FORCING ABSOLUTENESS AND REGULARITY PROPERTIES

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ABSTRACT. For a large natural class of forcing notions, we prove general equivalence theorems between forcing absoluteness statements, regularity properties, and transcendence properties over L and the core model K. We use our results to answer open questions from set theory of the reals.

1. INTRODUCTION & BACKGROUND

Forcing absoluteness statements have been investigated by Judah, Brendle, Halbeisen, Amir, Bagaria and others [18, 6, 12, 1, 3]. These statements of the form "Every Γ-statement is absolute between the ground model and its forcing extensions with $\mathbb{P}$" are typically independent of the axioms of ZFC, and can often be proved to be equivalent to statements about *regularity properties*. Typical equivalence theorems are:

Theorem 1.1 (Bagaria, Woodin, [2, 29]). Every $\mathbf{\Sigma}^1_3$-statement is absolute between the ground model and its Cohen forcing extensions if and only if every $\mathbf{\Delta}^1_2$ set has the Baire property.

Theorem 1.2 (Ikegami, [14]). Every $\mathbf{\Sigma}^1_3$-statement is absolute between the ground model and its Sacks forcing extensions if and only if every $\mathbf{\Delta}^1_2$-set either contains a perfect subset or is disjoint from a perfect set.

The mentioned regularity properties are in turn equivalent to *transcendence properties over* L. For instance, Judah and Shelah proved that the Baire property of all $\mathbf{\Delta}^1_2$-sets is equivalent to the transcendence statement "for all reals x, there is a Cohen real over $\mathrm{L}[x]$" [19]; similarly, Brendle and Löwe showed that the statement "every $\mathbf{\Delta}^1_2$ set either contains a perfect subset or is disjoint from a perfect set" is equivalent to "for all reals x, there is a real not in $\mathrm{L}[x]$" [8].

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In this paper, we shall prove a general abstract result underlying both Theorems 1.1 and 1.2, by connecting (for a large class of forcings $\mathbb{P}$) $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness, a regularity property at the $\mathbf{\Delta}^1_2$-level, and a transcendence property related to $\mathbb{P}$. The case of Cohen forcing might suggest that the right transcendence property is the existence of $\mathbb{P}$-generics, but this already fails in the case of Sacks forcing.[1] In order to deal with this situation, Brendle, Halbeisen and Löwe introduced the notion of *quasi-generic reals* [7]. In many cases of c.c.c. forcings (such as Cohen forcing), the notions of quasi-genericity and genericity coincide; in general, the existence of quasi-generics gives us the right transcendence property for our general theorem. We prove:

Theorem 1.3. For any forcing $\mathbb{P}$ in a large class of forcing notions[2], the following are equivalent:
(1) $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds,
(2) every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-measurable, and
(3) for any real a and $T \in \mathbb{P}$, there is a quasi-$\mathbb{P}$-generic real $x \in [T]$ over $\mathrm{L}[a]$.

We shall start by defining and investigating the basic concepts in § 2 and § 3. We then state and prove the main result of the paper (the precise version of Theorem 1.3) and its immediate consequences in § 4. Among the consequences is a general Solovay-style characterization theorem (in the tradition of [26]). In § 5, we move on to $\mathbf{\Sigma}^1_4$-absoluteness and prove the analogues of the results from § 4 under the assumption of appropriate large cardinal axioms. These proofs use some basic facts of inner model theory. In § 6, we give applications of our main results, answering an open question from [7]; finally, in § 7, we list a number of interesting open questions.

2. Basic concepts

From now on, we will work in ZFC. We assume that readers are familiar with the elementary theories of forcing and descriptive set theory. (For basic definitions not given in this paper, see [15, 22].) When we are talking about "reals", we mean elements of the Baire space or of the Cantor space.

In this section, we introduce the notions we will need for the rest. We start with introducing the forcing absoluteness we will focus on:

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[1] In the model after adding ω_1 many Cohen reals to L, every projective set either contains or is disjoint from a perfect set, but there is no Sacks real over L.

[2] We will give the precise class of forcings in Theorem 4.3. Also we will give precise definitions of the notions used here in § 2

Definition 2.1 ($\mathbf{\Sigma}^1_n$-$\mathbb{P}$-absoluteness)**.** Let $\mathbb{P}$ be a forcing notion and n be a natural number with $n \geq 1$. Then $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-*absoluteness* is the following statement:

> "for any Σ^1_n-formula φ, real r in V, and $\mathbb{P}$-generic filter G over V, $V \vDash \varphi(r)$ iff $V[G] \vDash \varphi(r)$".

Definition 2.2 (Projective forcings)**.** Let n be a natural number with $n \geq 1$. A partial order $\mathbb{P}$ is $\mathbf{\Sigma}^1_n$ (resp. $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$) if the sets P, $\leq_{\mathbb{P}}$, and $\perp_{\mathbb{P}}$ are $\mathbf{\Sigma}^1_n$ (resp. $\mathbf{\Pi}^1_n$, $\mathbf{\Delta}^1_n$), where $\mathbb{P} = (P, \leq_{\mathbb{P}})$ and $\perp_{\mathbb{P}}$ is the incompatibility relation in $\mathbb{P}$. We say $\mathbb{P}$ is *projective* if it is $\mathbf{\Sigma}^1_n$ for some $n \geq 1$.

Let n be a natural number with $n \geq 1$. A partial order $\mathbb{P}$ is *provably* Δ^1_n if there are Σ^1_n-formula ϕ and Π^1_n-formula ψ such that the statement "ϕ and ψ define the same partial order with the incompatibility relation" is provable in ZFC.

All typical forcings related to the regularity properties are provably Δ^1_2. In this paper, we are only interested in projective forcings.

