Can doxastic agents learn?
On the temporal structure of learning

Cédric Dégremont
cedric.uva@gmail.com

Nina Gierasimczuk
n.gierasimczuk@uva.nl

Institute for Logic, Language and Computation
University of Amsterdam

Formal learning theory constitutes an attempt to describe and explain the phenomenon of language acquisition. The considerations in this domain are also applicable in philosophy of science, where it can be interpreted as a theory of empirical inquiry. The main issue within this theory is to determine which classes of languages are learnable and how learnability is affected by e.g. restricting the learning functions, modifying the informativeness of the incoming data and changing the conditions of success of the learning process. All those directions focus on various properties of the process of *conjecture-change over time.* Treating "conjectures" as beliefs, we link the process of conjecture-change to doxastic update. Using this approach, we reconstruct and analyze the temporal aspect of learning in the context of temporal and dynamic logics of belief change.

The aim of connecting Learning Theory (LT) and modal logics of belief change is two-fold. By analyzing the temporal doxastic structure underlying formal learning theory, we provide additional insight into the semantics of inductive learning. By importing the ideas, problems and methodology from Learning Theory, logics of epistemic and doxastic change get enriched by new learning scenarios, i.e. those based not only on incorporation of new data but also on generalization, but they also gain new concepts and new problematic perspectives.

We will proceed as follows. In Sections 1 and 2 we introduce the basic formal notions of learning theory and modal logics of belief change. In Section 3 we propose a reduction of the learnability task to a generalized problem of DETL model checking. Furthermore, we prove a DETL representation result corresponding to an important theorem from Learning Theory, that characterizes learnability, namely Angluin's theorem. Then we step back and place notions of learning theory and doxastic temporal logic in a common perspective in order to compare them (Section 4). We focus both on the properties of agents and fine-grained notions of belief and knowledge. In Section 5 we consider an extension of the classical learning theoretic framework by introducing more agents and extending the protocols to include a possibility of communication between the agents. In the end we discuss consequences, possible extensions and profits that our work brings.

1 Learning Theory

Let us now introduce a framework that allows discussing the inductive inference scenarios. In the usual grammar inference the basis is what we call a "grammatical system". We will work with a slightly modified notion of "learning background", in order to get a generalized view, abstracting away from a particular learning model.

Definition 1.1. *A* learning background Lrn *is a triple* $\langle H, \Sigma, Name \rangle$*, where:*

- H *is a set of Turing Machines (hypotheses);*
- $Name : H \rightarrow \wp(\Sigma)$ *is a naming function.*

Naming functions assign names to sets of numbers. Formally, if $M \in H$, then $Name(M)$ is a set generated from M. To explicate what we mean by learning situation we need a learning function.

Definition 1.2. *Let $\langle H, \Sigma, Name \rangle$ be a learning background. A learning function is a function $L : \Sigma^* \rightharpoonup H$.*

In the next steps we define what it means for a learning function to successfully learn a class of sets. We will define three inequivalent conditions for learning: identification in the limit, finite identification and learning by erasing. Having them, we can define the "learning problem" to be:

- Is there a learning function L that successfully learns (wrt the appropriate learning condition) on a given learning background?

To get a more fine-grained picture, before we define the conditions for a learning function to be successful, we will focus on the data that learning functions learn from: environments.

Let us consider E, the set of all computably enumerable sets. Let $C \subseteq E$ be some class of c.e. sets. For each S in C we consider Turing machines h_n which generate S and in such a case we say that n is an index of S. The Turing machines will function as the conjectures that Learner makes. It is well-known that each S has infinitely many indices. Let us take I_S to be the set of all indices of the set S, i.e, $I_S = \{n | h_n \textit{ generates } S\}$.

Definition 1.3 (Environment)**.** *An environment of S is a surjective ω-sequence from S^ω*

Definition 1.4 (Notation)**.** *Let ε be an environment. We will use the following notation:*

- *ε_n is the n-th element of ε;*
- *$\varepsilon | n$ is the sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})$;*
- *$content(\varepsilon)$ is a set of elements that occur in ε;*
- *h_n will refer to a hypothesis, i.e., a finite description of a set, a Turing machine generating S;*
- *L is a learning function — a map from finite data sequences to indexes of hypotheses, $L : \Sigma^* \to \mathbb{N}$.*

Now we can provide definitions of different kinds of learnability. In each case, the structure of identifiability in the limit can be formulated by a chain of definitions that describe three levels of identification.

Definition 1.5 (Identification in the limit, LIM [9])**.** *We say that a learning function L:*

1. *identifies $S \in C$ in the limit on ε iff there is a number k, such that for co-finitely many m, $L(\varepsilon|m) = k$ and $k \in I_S$;*
2. *identifies $S \in C$ in the limit iff it identifies S in the limit on every ε for S;*
3. *identifies C in the limit iff it identifies in the limit every $S \in C$.*

The notion of identifiability can be strengthened in various ways. One radical case is to introduce a finite condition for identification.

Definition 1.6 (Finite identification, FIN)**.** *We say that a learning function L:*

1. *finitely identifies $S \in C$ on ε iff, when successively fed ε, at some point L outputs a single k, such that $k \in I_S$, and stops;*
2. *finitely identifies $S \in C$ iff it finitely identifies S on every ε for S;*
3. *finitely identifies C iff it finitely identifies every $S \in C$.*

Another, epistemically plausible, way to learn is learning by elimination of hypotheses that are implausible, e.g. hypotheses that are inconsistent with the incoming data. This paradigm is formalized in the framework of learning by erasing.

Definition 1.7 (Function stabilization)**.** *In learning by erasing we say that a function stabilizes to number k on environment ε iff for co-finitely many $n \in \mathbb{N}^*$:*

$$k = min\{\mathbb{N} - \{L(\varepsilon|1), \ldots, L(\varepsilon|n)\}\}.$$

Definition 1.8 (Learning by erasing, e-learning [10])**.** *We say that a learning function L:*

1. *learns $S \in C$ by erasing on ε iff L stabilizes to k on ε and $k \in I_S$;*
2. *learns $S \in C$ by erasing iff it learns by erasing S from every ε for S;*
3. *learns C by erasing iff it learns by erasing every $S \in C$.*

2 Modal Logics of Multi-agent Belief Change

In this paper we will be interested in two logical approaches to multi-agent belief change: the temporal approach [11, 7] and the dynamic approach [3].

2.1 The temporal approach

Temporal logics can be used to study the evolution of a system over time. Doxastic Epistemic Temporal Logics offer a global view of the evolution of a multi-agent system as events take place, focusing on the information that agents possess and what they believe.

2.1.1 Models

We interpret these logics on doxastic epistemic temporal forests [11]. Such logics could as well be interpreted on Interpreted Systems ([7], see [8] for a doxastic counterpart).

Definition 2.1 ($DETL$ Forests)**.** *A* doxastic epistemic temporal forest *($DETL$ forest for short) $\mathcal{H}$ is of the form $\langle W, \Sigma, H, (\leq_i)_{i\in N}, (\sim_i)_{i\in N}, V\rangle$, where W is a countable set of initial states, Σ is a countable set of events, $H \subseteq W(\Sigma^* \cup \Sigma^\omega)$ is closed under non-empty finite prefixes, for each $i \in N$, $\leq_i \subseteq H \times H$ is a well-founded pre-order on H, $\sim_i \subseteq H \times H$ is an equivalence relation and $V : Prop \to \wp(H)$. We write wh or $w\vec{e}$ (resp. $w\epsilon$) to denote some finite (respectively ω-) history starting in the state w.*

Let $\mathfrak{P} : s \mapsto (\{s\}(\Sigma^* \cup \Sigma^\omega)) \cap H$ for $s \in W$. Intuitively $\mathfrak{P}(w)$ is the *protocol* or *bundle* of sequences of events associated with w. We refer to the information of agent i at w by $\mathcal{K}_i[h] = \{h' \in H \mid h \sim_i h'\}$. We also write $\mathcal{B}_i[h] = Min_{\leq_i}\mathcal{K}_i[h]$, i.e the histories that i considers the most plausible at history h.

