Eventually Different Functions and Inaccessible Cardinals

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1 Introduction

Statements about regularity properties at the second level of the projective hierarchy such as "Every $\mathbf{\Delta}^1_2$ set of reals has property P" or "Every $\mathbf{\Sigma}^1_2$ set of reals has property P" are complicated enough not to be ZFC theorems (typically, they fail to hold in $\mathbf{L}$) and thus it is interesting to investigate their relative logical strength. The strongest such statement is "$\forall x(\omega_1^{\mathbf{L}[x]} < \omega_1)$" (or "$\omega_1$ is inaccessible by reals") which typically implies all of the above mentioned properties and the weakest nontrivial such statement is "$\forall x(\omega^\omega \backslash \mathbf{L}[x] \neq \varnothing)$" (which by [BL99, Theorem 7.1] is equivalent to "every $\mathbf{\Sigma}^1_2$ set of reals is Sacks-measurable").

Most of the regularity properties investigated in this context are derived from forcing notions, and the computation of relative logical strength has been done for many such properties, e.g., in [Sol70, JS89, BL99, BHL05]. In this paper, we continue this work by looking at the Baire property in the eventually different topology (cf. § 2) and the statements $\mathbf{\Sigma}^1_2(\mathbb{E})$ "every $\mathbf{\Sigma}^1_2$ set of reals has the Baire property in the eventually different topology" and $\mathbf{\Delta}^1_2(\mathbb{E})$ "every $\mathbf{\Delta}^1_2$ set of reals has the Baire property in the eventually different topology". Based on preliminaries on definability of ideals and

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forcing absoluteness (§ 3), we prove in § 4 that $\mathbf{\Sigma}^1_2(\mathbb{E})$ is equivalent to "ω_1 is inaccessible by reals" (Theorem 7). This result was not unexpected, as the present authors showed the same for the Baire property in the dominating topology in [BL99, Theorem 5.11], based on a combinatorial property of the Hechler ideal. In our proof here, we use the analogue of that property for eventually different forcing (Theorem 2). In § 5, we then move on to $\mathbf{\Delta}^1_2(\mathbb{E})$ and show that it fails in the ω_1-stage finite support iteration of Hechler forcing (Theorem 18). With this ingredient, we are then (§ 6) able to place $\mathbf{\Delta}^1_2(\mathbb{E})$ in the diagram of regularity statements on the second level of the projective hierarchy and prove all implications and non-implications.

The technical result leading to Theorem 18 (cf. Corollary 13 and Theorem 17) says that eventually different reals in the (iterated) Hechler extension are necessarily dominating reals. This may be of independent interest.

2 Eventually different forcing

The conditions of **eventually different forcing**, denoted by $\mathbb{E}$, are pairs $\langle s, F\rangle$, where $s \in \omega^{<\omega}$ is a finite sequence of natural numbers and F is a finite set of reals. We say that $\langle s, F\rangle \leq \langle t, G\rangle$ if and only if $t \subseteq s$, $G \subseteq F$ and for all $i \in \mathrm{dom}(s\backslash t)$ and all $g \in G$, we have that $s(i) \neq g(i)$. In [Łab96], Łabędzki discusses all basic properties of this forcing partial order, and we refer to this paper for details. Eventually different forcing is a c.c.c. and even σ-centered forcing that generates a topology $\mathcal{E}$ refining the standard topology on Baire space. Basic open sets of $\mathcal{E}$ are of the form $[s, F] = \{x \in \omega^\omega ; s \subseteq x \text{ and } \forall f \in F\ \forall n \geq |s|\ (x(n) \neq f(n))\}$ where $\langle s, F\rangle \in \mathbb{E}$. These sets are in fact closed in the standard topology and thus $\mathcal{E}$-clopen. Hence $\mathcal{E}$-open dense sets are F_σ, and $\mathcal{E}$-closed nowhere dense sets G_δ, in the standard topology. Therefore the $\mathcal{E}$-meager sets form an ideal $\mathcal{I}_\mathbb{E}$ which has a basis of $\mathbf{\Sigma}^0_3$ sets in the standard topology (cf. also [Łab96, Theorem 3.1]).

We fix some coding of the Borel sets by real numbers, and write B^M_c for the Borel set coded by c as interpreted in the model M and $\mathrm{B}_c := \mathrm{B}^V_c$. Note that the statement "c is a code for a Borel set in $\mathcal{I}_\mathbb{E}$" is absolute between models of set theory, allowing us to call a Borel code c $\mathcal{E}$**-meager** if B^M_c is meager in any model M of set theory. For a given model of set theory M, we write $\mathsf{EvD}(M)$ for the set of reals $\mathbb{E}$-generic over M. Since $\mathbb{E}$ is a c.c.c. forcing notion, we have the usual connection between $\mathcal{E}$-meager Borel codes and the notion of $\mathbb{E}$-genericity:

Lemma 1 ([Łab96, Theorem 3.3])**.** If M is a model of set theory and $x \in \omega^\omega$, then x is $\mathbb{E}$-generic over M if and only if for all $\mathcal{E}$-meager Borel codes $c \in M$, we have that $x \notin \mathrm{B}_c$.

Let $\langle f_\alpha : \alpha < \omega_1\rangle$ be a family of pairwise eventually different functions.

Let $E_\alpha := \{x \in \omega^\omega\,;\, \exists^\infty k \in \omega(x(k) = f_\alpha(k))\}$. Note that these sets are $\mathcal{E}$-nowhere dense.

Theorem 2 (Brendle)**.** If G is $\mathcal{E}$-meager and $\langle f_\alpha\,;\, \alpha < 2^\omega\rangle$ is a family of pairwise eventually different functions then the set $\{\alpha\,;\, E_\alpha \subseteq G\}$ is countable.