In some of our main results, we shall need a strengthening of the standard notion of properness for projective forcings:

Definition 2.3. A projective forcing $\mathbb{P}$ is *strongly proper* if for any countable transitive model M of a finite fragment of ZFC containing the real parameter in the formula defining $\mathbb{P}$, if $P^M, \leq^M_{\mathbb{P}}, \perp^M_{\mathbb{P}}$ are subsets of $P, \leq_{\mathbb{P}}, \perp_{\mathbb{P}}$ respectively, then for any condition p in P^M, there is an $(M, \mathbb{P})$-generic condition q below p, i.e., if $M \vDash$ "A is a maximal antichain in $\mathbb{P}$", then $A \cap M$ is predense below q.[3]

Here $(M, \mathbb{P})$-generic conditions are the same as $(X, \mathbb{P})$-generic conditions for countable elementary substructure X of $\mathcal{H}_\theta$: if $\mathbb{P}$ is projective, X is a countable elementary substructure of $\mathcal{H}_\theta$ for some enough large regular θ and M is the transitive collapse of X, then a condition p is $(M, \mathbb{P})$-generic iff it is $(X, \mathbb{P})$-generic in the usual sense. In particular, if $\mathbb{P}$ is projective and strongly proper, then $\mathbb{P}$ is proper.

All the typical examples of proper, $\mathbf{\Delta}^1_2$-forcings are strongly proper. But there is a proper, provably Δ^1_3-forcing which is not strongly proper (for the details, see the papers [5, 4] by Bagaria and Bosch).

[3]Although we will not explicitly mention the finite fragment of ZFC we will use for the definition of strong properness, it will be enough large so that we can proceed all the arguments in this paper as usual. From now on, we say "countable transitive models of ZFC" instead of "countable transitive models of a finite fragment of ZFC" for simplicity.

We use strong properness instead of properness, as it allows us to leave out the quantification "$\in \mathcal{H}_\theta$" which would increase the complexity of our statements in the relevant results (Proposition 2.17, Theorem 5.3, Theorem 5.6) beyond projective.

Next, we introduce a class of forcings containing all the tree-type forcings. A partial order $\mathbb{P}$ is *arboreal* if its conditions are perfect trees on ω (resp. 2) ordered by inclusion. But this class of forcings contains some trivial forcings such as $\mathbb{P} = \{{}^{<\omega}\omega\}$. We need the following stronger notion:

Definition 2.4. A partial order $\mathbb{P}$ is *strongly arboreal* if it is arboreal and the following holds:

$$(\forall T \in \mathbb{P})\ (\forall t \in T)\ T_t \in \mathbb{P},$$

where $T_t = \{s \in T \mid \text{either } s \subseteq t \text{ or } s \supseteq t\}$.

With strongly arboreal forcings, we can code generic objects by reals in the standard way: let $\mathbb{P}$ be strongly arboreal and G be $\mathbb{P}$-generic over V. Let $x_G = \bigcup\{\mathrm{stem}(T) \mid T \in G\}$, where $\mathrm{stem}(T)$ is the longest $t \in T$ such that $T_t = T$. Then x_G is a real and $G = \{T \in \mathbb{P} \mid x_G \in [T]\}$, where $[T]$ is the set of all infinite paths through T. Hence $V[x_G] = V[G]$. We call such real x_G a *$\mathbb{P}$-generic real over V*.

Almost all typical forcings related to regularity properties are strongly arboreal:

Example 2.5. (1) Cohen forcing ($\mathbb{C}$): let T_0 be ${}^{<\omega}\omega$. Consider the partial order $\big(\{(T_0)_s \mid s \in {}^{<\omega}\omega\}, \subseteq\big)$. Then this is strongly arboreal and equivalent to Cohen forcing.

(2) random forcing ($\mathbb{B}$): consider the set of all perfect trees T on 2 such that for any $t \in T$, $[T_t]$ has a positive Lebesgue measure, ordered by inclusion. Then this forcing is strongly arboreal and equivalent to random forcing.

(3) Hechler forcing ($\mathbb{D}$): for $(n, f) \in \mathbb{D}$, let

$$T_{(n,f)} = \Big\{t \in {}^{<\omega}\omega \mid \text{either } t \subseteq f \restriction n \text{ or } \Big(t \supseteq f \restriction n \text{ and } \big(\forall m \in \mathrm{dom}(t)\big)\ t(m) \geq f(m)\Big)\Big\}.$$

Then the partial order $(\{T_{(n,f)} \mid (n, f) \in \mathbb{D}\}, \subseteq)$ is strongly arboreal and equivalent to Hechler forcing.

(4) Mathias forcing: for a condition (s, A) of Mathias forcing, let

$$T_{(s,A)} = \{t \in {}^{<\omega}\omega \mid t \text{ is strictly increasing and } s \subseteq \mathrm{ran}(t) \subseteq s \cup A\}.$$

Then $\{T_{(s,A)} \mid (s, A) \text{ is a condition of Mathias forcing}\}$ is a strongly arboreal forcing equivalent to Mathias forcing.

(5) Sacks forcing, Silver forcing, Miller forcing, Laver forcing ($\mathbb{S}$, $\mathbb{V}$, $\mathbb{M}$, $\mathbb{L}$, respectively): these forcings can be naturally seen as strongly arboreal forcings.

We now introduce a general definition of a regularity property associated with an arbitrary arboreal forcing. Sets of reals with a regularity property should be approximated by some simple sets (e.g., Borel sets) modulo some "smallness" as Baire property and Lebesgue measurability. Therefore we first introduce "smallness" for each arboreal forcing by deciding a σ-ideal as follows:

Definition 2.6. Let $\mathbb{P}$ be an arboreal forcing. A set of reals A is $\mathbb{P}$-*null* if for any T in $\mathbb{P}$ there is a $T' \leq T$ such that $[T'] \cap A = \emptyset$. $N_{\mathbb{P}}$ denotes the set of all $\mathbb{P}$-null sets and $I_{\mathbb{P}}$ denotes the σ-ideal generated by $\mathbb{P}$-null sets.