2.1.2 Assumptions about the agents

Further assumptions about these structures will be considered in this paper.[1]

Definition 2.2 (Perfect Recall)**.** *We say that all agents in a group $G \subseteq N$ satisfy* Perfect Recall *iff $\forall i \in G\ \forall he, h'f \in H$ if $K_i[he] = K_j[h'f]$, then $K_i[h] = K_j[h']$. We write $PR(G)$.*

Definition 2.3 (Synchronicity)**.** *We say that all agents in a group $G \subseteq N$ satisfy* Synchronicity *iff $\forall i \in G\ \forall h, h' \in H$ if $K_i[h] = K_j[h']$, then $len[h] = len[h']$. We write $SYN(G)$.*

Definition 2.4 (E-Uniform No Miracles)**.** *Let $E \subseteq \Sigma$. We say that all agents in a group $G \subseteq N$ satisfies* E-Uniform No Miracles *iff $\forall i \in G\ \ \forall e_1, e_2 \in E$ if $\exists he_1, h'e_2$ such that $he_1 \sim_i h'e_2$ then $\forall je_1, j'e_2 \in W$ if $j \sim_i j'$ then $je_1 \sim_i j'e_2$. We write $E - UNM(G)$.*

Definition 2.5 (Perfect Observation)**.** *We say that all agents in a group $G \subseteq N$ satisfy* Perfect Observation *iff $\forall i \in G\ \ \forall he, h'f \in H$ if $K_i[he] = K_j[h'f]$, then $e = f$. We write $PO(G)$.*

Definition 2.6 (Preference Stability)**.** *We say that all agents in a group $G \subseteq N$ satisfies* Perfect Recall *iff $\forall i \in G \forall he, h'f \in H$ we have $he \leq_i h'f$ iff $h \leq_i h'$. We write $PS(G)$.*

2.1.3 A natural doxastic epistemic temporal language

Syntax Our dynamic epistemic temporal language of $\mathcal{L}_{DET}$ is defined by the following inductive syntax.

$$\phi := p \mid \neg\phi \mid \phi \vee \phi \mid K_i\phi \mid B_i\phi \mid \mathtt{A}\phi \mid \bigcirc^{-1}\phi \mid F\phi \mid P\phi \mid \forall\phi \mid$$

where i ranges over N, a over Σ, and p over proposition letters PROP. $K_i\phi$ ($B_i\phi$) reads i knows (resp. believes) that ϕ. F and P stand for future and past. $\forall\phi$ means: in all continuations ϕ. $H\phi := \neg P\neg\phi$.

[1]For these properties we often drop the label G when $G = N$ and we drop E whenever $E = \Sigma$.

Semantics $\mathcal{L}_{DET}$ is interpreted over pairs of nodes $w\vec{e}$, i.e. initial state together with a finite sequence of events and maximal histories $w\epsilon$ in our trees, such that $\vec{e}$ is a finite prefix of ϵ (cf. [12, 11]).

Definition 2.7 (Truth definition). *We only give the interesting clauses.*

$$
\begin{array}{lll}
\mathcal{H}, w\epsilon, w\vec{e} \Vdash K_i\phi & \text{iff} & \forall v\vec{f} \text{ if } v\vec{f} \sqsubseteq v\epsilon' \ \&\ v\vec{f} \in \mathcal{K}_i[w\vec{e}] \text{ then } \mathcal{H}, v\epsilon', v\vec{f} \Vdash \phi \\
\mathcal{H}, w\epsilon, w\vec{e} \Vdash B_i\phi & \text{iff} & \forall v\vec{f} \text{ if } v\vec{f} \sqsubseteq v\epsilon' \ \&\ v\vec{f} \in \mathcal{B}_i[w\vec{e}] \text{ we have } \mathcal{H}, v\epsilon', v\vec{f} \Vdash \phi \\
\mathcal{H}, w\epsilon, w\vec{e} \Vdash \mathtt{A}\phi & \text{iff} & \forall v\vec{f} \text{ if } v\vec{f} \sqsubseteq v\epsilon' \ \&\ v\vec{f} \in H \text{ then } \mathcal{H}, v\epsilon', v\vec{f} \Vdash \phi \\
\mathcal{H}, w\epsilon, w\vec{e} \Vdash \bigcirc^{-1}\phi & \text{iff} & \exists a \in \Sigma\ \exists \vec{f} \sqsubseteq \epsilon \text{ such that } \vec{f}.a = \vec{e} \text{ and } \mathcal{H}, w\epsilon, w\vec{f} \Vdash \phi \\
\mathcal{H}, w\epsilon, w\vec{e} \Vdash F\phi & \text{iff} & \exists \vec{g} \in \Sigma^*\ \exists \vec{f} \sqsubseteq \epsilon \text{ such that } \vec{f} = \vec{e}\vec{g} \text{ and } \mathcal{H}, w\epsilon, w\vec{f} \Vdash \phi \\
\mathcal{H}, w\epsilon, w\vec{e} \Vdash P\phi & \text{iff} & \exists \vec{g} \in \Sigma^*\ \exists \vec{f} \sqsubseteq \epsilon \text{ such that } \vec{f}\vec{g} = \vec{e} \text{ and } \mathcal{H}, w\epsilon, w\vec{f} \Vdash \phi \\
\mathcal{H}, w\epsilon, w\vec{e} \Vdash \forall\phi & \text{iff} & \forall h\,' \in \mathfrak{P}(w) \text{ s.t. } \vec{e} \sqsubseteq h \text{ we have } \mathcal{H}, wh', w\vec{e} \Vdash \phi
\end{array}
$$

2.2 The dynamic approach

The dynamic approach of Dynamic Doxastic and Dynamic Epistemic Logics (*DDL* and *DEL* for short) considers belief change as a step by step operation on models.

Definition 2.8 (Epistemic-Plausibility and Event Models, Product Update).

- *An* epistemic plausibility model *(EP for short)* $\mathcal{M}$ *is of the form* $\langle W, (\sim_i)_{i\in N}, (\leq_i)_{i\in N}, V\rangle$ *where* $W \neq \emptyset$*, for each* $i \in N$*,* $\sim_i$ *is a binary reflexive and euclidean relation on* W*,* $\leq_i$ *is a pre-order on* W *and* $V : Prop \to \wp(W)$*. We let* $\mathcal{K}_i[w] = \{v \in W \mid w \sim_i v\}$ *and* $\mathcal{B}_i[w] = Min_{\leq_i}\mathcal{K}_i[w]$*.*
- *An* event model $\epsilon = \langle E, (\sim_i)_{i\in N}, \mathtt{pre}\rangle$ *has* $E \neq \emptyset$*, and for each* $i \in N$*,* $\sim_i$ *is a relation on* W*. Finally, there is a precondition map* $\mathtt{pre} : E \to \mathcal{L}_{DL}$*, where* $\mathcal{L}_{DL}$ *is some doxastic language.*[2]*.*
- *The* product update $\mathcal{M}\otimes\epsilon$ *of an epistemic model* $\mathcal{M} = \langle W, (\sim'_i)_{i\in N}, V\rangle$ *with an event model* ϵ *is the model* $\langle E, (\sim_i)_{i\in N}, \mathtt{pre}\rangle$*, whose worlds are pairs* (w, e) *with the world* w *satisfying the precondition of the event* e*, and accessibilities defined as:*

$$(w, e) \sim'_i (w', e') \text{ iff } e \sim_i e', w \sim_i w'$$

An *EP* model describes what agents currently believe and know, while product update creates the new doxastic epistemic situation after some information event has taken place.[3]

Recently *DEL* borrowed the crucial idea of *protocol* from the temporal approach. A *protocol* P maps states in an *EP* model to sets of finite sequences of pointed event models closed under taking prefixes. This defines the admissible runs of some informational process: not every observation may be available, or appropriate. We let $\mathfrak{E}$ be the class of all pointed plausibility event models. Let $Prot(\mathfrak{E}) = \{\mathtt{P} \subseteq (\mathfrak{E}^* \cup \mathfrak{E}^\omega) \mid \mathtt{P}$ is closed under finite prefixes $\}$ be the co-domain of protocols, it is the class of all sets of sequences (infinite and finite) of pointed plausibility event models closed under taking finite prefixes. Given some $\epsilon \in \mathfrak{E}^\omega$ we often refer the generated *EP* model $M^{\varepsilon|m}$ to mean $\mathcal{M} \otimes \epsilon_1 \otimes \dots \epsilon_m$.

Definition 2.9. *Given an EP model* $\mathcal{M}$*. A local protocol for* $\mathcal{M}$ *is a function* $P : |\mathcal{M}| \to Prot(\mathfrak{E})$*.*

2.3 Dynamic doxastic language

We first look at a core language that matches dynamic belief update.

Syntax Our dynamic doxastic language $\mathcal{L}_{DDE}$ is defined as follows:

$$\phi := p \mid \neg\phi \mid \phi \vee \phi \mid \langle\leq_i\rangle\phi \mid \langle i\rangle\phi \mid \mathtt{E}\phi \mid \langle\epsilon, \mathbf{e}\rangle\phi$$

where i ranges over over N, p over a countable set of proposition letters *Prop*, and $(\epsilon, \mathbf{e})$ ranges over a suitable set of symbols for event models.

[2]In our case the $\langle\epsilon, \mathbf{e}\rangle$-free fragment of the dynamic doxastic language $\mathcal{L}_{DDE}$ defined in the next subsection

[3]Illustrations of the strength of this simple mechanism are in [2].

Semantics Here is how we interpret the $DDE(L)$ language. A pointed event model is an event model plus some distinguished element of its domain. To economize on notation we use event symbols in the semantic clause. Also, we write $\mathtt{pre}(e)$ for $\mathtt{pre}_\epsilon(e)$ when it is clear from context.

