$$\begin{aligned}
&P(H(\phi);\sigma;\gamma) = Prob(w_i = z \text{ for some } i) = \\
&= Prob(w_1 = z) + Prob(w_1 \neq z \;\&\; w_2 = z) + \ldots + \\
&+ Prob(w_1 \neq z \;\&\; w_2 \neq z \;\&\; \ldots \;\&\; w_{p-1} \neq z \;\&\; w_p = z) = \\
&= 1/q + (q-1)/q \cdot 1/(q-1) + \ldots + (q-1)/q \cdot (q-2)/(q-1) \cdot \ldots \\
&\ldots \cdot 1/(q-p+1) = p/q
\end{aligned}$$

Since this holds for any σ,

$$V(\phi) = \sup_\gamma \inf_\sigma P(H(\phi);\sigma;\gamma) \geq p/q$$

On the other hand, consider the behavioral strategy β for Player **I** which selects the value of z among $y_1 \dots y_q$ with uniform probability, and let τ be any pure strategy for Player **II**.
Then, τ fixes, independently from **I**'s choice of z, some values of $w_1 \dots w_p \in \{y_1 \dots y_q\}$, and

$$P(H(\phi);\beta;\tau) = Prob(z \in \{w_1 \dots w_p\}) = \sum_{i=1}^{p} Prob(z = w_i) = p/q$$

Thus,

$$\exists\beta\forall\tau P(H(\phi);\beta;\tau) \leq p/q$$

and therefore, by the Minimax Theorem,

$$\forall\gamma\exists\sigma P(H(\phi);\sigma;\beta) \leq p/q$$

But then $V(\phi) \leq r$ and, in conclusion,

$$V(\phi) = p/q$$

- $\{r \in \mathbb{R} : V(\phi) = r \text{ for some } \phi\} \subseteq \mathbb{Q} \cap [0,1]$:
Since the model is finite, in $H^{\mathcal{M}}(\phi)$ there exists a finite set $\{\sigma_1, \sigma_2, \dots \sigma_k\}$ of all pure strategies for Player **I**, and a finite set $\{\tau_1, \tau_2, \dots \tau_t\}$ of all pure strategies for Player **II**.

Now, let us consider all uniform behavioral strategies γ for Player **II**, or, more precisely, the corresponding distributions of pure strategies γ^*.

It has already been seen that not all distributions derive from a behavioral strategy: more precisely, if γ is required to be uniform then, for all partial plays $(p_1 \dots p_i)$ and $(p'_1 \dots p'_i)$ with

$$p_i = (\exists x \backslash V \psi, s), p'_i = (\exists x \backslash V \psi, s') \text{ for the same instance of } \exists x \backslash V \psi;$$
$$s(x_i) = s'(x_i) \text{ for all } x_i \in V$$

it must hold that, for all $m \in M$,

$$\gamma_i(p_1 \dots p_i)(\psi, s[m/x]) = \sum \{\gamma^*(\tau_i) : \tau_i(p_1 \dots p_i) = (\psi, s[m/x])\} =$$
$$= \sum \{\gamma^*(\tau_i) : \tau_i(p'_1 \dots p'_i) = (\psi, s'[m/x])\} = \gamma_i(p'_1 \dots p'_i[m/x])$$

Since the model is finite, there exist only finitely many possible partial plays $(p_1 \dots p_i)$ and $(p'_1 \dots p'_i)$ as above, and therefore the requirement that a vector

$$\bar{\gamma}^* = \begin{pmatrix} \gamma^*(\tau_1) \\ \gamma^*(\tau_2) \\ \dots \\ \gamma^*(\tau_t) \end{pmatrix}$$

corresponds to an uniform behavioral strategy γ can be expressed by a linear equation

$$A\bar{\gamma}^* = c$$

for a suitable matrix A and for a vector c *with rational coefficients.*

Then the value $V(\phi)$ is the result of the following linear programming problem:

maximize v, with respect to the variables $(v, \lambda_1, \dots \lambda_t)$,

and under the conditions
$$\begin{cases} \sum_{i=1}^{t} \lambda_1 = 1; & \\ \sum_{i=1}^{t} \lambda_i P(H(\phi); \sigma_j; \tau_i) \geq v, & \text{for all } j = 1 \dots k; \\ A(\lambda_1, \dots \lambda_t)^T = c; & \\ \lambda_i \geq 0, & \text{for all } i = 1 \dots t. \end{cases}$$

where the tuple $(\lambda_1, \dots \lambda_t)$ represents the probability distribution over pure strategies induced by a uniform behavioral strategy γ.