Example 2.7. (1) Cohen forcing $\mathbb{C}$: $\mathbb{C}$-null sets are the same as nowhere dense sets and $I_{\mathbb{C}}$ is the meager ideal.

(2) random forcing $\mathbb{B}$: $\mathbb{B}$-null sets are the same as Lebesgue null sets and $I_{\mathbb{B}}$ is the Lebesgue null ideal.

(3) Hechler forcing $\mathbb{D}$: $\mathbb{D}$-null sets are the same as nowhere dense sets in the dominating topology, i.e., the topology generated by $\{[s, f] \mid (s, f) \in \mathbb{D}\}$ where

$$[s, f] = \{x \in {}^{\omega}\omega \mid s \subseteq x \text{ and } (\forall n \geq \operatorname{dom}(s))\ x(n) \geq f(n)\}.$$

Hence $I_{\mathbb{D}}$ is the meager ideal in the dominating topology.

(4) Mathias forcing: a set of reals A is Mathias-null iff $\{\operatorname{ran}(x) \mid x \in A \cap A_0\}$ is Ramsey null or meager in the Ellentuck topology, where A_0 is the set of strictly increasing infinite sequences of natural numbers. Also, Mathias-null sets form a σ-ideal by a standard fusion argument.

(5) Sacks forcing $\mathbb{S}$: in this case, $I_{\mathbb{S}} = N_{\mathbb{S}}$ by a standard fusion argument. The ideal $I_{\mathbb{S}}$ is called the Marczewski ideal and often denoted by s_0.

As with Sacks forcing, all the typical non-ccc tree-type forcings admitting a fusion argument satisfy the equation $I_{\mathbb{P}} = N_{\mathbb{P}}$. Since $I_{\mathbb{P}}$ is Borel generated for any ccc arboreal forcing, the condition $(**)$ in Theorem 4.4 (which we will state in § 4) holds for all the typical tree-type strongly arboreal forcings.

Now we introduce the regularity property for each arboreal forcing:

Definition 2.8. Let $\mathbb{P}$ be arboreal. A set of reals A is $\mathbb{P}$-*measurable* if for any T in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in I_{\mathbb{P}}$ or $[T'] \setminus A \in I_{\mathbb{P}}$.

As we expect, $\mathbb{P}$-measurability coincides with the known regularity property for $\mathbb{P}$ when $\mathbb{P}$ is ccc:

Proposition 2.9. Let $\mathbb{P}$ be a strongly arboreal, ccc forcing and let P be a set of reals. Then P is $\mathbb{P}$-measurable iff there is a Borel set B such that $P \triangle B \in I_{\mathbb{P}}$.

Proof. The direction from right to left follows from the fact that every Borel set of reals is $\mathbb{P}$-measurable which will be proved in Lemma 3.5.

For the other direction, suppose P is $\mathbb{P}$-measurable and we will find a Borel set approximating P modulo $I_{\mathbb{P}}$. Since P is $\mathbb{P}$-measurable, the set $D = \{T \in \mathbb{P} \mid \text{ either } [T] \cap P \in I_{\mathbb{P}} \text{ or } [T] \setminus P \in I_{\mathbb{P}}\}$ is dense. We take a maximal antichain A in D and define $B = \bigcup\{[T] \mid T \in A \text{ and } [T] \setminus P \in I_{\mathbb{P}}\}$. Then since A is countable, B is Borel and $P \triangle B \in I_{\mathbb{P}}$ because D is dense. ■

This argument does not work for non-ccc forcings such as Sacks forcing.[4] But $\mathbb{P}$-measurability is almost the same as the regularity properties for non-ccc forcings $\mathbb{P}$, e.g., for Mathias forcing, a set of reals A is Mathias-measurable iff $\{\mathrm{ran}(x) \mid x \in A \cap A_0\}$ is completely Ramsey (or has the Baire property in the Ellentuck topology), where A_0 is the set of all strictly increasing infinite sequences of natural numbers. Also, for Sacks forcing, the following holds:

Proposition 2.10 (Brendle-Löwe)**.** Let $\mathbf{\Gamma}$ be a topologically reasonable pointclass, i.e., it is closed under continuous preimages and any intersection between a set in $\mathbf{\Gamma}$ and a closed set. Then every set in $\mathbf{\Gamma}$ is $\mathbb{S}$-measurable iff every set in $\mathbf{\Gamma}$ has the Bernstein property.[5]

Proof. See [8, Lemma 2.1]. □

Next we introduce a technical ideal ${I_{\mathbb{P}}}^*$ which we need later:

Definition 2.11. Let $\mathbb{P}$ be an arboreal forcing. A set of reals A is in ${I_{\mathbb{P}}}^*$ if for any T in $\mathbb{P}$ there is a $T' \leq T$ such that $[T'] \cap A$ is in $I_{\mathbb{P}}$.

Question 2.12. Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Can we prove $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$?

[4] For example, assuming every $\mathbf{\Pi}^1_1$-set has the perfect set property, every $\mathbf{\Sigma}^1_1$-set of reals has the Bernstein property (i.e., either it contains a perfect or there is a perfect set disjoint from the set) but for a $\mathbf{\Sigma}^1_1$-set of reals A, A is approximated by a Borel set modulo $I_{\mathbb{S}}$ iff A is Borel. This is because $I_{\mathbb{S}}$ restricted to analytic sets (or co-analytic sets) is the set of all countable sets of reals.

[5] In general, the Bernstein property does not imply $\mathbb{S}$-measurability while the converse is true. By using the axiom of choice, we can construct a set of reals which is not $\mathbb{S}$-measurable and has the Bernstein property.

We give some easy observations concerning to Question 2.12:

Lemma 2.13. Let $\mathbb{P}$ be strongly arboreal forcing.