Definition 2.10 (Truth definition)**.** *We only give the interesting clauses.*

$$\begin{array}{lll} \mathcal{M}, w \Vdash \langle \leq_i \rangle \phi & \text{iff} & \exists v \ \text{ such that } w \preceq_i v \text{ and } \mathcal{M}, v \Vdash \phi \\ \mathcal{M}, w \Vdash K_i \phi & \text{iff} & \forall v \ \text{ such that } v \in \mathcal{K}_i[w] \text{ we have } \mathcal{M}, v \Vdash \phi \\ \mathcal{M}, w \Vdash B_i \phi & \text{iff} & \forall v \ \text{ such that } v \in \mathcal{B}_i[w] \text{ we have } \mathcal{M}, v \Vdash \phi \\ \mathcal{M}, w \Vdash \mathtt{E} \phi & \text{iff} & \exists v \ \in W \text{ such that } \mathcal{M}, v \Vdash \phi \\ \mathcal{M}, w \Vdash \langle \epsilon, \mathbf{e} \rangle \phi & \text{iff} & \mathcal{M}, w \Vdash \mathtt{pre}(e) \text{ and } \mathcal{M} \times \epsilon, (w, e) \Vdash \phi \end{array}$$

We make use of the usual abbreviations.

2.4 Connection between the temporal and the dynamic approach

There is a connection between the two approaches. In fact the Product Updaters of the dynamic approach are just one interesting type of doxastic (temporal) agents. Indeed Iterated Product Update of an epistemic plausibility model according to a uniform line protocol P generates doxastic epistemic temporal forests that validate particular doxastic temporal properties. We use the following construction:

Definition 2.11 ($DETL$ forest generated by a DDL protocol)**.** *Each initial epistemic plausibility model* $\mathcal{M} = \langle W, (\sim_i^{\mathcal{M}})_{i \in N}, (\leq_i^{\mathcal{M}})_{i \in N}, V^{\mathcal{M}} \rangle$ *and each local protocol P yields a* generated $DETL$ forest $\mathcal{H}$ *is of the form* $\langle W^{\mathcal{H}}, \Sigma, H, (\leq_i)_{i \in N}, (\sim_i)_{i \in N}, V \rangle$ *as follows:*

- *Let* $\Sigma := \bigcup_{w \in W} \bigcup_{n \in \omega} P(w)(n)$.
- *Let* $W^{\mathcal{H}} := |\mathcal{M}|$, $H_1 = W^{\mathcal{H}}$ *and for each* $1 < n < \omega$, *let* $H_{n+1} := \{(we_1 \dots e_{n+1}) \mid (we_1 \dots e_n) \in H_n, \mathcal{M} \otimes \epsilon_1 \otimes \dots \otimes \epsilon_n \Vdash \mathtt{pre}_n(e_{n+1}) \text{ and } e_1 \dots e_{n+1} \in P(w)\}$.
  Finally let $H = \bigcup_{1 \leq k < \omega} H_k$.
- *If* $h, h' \in W^{\mathcal{H}}$, *then* $h \sim_i h'$ *iff* $h \sim_i^{\mathcal{M}} h'$.
- *If* $h, h' \in W^{\mathcal{H}}$, *then* $h \leq_i h'$ *iff* $h \leq_i^{\mathcal{M}} h'$.
  Finally information partitions and plausibility are defined according to Product Update.
- *For* $1 < k \leq m$, $he \sim_i h'e'$ *iff* $he, h'e' \in H_k$, $h \sim_i h'$, e *and* e' *are pointed event-model from the same event model and* $e \sim_i e'$ *in their event model.*
- *For* $1 < k \leq m$, $he \leq_i h'e'$ *iff* $he, h'e' \in H_k$ *and* $h \leq_i h'$
- *Finally, set* $wh \in V(p)$ *iff* $w \in V^{\mathcal{M}}(p)$.

We conclude by mentioning an important representation theorem that we will later make use of. First we introduce the following notion:

Definition 2.12 (Propositional Stability)**.** *We say that a forest satisfies propositional stability iff for all* $h, he \in H$ *we have* $p \in V(he)$ *iff* $p \in V(h)$.

Theorem 2.13 (van Benthem et al. [5])**.** *An* ETL*-model* $\mathcal{H}$ *is isomorphic to the forest generated by the sequential product update of an epistemic model according to some state-dependent* DEL*-protocol iff it satisfies* $PR(N)$, $UNM(N)$, $SYN(G)$ *and Propositional Stability.*

A simple way to update doxastic epistemic model in a DEL style is to actually update the epistemic relation according to Product Update while leaving the plausibility relation unmodified.[4]

Let us know mention an important corollary:

Corollary 2.14. *A* $DETL$*-model is isomorphic to the forest generated by the sequential product update of a doxastic epistemic model according to some state-dependent* DEL*-protocol iff it satisfies* $PR(N)$, $UNM(N)$, $SYN(G)$, *Propositional Stability and* $PS(N)$.

A final piece of notation: given a $DETL$ (resp. EP) model $\mathcal{H}$ (resp. $\mathcal{M}$) we write $F(\mathcal{H})$ (resp. $F(\mathcal{M})$) to denote the frame of the model, i.e. the model without the valuation function.

[4]More subtle policy exists, see e.g. [4].

3 DETL reductions and representation theorems for learnability.

The aim of this section is to give a first result bridging Learning Theory and Dynamic Epistemic Temporal Logics. In particular we prove that the problem of checking whether a class of sets is finitely identifiable can be reduced to the model-checking problem of $\mathcal{L}_{DET}$ on Doxastic Epistemic Temporal Forests.

In general Learning Situations will be captured by doxastic epistemic temporal structures. In the case of Set Learning, they can be more specifically and accurately captured by a doxastic epistemic model and a local protocol. We start by giving a formal definition of such a construction.

3.1 Protocols that correspond to set learning

Given a countable class of countable sets $\Omega = \{S_1, \ldots, S_n, \ldots\}$ a Set Learning Situation is a triple $Sit_\Omega := \langle \Omega, \mathfrak{S}, L\rangle$ where $\mathfrak{S} = \bigcup_{S\in\Omega} S$ and L is the identity map. In words environments will enumerate elements from the sets in Ω (see Section 1). Given the Set Learning Situation $Sit_\Omega = \langle \Omega, \mathfrak{S}, L\rangle$ we can construct an initial epistemic model $\mathcal{M}_\Omega$ and a local protocol P_Ω.

Initial epistemic model. Our initial epistemic model $\mathcal{M}^{Sit}_\Omega = \langle W_\Omega, \sim, V\rangle$ where $W_\Omega = \Omega$, $\sim = W \times W$ and for each set $S_n \in \Omega$, we set $V(i_n) = \{S_n\}$. In words, we identify states of the model with sets. We also assume that our agent does not have any particular initial information and for each state S_n, i_n is some nominal for S_n.

Class of event models. For each $e \in \mathfrak{S}$, we have a corresponding event model $\mathfrak{e} = \langle \{e\}, \sim_e\rangle$ where $\sim_e = (e, e)$. In words we assume that our agent has perfect observational powers.

Given a set S, we write $\mathfrak{E}(S) = \{(\mathfrak{e}, e) \mid e \in S\}$. We are now ready to define our local protocol.

Local protocol. Given a state $s \in W$, our protocol P_Ω will authorize at s any ω-sequence that enumerate s. Formally: for every $s \in W$, $P_\Omega(s)$ is the smallest subset of $((\mathfrak{E}(\mathfrak{S}))^* \cup (\mathfrak{E}(\mathfrak{S}))^\omega)$ that contains $\{f : \omega \to \mathfrak{E}(s) \mid f \text{ is surjective }\}$ and that is closed under non-empty finite prefixes.

3.2 DETL Reduction of Learning Problems

Definition 3.1 (Stabilization). *j's belief (resp. knowledge) about the initial state stabilizes to w on the history $v\epsilon$ iff there is a finite prefix $e^* \sqsubset \epsilon$ such that for any finite sequence e' such that $e^* \sqsubseteq e' \sqsubset \epsilon$ we have for all histories sh if $sh \in \mathcal{B}_j[ve']$ then $s = w$ (resp. for $\mathcal{K}_j[ve']$).*

Given a collection of sets $\Omega = \{S_1, \ldots, S_n, \ldots\}$ we let $\textsc{prop}_\Omega = \{i_n \mid S_n \in \Omega\}$. In what follows we indicate one of the possible logical reductions of the problem of finite identifiability. We reduce it to the model checking of an hybrid extension of $\mathcal{L}_{DET}$ that we lack the space to introduce (see Appendix for details). The following holds:

Proposition 3.2. *The following are equivalent:*

1. *Ω is finitely identifiable.*
2. *For all $s \in W_\Omega$ and $\epsilon \in P_\Omega$ the learner's knowledge about the initial state stabilizes to s on $s\epsilon$.*[5]
3. *$\mathcal{F}or(F(\mathcal{M}^{Sit}_\Omega), V_\Omega, P_\Omega) \Vdash \mathtt{A}(\bigcirc^{-1}\bot \rightarrow \downarrow x.\forall FKH(\bigcirc^{-1}\bot \rightarrow x)\ )$.*

One can thus reduce Finite Identifiability to Model Checking of an hybrid doxastic epistemic temporal language. But there are other interesting directions. Below we use another approach that gives more intuitive $DETL$ reduction. We let $\textsc{nom}$ be a set of propositionally stable nominals, i.e. for each $i \in \textsc{nom}$, and for each valuation V, $V(i) \in W$ and whose truth conditions is given by $\mathcal{H}, w\epsilon, w\vec{e} \Vdash i$ iff $V(i) = w$. We can now reduce various Learnability problems to the (extended) validity problem of some $DETL$ formulas, i.e. we specify $DETL$ conditions that must be validated by a protocol to guarantee learnability.