Proof. [Łab96, Theorem 4.7] Q.E.D.

This theorem is the main ingredient of the proof that the additivity of the meager ideal in $\mathcal{E}$ is $\aleph_1$. Its combinatorics is based on the construction in [Bre95, Theorem 2.1]

3 Preliminaries

We shall work in the general framework introduced by Ikegami [Ike09] which we briefly review: We call a forcing notion $\mathbb{P}$ **arboreal** if its conditions are (isomorphic to) a set of perfect trees ordered by inclusion and we call it **strongly arboreal** if for any $T \in \mathbb{P}$ and any $t \in T$, we have that $\{s \in T\,;\, s \subseteq t \vee t \subseteq s\} \in \mathbb{P}$. For arboreal forcings, we say that a set $X \subseteq \omega^\omega$ is **$\mathbb{P}$-null** if for any $T \in \mathbb{P}$ there is some $S \in \mathbb{P}$ such that $S \leq T$ and $[S] \cap X = \varnothing$. We let $\mathcal{I}_\mathbb{P}$ be the σ-ideal generated by the $\mathbb{P}$-null sets. Using $\mathcal{I}_\mathbb{P}$, we call a set X **$\mathbb{P}$-measurable** if for any $T \in \mathbb{P}$ there is an $S \in \mathbb{P}$ with $S \leq T$ such that either $[S] \cap X \in \mathcal{I}_\mathbb{P}$ or $[S]\backslash X \in \mathcal{I}_\mathbb{P}$. We define an ideal $\mathcal{I}^*_\mathbb{P} \subseteq \mathcal{I}_\mathbb{P}$ by $\mathcal{I}^*_\mathbb{P} := \{X\,;\, \forall T \in \mathbb{P} \exists S \in \mathbb{P}(S \leq T \wedge [S] \cap X \in \mathcal{I}_\mathbb{P}\}$. If $\boldsymbol{\Gamma}$ is a pointclass and $\mathbb{P}$ is an arboreal forcing, we write $\boldsymbol{\Gamma}(\mathbb{P})$ for the statement "Every set in $\boldsymbol{\Gamma}$ is $\mathbb{P}$-measurable".

For all classical forcing notions (Cohen, random, Hechler, Laver, Miller, Sacks, etc.), $\mathbb{P}$-measurability coincides with the natural notion of measurability. It is easy to see that for eventually different forcing $\mathbb{E}$, the ideal $\mathcal{I}_\mathbb{E}$ is exactly the ideal of $\mathcal{E}$-meager sets (thus allowing us to use the same notation) and being $\mathbb{E}$-measurable coincides with having the $\mathcal{E}$-Baire property.

In joint work with Halbeisen, the present authors introduced a notion of quasigenericity in [BHL05, §1.5]: given a model of set theory M, an ideal $\mathcal{I}$ and a real r, we say that r is **$\mathcal{I}$-quasigeneric** over M if for all Borel codes $c \in M$ such that $\mathrm{B}_c \in \mathcal{I}$, we have that $r \notin \mathrm{B}_c$.[1] For classical c.c.c. forcing notions $\mathbb{P}$ (Cohen, random, Hechler, eventually different, etc.), $\mathcal{I}_\mathbb{P}$-quasigenericity agrees with $\mathbb{P}$-genericity (cf. Lemma 1 for eventually different forcing). These are particular instances of a general fact [Ike09, Proposition 2.17].

Recall that if $\boldsymbol{\Gamma}$ is a projective pointclass, we say **$\boldsymbol{\Gamma}$-$\mathbb{P}$-absoluteness** holds if for every sentence φ in $\boldsymbol{\Gamma}$ with parameters in V, $V \models \varphi$ iff $V^\mathbb{P} \models \varphi$.

[1] Note that we are presupposing that "$\mathrm{B}_c \in \mathcal{I}$" is absolute between models of set theory. This is the case by Shoenfield absoluteness if $\mathcal{I}$ is sufficiently definable, for instance if it is $\boldsymbol{\Sigma}^1_2$ on $\boldsymbol{\Sigma}^1_1$.

Theorem 3 ([Ike09, Theorem 4.3])**.** If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c\,;\, c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2$, then the following are equivalent:

(i) $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness,

(ii) every $\mathbf{\Delta}^1_2$ set is $\mathbb{P}$-measurable, and

(iii) for every real x and every $T \in \mathbb{P}$, there is a $\mathcal{I}^*_{\mathbb{P}}$-quasigeneric real in $[T]$ over $\mathbf{L}[x]$.

Theorem 4 ([Ike09, Theorem 4.4])**.** If $\mathbb{P}$ is a proper and strongly arboreal forcing notion such that $\{c\,;\, c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}^*_{\mathbb{P}}\}$ is $\mathbf{\Sigma}^1_2$ and $\mathcal{I}_{\mathbb{P}}$ is Borel generated, then the following are equivalent:

(i) every $\mathbf{\Sigma}^1_2$ set is $\mathbb{P}$-measurable, and

(ii) for every real x, the set $\{y\,;\, y$ is not $\mathcal{I}^*_{\mathbb{P}}$-quasigeneric over $\mathbf{L}[x]\}$ belongs to $\mathcal{I}^*_{\mathbb{P}}$.

Since the ideal $\mathcal{I}^*_{\mathbb{P}} \supseteq \mathcal{I}_{\mathbb{P}}$ is equal to $\mathcal{I}_{\mathbb{P}}$ for c.c.c. forcing notions [Ike09, Lemma 2.13] and since we only deal with c.c.c. forcing, we can ignore the difference between $\mathcal{I}^*_{\mathbb{P}}$ and $\mathcal{I}_{\mathbb{P}}$.