(1) The ideal $I_{\mathbb{P}}$ is a subset of ${I_{\mathbb{P}}}^*$.

(2) A set of reals A is $\mathbb{P}$-measurable iff for any T in $\mathbb{P}$ there is a $T' \leq T$ such that either $[T'] \cap A \in {I_{\mathbb{P}}}^*$ or $[T'] \setminus A \in {I_{\mathbb{P}}}^*$ holds. Hence we get the same notion of measurability even if we replace $I_{\mathbb{P}}$ by ${I_{\mathbb{P}}}^*$ in the definition of $\mathbb{P}$-measurability.

(3) If $\mathbb{P}$ is ccc, then $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$.

(4) If $I_{\mathbb{P}} = N_{\mathbb{P}}$, then $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$. Hence $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$ for any typical tree-type strongly arboreal forcing admitting a fusion argument.

(5) (Brendle) Suppose $\mathbb{P}$ satisfies the following condition: for any maximal antichain $\mathcal{A}$ in $\mathbb{P}$, there is a maximal antichain $\mathcal{A}'$ such that for any two elements T, T' of $\mathcal{A}'$, $[T]$ and $[T']$ are disjoint and $\mathcal{A}'$ refines $\mathcal{A}$, i.e., for any T' in $\mathcal{A}'$ there is a T in $\mathcal{A}$ with $T' \subseteq T$. Then $I_{\mathbb{P}} = {I_{\mathbb{P}}}^*$.

Sacks forcing is a typical example of the condition in (5). But we do not know of any strongly arboreal $\mathbb{P}$ satisfying the condition but which are neither ccc nor satisfy $I_{\mathbb{P}} = N_{\mathbb{P}}$.

Proof. We will prove only (5). The rest are straightforward. Suppose $\mathbb{P}$ satisfies the above condition and let A be in ${I_{\mathbb{P}}}^*$. We prove A is in $I_{\mathbb{P}}$. Since A is in ${I_{\mathbb{P}}}^*$, the set of all Ts in $\mathbb{P}$ such that $[T] \cap A \in I_{\mathbb{P}}$ is dense in $\mathbb{P}$. Hence we can take a maximal antichain $\mathcal{A}$ contained in this set. By the condition, we may assume for any two distinct elements T_1, T_2 of $\mathcal{A}$, $[T_1]$, $[T_2]$ are pairwise disjoint. For each T in $\mathcal{A}$, $[T] \cap A \in I_{\mathbb{P}}$. So we can pick $\{N_{n,T} \mid n \in \omega\}$ such that each $N_{n,T}$ is $\mathbb{P}$-null and $\bigcup_{n\in\omega} N_{n,T} = [T] \cap A$. Let $N_n = \bigcup_{T\in\mathcal{A}} N_{n,T}$ for each $n \in \omega$. Since $A = \bigcup_{n\in\omega} N_n$, the proof is complete if we prove the following

Claim 2.14. For each $n \in \omega$, N_n is $\mathbb{P}$-null.

Proof of Claim 2.14. Take any T' in $\mathbb{P}$. Since $\mathcal{A}$ is a maximal antichain, we can take a $T \in \mathcal{A}$ such that T and T' are compatible. Take a common extension T''. Then $[T''] \cap N_n = [T''] \cap N_{n,T}$ because of the property of $\mathcal{A}$. But we know that $N_{n,T}$ is $\mathbb{P}$-null. Hence we can take a further extension of T'' disjoint from N_n. □

■

Next, we introduce quasi-$\mathbb{P}$-genericity for arboreal forcings $\mathbb{P}$ and compare it with $\mathbb{P}$-genericity. Quasi-generic reals are obvious generalization of Cohen reals and random reals:

Definition 2.15. Let $\mathbb{P}$ be arboreal and M be a transitive model of ZFC. A real x is *quasi-$\mathbb{P}$-generic over M* if for any Borel code c in M

with $B_c \in I_{\mathbb{P}}{}^*$, x is not in B_c, where B_c is the decoded Borel set from c.

Example 2.16. (1) Cohen forcing ($\mathbb{C}$): quasi-$\mathbb{C}$-generic reals are the same as Cohen reals by definition. Hence quasi-$\mathbb{C}$-genericity coincides with $\mathbb{C}$-genericity.

(2) random forcing ($\mathbb{B}$): quasi-$\mathbb{B}$-generic reals are the same as random reals by definition. Hence quasi-$\mathbb{B}$-genericity coincides with $\mathbb{B}$-genericity.

(3) Hechler forcing ($\mathbb{D}$): quasi-$\mathbb{D}$-generic reals are the same as Hechler reals. Hence quasi-$\mathbb{D}$-genericity coincides with $\mathbb{D}$-genericity.

(4) Sacks forcing ($\mathbb{S}$): if M is an inner model of ZFC, quasi-$\mathbb{S}$-generic reals over M are the reals which are not in M because any Borel set in $I_{\mathbb{S}}{}^* = N_{\mathbb{S}}$ is countable and this is also true in M if the code is in M by Shoenfield absoluteness. Therefore, quasi-$\mathbb{S}$-genericity does not coincide with $\mathbb{S}$-genericity.

The last example explains the difference between genericity and quasi-genericity and shows that the equivalence for Sacks forcing we mentioned in the introduction is a special case of Theorem 4.3 which we will prove later.[6]

As is expected, genericity implies quasi-genericity for all the typical strongly arboreal forcings and the converse is true for most ccc forcings:

Proposition 2.17. Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably Δ^1_2 forcing. Then

(1) The set $\{c \mid B_c \in I_{\mathbb{P}}{}^*\}$ is Π^1_2. Hence the statement "c codes a Borel set in $I_{\mathbb{P}}{}^*$" is absolute between inner models of ZFC.

(2) If M is a transitive model of ZFC and a real x is $\mathbb{P}$-generic over M, then x is quasi-$\mathbb{P}$-generic over M.