We are going to proceed with an attempt to find a uniform representation of learning types. Here is the general scheme of our uniform representation of learning types:

A DETL frame $F(\mathcal{H})$ satisfies Learning Condition iff
`[Specification of a procedure of choosing the current belief]`
$F(\mathcal{H}) \Vdash i \rightarrow$ `[Quantifier]` F `[Epistemic Temporal Condition]`i.

[5]in the generated forest $\mathcal{F}or(F(\mathcal{M}^{Sit}_\Omega), V_\Omega, P_\Omega)$.

The most straight-forward is the characteristics of finite identifiability.

$$\text{A DETL frame } F(\mathcal{H}) \text{ satisfies FIN iff } F(\mathcal{H}) \Vdash i \rightarrow \forall FKi$$

Learner can finitely identify a class iff for all elements i of the class if i holds, then in the future Learner will know that i.

Further extension of the validity approach demands more expressive power, namely we need to express the existence of an appropriate belief-choosing procedure, which leads to Dyadic Second Order quantification. If we skip the certainty condition, we get the characteristics of learning by erasing.

$$\text{A DETL frame F}(\mathcal{H}) \text{ satisfies ERASE iff } \exists \leq \ F(\mathcal{H}[\leq]) \Vdash i \rightarrow \forall FGBi\ .$$

The effectiveness of this procedure, in the presence of uncertainty, is guaranteed by the existence of an underlying preference ordering. The temporal condition is weakened, since Learner can not be guaranteed certainty. The success is defined as a stabilization to a correct hypothesis.

In general if we allow some freedom in defining beliefs, we can make an attempt to formalize computable identification in the limit.

$$\text{A DETL frame F}(\mathcal{H}) \text{ satisfies Comp-LIM iff } \exists\ \mathfrak{B}\text{-Algorithm } F(\mathcal{H}[\mathfrak{B}]) \Vdash i \rightarrow \forall FGBi\ .$$

In this expression the B-Algorithm is an effective procedure that at each step of the procedure computes the current belief. If we do not pose the restriction to computable functions we get the general identification in the limit. In general we can make further substitution to our general scheme and see what happens. Let us consider the following example.

$$\text{Property of F}(\mathcal{H}) \text{ iff } \exists \leq \ F(\mathcal{H}[\leq])i \rightarrow \exists FGBi$$

Here, we again take a preference ordering to determine the current belief, but we only require that the convergence happens only for some environments. We can immediately see that this is an overuse of the scheme. To guarantee an "honest" convergence, we have to require that it happens for all allowed sequences of events. Otherwise we have to deal with a situation in which the correct answer is "communicated" to the learner by a particular sequence encoding the answer.

3.3 Characterizing protocols that guarantee learnability

We now prove representation theorems that characterize classes of $DETL$ models in which learnability is guaranteed. We start by giving two intuitive results and then we move to give a $DETL$ counterpart of Angluin's Theorem.

Proposition 3.3. *A synchronous, perfect recall, perfect observation $DETL$ model $\langle W, \Sigma, H, \sim, \leq, V\rangle$ satisfies* finite identifiability *whenever for all $w \in W$ and history $wh \in H \cap \Sigma^\omega$, there is some natural number $n \in \omega$ such that for every $v \neq w$ such that $v \in W$ and for every $wh' \in H \cap \Sigma^\omega$ we have $(h|n) \neq (h'|n)$*

Proposition 3.4. *A permutation closed, synchronous, perfect recall, perfect obervation $DETL$ model $\langle W, \Sigma, H, \sim, \leq, V\rangle$ based on a finite state space satisfies* finite identifiability *whenever for all $w \in W$ for all $v \neq w$ there is some event $a \in (\Sigma \cap \mathfrak{P}(w) - \mathfrak{P}(v))$.*

We now turn to a DETL counterpart to one of the most important results in learning theory: Angluin's theorem. This theorem characterizes learnable classes of sets.

Theorem 3.5 (Angluin [1])**.** *A class of sets C is identifiable in the limit iff for all $S \in C$ there is a finite $D_S \subseteq S$ such that for all $S' \in C$, if $S \neq S'$ and $D_S \in S'$, then $S' \not\subseteq S$.*

The next result is proved using once more the concept of a DEL-generated forest. Before we state the result, let us introduce the following definitions:

Set-driven A local protocol P for $\mathcal{M}$ is set-driven iff $\forall w \exists S_w \subseteq \mathbb{N}$ such that $\forall \varepsilon \in P(w)\ content(\varepsilon) = S_w$.

A-condition for protocols A local protocol P satisfies the A-condition iff

$$\forall w \exists e \in P(w) \cap \Sigma^* \forall w \neq v (e \in P(v) \implies P(v) \not\subset P(w)).$$

Finite identifiability of the incomparable A local protocol P satisfies the condition of finite identifiability of the incomparable sets iff states whose image under P are $\subseteq$-incomparable constitute finitely identifiable classes.

Let us assume that a local protocol P satisfies finite identifiability of the incomparable. The we can show the following equivalence.

Theorem 3.6. *A state space W together with a set-driven local protocol P satisfies A-condition iff there is a preference ordering $\leq$ on W and an epistemic plausibility frame $M = (W, \sim, \leq)$, where $\sim = W \times W$ such that*

(#) for all $w \in W$ and for all $\varepsilon \in P(w)$ there is some $n \in \omega$ such that for every $m > n$, $w \in |M^{\varepsilon|m}|$ and w is the $\leq$-minimum of $|M^{\varepsilon|m}|$ in the generated doxastic model $M^{\varepsilon|m}$.

4 Notions from LT vs Notions from DTL

Learners vs Doxastic Agents. In Learning Theory the agent is usually assumed to be an arbitrary computably enumerable function that transforms finite sequences of data into conjectures about the underlying rules. This setting is very general and allows for analyzing learning settings that are in fact epistemically quite improbable. Various restrictions has been put on the learners to see how different "impairments" affect learning. Let us briefly discuss some of them.

Consistency Learner's answers are always consistent with the up-to-now given data.

Conservatism Learner changes his conjecture only when it is necessary, i.e. only when the last piece of data contradicts the previous conjecture.

Set-drivenness Let L be a learning function. L is set driven iff for all $t, s \in SEQ$, if $set(t) = set(s)$, then $L(t) = L(s)$.

The correspondence between Learners and doxastic agents can be established via different epistemic interpretations of the learning situation. We present our conjectures in Table 1.

DETL notions	LT properties
hard incoming information	consistent learning
preference stability	conservative learning
uniform no miracles	set-driven learning

Table 1: DETL Notions vs Properties of Learners

Non-introspective knowledge. Notions of knowledge and belief are not explicitly involved in Formal Learning Theory. However they are a by-product of its analysis of inductive inference. In a successful identification in the limit the learner stabilizes to a correct hypothesis (belief). But, the process leads to something more than belief. LT shows that for some classes of problems there are *procedures* of belief change that guarantee success. After converging Learner's beliefs are safe, they will not change under any true information. LT thus stress the operational aspect of a successful belief-change. The process results therefore in a state of "non-introspective knowledge". In $DETL$ terms $\not\Vdash \forall FGB\phi \ \rightarrow K\forall FGB\phi$.

5 Multi-agent learning

Let us now have a look at learning from the perspective of multi-agents logics of belief change. What if more than one agents are learning together? In some epistemic scenarios, like *Muddy Children* [7], the fact that the agents are publicly sharing something about their doxastic state is crucial. To illustrate this let us take a pointed epistemic model that specifies the Muddy Children scenario with at least two muddy children. Imagine that for every child we have an atomic formula p_i, indicating that child i is muddy. Let us compare two protocols.

1. The father keeps announcing $\bigvee_i p_i$. The protocol maps each state in which at least one of the children is muddy to the prefix-closure of $\{!\bigvee_i p_i\}^\omega$ and to the empty set otherwise.

2. Similarly, the father keeps announcing $\bigvee_i p_i$ in muddy state, but before each announcement the children publicly announce their belief.

To define formally the second type of protocol we introduce an abstract, model-independent concept of belief announcement. We thus start by extending our set of events so that it includes the event $!B$ which intuitively corresponds the fact that the agents are making their beliefs public. $!B$ is treated as a special kind of event-model for which Product Update is defined as follows.