In other words, the problem of calculating $V(\phi)$ is equivalent to the problem of finding the maximum of the linear function z in a $t+1$-dimensional polytope described by the above linear inequalities and equalities with rational coefficients.
It is then clear that the maximum is always reached at one of the vertices of the polytope[2]; but since the linear inequalities have rational coefficients, the coordinates of these vertices are also rational, and thus the value of

[2] This is also the basis of the *simplex method* for solving linear optimization problems.

our target function z at this point will also be rational.

Moreover, the value function always assumes values between 0 and 1, and this concludes the proof.

This for finite models; instead, ([5], §3.5) shows a sentence ϕ such that, for every $r \in \mathbb{R} \cap [0,1]$ there exists an infinite model $\mathcal{M}_r$ in which $V_\mu(\phi) = r$. The idea is to build ϕ so that the game $H(\phi)$ is as follows:

1. Player **II** chooses two points a and b on the unit circumference such that the arc $\widehat{ab}$ (obtained starting from a, and moving clockwise until b is reached) is long exactly $r/2\pi$;
2. Player **I** chooses a point c on the circumference, without knowing a and b;
3. Player **II** wins the game if and only if c lies in the arc $\widehat{ab}$; otherwise, Player **I** wins.

It is not difficult to write ϕ and $\mathcal{M}_r$ explicitly, and to verify that the value of ϕ in $\mathcal{M}_r$ is precisely r.
However, there are difficulties in extending the approach described in this work to infinite models: since the Minimax Theorem does not hold, some clauses of Theorem 2 fail, and moreover the very notion of game value loses much of its interest, as von Neumann himself observes in [12].
□

7 The values of first-order formulas

Let ϕ be a first-order formula. What can one say, in general, about $V_\mu(\phi)$?
The next theorem shows that, in this case, the value of ϕ is the relative size of the biggest subteam of μ which satisfies ϕ:

Theorem 4

Let ϕ be a first-order formula with $FV(\phi) = \{x_1 \dots x_n\}$, let $\mathcal{M}$ be a finite model and let μ be a probabilistic team with $dom(\mu) = FV(\phi)$.
Then

$$V_\mu^{\mathcal{M}}(\phi) = \sum_{s \models_{FO} \phi} \mu(s)$$

that is, the value of ϕ is the probability, under the distribution μ, that a random assignment satisfies classically ϕ.

Proof:
The proof is by structural induction on ϕ:

- ϕ is a literal:
  In this case, the result has already been proved.

- $\phi = \psi \wedge \theta$: In this case,

$$V_\mu(\phi) = \inf\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\} =$$
$$= \inf\left\{p \sum_{s \models_{FO} \psi} \xi_1(s) + (1-p) \sum_{s \models_{FO} \theta} \xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu\right\}$$

For every assignment s, let λ_s be the fraction of the weight $\mu(s)$ which is assigned to ξ_1, that is,

$$\lambda_s = \frac{p\xi_1(s)}{\mu(s)}$$

Then, it is easy to verify that

$$p = \sum_s \mu(s)\lambda_s$$

and that

$$\xi_1(s) = \frac{\lambda_s \mu(s)}{p};$$
$$\xi_2(s) = \frac{(1-\lambda_s)\mu(s)}{1-p}.$$

Then, every decomposition of μ in $p\xi_1 + (1-p)\xi_2$ is determined by the values of the λ_s; and moreover, every family of values $\lambda_s \in [0,1]$ corresponds to an unique linear decomposition.

Thus,

$$V_\mu(\psi \wedge \theta) = \inf\left\{\sum_{s \models_{FO} \psi} p\xi_1(s) + \sum_{s \models_{FO} \theta} (1-p)\xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu\right\} =$$
$$= \inf\left\{\sum_{s \models_{FO} \psi} \lambda_s \mu(s) + \sum_{s \models_{FO} \theta} (1-\lambda_s)\mu(s) : \lambda_s \in [0,1] \text{ for all } s\right\}$$

The infimum is then obtained by letting $\lambda_s = 1$ for all s such that $s \not\models_{FO} \psi$ and $\lambda_s = 0$ for all s such that $s \models_{FO} \psi$ but $s \not\models_{FO} \theta$; the choice of λ_s for the remaining s does not make any difference, and

$$V_\mu(\psi \wedge \theta) = \sum_{s \models_{FO} \psi \wedge \theta} \mu(s_i)$$

as required.