(3) Suppose $\mathbb{P}$ is also provably ccc, i.e., there is a formula ϕ defining $\mathbb{P}$ and the statement "ϕ is ccc" is provable in ZFC. Then if M is an inner model of ZFC and a real x is quasi-$\mathbb{P}$-generic over M, then x is $\mathbb{P}$-generic over M.

Proof. See § 3. □

In [31], Zapletal starts from a σ-ideal I on a Polish space X and considers the quotient of the set of all Borel sets in X modulo I and develops the general theory of this forcing as a Boolean algebra. Let us compare his setting with our setting:

[6]It is easy to check the condition $(*)$ in Theorem 4.3 for Sacks forcing by noting that the ideal $I_{\mathbb{S}}$ restricted to Borel sets is the ideal of countable sets as we mentioned in the last example.

Proposition 2.18. Suppose $\mathbb{P}$ is a strongly arboreal, proper forcing. Then the map $i\colon \mathbb{P} \to \big(\mathbf{B}/I_{\mathbb{P}}{}^*\big) \setminus \{0\}$ defined by

$$i(T) = \text{ the equivalence class represented by } [T],$$

is a dense embedding, where $\mathbf{B}$ denotes the set of all Borel sets of the reals and $\mathbf{B}/I_{\mathbb{P}}{}^*$ is the quotient Boolean algebra via $I_{\mathbb{P}}{}^*$.

Hence, our situation is a special case of Zapletal's.[7]

Proof. See § 3. □

3. $\mathbb{P}$-measurability and $\mathbb{P}$-Baireness

In this section, we shall prove the propositions listed in § 2. In order to do so, we first consider the connection between $\mathbb{P}$-measurability and a property called $\mathbb{P}$-Baireness (which was implicitly introduced by Feng-Magidor-Woodin [11]). This connection will allow us to characterize $I_{\mathbb{P}}{}^*$ in terms of Banach-Mazur games, which plays an essential role in the proof of Proposition 2.17.

Let $\mathbb{P}$ be a partial order. The *Stone space of* $\mathbb{P}$ (denoted by $\mathrm{St}(\mathbb{P})$) is the set of ultrafilters of $\mathbb{P}$ equipped with the topology generated by $\{O_p \mid p \in \mathbb{P}\}$, where $O_p = \{u \in \mathrm{St}(\mathbb{P}) \mid u \ni p\}$.

For example, if $\mathbb{P}$ is Cohen forcing ($\mathbb{C}$), then $\mathrm{St}(\mathbb{C})$ is homeomorphic to the Baire space ${}^\omega\omega$.

Dense sets in $\mathbb{P}$ are the same as open dense subsets in $\mathrm{St}(\mathbb{P})$: if D is a dense subset of $\mathbb{P}$, then the set $\bigcup\{O_p \mid p \in D\}$ is open dense in $\mathrm{St}(\mathbb{P})$. Conversely, if U is an open dense subset of $\mathrm{St}(\mathbb{P})$, then $\{p \in \mathbb{P} \mid O_p \subseteq U\}$ is a dense open subset of $\mathbb{P}$.

Next, we will talk about meagerness and the Baire property in $\mathrm{St}(\mathbb{P})$. The first observation we should make is that this is not nonsense:

Lemma 3.1. Let $\mathbb{P}$ be a partial order. Then for any $p \in \mathrm{St}(\mathbb{P})$, O_p is not meager.

Proof. Take any $p \in \mathbb{P}$ and let $\{U_n \mid n \in \omega\}$ be a countable set of open dense subsets of $\mathrm{St}(\mathbb{P})$. We would like to prove that the intersection $\bigcap_{n\in\omega} U_n$ with O_p is nonempty. But this is just the Rasiowa-Sikorsky Theorem or finding a generic object G over a countable structure containing $\mathbb{P}$ with $p \in G$. ■

[7]In [31, Corollary 2.1.5], Zapletal proved a more general result. His I is essentially the same as our $I_{\mathbb{P}}{}^*$ and if we use $b_n = |\dot{x}_{gen}(\check{n}) = 1|$ $(n \in \omega)$ instead of b_t $(t \in {}^{<\omega}2)$ for the generators of C, then Zapletal's I is exactly the same as our $I_{\mathbb{P}}{}^*$ on Borel sets.

Before defining $\mathbb{P}$-Baireness, let us see the connection between Baire measurable functions from $\mathrm{St}(\mathbb{P})$ to the reals and $\mathbb{P}$-names for a real.

Let X, Y be topological spaces. Then a function $f\colon X \to Y$ is *Baire measurable* if for any open set U in Y, $f^{-1}(U)$ has the Baire property in X. Baire measurable functions are the same as continuous functions modulo meager sets: let X, Y be topological spaces and assume Y is second countable. Then it is fairly easy to see that a function $f\colon X \to Y$ is Baire measurable iff there is a comeager set D in X such that $f \restriction D$ is continuous.

There is a natural correspondence between Baire measurable functions from $\mathrm{St}(\mathbb{P})$ to the reals and $\mathbb{P}$-names for a real:

Lemma 3.2 (Feng-Magidor-Woodin)**.** Let $\mathbb{P}$ be a partial order.

(1) If $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$ is a Baire measurable function, then

$$\tau_f = \big\{ (\check{(m,n)}, p) \mid O_p \setminus \{u \in \mathrm{St}(\mathbb{P}) \mid f(u)(m) = n\} \text{ is meager}\big\}$$

is a $\mathbb{P}$-name for a real.

(2) Let τ be a $\mathbb{P}$-name for a real. Define f_τ as follows. For $u \in \mathrm{St}(\mathbb{P})$ and $m, n \in \omega$,

$$f_\tau(u)(m) = n \iff (\exists p \in u)\ p \Vdash \tau(\check{m}) = \check{n}.$$

Then the domain of f_τ is comeager in $\mathrm{St}(\mathbb{P})$ and f_τ is continuous on the domain. Hence it can be uniquely extended to a Baire measurable function from $\mathrm{St}(\mathbb{P})$ to the reals modulo meager sets.