Definition 5.1. *Let $\mathcal{M} = \langle W, (\sim_i)_{i\in N}, (\leq_i)_{i\in N}, V\rangle$ be a doxastic epistemic model. The result of updating $\mathcal{M}$ by $!B$ is the model $\mathcal{M}\otimes !B := \langle W\{!B\}, (\sim_i)^{!B}_{i\in N}, (\leq_i)^{!B}_{i\in N}, V^{!B}\rangle$ where $W\{!B\} = \{w!B | w \in W\}$ and*

- $w!B \sim^{!B}_j w'!B$ *iff* $(w \sim_j w' \wedge \forall i \in N\ B_i[w] = B_i[w'])$
- $w!B \leq^{!B}_j w'!B$ *iff* $w \leq_j w'$. *And finally* $w!B \in V^{!B}(p)$ *iff* $w \in V(p)$.

This special event has a natural counterpart in DETL style frameworks.

Definition 5.2. *$!B$ is the* Belief Announcement *event iff for all h, h' we have $h!B \sim_j h'!B$ iff $(h \sim_j h' \wedge \forall i \in N\ B_i[h] = B_i[h'])$.*

It is now possible to define the transformation of protocol we have mentioned at the beginning of this section. We leave the definition to the appendix and rather go on to present our findings. Simply note that $P^{!B}$ stand transformation of P as explained in our starting example. The following proposition extends the representations results of Section 2 to include Belief Announcement and will be useful to prove our next result.

Proposition 5.3. *Let $\mathcal{H}$ be an $DETL$-model. The following are equivalent:*

1. *$\mathcal{H}$ is isomorphic to the forest generated by the sequential product update of some epistemic plausibility model according to the $!B$ transformation of some state-dependent DEL-protocol*

2. *$\mathcal{H}$ satisfies $\{!B\}/\Sigma$-alternation, $PR(N)$, $\Sigma - UNM(N)$, $SYN(N)$, Propositional Stability, $BI(N)$ $PS(N)$ and $!B$ is the* Belief Announcement *event.*[6]

This representation result will be useful since it allows to talk of such DEL-generated forests as particular of $DETL$ models. The next theorem proves that in typical DEL situations if agents starts with the same background information and have the same observational powers then allowing the agents to announce their beliefs does not allow them to learn anything more.

Theorem 5.4. *Whenever agents have the same background information and the same observational powers, then there is no knowledge gain by forcing announcement of beliefs between each step. Formally, let $\mathcal{M}$ be a doxastic epistemic model that satisfies $SII(N)$ and P a local protocol for $\mathcal{M}$ such that P satisfies $SOC(N)$. Let $\mathcal{F}or(\mathcal{M}, P)[(\sim_i)_{i\in N}]$ be the doxastic epistemic forest generated by $\mathcal{M}$ and P and $\mathcal{F}or(\mathcal{M}, !B(p))[(\sim^{!B}_i)_{i\in N}]$ be the doxastic epistemic forest generated by $\mathcal{M}$ and $!B(p)$.*

We have $h \sim_i h'$ iff $!B(h) \sim^{!B}_i !B(h')$ for all $i \in N$.

The following corollary is of interest from the perspective of learning theory

Corollary 5.5. *Given some EP model $\mathcal{M}$ in which the agents have the same background information and a protocol P for $\mathcal{M}$ in which agents have same observational powers. $\forall j \in N$, $\forall w \in |\mathcal{M}|$ and $\forall \epsilon \in P(w)$, j's belief stabilizes on the same initial state of the world in the model without announcement of beliefs ($\mathcal{F}or(\mathcal{M}, p)$) and in the model with announcement of beliefs ($\mathcal{F}or(\mathcal{M}, !B(p))$).*

[6] The missing notions are defined in the appendix.

6 Conclusion

This work provides a comparison of the notions of learning theory, doxastic temporal logic and dynamic doxastic logic. We show than the problem of learnability can either be reduced to the model checking problem or to the validity problem of some doxastic temporal language. We provide a main theorem characterizing identifiability in the limit in terms of properties of temporal protocols. We also introduce multi-agent identification, and give conditions under which the possibility of communication between the agents does not influence their individual learning process.

We have shown that the two prominent approaches, learning theory and doxastic temporal logic, can be joined in order to describe the notions of belief and knowledge involved in inductive inference. Also, our representation of hypothesis space and environments gives an additional interesting application for the theory of protocols. We believe that bridging the two approaches together benefits both sides. For formal learning theory, to create a logic for it is to provide additional insight into the semantics of inductive learning. For logics of epistemic and doxastic change, it enriches their present scope with different learning scenarios, i.e. those based not only on the incorporation of new data but also on generalization. Further issues that we are interested in are: extending our approach to other types of identification, e.g. identification of functions; finding modal framework for learning from both positive and negative information; studying systematically the effects of different constraints on protocols.

References

[1] Dana Angluin. Inductive inference of formal languages from positive data. *Information and Control*, 45(2):117–135, May 1980.

[2] Alexandru Baltag and Lawrence S. Moss. Logics for Epistemic Programs. *Synthese*, 139(2):165–224, 2004.

[3] Alexandru Baltag, Lawrence S. Moss, and Slawomir Solecki. The logic of public announcements, common knowledge, and private suspicions. In *TARK '98: Proceedings of the 7th conference on Theoretical aspects of rationality and knowledge*, pages 43–56, San Francisco, CA, USA, 1998. Morgan Kaufmann Publishers Inc.

[4] Alexandru Baltag and Sonja Smets. A qualitative theory of dynamic interactive belief revision. In Giacomo Bonanno, Wiebe van der Hoek, and Michael Wooldridge, editors, *Logic and the Foundation of Game and Decision Theory (LOFT7)*, volume 3 of *Texts in Logic and Games*, pages 13–60. Amsterdam University Press, 2008.

[5] J. van Benthem, Jelle Gerbrandy, Tomohiro Hoshi, and Eric Pacuit. Merging frameworks for interaction: DEL and ETL. *Journal of Philosophical Logic*, 2009.

[6] B. ten Cate. *Model theory for extended modal languages.* PhD thesis, University of Amsterdam, 2005. ILLC Dissertation Series DS-2005-01.

[7] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge.* MIT Press, Cambridge, 1995.

[8] Nir Friedman and Joseph Y. Halpern. Modeling belief in dynamic systems, part i: foundations. *Artificial Intelligence*, 95(2):257–316, 1997.

[9] E.M. Gold. Language identification in the limit. *Information and Control*, 10:447–474, 1967.

[10] Steffen Lange, Rolf Wiehagen, and Thomas Zeugmann. Learning by erasing. In *Proc. 7th Int. Workshop on Algorithmic Learning Theory*, Lecture Notes in Artificial Intelligence, pages 228–241. Springer-Verlag, 1996.

[11] Rohit Parikh and Ramaswamy Ramanujam. A knowledge based semantics of messages. *Journal of Logic, Language and Information*, 12(4):453–467, 2003.

[12] Ron van der Meyden and Ka-shu Wong. Complete axiomatizations for reasoning about knowledge and branching time. *Studia Logica*, 75(1):93–123, 2003.

Appendix: Proofs

Proof of Corollary 2.14

Proof. Since the epistemic part Theorem 2.13 of is unchanged we have only have to do two things.

1. Prove soundness of $PS(N)$
2. Fix the construction for the plausibility ordering

For soundness of $PS(N)$. Assume that in a $DETL$ forest generated by a protocol we have $whe, vh'e' \in$ and $wh \leq_i vh'$. It follows by Definition 2.11 that $whe \leq_i vh'e'$.

For the construction we prove that $whe \leq_i vh'e'$ iff $whe \leq_i^{\mathcal{F}} vh'e'$. The base follows from Definition 2.11. Now for the inductive step. From left to right assume that $whe \leq_i vh'e'$ in the ETL model. It follows by $PS(i)$ that $wh \leq_i vh'$. But by IH this means that $wh \leq_i^{\mathcal{F}} vh'$. But then by Product Update (cf. Definition 2.8) we have $whe \leq_i^{\mathcal{F}} vh'e'$. From right to left assume that $whe \leq_i^{\mathcal{F}} vh'e'$. We have then by definition of Product Update $wh \leq_i^{\mathcal{F}} vh'$ but then by IH, $wh \leq_i vh'$ in the ETL model and thus by $PS(i)$, $whe \leq_i vh'e'$. □

Proof of Proposition 3.2 In this paper it will be useful to work with an hybrid extension of the $\mathcal{L}_{DET}$ language. In which cases the language, which gains two more clauses $\phi := p|x|\ldots|\downarrow x.\phi$ (where x ranges over a countable set of node variables SVAR), is interpreted as before but together with an assignment function $g : \text{SVAR} \to H$. The intuition is that x can be used to name some node in the tree that can be refer to as a propositional letter and that $\downarrow x$ is the action of setting the variable x to name the current node. More details in [6]. We only give the clauses for which the addition of g matters:

$$\begin{array}{lll} \mathcal{H}, w\epsilon, w\vec{e}, g \Vdash x & \text{iff} & g(x) = w\vec{e} \\ \mathcal{H}, w\epsilon, w\vec{e}, g \Vdash \downarrow x.\phi & \text{iff} & \mathcal{H}, w\epsilon, w\vec{e}, g[g(x) := w\vec{e}] \Vdash \phi \end{array}$$