Suppose that $\mathbf{\Gamma}$ is a projective pointclass. A σ-ideal $\mathcal{I}$ is called $\mathbf{\Gamma}$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subseteq 2^\omega \times \omega^\omega$, the set $\{y \in 2^\omega\,;\, A_y \in \mathcal{I}\}$ is in $\mathbf{\Gamma}$ (where $A_y := \{x\,;\, \langle y, x\rangle \in A\}$ is the vertical section at y). The notion of being $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ is a crucial property of ideals, as discussed in [Zap08, §3.8]. Most ideals occurring in nature are $\mathbf{\Delta}^1_2$ on $\mathbf{\Sigma}^1_1$. The ideal of $\mathcal{E}$-meager sets is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ [Zap08, Proposition 3.8.12].

Lemma 5. Assume $\mathcal{I}$ is a $\mathbf{\Sigma}^1_2$ on $\mathbf{\Sigma}^1_1$ σ-ideal which has a $\mathbf{\Sigma}^0_\alpha$ basis for some α. Then $\{c\,;\, c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}\}$ is $\mathbf{\Sigma}^1_2$.

Proof. Let $A \subseteq 2^\omega \times \omega^\omega$ be a universal $\mathbf{\Sigma}^0_\alpha$ set. Then $\mathrm{B}_c \in \mathcal{I}$ iff there is x such that $A_x \in \mathcal{I}$ and $\mathrm{B}_c \subseteq A_x$. By assumption, the first statement is $\mathbf{\Sigma}^1_2$, and the second is obviously $\mathbf{\Pi}^1_1$. Q.E.D.

A similar argument, using a universal analytic set instead, shows that if $\mathcal{I}$ is $\mathbf{\Delta}^1_2$ on $\mathbf{\Sigma}^1_1$, then $\{c\,;\, c$ is a Borel code and $\mathrm{B}_c \in \mathcal{I}\}$ is $\mathbf{\Pi}^1_2$.

4 $\mathbf{\Sigma}^1_2$ sets

We start by proving a "Judah-Shelah-style characterization" connecting the $\mathcal{E}$-Baire property of all sets in $\mathbf{\Delta}^1_2$ and the existence of generics.

Theorem 6. The following are equivalent:

(i) $\mathbf{\Sigma}^1_3$-$\mathbb{E}$-absoluteness,

(ii) Every $\mathbf{\Delta}^1_2$ set has the $\mathcal{E}$-Baire property (i.e., $\mathbf{\Delta}^1_2(\mathbb{E})$), and

(iii) for every x, there is an $\mathbb{E}$-generic over $\mathbf{L}[x]$.

Proof. This follows immediately from Theorem 3, keeping in mind that in the case of $\mathbb{E}$, having the $\mathcal{E}$-Baire property and being $\mathbb{E}$-measurable are the same, that $\mathcal{I}^*_{\mathbb{E}} = \mathcal{I}_{\mathbb{E}}$, that $\mathcal{I}_{\mathbb{E}}$-quasigenericity and $\mathbb{E}$-genericity are the same (Lemma 1), that $\mathcal{I}_{\mathbb{E}}$ has a basis consisting of $\mathbf{\Sigma}^0_3$ sets, that it is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ [Zap08, Proposition 3.8.12], and that $\{c\,;\, c \text{ is a Borel code and } \mathrm{B}_c \in \mathcal{I}_{\mathbb{E}}\}$ therefore is $\mathbf{\Sigma}^1_2$ by Lemma 5. Q.E.D.

Here is the characterization of the $\mathcal{E}$-Baire property of the $\mathbf{\Sigma}^1_2$ sets.

Theorem 7. The following are equivalent:

(i) Every $\mathbf{\Sigma}^1_2$ set has the $\mathcal{E}$-Baire property (i.e., $\mathbf{\Sigma}^1_2(\mathbb{E})$),

(ii) for every x, the set of $\mathbb{E}$-generics over $\mathbf{L}[x]$ is $\mathcal{E}$-comeager, and

(iii) ω_1 is inaccessible by reals.

Proof. The equivalence of (i) and (ii) follows from Theorem 4.

"(iii)⇒(ii)": If $\omega_1^{\mathbf{L}[x]}$ is countable, then there are at most countably many codes for $\mathcal{E}$-meager sets in $\mathbf{L}[x]$. By Lemma 1, $\omega^\omega \backslash \mathsf{EvD}(\mathbf{L}[x]) = \bigcup\{\mathrm{B}_c\,;\, c \in \mathbf{L}[x] \text{ is an } \mathcal{E}\text{-meager Borel code}\}$ which now is a countable union of $\mathcal{E}$-meager sets, and thus $\mathcal{E}$-meager. Consequently, $\mathsf{EvD}(\mathbf{L}[x])$ is $\mathcal{E}$-comeager.

"(ii)⇒(iii)": Towards a contradiction, let $\omega_1^{\mathbf{L}[x]} = \omega_1$ for some fixed x. In $\mathbf{L}[x]$, there is a family $\langle f_\alpha\,;\, \alpha < \omega_1\rangle$ of pairwise eventually different functions. Recall that the sets $E_\alpha := \{y \in \omega^\omega\,;\, \exists^\infty k (y(k) = f_\alpha(k))\}$ are $\mathcal{E}$-nowhere dense in $\mathbf{L}[x]$. Therefore, $\omega^\omega \backslash \mathsf{EvD}(\mathbf{L}[x])$ must contain all (i.e., uncountably many) of these sets E_α. By Theorem 2, $\omega^\omega \backslash \mathsf{EvD}(\mathbf{L}[x])$ cannot be $\mathcal{E}$-meager, and thus $\mathsf{EvD}(\mathbf{L}[x])$ cannot be $\mathcal{E}$-comeager. Q.E.D.