(3) If $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$ is a Baire measurable function, then f_{τ_f} and f agree on a comeager set in $\mathrm{St}(\mathbb{P})$. Also, if τ is a $\mathbb{P}$-name for a real, then $\Vdash \tau_{f_\tau} = \tau$.

Proof. See [11, Theorem 3.2]. □

Recall that we have defined a generic real x_G from a generic object G for any strongly arboreal forcing $\mathbb{P}$. Let $\dot{x_G}$ be a canonical $\mathbb{P}$-name for x_G.

Example 3.3. Let $\mathbb{P}$ be strongly arboreal. Then $f_{\dot{x_G}}(u)(m) = n$ iff there is a T in u such that $\mathrm{stem}(T)(m) = n$. Hence $f_{\dot{x_G}}(u) = \bigcup\{\mathrm{stem}(T) \mid T \in u\}$ for $u \in \mathrm{dom}(\pi)$ as we expect.

Now we define the property $\mathbb{P}$-Baireness. Let $\mathbb{P}$ be a partial order and A be a set of reals. Then A is $\mathbb{P}$-*Baire* if for any Baire measurable function $f\colon \mathrm{St}(\mathbb{P}) \to {}^{\omega}\omega$, $f^{-1}(A)$ has the Baire property in $\mathrm{St}(\mathbb{P})$. It is easy to see that every Borel set of reals is $\mathbb{P}$-Baire for any $\mathbb{P}$ by the same argument as for the Baire property.

Example 3.4. Let $\mathbb{C}$ be Cohen forcing. A set of reals A is $\mathbb{C}$-Baire iff $f^{-1}(A)$ has the Baire property for any *continuous* function $f\colon {}^{\omega}\omega \to {}^{\omega}\omega$.

Proof. As we have seen in the beginning of this section, $\mathrm{St}(\mathbb{C})$ is homeomorphic to the Baire space ${}^{\omega}\omega$. In the Baire space, any G_δ comeager set is homeomorphic to the whole space. Hence we can replace Baire measurable functions by continuous functions in the definition of $\mathbb{C}$-Baireness. ■

Before talking about the relation between $\mathbb{P}$-measurability and $\mathbb{P}$-Baireness, let us mention the connection between $\mathbb{P}$-Baireness and universally Baireness. A set of reals A is *universally Baire* if for any compact Hausdorff space X and any continuous function $f\colon X \to {}^{\omega}\omega$, $f^{-1}(A)$ has the Baire property in X. A set of reals A is universally Baire iff A is $\mathbb{P}$-Baire for any partial order $\mathbb{P}$. (This is essentially proved in [11].)

Recall that ${I_{\mathbb{P}}}^*$ is a technical ideal introduced in Definition 2.11 which is the same as $I_{\mathbb{P}}$ for most cases.

Lemma 3.5 ($\mathbb{P}$-measurability vs. $\mathbb{P}$-Baireness)**.** Let $\mathbb{P}$ be a strongly arboreal, proper forcing and A be a set of reals. Then

(1) A is in ${I_{\mathbb{P}}}^*$ iff $f_{\dot{x_G}}^{-1}(A)$ is meager in $\mathrm{St}(\mathbb{P})$, and

(2) A is $\mathbb{P}$-measurable iff $f_{\dot{x_G}}^{-1}(A)$ has the Baire property in $\mathrm{St}(\mathbb{P})$. In particular, if A is $\mathbb{P}$-Baire, then A is $\mathbb{P}$-measurable. Hence every Borel set if $\mathbb{P}$-measurable.

Note that $\mathbb{P}$-measurability does not imply $\mathbb{P}$-Baireness in general.[8]

Proof of Lemma 3.5. Let $\pi = f_{\dot{x_G}}$ for abuse of notation.

The following are useful for the proof:

Claim 3.6. (a) For T in $\mathbb{P}$ and $u \in \mathrm{dom}(\pi)$, if $T \in u$, then $\pi(u) \in [T]$.

(b) For T in $\mathbb{P}$, the converse of (a) holds for comeager many u in $\mathrm{dom}(\pi)$.

Proof of Claim 3.6. (a) Suppose $T \in u$. We prove $\pi(u) \restriction n \in T$ for each $n \in \omega$. Fix a natural number n. Then by Example 3.3, there is a T' in u such that $\mathrm{stem}(T') \supseteq \pi(u) \restriction n$. Since both T and T' are in u, they are compatible, especially $\mathrm{stem}(T') \in T$ (otherwise $[T] \cap [T'] = \emptyset$). Hence $\pi(u) \restriction n \in T$.

(b) Take any T in $\mathbb{P}$. Then the set $D = \{T' \in \mathbb{P} \mid T' \subseteq T \text{ or } [T'] \cap [T] = \emptyset\}$ is dense in $\mathbb{P}$. (Take any T'. If $T' \not\subseteq T$, then there is a $t' \in T' \setminus T$. By strong arborealness of $\mathbb{P}$, $T'_{t'} \in \mathbb{P}$ and $[T'_{t'}] \cap [T] = \emptyset$.) Since D is dense, the set $\{u \mid u \cap D \neq \emptyset\}$ is dense open in $\mathrm{St}(\mathbb{P})$. Hence

[8] For example, if A is a Σ^1_2 (lightface) set of reals universal for $\boldsymbol{\Sigma}^1_2$ (boldface) sets of reals and if every Σ^1_2 (lightface) set of reals has the Baire property but there is a $\boldsymbol{\Sigma}^1_2$ (boldface) set of reals without the Baire property, then A is $\mathbb{C}$-measurable by Proposition 2.9, but A is not $\mathbb{C}$-Baire by Example 3.4.

it suffices to show that if u is in $\mathrm{dom}(\pi)$, $u \cap D \neq \emptyset$ and $\pi(u) \in [T]$, then $T \in u$. Suppose $T \notin u$. Then since $u \cap D \neq \emptyset$, there is a $T' \in u$ such that $[T'] \cap [T] = \emptyset$. By (a), $\pi(u) \in [T']$, hence $\pi(u) \notin [T]$, a contradiction. □

(1) We prove the direction from left to right.