- $\phi = \psi \vee \theta$: The proof is very similar to that for the conjunction: the supremum

$$\sup\{pV_{\xi_1}(\psi) + (1-p)V_{\xi_2}(\theta) : p\xi_1 + (1-p)\xi_2 = \mu\} =$$

$$= \sup\left\{ p \sum_{s \models_{FO} \psi} \xi_1(s) + (1-p) \sum_{s \models_{FO} \theta} \xi_2(s) : p\xi_1 + (1-p)\xi_2 = \mu \right\} =$$

$$= \sup\left\{ \sum_{s \models_{FO} \psi} \lambda_s \mu(s) + \sum_{s \models_{FO} \theta} (1-\lambda_s)\mu(s) : \lambda_s \in [0,1] \text{ for all } s \right\}$$

is reached by letting $\lambda_s = 1$ for all s such that $s \models_{FO} \psi$, and $\lambda_s = 0$ and all s such that $s \models_{FO} \theta$; as a consequence,

$$V_\mu(\psi \vee \theta) = \sum_{s \models_{FO} \psi \vee \theta} \mu(s)$$

- $\phi = \forall x \psi$: By definition,

$$V_\mu(\forall x \psi) = \inf_F V_{\mu[F/x]}(\psi) = \inf_F \sum_{s[m/x] \models_{FO} \psi} \mu(s) \cdot (F(s))(m)$$

The infimum can be reached as follows: given an assignment s, if there exists a $c \in M$ such that $s[c/x] \not\models_{FO} \psi$, let F satisfy

$$F(s)(m) = \begin{cases} 1 & \text{if } m = c; \\ 0 & \text{otherwise.} \end{cases}$$

If instead $s[c/x]$ satisfies ψ for all c, the choice of the distribution $F(s)$ has no importance, since $\sum_{c \in M} \mu(s) \cdot F(s)(m) = \mu(s)$. In conclusion,

$$V_\mu(\forall x \psi) = \sum_{s \models_{FO} \forall x \psi} \mu(s)$$

as required.

- $\phi = \exists x \psi$: The proof is as for the universal quantifier: we have that

$$V_\mu(\exists x \psi) = \sup_F V_{\mu[F/x]}(\psi) = \sup_F \sum_{s[m/x] \models_{FO} \psi} \mu(s) \cdot F(s)(m)$$

The supremum is reached as follows: for every s, if there exists a $c \in M$ such that $s[c/x] \models_{FO} \psi$ then let

$$F(s)(m) = \begin{cases} 1 & \text{if } m = c; \\ 0 & \text{otherwise.} \end{cases}$$

If this is not the case, the choice of $F(s_i)$ is again of no consequence, and

$$V_\mu(\exists x \psi) = \sum_{s \models_{FO} \exists x \psi} \mu(s)$$

- $\phi = \sim\psi$: As the law of the excluded middle holds in first-order logic,

$$V_\mu(\sim \psi) = 1 - V_\mu(\psi) = 1 - \sum_{s \models_{FO} \psi} \mu(s) = \sum_{s \models_{FO} \neg\psi} \mu(s)$$

  This concludes the proof.

□

8 The value of dependence atomic formulas

In this section, dependence atomic formulas $=(t_1 \dots t_n)$, meaning "The value of t_n is determined by the values of $t_1 \dots t_{n-1}$", will be taken in exam.
These formulas, as said before, can be defined as

$$=(t_1 \dots t_n) := \exists y_1 \dots y_{n-1} (\exists y_n \backslash \{t_1 \dots t_n\}) \bigwedge_{i=1}^{n} (y_i = t_i)$$

Then, the next theorem shows that the value of dependence formulas is the relative size of the biggest subteam of the probabilistic team μ which satisfies the dependency relation:

Theorem 5

$$V_\mu(=(t_1 \dots t_n)) = \sup_{\{B_i\}} \sum_{s \in B_i} \mu(s)$$

where the B_i are the maximal sets of assignments which satisfy the dependence condition, that is,

$$s, s' \in B_i, t_i\langle s \rangle = t_i\langle s' \rangle \text{ for } i = 1 \dots n-1 \Rightarrow t_n\langle s \rangle = t_n\langle s' \rangle$$

Proof:
By the Minimax Theorem,

$$V_\mu(=(t_1 \dots t_n)) = \sup_\gamma \inf_\sigma P(H_\mu(=(t_1 \dots t_n)); \sigma; \gamma) = \inf_\beta \sup_\tau P(H_\mu(=(t_1 \dots t_n)); \beta; \tau) =$$

$$= \inf_\beta \sup_\tau P(H_\mu(\exists y_1 \dots \exists y_{n-1} (\exists y_n \backslash \{y_1 \dots y_{n-1}\}) \bigwedge_{i=1}^{n} y_i = t_i); \beta; \tau)$$

An optimal pure strategy τ for Player **II** in this game fixes a function $f : dom(\mathcal{M})^{n-1} \to dom(\mathcal{M})$: if the assignment s is extracted, it will choose the elements $s\langle t_1 \rangle \dots s\langle t_{n-1} \rangle$ for $y_1 \dots y_{n-1}$, and then it will compute the value of y_n by applying the function f to these values.