5 $\mathbf{\Delta}^1_2$ sets

In this section we compare the $\mathcal{E}$-Baire property of $\mathbf{\Delta}^1_2$ sets with measurability and the Baire property of $\mathbf{\Sigma}^1_2$ sets. We first show that the statement that all $\mathbf{\Sigma}^1_2$ are Lebesgue measurable, $\mathbf{\Sigma}^1_2(\mathbb{B})$, implies $\mathbf{\Delta}^1_2(\mathbb{E})$.

The partial order $\mathbb{LOC}$ of **localization forcing** consists of all pairs $\langle \sigma, F\rangle$ such that $\sigma \in ([\omega]^{<\omega})^{<\omega}$ is a finite sequence with $|\sigma(n)| = n$ for all $n < |\sigma|$ and F is a finite set of reals with $|F| \leq |\sigma|$. The order is given by $\langle \tau, G\rangle \leq \langle \sigma, F\rangle$ iff $\tau \supseteq \sigma$, $G \supseteq F$, and $f(n) \in \tau(n)$ for all $f \in F$ and all $n \in |\tau| \backslash |\sigma|$. The forcing $\mathbb{LOC}$ is c.c.c. and even σ-linked (but not σ-centered) and adds a generic **slalom** $\varphi \in ([\omega]^{<\omega})^\omega$ given by $\varphi = \bigcup\{\sigma\,;\, \langle \sigma, F\rangle \in G \text{ for some } F\}$ where G is the generic filter over V. The slalom φ **localizes** the ground model reals in the sense that $f(n) \in \varphi(n)$ for almost all n for all reals f from V.

Lemma 8. The product $\mathbb{LOC} \times \mathbb{C}$ adds an $\mathbb{E}$-generic real. In particular $\mathbb{E} \lessdot \mathbb{LOC} \times \mathbb{C}$.

Proof. Let φ be the $\mathbb{LOC}$-generic real. Since $\mathbb{LOC} \times \mathbb{C} \cong \mathbb{LOC} \star \dot{\mathbb{C}}$, we may think of Cohen forcing $\mathbb{C}$ as adding a generic real c over $V[\varphi]$. Furthermore, we may think of $\mathbb{C}$ as being the order of finite partial functions s with $s(n) \notin \varphi(n)$ for all $n < |s|$. That is, $c(n) \notin \varphi(n)$ for all n. We claim that this c is $\mathbb{E}$-generic over V.

Let $D \subseteq \mathbb{E}$ be open dense. Let $(\langle \sigma, F \rangle, s) \in \mathbb{LOC} \times \mathbb{C}$. Without loss we may assume $|\sigma| = |s|$. By our stipulation in the preceding paragraph this means that $s(n) \notin \sigma(n)$ for all $n < |s|$. We need to find $(\langle \tau, G \rangle, t) \leq (\langle \sigma, F \rangle, s)$ with $|\tau| = |t|$ and $\langle t, G \rangle \in D$. To see that this suffices note that such $(\langle \tau, G \rangle, t)$ necessarily forces that $t \subseteq \dot{c}$ and $\dot{c}(n) \neq f(n)$ for all $n \geq |t|$ and $f \in G$.

Clearly, $\langle s, F \rangle \in \mathbb{E}$. There is $\langle t, G \rangle \leq \langle s, F \rangle$ with $\langle t, G \rangle \in D$. By extending t, if necessary, we may assume that $|t| \geq |G|$. Next extend σ to τ such that $|\tau| = |t|$ and for all n with $|s| \leq n < |t|$ and all $f \in F$ we have $f(n) \in \tau(n)$ and $t(n) \notin \tau(n)$. This is possible because $t(n) \neq f(n)$ for all $f \in F$ and all such n. Then $(\langle \tau, G \rangle, t) \leq (\langle \sigma, F \rangle, s)$ is as required. Q.E.D.

In the following, we shall be interested in the statement "for all reals x, there is a $\mathbb{LOC}$-generic over $\mathbf{L}[x]$". Using Ikegami's general methods, we can prove that this is equivalent to the statement $\mathbf{\Delta}^1_2(\mathbb{LOC})$, i.e., every $\mathbf{\Delta}^1_2$ set is $\mathbb{LOC}$-measurable, as in Theorem 6. We shall not go into details here, and just use the notation $\mathbf{\Delta}^1_2(\mathbb{LOC})$ as a shorthand for the statement "for all reals x, there is a $\mathbb{LOC}$-generic over $\mathbf{L}[x]$".

Corollary 9. $\mathbf{\Delta}^1_2(\mathbb{LOC})$ implies $\mathbf{\Delta}^1_2(\mathbb{E})$.

Proof. It is easy to see that $\mathbb{LOC}$ adds a Cohen real. Therefore $\mathbb{E} \lessdot \mathbb{LOC} \star \dot{\mathbb{LOC}}$ by Lemma 8. In particular, if for all x there is a $\mathbb{LOC}$-generic over $\mathbf{L}[x]$, then for all x there is an $\mathbb{E}$-generic over $\mathbf{L}[x]$, and $\mathbf{\Delta}^1_2(\mathbb{E})$ follows by Theorem 6. Q.E.D.

Lemma 10. $\mathbf{\Delta}^1_2(\mathbb{LOC})$ is equivalent to $\mathbf{\Sigma}^1_2(\mathbb{B})$, the statement "all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable".