Proof. We sketch the proof of 1 ⇒ 2. We prove the contrapositive. Assume that there is a state $s \in W_\Omega$ and ω-sequence $\epsilon \in P_\Omega(s)$ such that agent's knowledge does not stabilize to s on ϵ. There are two cases. Case 1: The learner stabilizes to another state, but by construction of $P_\Omega(s)$ and definition of a generated DEL-forest for every finite prefix $\vec{e} \sqsubset \epsilon$, $s\vec{e} \in \mathcal{K}[s\vec{e}]$, contradiction. So we are in the other case. Case 2: After each finite prefix $\vec{e} \sqsubset \epsilon$, there is at least a state different from s that remains epistemically possible. Since generated DEL forest satisfies Perfect Recall (Theorem 2.13), it follows that there is some state $s \neq t$ that remains epistemically possible after each finite prefix $\vec{e} \sqsubset \epsilon$. But by construction of $P_\Omega(s)$ this is only possible if $s \subset t$. Now assume for contradiction that Ω is Finitely Identifiable. It follows that the learner stops after some finite prefix $e^* \sqsubset \epsilon$. There are two possibilities. Case 1: Learner select s after e^*. But since $s \subset t$, e^* is the finite prefix of some environment ϵ_t for t. So assume that Nature chooses t and ϵ_t, then learner will stop after e^* and select s. But this contradicts the fact that Ω is Finitely Identifiable. Case 2: Learner select another state than s. But then the learner fails to identify s on ϵ. Contradiction.

We now sketch the proof of 2 ⇒ 3. We prove the contrapositive. Assume that $\mathcal{F}or(F(\mathcal{M}_\Omega^{Sit}), V_\Omega, P_\Omega) \nVdash \mathtt{A}(\bigcirc^{-1}\bot \to \downarrow x.\forall FKH(\bigcirc^{-1}\bot \to x))$. But this means that we some history that satisfies $\bigcirc^{-1}\bot$, i.e. by truth condition, some initial state in $w \in W_\Omega$ such that for some $\epsilon \in P_\Omega$ and for all finite prefix $e^* \sqsubset \epsilon$ we have $\mathcal{F}or(F(\mathcal{M}_\Omega^{Sit}), V_\Omega, P_\Omega)w, w\epsilon, we^*, g[g(x) := w] \nVdash KH(\bigcirc^{-1}\bot \to x))$. By truth condition of K and $H(\bigcirc^{-1}\bot \to x))$ this means that there is some history $vh \in \mathcal{K}[we^*]$ such that $v \neq w$. But this means that Learner's knowledge does not stabilize to w on $w\epsilon$ in $\mathcal{F}$ or$(F(\mathcal{M}_\Omega^{Sit}), V_\Omega, P_\Omega)$. Concluding this direction. □

Proof of Proposition 3.3

Proof. From right to left. Take an arbitrary w. We have by assumption some $n \in \omega$ such that for every $v \neq w$ such that $v \in W$ and for every $wh' \in H \cap \Sigma^\omega$ we have $(h|n) \neq (h'|n)$. We prove that $w(h|n) \not\sim v(h'|n)$ by an inductive argument. Indeed assume that they are in the same information partition. Then by Perfect Observation the last events were the same. But PR we also have that the nodes right before were also in the same information partition so we can iterate this argument and apply Perfect Observation all the way down, proving that $(h|n) \neq (h'|n)$. Contradiction. □

Proof of Proposition 3.4

Proof. From right to left. Take an arbitrary w and $\epsilon(w)$. For each $v \neq w$ we have some a that occurs by Permutation Closure on $\epsilon(w)$. We refer to it as $a_w(v)$. Since W is finite so is $\{\epsilon^{-1}(a_w(v))|v \neq w\}$. We can thus take the least upper bound of the previous set, call it m. Now assume that some state $v \neq w$ is still considered possible at $\epsilon(w)(m+1)$. By the argument in previous proof, it means that there is an environment in $P(v)$ in which also $a_w(v)$ occurs. Contradiction. □

Proof of Theorem 3.6

Proof. ($\Rightarrow$) Let us assume that W, P satisfies A-condition, well-foundedness and finite identifiability of the incomparable. Let us define the preference ordering $\leq$ in the following way:

$$v \leq w \text{ iff } P(v) \subseteq P(w).$$

Note by the fact that we deal with a doxastic epistemic model and protocol that correspond to a set learning situation we have that $v \simeq w$ iff $v = w$, where $v \simeq w$ iff $v \leq w$ and $w \leq v$ (1).

We have to show that $\leq$ satisfies (#). Let us then take a hypothesis $w \in W$ and choose one environment for w, i.e. a particular $\varepsilon \in P(w)$. We show that there is some $n \in \mathbb{N}$ such that for every $m > n$, $w \in |M^{\varepsilon|m}|$ and w is the $\leq$-minimum of $|M^{\varepsilon|m}|$ in the generated doxastic model $M^{\varepsilon|m}$.

To show that we have to consider all $v \neq w$ such that $v \leq w$ or such that v is $\leq$-incomparable to w. We show that there is a finite stage of the epistemic update at which v is eliminated, i.e. w is the $\leq$-minimal element of $|M^{\varepsilon|m}|$.

Let us first take $v \in W$ such that $v \leq w$. By (1), if $v \leq w$ then $P(v) \subset P(w)$. Then there is a sequence $e \in \Sigma$ such that $e \in P(w)$ but $e \notin P(W)$. And since protocols allow environments that enumerate all and only elements from the set S_w, e will appear at some point and at which v will be eliminated as inconsistent with e.

The fact that all $v \leq w$ are going to be eliminated at some finite stage is guaranteed by the fact that the protocol satisfies the A-condition, i.e. there is no $w \in W$ such that for all $e \in P(w) \cap \Sigma^*$ there is $v \in W$ such that $v \neq w$ and $P(v) \subset P(w)$, which implies that for each $w \in W$ there is only finite number of $v \in W$, such that $v \leq w$.[7]

If v is $\leq$-incomparable to w, then $P(v) \nsubseteq P(w)$ and $P(w) \nsubseteq P(v)$. Therefore there is an event $e \in \Sigma$ such that $e \in P(w)$ and $e \notin P(v)$. And since protocols allow environments that enumerate all and only elements from the set S_w, e will appear at some point and at which v will be eliminated as inconsistent with e.

Moreover, all $v \in W$ such that v is $\leq$-incomparable to w will be eliminated at some finite stage by assumption of finite identifiability of the incomparable.[8]

Therefore we can conclude that at some finite stage m, all $v \in W$ that are either $\leq$-smaller that w or are $\leq$-incomparable to w will get eliminated, leaving w the smallest state in $|M^{\varepsilon|m}|$.

($\Leftarrow$) Assume that there is a preference ordering on W, such that it satisfies (#).

To see that the underlying protocols satisfy A-condition for each $w \in W$ we take $\varepsilon_w \in P(w)$ and, from the assumption, for each ε_w there is n such that for all $m \geq n$, $M^{\varepsilon_w|m} = M'$ and in w is minimal wrt $\leq$ in M'. Let us take $\varepsilon_w|m = \sigma_w$. Since for each w, σ_w is finite it is enough to show that for all $v \in W$ such that $v \neq w$ if $\sigma_w \in P(v)$ then $P(v) \nsubseteq P(w)$.

For a contradiction assume that there is $v \in W$ such that $\sigma_w \in P(v) \wedge P(v) \subset P(w)$. Let $\tau \in P(v)$ such that $\tau|length(\sigma_w) = \sigma_w$ (there is such because $\tau \in P(v)$). From the assumption, M^τ converges to a model that has w as minimal wrt to $\leq$. But $v \neq w$, so we have that for one environment v, namely $\tau \in P(v)$, M^τ will converge to a model with w as the minimal and not v. Contradiction.

□

[7]A counterexample is the class of sets $C = \{\{1\}, \{1,2\}, \{1,2,3\}, \ldots, \mathbb{N}\}$. Using the chosen preference relation $\mathbb{N}$ cannot be identified.

[8]Otherwise the class of sets $C = \{Even, Even - \{2\} \cup \{3\}, Even - \{4\} \cup \{5\}, \ldots\}$ is allowed, and it is clear we cannot get the 'Even' set to become the $\leq$-minimal after any finite number of steps.

Definitions omitted in Section 5. In Section 5 we omitted these lengthy definition which are necessary to understand the proofs to follow.

First we define the transformation of a sequence of events and then of protocol as we have mentioned at the beginning of Section 5.