We first show that $\pi^{-1}(A)$ is meager if A is in $N_{\mathbb{P}}$. If A is in $N_{\mathbb{P}}$, then the set $D = \{T \mid [T] \cap A = \emptyset\}$ is dense in $\mathbb{P}$. Hence the set of all $u \in \mathrm{dom}(\pi)$ with $u \cap D \neq \emptyset$ is comeager. But if u is in the comeager set, then there is a $T \in u \cap D$ and by Claim 3.6 (a), $\pi(u) \in [T]$ and $[T] \cap A = \emptyset$, in particular $\pi(u) \notin A$. Therefore $\pi^{-1}(A)$ is meager.

We have seen that $\pi^{-1}(A)$ is meager assuming A is in $N_{\mathbb{P}}$. Since $I_{\mathbb{P}}$ is the σ-ideal generated by sets in $N_{\mathbb{P}}$, $\pi^{-1}(A)$ is meager for all A in $I_{\mathbb{P}}$.

We show that $\pi^{-1}(A)$ is meager if A is in ${I_{\mathbb{P}}}^*$. Since A is in ${I_{\mathbb{P}}}^*$, the set $D' = \{T \mid [T] \cap A \in I_{\mathbb{P}}\}$ is dense in $\mathbb{P}$. We use the following well-known fact:

Fact 3.7. Let X be a topological space and A be a subset of X. Then $\left(\bigcup\{U \mid U \text{ is open and } U \cap A \text{ is meager }\}\right) \cap A$ is meager.

Proof of Fact 3.7. See [20, Theorem 8.29]. □

Since D' is dense, $\bigcup\{O_T \mid T \in D'\}$ is open dense. By the above fact, it suffices to prove that $O_T \cap \pi^{-1}(A)$ is meager for any T in D'.

Take any T in D'. By the definition of D', we know that $[T] \cap A$ is in $I_{\mathbb{P}}$. Hence $\pi^{-1}([T] \cap A)$ is meager in $\mathrm{St}(\mathbb{P})$. But by Claim 3.6 (a), $O_T \cap \pi^{-1}(A) \subseteq \pi^{-1}([T] \cap A)$. Therefore, $O_T \cap \pi^{-1}(A)$ is meager as we desired.

Next, we see the direction from right to left. Suppose $\pi^{-1}(A)$ is meager. Take any T in $\mathbb{P}$ and we will find an extension T' of T such that $[T'] \cap A$ is in $I_{\mathbb{P}}$. Since $\pi^{-1}(A)$ is meager, then there is a sequence $\langle U_n \mid n \in \omega\rangle$ of open dense sets in $\mathrm{St}(\mathbb{P})$ such that $\bigcap_{n\in\omega} U_n \cap \pi^{-1}(A) = \emptyset$. For each $n \in \omega$, let $D_n = \{S \in \mathbb{P} \mid O_S \subseteq U_n\}$. Since U_n is open dense in $\mathrm{St}(\mathbb{P})$, D_n is dense open in $\mathbb{P}$. We choose a sequence $\langle \mathcal{A}_n \mid n \in \omega\rangle$ of maximal antichains such that $\mathcal{A}_n \subseteq D_n$, for each element S of $\mathcal{A}_n$, the length of $\mathrm{stem}(S)$ is greater than n, and $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$, i.e., every element of $\mathcal{A}_{n+1}$ is below some element in $\mathcal{A}_n$.

Now we use the properness of $\mathbb{P}$ to treat each $\mathcal{A}_n$ as "countable". Let θ be a sufficiently large regular cardinal and X be a countable elementary substructure of $\mathcal{H}_\theta$ such that $\mathbb{P}, T, \langle \mathcal{A}_n \mid n \in \omega\rangle$ are in X. By properness, there is an $(X, \mathbb{P})$-generic condition T' below T. We show that $[T'] \cap A$ is in $I_{\mathbb{P}}$, which will complete the proof of (1).

Consider the set $B = \bigcap_{n\in\omega} \bigcup\{[S] \mid S \in \mathcal{A}_n \cap X\} \setminus \bigcup_{n\in\omega}\{[S] \cap [S'] \mid S, S' \in \mathcal{A}_n \cap X \text{ and } S \neq S'\}$. So B is the set of all xs uniquely deciding which condition from $\mathcal{A}_n$ contains it for each n. By the property of $\langle \mathcal{A}_n \mid n \in \omega \rangle$, it will generate a filter coming from elements in $\mathcal{A}_n$s. The point is that any ultrafilter u extending that filter satisfies $\pi(u) = x$, the given element, and that u is in U_n for each n. This will play a role for the argument.

Now we claim $[T'] \setminus B \in I_{\mathbb{P}}$ and $B \cap A = \emptyset$. We will be done if we prove them. The fact that $[T'] \setminus B \in I_{\mathbb{P}}$ follows from the fact that $\{S \mid S \in \mathcal{A}_n \cap X\}$ is predense below $[T']$ for each n because T' is $(X, \mathbb{P})$-generic and from that $[S] \cap [S'] \in I_{\mathbb{P}}$ for each $S, S' \in \mathcal{A}_n \cap X$ with $S \neq S'$ because $\mathcal{A}_n$ is an antichain, and from that $\mathcal{A}_n \cap X$ is countable for each n.