Proof. By Bartoszyński's characterization of additivity of measure [BJ95, Theorems 2.3.11 and 2.3.12], $\mathbf{\Delta}^1_2(\mathbb{LOC})$ implies $\mathbf{\Sigma}^1_2$-Lebesgue measurability. The same characterization yields that $\mathbf{\Sigma}^1_2(\mathbb{B})$ implies that for all x, there is a slalom localizing all reals in $\mathbf{L}[x]$. By [Bre06, Lemma 2.1], this in turn entails that there is is a $\mathbb{LOC}$-generic over $\mathbf{L}[x]$ for all x, that is, $\mathbf{\Delta}^1_2(\mathbb{LOC})$ holds. Q.E.D.

Corollary 11. $\mathbf{\Sigma}^1_2(\mathbb{B})$ implies $\mathbf{\Delta}^1_2(\mathbb{E})$.

Proof. This follows from the two preceding results. Q.E.D.

After Lebesgue measurability, we are now considering the (regular) Baire property. The statement $\mathbf{\Sigma}^1_2(\mathbb{C})$, i.e., "all $\mathbf{\Sigma}^1_2$ sets have the Baire property (in the ordinary topology)" is equivalent to $\mathbf{\Delta}^1_2(\mathbb{D})$, i.e., "all $\mathbf{\Delta}^1_2$ sets have the Baire property in the dominating topology by [BL99, Theorem 5.8]. The rest of this section will contain the proof that this statement does not imply $\mathbf{\Delta}^1_2(\mathbb{E})$.

For technical purposes, we consider a slight variant of standard Hechler forcing: conditions of our $\mathbb{D}$ are trees $T \subseteq \omega^{<\omega}$ such that for any $s \in T$ beyond the stem, $s\hat{\ }n$ belongs to T for almost all n. Obviously $\mathbb{D}$ is a σ-centered forcing notion which adds a dominating real. So $\mathbb{D} \times \mathbb{C}$ adds a standard Hechler real [BJ95, Corollary 3.5.3]. On the other hand, it is easy to see that standard Hechler forcing adds a $\mathbb{D}$-generic. (In fact, if $d \in \omega^\omega$ is a standard Hechler generic satisfying $d(n) > n$ for all n, then $d' \in \omega^\omega$ given recursively by $d'(0) = d(0)$ and $d'(n+1) = d(d'(n))$ is a $\mathbb{D}$-generic.) Thus, the finite support iterations of the two partial orders have the same properties. In particular, the finite support iteration of $\mathbb{D}$ forces $\mathbf{\Delta}^1_2(\mathbb{D})$.

The reason we use $\mathbb{D}$ is that this makes the rank analysis of Hechler forcing (which is originally due to Baumgartner and Dordal [BD85]) a bit simpler. Recall that, if $s \in \omega^{<\omega}$ and φ is a statement of the forcing language, we say s **forces** φ if there is $T \in \mathbb{D}$ with stem s such that T forces φ. Next, define the rank ρ_φ by:

$$\rho_\varphi(s) = 0 \iff s \text{ forces } \varphi$$
$$\rho_\varphi(s) \leq \alpha \iff \exists^\infty n \quad \rho_\varphi(s\hat{\ }n) < \alpha$$

Say that s **favors** φ if $\rho_\varphi(s) < \infty$. The following are well-known and easy:

(i) A sequence can force at most one of φ and $\neg\varphi$.

(ii) Each sequence favors at least one of φ and $\neg\varphi$.

(iii) A sequence s forces φ iff s does not favor $\neg\varphi$.

(iv) A sequence s favors φ iff for all T with stem s there is $U \leq T$ such that $U \Vdash \varphi$.

We prove (and this is the main technical result of this section):

Theorem 12. Let W be a c.c.c. extension of V with the property that for all infinite partial functions $x : \omega \to \omega$ in W which are not dominating over V, there are infinite partial functions $\{x_n \,;\, n \in \omega\}$ in V such that whenever $y \in \omega^\omega \cap V$ is infinitely often equal to all x_n, then y is infinitely often equal to x. Then, if d is $\mathbb{D}$-generic over W, for all infinite partial functions $x : \omega \to \omega$

in $W[d]$ which are not dominating over V, there are infinite partial functions $\{x_n\,;\,n\in\omega\}$ in V such that whenever $y\in\omega^\omega\cap V$ is infinitely often equal to all x_n, then y is infinitely often equal to x.

Corollary 13 (Dichotomy for Hechler forcing)**.** Let d be a Hechler real over V and let $x\in V[d]$ be a real. Then

(i) either x is dominating over V

(ii) or x is not eventually different over V.

Proof. Apply the theorem with $V=W$. Q.E.D.

The proof of Theorem 12 splits into three cases (see below); in Case 1 and Case 2, we find a ground model real that is infinitely often equal to x, and in Case 3, we can prove that x is dominating. In fact, we believe a different dichotomy (Conjecture 14) holds as well:

Conjecture 14. Let $x\in V[d]$ be a new real. Then

(i) either there is a dominating real (over V) in $V[x]$

(ii) or $V[x]$ is a Cohen extension of V.

However, the split would occur along different lines than in the proof of Theorem 12. It can be shown that in Case 2 of the proof of Theorem 12, $V[x]$ does contain a dominating real. We could strengthen Conjecture 14 to:

Conjecture 15. Let $x\in V[d]$ be a new real. Then

- either $V[x]$ is a Hechler extension of V
- or $V[x]$ is a Cohen extension of V.

Note that the original dichotomy Corollary 13 generalizes to iterated Hechler extensions (cf. Theorem 17). This could not be true for Conjecture 15, because $\mathbb{D}\star\dot{\mathbb{D}}\not\cong\mathbb{D}$ by [Paw86].