The optimal behavioral strategy β^e for Player **I** is also easy to find out, as the only choice of Player **I** in this game is which one of the conjuncts to verify:

if for some $i \in 1 \dots n$ it holds that $y_i \neq t_i$, Player **I** can select the corresponding conjunct and win, and otherwise Player **II** wins.
Thus,

$$V_\mu(=(t_1 \dots t_n)) = \sup_f \sum_{s \in B(f)} \mu(s)$$

where

$$B(f) = \{s : f(s\langle t_1\rangle, \dots s\langle t_{n-1}\rangle) = s\langle t_n\rangle\}$$

In order to prove the desired result, it then suffices to verify that for every choice of f the set $B(f)$ is contained in one of the maximal sets B_i and that each B_i can be written as $B(f)$ for some f.

This is trivial: as every $B(f)$ satisfies the dependency condition, it is contained in some B_i. Moreover, since every B_i satisfies the dependency condition, it is possible to define a function f_i as

$$f_i(a_1 \dots a_{n-1}) = \begin{cases} s\langle t_n\rangle & \text{if } \exists s \in B_i \text{ s.t. } s\langle t_i\rangle = a_i, i = 1 \dots n-1; \\ \text{some fixed } a_0 & \text{otherwise.} \end{cases}$$

By definition, $B_i \subseteq B(f_i)$; but B_i is maximal, and therefore $B_i = B(f_i)$, as required.
□

9 Approximate Functional Dependency in Database Theory

The concept of functional dependency is also one of the main tools of Database Theory [4], and its definition corresponds exactly to Väänänen's interpretation of the dependence atomic formulas:

Definition 10
Given a relation $r \subseteq A_1 \times \dots \times A_k$, and two attribute sets $X, Y \subseteq \{A_1, \dots A_k\}$, it is said that Y is functionally dependent from X if and only if, for all the tuples $u, v \in r$,

$$\pi_i(u) = \pi_i(v) \ \forall A_i \in X \Rightarrow \pi_j(u) = \pi_j(v) \ \forall A_j \in Y$$

where $\pi_i(u)$ is the i-th element of the tuple u.

In this case, one can write that

$$r \models_{DT} X \rightarrow Y$$

This dependency relation satisfies *Armstrong's Axioms*:

Axiom of reflexivity: If $X \supseteq Y$, then, for all r, $r \models_{DT} X \rightarrow Y$;

Axiom of augmentation: If $r \models_{DT} X \to Y$, then, for all Z, $r \models_{DT} X \cup Z \to Y \cup Z$;

Axiom of transitivity: If $r \models_{DT} X \to Y$ and $r \models_{DT} Y \to X$ then $r \models_{DT} X \to Z$.

which are also known to be complete, in the sense that, given a set $\mathcal{F}$ of dependency conditions, the condition $X \to Y$ is derivable from $\mathcal{F}$ if and only if every relation which satisfies all dependencies in $\mathcal{F}$ satisfies $X \to Y$.

Some measures of *Approximate Functional Dependency* have been introduced, one of the most commonly used ones being the g_3 measure of Kivinen and Mannila ([9], [11], [8]):

Definition 11 (g_3 measure)
Let $X \to Y$ be a functional dependency, and let r be a relation over the attribute set R.
Then $G_3(X \to Y, r)$ is the minimum number of tuples that one must remove from r in order to obtain a relation s satisfying $X \to Y$, that is,

$$G_3(X \to Y, r) = |r| - \max\{|r'| : r' \subseteq r, r' \models_{DT} X \to Y\}$$

Then, the g_3 measure is defined as

$$g_3(X \to Y, r) = G_3(X \to Y, r)/|r|$$

This definition is quite similar to the semantics of our dependence operator. This intuition is formalized by the next theorem:

Theorem 6 *Let r be a relation over $A_1 \times \ldots \times A_n$, and let μ be the corresponding probabilistic team over $\{x_1 \ldots x_n\}$ that is,*

$$\mu(s) = \begin{cases} 1/|r| & \text{if } \langle s(x_1), \ldots, s(x_n)\rangle \in r; \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all functional dependencies of the form

$$\{A_{i_1} \ldots A_{i_{q-1}}\} \to \{A_q\}$$

it holds that

$$g_3(\{A_{i_1} \ldots A_{i_{q-1}}\} \to \{A_q\}, r) = 1 - V_\mu(=(x_{i_1}, \ldots x_{i_q}))$$

Proof:
As it is known,

$$V_\mu(=(x_{i_1} \ldots x_{i_q})) = \max_{B_j} \sum_{s \in B_j} \mu(s)$$

where $\{B_1, B_2, \ldots B_k\}$ are all maximal sets of assignments which satisfy the dependency condition $=(x_{i_1}, \ldots x_{i_q})$.

Therefore,

$$V_\mu(=(x_{i_1}\dots x_{i_q})) = \max_{B_j}\sum_{s\in B_j}\mu(s) =$$
$$= \max_{B_j}\sum\{1/|r| : s \in B_j, \langle s(x_1),\dots s(x_n)\rangle \in r\} =$$
$$= 1/|r| \max_{B_j} |\{s \in B_j, \langle s(x_1),\dots s(x_n)\rangle \in r\}| =$$
$$= 1/|r| \max\{|r'| : r' \subseteq r, s \models_{DT} \{A_1\dots A_{q-1}\} \to \{A_q\}\}$$

where the last equivalence follows from the fact that every subset of r satisfying the dependence condition corresponds to a subset of some B_j.

In conclusion,

$$V_\mu(=(x_{i_1}\dots x_{i_q})) = 1 - g_3(\{A_{i_1}\dots A_{i_{q-1}}\} \to \{A_{i_q}\}, r)$$

as required.
□

10 Further work

a) Dynamic Dependence Logic

In this work, the atomic dependence formulas $=(t_1\dots t_n)$ have been interpreted as shorthands for the corresponding DF-Logic formulas.
Because of this, it is obvious that

$$V_\mu(=(t_1\dots t_n)) = V_\mu(\exists y_1\dots y_{n-1}(\exists y_n\backslash\{y_1\dots y_{n-1}\})\bigwedge_i(y_i = t_i))$$

in all finite models $\mathcal{M}$ and for all probabilistic teams μ.

However, although in the non-probabilistic framework it is true that

$$(\exists x_n\backslash\{x_1\dots x_{n-1}\})\psi \equiv \exists x_n(=(x_1\dots x_n)\wedge\psi)$$

this equivalence, in general, does not carry over to probabilistic dependence logic - for example, in [5] it is shown that, for some team μ,

$$V_\mu((\exists z\backslash\{\})(=(y)\wedge x = z)) < V_\mu(\exists z(=(z)\wedge =(y)\wedge x = z))$$

The problem lies in the fact that the value of the conjunction of two dependence atomic formulas is not, in general, the measure of the biggest subteam of μ satisfying both of them, since the interpretation of $\psi\wedge\theta$ is "Player **I** decides whether to verify ψ or θ" rather than "Player **I** verifies ψ; if it turns out to be true, he verifies θ too".

Introducing this new kind of conjunction, which semantically seems to correspond to the sequential conjunction of [1], would not increase the expressive power of the logic, but would allow us to recover the above equivalence and, more importantly, to express complex patterns of dependence and independence in terms of conjunctions atomic dependence formulas.

The last part of [5] sketches (although with a minor mistake, for which a solution has already been found) how to adapt the machinery described in this article to the resulting "Dynamic Probabilistic Dependence Logic", and further investigation on this matter is underway.

b) (Probabilistic) Dependence Logic and Database Theory

The link between Dependence Logic and Database Theory runs certainly deeper than what was hinted to in this article.

For example, one of the problems in database theory and data mining is the search, given a relation r, of a minimal set of dependence relations which entail all functional dependencies occurring in r - in effect, Kivinen and Mannila introduced their measures of approximate functional dependency as a tool for searching efficiently such minimal sets [9].

This and similar questions could, in the author's opinion, benefit from a thorough investigation on the model theory of Dependence Logic and Probabilistic Dependence Logic.

c) Infinite models

One of the main drawbacks of the analysis described in this work is that it only holds for finite models.
As the Minimax Theorem seems to be an essential requisite of our adaptation of Hodges' semantics, there is little hope to transfer these results to general infinite models; however, it may be worthwhile to attempt to define "well-behaved" classes of infinite models, possibly by considering limits of directed chains of finite models.

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