Definition 6.1. *Given a finite sequence of events $e_1 \dots e_n \in E^*$, let $!B(e_1 \dots e_n)$ denotes the sequence $!Be_1!Be_2 \dots !Be_n!B$. In particular $!B(\lambda) = !B$. Given an ω-sequence ϵ we define $!B(\epsilon)$ as follows:*

$$!B(\epsilon)(n) = \begin{cases} \epsilon(n/2) & \textit{if } n \textit{ is even.} \\ !B & \textit{otherwise} \end{cases}$$

Definition 6.2 ($!B$ transformation of a protocol)**.** *Given a EP model $\mathcal{M}$ and a local protocol P for $\mathcal{M}$, the $!B$ transformation of P is the function $P^{!B}$ such that for each w, $P^{!B}(w)$ is the smallest set of sequences such that $!B(h) \in P^{!B}(w)$ iff $h \in P(w)$ and $P^{!B}(w)$ is closed under non-empty finite-prefixes.*

Definition 6.3 ($DETL$ model generated by $!B$ transformation of a protocol)**.** *Due to the alternation, we only have to add the following elements to the construction of Definition 2.11.*

- *For $1 < k \leq m$, $h!B \sim_i h'!B$ iff $h!B, h'!B \in H_k$, $h \sim_i h'$ and for all $j \in N$, $\mathcal{B}_j[h] = \mathcal{B}_j[h']$.*
- *For $1 < k \leq m$, $h!B \leq_i h'!B$ iff $h!B, h'!B \in H_k$ and $h \leq_i h'$*

Definition 6.4. *We say that a $DETL$ forest $\mathcal{H}$ satisfies Σ/Σ'-alternation iff for all history $whe, vh'f \in H$ we have ($e \in \Sigma$ iff $f \in \Sigma'$) iff $len(whe) = len(vh'f)$ and moreover there is an history of the form $we \in H$ such that $e \in \Sigma$.*

We get the following result as corollary of Corollary 2.14.

Proof of Proposition 5.3

Proof. **Necessity** ($1 \implies 2$)**.** We show that the given conditions are satisfied by any $DETL$ model generated through successive Product Updates following $!B$-transformation of a protocol.

This direction of the proof is easy so we only give the details for the other direction.

Sufficiency ($2 \implies 1$)**.** Given a $DETL$ model $\mathcal{H}$ satisfying the stated conditions, we show how to construct an epistemic plausibility model $\mathcal{M}$ and protocol P such that $\mathcal{F}$ is isomorphic to $\mathcal{F}or(\mathcal{M}, !B(P))$.

Construction We first construct the initial model, then we construct our protocol.

Here is the initial epistemic plausibility model $\mathcal{M}_0 = \langle W_0, (\sim_i^0)_{i \in N}, (\leq_i^0)_{i \in N}, \hat{V}_0 \rangle$:

- $W_0 := \{w \in W \mid len(h) = 1\}$.
- Set $w \sim_i^0 v$ iff $w \sim_i v$.
- Set $w \leq_i^0 v$ iff $w \leq_i v$.
- For every $p \in Prop$, $\hat{V}_0(p) = V(p) \cap W$.

Now for the protocol. By SYN(N), Σ-UNM and $\{!B\}/\Sigma$-alternation we can safely construct event models from synchronous slices of of the $DETL$ model $\mathcal{H}$. Now we construct the j-th event model $\epsilon_j = \langle E_j, (\preceq_i^j)_{i \in N}, \mathtt{pre}_j \rangle$.

If j is *odd*, then:

- $E_j := \{!B \in \{!B\} \mid$ there is a history $wh!B \in H$ with $len(h) = j\} = \{!B\}$

If j is *even*, then:

- $E_j := \{e \in \Sigma \mid$ there is a history $he \in H$ with $len(h) = j\}$
- Set $a \sim_i^j b$ iff there are $ha, h'b \in H$ such that $len(h) = len(h) = j$ and $ha \sim_i h'b$.

- For each $e \in E_j$, let $\mathtt{pre}_j(e) = \top$

We now define the protocol !B(P) as follows. For every $w \in W_0$, every $n \in \omega$ and ϵ with $len(\epsilon) = n$, we let

1. $\epsilon|n-1!B \in P(w)$ iff $w\epsilon \in H$, whenevern is odd
2. $\epsilon|n-1(E_n, \epsilon(n)) \in P(w)$ iff $w\epsilon \in H$, whenever n is even

Finally we let P we define inductively from $!B$ to recover the sequences by eliminating $!B$ events which is possible by $\{!B\}/\Sigma$-alternation.

Now we show that the construction is correct in the following sense:

Claim 6.5 (Correctness). *Let $\sim$ be the epistemic relation in the given doxastic epistemic temporal model. Let $\sim^{\mathcal{F}}$ be the epistemic relation in the forest model induced by Product Update over the just constructed plausibility model $\mathcal{M}_0$ and the constructed protocol $!B(P)$ we have:*

$$h \sim h' \text{ iff } h \sim^{\mathcal{F}} h' \wedge h \leq h' \text{ iff } h \leq h'$$

Proof of the claim. The proof is by induction on the length of histories (which is possible by $SYN(N)$. The case for the plausibility ordering follows easily from Corollary 2.14. Now we consider the case of the epistemic relation. The base case is obvious from the construction of our initial model $\mathcal{M}_0$.

Now comes the induction step:

From $DETL$ to $\mathcal{F}or(\mathcal{M}, !B(P))$. Due to $\{!B\}/\Sigma$-alternation, we are in one of two cases depending the parity of the length of the histories we are considering.

The even case if as in Theorem 2.13. Assume it is odd. Now assume that $h_1!B \sim_j h_2!B$ (1). It follows from Perfect Recall that $h_1 \sim_j h_2$ (2). But from IH we have then $h_1 \sim_j^{\mathcal{F}} h_2$ (3). From the assumption that $!B$ is *Belief Announcement* event and (1) it follows that $\forall i \in N\ \mathcal{B}_i[h_1] = \mathcal{B}_i[h_2]$), but by definition of $\mathcal{B}_i$ and IH, it follows that $\forall i \in N\ \mathcal{B}_i^{\mathcal{F}}[h_1] = \mathcal{B}_i^{\mathcal{F}}[h_2]$) (4). But by (3), (4) and Product Update for $!B$ we have $h_1!B \sim_j^{\mathcal{F}} h_2!B$.

From $\mathcal{F}or(\mathcal{M}, !B(P))$ to $DETL$. Again due to $\{!B\}/\Sigma$-alternation, we are in one of two cases depending the parity of the length of the histories we are considering.

The even case if as in Theorem 2.13. Assume it is odd. Now assume that $h_1!B \sim_j^{\mathcal{F}} h_2!B$ by the definition of Product Update for $!B$ we have $h_1 \sim_j^{\mathcal{F}} h_2$ (5) and $\forall i \in N\ \mathcal{B}_i^{\mathcal{F}}[h_1] = \mathcal{B}_i^{\mathcal{F}}[h_2]$) (6). From (5) and IH we have $h_1 \sim_j h_2$ (7). From $\forall i \in N\ \mathcal{B}_i^{\mathcal{F}}[h_1] = \mathcal{B}_i^{\mathcal{F}}[h_2]$) (8) and IH and $\mathcal{B}_i$ we have $\forall i \in N\ \mathcal{B}_i[h_1] = \mathcal{B}_i[h_2]$) (9). But (9), (6) and he assumption that $!B$ is *Belief Announcement* event we have $h_1!B \sim_j h_2!B$. □

Proof of Theorem 5.4 In the proof of this theorem we will need the two following Lemmas.

Lemma 6.6 (Same info Lemma). *Let $\mathcal{H}$=$\langle W, \Sigma, H, (\leq_i)_{i\in N}, (\sim_i)_{i\in N}, V\rangle$ be a doxastic epistemic model satisfying $SOC(i,j)$, $PR(i,j)$, $SYN(i,j)$ and $SII(i,j)$, it follows that for all $h', h \in H$, we have $h \sim_i h'$ iff $h \sim_j h'$.*

Proof. The proof is by induction on the length of h, h'. The proof by induction is justified by $SYN(i,j)$.

Base case is immediate by $SII(i,j)$.