With respect to the eventually dominating order $\leq^*$ on the Baire space ω^ω, one may consider three different kinds of eventually different reals: bounded reals, unbounded reals which are not dominating, and reals which are dominating. E.g., random forcing $\mathbb{B}$ adds a bounded eventually different real and, since $\mathbb{B}$ is ω^ω-bounding, there are no other kinds of eventually different reals. By Corollary 13, $\mathbb{D}$ adds a dominating (and thus necessarily eventually different) real, but no other eventually different reals. Finally, $\mathbb{E}$ adds an eventually different real which is unbounded but not dominating and, again, there are no other kinds of eventually different reals. This is so because a(n iteration of) σ-centered forcing cannot add a bounded eventually different real. (The proof for this is similar to, but easier than, the arguments in the proof of Theorem 12.)

Proof of Theorem 12. Let $\dot{x}$ be a $\mathbb{D}$-name for x. Call $s \in \omega^{<\omega}$ *very good* if there is an infinite partial function $x_s : \omega \to \omega$ in W which is not dominating over V such that s favors $\dot{x}(k) = x_s(k)$ for all $k \in \mathrm{dom}(x_s)$. By "not-dominating" we mean, of course, that there is $z \in \omega^\omega \cap V$ such that $x_s(k) \leq z(k)$ for infinitely many $k \in \mathrm{dom}(x_s)$. With respect to this notion we introduce a rank $\mathrm{rk}_{\dot{x}}$ exactly as before:

$$\mathrm{rk}_{\dot{x}}(s) = 0 \iff s \text{ is very good}$$
$$\mathrm{rk}_{\dot{x}}(s) \leq \alpha \iff \exists^\infty n \quad \mathrm{rk}_{\dot{x}}(s\hat{\ }n) < \alpha$$

We say that s is *good* if $\mathrm{rk}_{\dot{x}}(s) < \infty$. Otherwise s is *not good.*

Case 1. *All s are good.*

This is the easiest case. By assumption, there is $\{x_n\,;\, n \in \omega\} \in V$ such that whenever $y \in \omega^\omega \cap V$ is infinitely often equal to all x_n, then y is infinitely often equal to all x_s for very good s. So, choose $y \in \omega^\omega \cap V$ which is infinitely often equal to all x_n. We claim that the trivial condition forces that y is infinitely often equal to $\dot{x}$.

For indeed, let k_0 and $T \in \mathbb{D}$ be given. Let s be its stem. By assumption, $\mathrm{rk}_{\dot{x}}(s) < \infty$. Thus, by replacing T with a stronger condition if necessary, we may assume without loss of generality that $\mathrm{rk}_{\dot{x}}(s) = 0$, i.e., s is very good. Choose $k \geq k_0$ such that $y(k) = x_s(k)$. Since s favors $\dot{x}(k) = x_s(k)$, there is a $U \leq T$ such that $U \Vdash \dot{x}(k) = y(k)$, as required. $\dashv$

Hence we may assume some s is not good. Then we can easily construct a condition T with stem s such that all $t \in T$ extending s are not good. We now work below the condition T. Call such t *not bad* if there are infinite partial functions $y_t, f_t : \omega \to \omega$ with the same domain in W such that y_t is not dominating over V, f_t is one-to-one, and $s\hat{\ }f_t(k)$ favors $\dot{x}(k) = y_t(k)$ for all $k \in \mathrm{dom}(y_t)$. Define the rank $\mathrm{Rk}_{\dot{x}}$ as before:

$$\mathrm{Rk}_{\dot{x}}(t) = 0 \iff t \text{ is not bad}$$
$$\mathrm{Rk}_{\dot{x}}(t) \leq \alpha \iff \exists^\infty n \quad \mathrm{Rk}_{\dot{x}}(t\hat{\ }n) < \alpha$$

We say that t is *not very bad* if $\mathrm{Rk}_{\dot{x}}(t) < \infty$. Otherwise t is *very bad.*

Case 2. *All $t \in T$ are not very bad.*

Again, there is $\{x_n\,;\, n \in \omega\} \in V$ such that whenever $y \in \omega^\omega \cap V$ is infinitely often equal to all x_n, then y is infinitely often equal to all y_t for t which is not bad. Choose $y \in \omega^\omega \cap V$ which is infinitely often equal to all x_n. We claim that T forces that y is infinitely often equal to $\dot{x}$.

Assume k_0 and $U \leq T$ are given. Let t be the stem of U. By assumption, $\mathrm{Rk}_{\dot{x}}(t) < \infty$. Without loss of generality, $\mathrm{Rk}_{\dot{x}}(t) = 0$. Choose $k \geq k_0$

such that $y(k) = y_t(k)$ and $s^\frown f_t(k)$ belongs to U. Since $s^\frown f_t(k)$ favors $\dot{x}(k) = y_t(k)$, there is a $U' \leq U$ such that $U' \Vdash \dot{x}(k) = y(k)$. $\dashv$

Case 3. *Some $t \in T$ is very bad.*

Construct a condition $U \leq T$ with stem t such that all $u \in U$ extending t are very bad. We claim that U forces that $\dot{x}$ is a dominating real over V.

To see this let $z \in \omega^\omega \cap V$ and $U' \leq U$. We need to find k_0 and $U'' \leq U'$ such that $U'' \Vdash \dot{x}(k) \geq z(k)$ for all $k \geq k_0$.

Let $t' = \mathrm{stem}(U')$. For $u' \in U'$, define the partial function $x_{u'}$ by $x_{u'}(k) = \min\{\ell\,;\, u'$ favors $\dot{x}(k) = \ell\}$ if the latter set is non-empty; otherwise $x_{u'}(k)$ is undefined. Note that, since u' is not (very) good, $x_{u'}$ either has finite domain or dominates V. Therefore there is k_0 such that for all $k \geq k_0$, either $x_{t'}(k)$ is undefined or $x_{t'}(k) \geq z(k)$.

Similarly, for $u' \in U'$, define $y_{u'}$ by $y_{u'}(k) = \min\{\ell\,;$for some n, we have $u'^\frown n \in U'$ and $u'^\frown n$ favors $\dot{x}(k) = \ell\}$ if the latter set is non-empty; otherwise $y_{u'}$ is undefined. Again, since u' is (very) bad, it is easy to see that $y_{u'}$ either has finite domain or dominates V.

Now we recursively construct $U'' \leq U'$ with $\mathrm{stem}(U'') = t'$, as well as numbers $k_{u'}$ for all $u' \in U''$.

First put t' into U'' and fix $k_{t'} \geq k_0$ such that for all $k \geq k_{t'}$, either $y_{t'}(k)$ is undefined or $y_{t'}(k) \geq z(k)$. Next, put $t'^\frown n$ into U'' if for all k with $k_0 \leq k < k_{t'}$, whenever $t'^\frown n$ favors $\dot{x}(k) = \ell$, then $\ell \geq z(k)$. This defines the successor level of t' because for each such k and each $\ell < z(k)$, there are only finitely many n such that $t'^\frown n$ favors $\dot{x}(k) = \ell$. By replacing the trees $U'_{t'^\frown n}$ by appropriate subtrees if necessary, we may assume without loss of generality that $U'_{t'^\frown n}$ forces $\dot{x}(k) \geq z(k)$ for all k with $k_0 \leq k < k_{t'}$. Thus, U'' will also force this. Notice that $x_{t'^\frown n} \geq y_{t'}$ everywhere so that $x_{t'^\frown n}(k) \geq z(k)$ for all $k \geq k_{t'}$.

In general, assume u' has been put into U''. Fix $k_{u'} \geq k_{u' \restriction (|u'|-1)}$ such that for all $k \geq k_{u'}$, either $y_{u'}(k)$ is undefined or $y_{u'}(k) \geq z(k)$. Next, put $u'^\frown n$ into U'' if for all k with $k_{u' \restriction (|u'|-1)} \leq k < k_{u'}$, whenever $u'^\frown n$ favors $\dot{x}(k) = \ell$, then $\ell \geq z(k)$. Since $x_{u'}(k) \geq z(k)$ for all $k \geq k_{u' \restriction (|u'|-1)}$, this indeed defines the successor level of u'. Again we may assume that $U'_{u'^\frown n}$ forces $\dot{x}(k) \geq z(k)$ for all k with $k_{u' \restriction (|u'|-1)} \leq k < k_{u'}$. Notice again that $x_{u'^\frown n} \geq y_{u'}$ everywhere so that $x_{u'^\frown n}(k) \geq z(k)$ for all $k \geq k_{u'}$.

This completes the construction of U'', and it is immediate from the construction that U'' forces $\dot{x}(k) \geq z(k)$ for all $k \geq k_0$. $\dashv$

This completes the proof of the theorem. Q.E.D.

The following is proved by a standard argument.

Lemma 16. Let γ be a limit ordinal. Assume $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\,;\,\alpha < \gamma)$ is a finite support iteration of c.c.c. forcing such that for all $\alpha < \gamma$ the following holds:

> For every $\mathbb{P}_\alpha$-name $\dot{x} : \omega \to \omega$ for an infinite partial function which is not dominating over V, there are infinite partial functions $x_n : \omega \to \omega$, $n \in \omega$, in V such that whenever $y \in \omega^\omega \cap V$ is infinitely often equal to all x_n, then y is forced to be infinitely often equal to $\dot{x}$. $(\star_\alpha)$

Then $(\star_\gamma)$ holds as well.

Proof. If $\mathrm{cf}(\gamma) > \omega$, then no new real number occurs at stage γ, and so the claim is trivially true. Therefore we can assume that $\mathrm{cf}(\gamma) = \omega$. Since an iteration of length γ with $\mathrm{cf}(\gamma)$ is isomorphic to one of length ω, we can without loss of generality assume that $\gamma = \omega$.

Let $\dot{x}$ be a $\mathbb{P}_\omega$-name for an infinite partial function. Assume the trivial condition forces that $\dot{x}$ is not dominating over V and fix $n < \omega$. In the $\mathbb{P}_n$-generic extension V_n, define a partial function x_n by $x_n(k) = \min\{\ell\,; \text{there is a } p \text{ in the remainder forcing } \mathbb{P}_\omega/\mathbb{P}_n \text{ such that } p \Vdash \dot{x}(k) = \ell\}$ if this set is non-empty; otherwise $x_n(k)$ is undefined. Notice that x_n is an infinite partial function, and that it cannot be dominating over V.

In the ground model V, we have $\mathbb{P}_n$-names $\dot{x}_n$ for all the x_n. By $(\star_n)$, we can find a countable family $\{y_m\,;\,m \in \omega\}$ such that whenever $y \in \omega^\omega \cap V$ is infinitely often equal to all y_m, then y is forced to be infinitely often equal to all $\dot{x}_n$. We show that such a y is also forced to be infinitely often equal to $\dot{x}$.

Fix k_0 and $p \in \mathbb{P}_\omega$. Let n be such that $p \in \mathbb{P}_n$. Step into V_n where the generic contains p. Fix $k \geq k_0$ such that $x_n(k) = y(k)$. Let q be a condition in the remainder forcing which forces $\dot{x}(k) = x_n(k)$. Then clearly $r\hat{\ }\dot{q}$ forces $\dot{x}(k) = y(k)$ for some $r \leq p$ in $\mathbb{P}_n$, as required. Q.E.D.

Theorem 17 (Dichotomy for iterated Hechler forcing)**.** Let $(\mathbb{P}_\alpha, \dot{\mathbb{D}}_\alpha\,;\,\alpha < \gamma)$ be a finite support iteration of Hechler forcing. Let x be a real in the $\mathbb{P}_\gamma$-generic extension. Then

(i) either x is dominating over V

(ii) or x is not eventually different over V.