Induction step. Assume that $ve_1 \dots e_{n+1} \sim_i we'_1 \dots e'_{n+1}$ (a). By $PR(i)$ we have we have $ve_1 \dots e_n \sim_i we'_1 \dots e'_n$ (b). But then by IH we have $ve_1 \dots e_n \sim_j we'_1 \dots e'_n$ (c). From (b), (c), (a) and $SOC(i,j)$ it follows that $ve_1 \dots e_{n+1} \sim_j we'_1 \dots e'_{n+1}$. Other direction is of course identical. □

Lemma 6.7 (Inter-model No Miracles Lemma). *Let $\mathcal{M}$ be a doxastic epistemic model. Let P be local protocol for $\mathcal{M}$. Let $\mathcal{F}or(\mathcal{M}, p) = \langle W, \Sigma, H, (\leq_i)_{i\in N}, (\sim_i)_{i\in N}, V\rangle$ be the doxastic epistemic forest generated by $\mathcal{M}$ and P and $\mathcal{F}or(\mathcal{M}, !B(p)) = \langle W, \Sigma \cup \{!B\}, H^{!B}, (\leq_i)_{i\in N}, (\sim_i^{!B})_{i\in N}, V\rangle$ be the doxastic epistemic forest generated by $\mathcal{M}$ and $!B(P)$. If $we_1 \dots e_{n+1} \sim_i ve'_1 \dots e'_{n+1}$ and $w!B(e_1 \dots e_n) \sim_i^{!B} v!B(e'_1 \dots e'_n)$, then $w!B(e_1 \dots e_n)e_{n+1} \sim_i^{!B} v!B(e'_1 \dots e'_n)e_{n+1}$*

Proof. By hypothesis $we_1 \dots e_{n+1} \sim_i ve'_1 \dots e'_{n+1}$. But then by the definition of Product Update we have $e_{n+1} \sim_i e'_{n+1}$. By definition of $!B(P)$ it follows that $e_{n+1} \sim_i^{!B} e'_{n+1}$ (1). Now by hypothesis we have $w!B(e_1 \dots e_n) \sim_i^{!B} v!B(e'_1 \dots e'_n)$ (2). But by (1), (2) and Product Update we have $w!B(e_1 \dots e_n)e_{n+1} \sim_i^{!B} v!B(e'_1 \dots e'_n)e_{n+1}$. □

We can now start with the proof of the Theorem 5.4.

Proof. Proof is by induction on the length of the history, which is allowed Synchronicity[9] We start by the easy direction: From right to left.

Base case Assume that $!B(w) \sim_i^{!B} !B(v)$. It follows by PR(N) that $w \sim_i^{!B} v$. But by construction the initial models are identical, thus $w \sim_i v$.

Induction step. Assume that $w!B(e_1 \dots e_n e_{n+1}) \sim_i^{!B} v!B(e'_1 \dots e'_n e'_{n+1})$. It follows by Perfect Recall that $w!B(e_1 \dots e_n)e_{n+1} \sim_i^{!B} v!B(e'_1 \dots e'_n)e'_{n+1}$ (1). By Perfect Recall, it follows from (1) that $w!B(e_1 \dots e_n) \sim_i^{!B} v!B(e'_1 \dots e'_n)$ (2). By IH and (2) we have $we_1 \dots e_n \sim_i ve'_1 \dots e'_n$ (3). But then by Lemma 6.7, (1) and (3) it follows that $we_1 \dots e_{n+1} \sim_i ve'_1 \dots e'_{n+1}$.

Now for the more interesting direction: From left to right. We re-start counting of propositions.

Base case. Assume that $v \sim_i w$ (1). We prove that $v!B \sim_i^{!B} w!B$. Take an arbitrary agent j. From (1) we have $K_i[v] = K_i[w]$ (2). By $SII(i,j)$ and (2) it follows that:

$$K_j[w] = K_i[w] = K_i[v] = K_j[v] \quad (3)$$

But then by $BI(j)$ and (3) we have $B_j[w] = B_j[v]$ (4). Since j was arbitrary it follows from (4), (2) "$!B$ is the *Belief Announcement* event" that $w!B \sim_i^{!B} v!B$.

Induction step. Assume that $ve_1 \dots e_{n+1} \sim_i we'_1 \dots e'_{n+1}$ (5). We prove that $w!B(e_1 \dots e_{n+1}) \sim_i^{!B} v!B(e'_1 \dots e'_{n+1})$. First of all it follows from (5) and $PR(i)$ that $ve_1 \dots e_n \sim_i we'_1 \dots e'_n$ (6). But then by (6) and (IH) we have $w!B(e_1 \dots e_n) \sim_i^{!B} v!B(e'_1 \dots e'_n)$ (7).

Now take an arbitrary $j \in N$. By (6) and Lemma 6.6 it follows that $ve_1 \dots e_n \sim_j we'_1 \dots e'_n$ (8). By IH and (8) it follows that $v!B(e_1 \dots e_n) \sim_j^{!B} w!B(e'_1 \dots e'_n)$ (9). By (5) and Lemma 6.6 it follows that $ve_1 \dots e_{n+1} \sim_j we'_1 \dots e'_{n+1}$ (10).

Now from (10), (9) and $\Sigma - UNM_j$ that $v!B(e_1 \dots e_n)e_{n+1} \sim_j^{!B} w!B(e'_1 \dots e_n)e'_{n+1}$ (11). Similarly from (5), (7) and UNM_i we have $v!B(e_1 \dots e_1)e_{n+1} \sim_j^{!B} w!B(e'_1 \dots e_n)e'_{n+1}$ (12). By $BI(j)$ and (11) it follows that $B_j[v!B(e_1 \dots e_1)e_{n+1}] = B_j[w!B(e'_1 \dots e_n)e'_{n+1}]$ (13). Since j was arbitrary it follows from (13), (12) and "$!B$ is the *Belief Announcement* event" that $v!B(e_1 \dots e_{n+1}) \sim_i^{!B} w!B(e'_1 \dots e_n e'_{n+1})$ □

Proof of Corollary 5.5 We start by proving the following Corollary which will make it very easy to prove Corollary 5.5.

Corollary 6.8. *let $\mathcal{M}$ be a doxastic epistemic model that satisfies $SII(N)$ and P a local protocol for $\mathcal{M}$ such that P satisfies $SOC(N)$. Let $\mathcal{F}or(\mathcal{M}, p) = \langle W, \Sigma, H, (\leq_i)_{i \in N}, (\sim_i)_{i \in N}, V\rangle$ be the doxastic epistemic forest generated by $\mathcal{M}$ and p and $\mathcal{F}or(\mathcal{M}, !B(p)) = \langle W, \Sigma \cup \{!B\}, H^{!B}, (\leq_i)_{i \in N}, (\sim_i^{!B})_{i \in N}, V\rangle$ be the doxastic epistemic forest generated by $\mathcal{M}$ and $!B(p)$.*

We have $\exists h = wh' \in B_i[ve_1 \dots e_n]$ iff $\exists h_2 = wh_3 \in B_i^{!B}[!B(ve_1 \dots e_n)]$.

Proof. The proof is by induction on the length of $ve_1 \dots e_n$ which is allowed by the assumption of Synchronicity[10].

Base case. We prove both direction simultaneously. By construction we have for all $v, w \in W$ $w \leq_i v$ iff $w!B \leq_i v!B$. Since by Theorem 5.4 we have $w \sim_i v$ iff $w!B \sim_i v!B$. It follows that for all $v, w \in W$ we have $w \in B_i[v] = Min_{\leq_i} K_i[v]$ iff $w!B \in B_i[v!B] = Min_{\leq_i} K_i[v!B]$.

Induction step. From left to right. Assume that there is some history $wh \in B_i[ve_1 \dots ve_{n+1}]$ (1). It follows that $wh \in K_i[ve_1 \dots ve_{n+1}]$, i.e. $wh \sim_i ve_1 \dots ve_{n+1}$ (2). But then it follows by Theorem 5.4 that $w!B(h) \sim_i v!B(e_1 \dots ve_{n+1})$ (3). No assume that for contradiction that for every h' of the form $w!B(h_2)$, we have $h' \notin B_i[!B(ve_1 \dots e_{n+1})]$ (4). It follows that for every such h' we have some history $s!B(h_3) \in K_i[!B(ve_1 \dots e_{n+1})]$ (5) with $s \neq w$ (6) and $s!B(h_3) <_i h'$ (7). It is easy to check

[9] In the following proof the usage of properties such as Synchronicity is justified by Corollary 2.14 when talking about $\mathcal{F}or(\mathcal{M}, p)$ and by Proposition 5.3 when talking about $\mathcal{F}or(\mathcal{M}, !B(p))$. We drop further reference to this two results in the proof.

[10] Same remark as for the previous proof.

that $len(sh_3) = len(h')$ (8). But then by (7), (8) and Preference Stability we have $s <_i^{!B} w$ (9). By construction it follows that $s < w$ (10). But by (5) and Theorem 5.4 we have $sh_3 \in K_i[ve_1 \dots e_{n+1}]$ (11). But then by Preference Stability we have $sh_3 < w(h_2)$ (12). But then by definition of B_i we have $w(h_2) \notin B_i[ve_1 \dots ve_{n+1}]$ (13). But since h_2 was arbitrary we have in particular $wh \notin B_i[ve_1 \dots ve_{n+1}]$ (14). Contradicting (1). Thus by reduction there is some history h' of the form $w!B(h_2)$ such that $h' \in B_i[!B(ve_1 \dots e_{n+1})]$ (15).

The other direction is similar. □

The proof Corollary 5.5 is now easy.

Proof. Assume that i stabilizes on $v \in W$ after the sequence $we_1 \dots e_n$. Then for every h which extends $we_1 \dots e_n$, all histories in $B_i[h]$ starts with v. But then for every h by the preceding corollary at $!B(h)$, i believes only in histories starting with v. The other direction is similar. □