More explicitly, if x is not dominating over V, then there are infinite partial functions $x_n : \omega \to \omega$, $n \in \omega$, such that whenever $y \in \omega^\omega \cap V$ is infinitely often equal to all x_n, then y is infinitely often equal to x as well.

Proof. By induction on γ. The case $\gamma = 1$ is Corollary 13. More generally, if $\gamma = \delta + 1$ is a successor, apply Theorem 12 with W being the $\mathbb{P}_\delta$-generic extension of V. If γ is a limit, apply Lemma 16. Q.E.D.

Theorem 18. Let G be $\mathbf{L}$-generic for the ω_1-stage finite support iteration of Hechler forcing. Then in $\mathbf{L}[G]$, $\mathbf{\Delta}^1_2(\mathbb{D})$ holds while $\Delta^1_2(\mathbb{E})$ fails.

Proof. By Theorem 17, it is immediate that the finite support iteration of Hechler forcing does not add an $\mathbb{E}$-generic over the ground model V. Thus, if the ground model is $\mathbf{L}$, then in the generic extension after adding ω_1 Hechler reals, there are no $\mathbb{E}$-generics over $\mathbf{L}$. This implies the failure of $\Delta^1_2(\mathbb{E})$ by Theorem 6. Q.E.D.

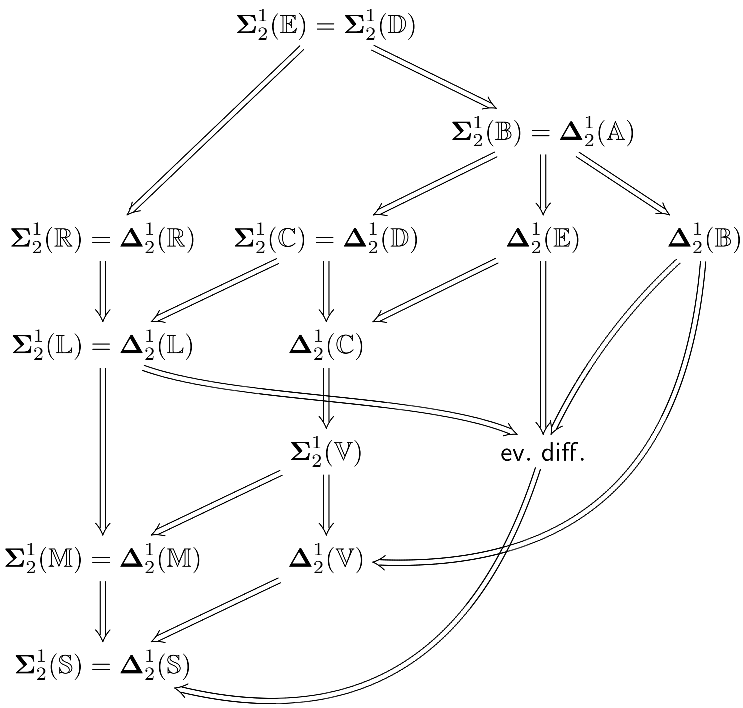

FIGURE 1.

6 Conclusions

Our two main results, Theorems 7 and 18, are enough to place the two statements $\mathbf{\Sigma}^1_2(\mathbb{E})$ and $\mathbf{\Delta}^1_2(\mathbb{E})$ in the diagram of regularity statements, as it has been developed by other work. In the diagram given in Figure 1, the letters $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$, $\mathbb{L}$, $\mathbb{M}$, $\mathbb{R}$, $\mathbb{S}$, and $\mathbb{V}$ stand for Amoeba, random, Cohen, Hechler, eventually different, Laver, Miller, Mathias, Sacks, and Silver forcing, respectively. The notation ev. diff. stands for "for every x, there is an eventually different real over $\mathbf{L}[x]$". All implications and non-implications not involving $\mathbb{E}$ have been known before this paper, and in the following we'll give arguments for all implications and non-implications involving $\mathbf{\Delta}^1_2(\mathbb{E})$. In the arguments, we shall freely use Theorem 6 and its analogues for other forcings:

The statement "$\mathbf{\Delta}^1_2(\mathbb{E}) \Rightarrow \mathsf{ev.diff.}$" is trivial, and "$\mathbf{\Delta}^1_2(\mathbb{E}) \Rightarrow \mathbf{\Delta}^1_2(\mathbb{C})$" is easy because $\mathbb{E}$ adds Cohen reals. Corollary 11 yields "$\mathbf{\Sigma}^1_2(\mathbb{B}) \Rightarrow \mathbf{\Delta}^1_2(\mathbb{E})$", and Theorem 18 shows that the Hechler model witnesses "$\mathbf{\Delta}^1_2(\mathbb{D}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{E})$".

Note that neither random nor Mathias forcing add Cohen reals, so that the random model is a model of $\mathbf{\Delta}^1_2(\mathbb{B}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{C})$ and the Mathias model is a model of $\mathbf{\Delta}^1_2(\mathbb{R}) \wedge \neg\mathbf{\Delta}^1_2(\mathbb{C})$. Since every $\mathbb{E}$-generic defines a Cohen real, these two models witness "$\mathbf{\Delta}^1_2(\mathbb{B}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{E})$" and "$\mathbf{\Delta}^1_2(\mathbb{R}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{E})$", respectively.

Finally, the iteration of $\mathbb{E}$ adds neither dominating nor random reals, and thus the eventually different model witnesses both "$\mathbf{\Delta}^1_2(\mathbb{E}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{L})$" and "$\mathbf{\Delta}^1_2(\mathbb{E}) \not\Rightarrow \mathbf{\Delta}^1_2(\mathbb{B})$".

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