To prove $B \cap A = \emptyset$, take any element x from B. As we mentioned above, for each $n \in \omega$, there is a unique element S_n in $\mathcal{A}_n \cap X$ with $x \in [S_n]$. Since $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$, $S_{n+1} \leq S_n$ for each n. Hence the set $\{S_n \mid n \in \omega\}$ generate a filter F_x. Take any ultrafilter u extending F_x. We claim that $\pi(u) = x$ and $u \in U_n$ for each n. By the property of $\langle \mathcal{A}_n \mid n \in \omega \rangle$, the length of $\mathrm{stem}(S_n)$ is greater than n. Hence, by Example 3.3, $\pi(u)$ is already decided to be x by S_ns. The fact that $u \in U_n$ for each n follows from the fact that $S_n \in \mathcal{A}_n \subseteq D_n$ and the definition of D_n.

Since we have assumed that $\bigcap_{n \in \omega} U_n \cap \pi^{-1}(A) = \emptyset$, x does not belong to A because $x = \pi(u) \in U_n$ for each n by Claim 3.6. Hence we have seen $B \cap A = \emptyset$ as we desired.

(2) For left to right, we assume A is $\mathbb{P}$-measurable. Then the set $D = \{T \in \mathbb{P} \mid \text{ either } [T] \cap A \in I_{\mathbb{P}} \text{ or } [T] \setminus A \in I_{\mathbb{P}}\}$ is dense. Then the set $U = \bigcup\{O_T \mid T \in D\}$ is open dense in $\mathrm{St}(\mathbb{P})$. Let $U_1 = \bigcup\{O_T \mid [T] \setminus A \in I_{\mathbb{P}}\}$, $U_2 = \bigcup\{O_T \mid [T] \cap A \in I_{\mathbb{P}}\}$. Then $U = U_1 \cup U_2$. By Lemma 2.13 (1), Lemma 3.1, Claim 3.6 (a), and (1) in this lemma, $U_1 \cap U_2 = \emptyset$. Hence, it suffices to show that $U_1 \setminus \pi^{-1}(A)$, $U_2 \cap \pi^{-1}(A)$ are meager because that will imply $U_1 \triangle \pi^{-1}(A)$ is meager.

We will only see that $U_2 \cap \pi^{-1}(A)$ is meager. The case for $U_1 \setminus \pi^{-1}(A)$ being meager is similar. By Fact 3.7, it suffices to see that $O_T \cap \pi^{-1}(A)$ is meager when $[T] \cap A \in I_{\mathbb{P}}$. But if $[T] \cap A \in I_{\mathbb{P}}$, then $O_T \cap \pi^{-1}(A) \subseteq \pi^{-1}([T] \cap A)$ and $\pi^{-1}([T] \cap A)$ is meager by Claim 3.6 (a), Lemma 2.13 (1), and (1) in this lemma. Hence we are done.

Now we see the direction from right to left. Assume $\pi^{-1}(A)$ has the Baire property in $\mathrm{St}(\mathbb{P})$. Then there are open sets U_1, U_2 such that $U_1 \triangle \pi^{-1}(A)$, $U_2 \triangle \pi^{-1}({}^{\omega}\omega \setminus A)$ are meager. By Lemma 3.1, $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2$ is open dense in $\mathrm{St}(\mathbb{P})$. Let $D_i = \{T \in \mathbb{P} \mid O_T \subseteq U_i\}$ for $i = 1, 2$. Then $D_1 \cup D_2$ is dense in $\mathbb{P}$. Hence by Lemma 2.13

(2), it suffices to prove that $[T] \setminus A \in I_{\mathbb{P}}{}^*$ for each T in D_1 and that $[T] \cap A \in I_{\mathbb{P}}{}^*$ for each T in D_2.

We only prove $[T] \setminus A \in I_{\mathbb{P}}{}^*$ for each T in D_1. By (1) in this Lemma, it is enough to see that $\pi^{-1}([T] \setminus A)$ is meager in $\mathrm{St}(\mathbb{P})$. But by Claim 3.6 (b), $\pi^{-1}([T] \setminus A)$ is almost the same as $O_T \setminus \pi^{-1}(A)$. Since T is in D_1, by the definition of U_1, $O_T \setminus \pi^{-1}(A)$ is meager. This completes the proof of (2). ■

Note that if $\mathbb{P}$ satisfies the condition in Lemma 2.13 (5), then we do not need the properness of $\mathbb{P}$ for the proofs of Lemma 3.5.

Now we are ready to prove Proposition 2.17 and Proposition 2.18. We first see the proof of Proposition 2.18:

Proof of Proposition 2.18. First we see that the map i is well-defined, i.e., $[T]$ is not in $I_{\mathbb{P}}{}^*$ for each T in $\mathbb{P}$. If it were in $I_{\mathbb{P}}{}^*$, then by Lemma 3.5 (1), $\pi^{-1}([T])$ would be meager and $O_T \subseteq \pi^{-1}([T])$ by Claim 3.6 (a). Hence O_T must be meager, which contradicts Lemma 3.1. Therefore $[T]$ is not in $I_{\mathbb{P}}{}^*$.

It is clear that if $T_1 \leq T_2$, then $i(T_1) \leq i(T_2)$. To show the converse, assume $T_1 \not\leq T_2$ and we prove that $i(T_1) \not\leq i(T_2)$. Since $T_1 \not\leq T_2$, there is a $t \in T_1$ which is not in T_2. By strong arborealness of $\mathbb{P}$, $(T_1)_t \in \mathbb{P}$ and $[(T_1)_t] \cap [T_2] = \emptyset$. Hence $i((T_1)_t) \not\leq i(T_2)$. Since $(T_1)_t \leq T_1$, $i((T_1)_t) \leq i(T_1)$. Therefore, $i(T_1) \not\leq i(T_2)$.

So it suffices to see that $i``\mathbb{P}$ is dense in $\big(\mathbf{B}/I_{\mathbb{P}}{}^*\big) \setminus \{0\}$. Let B be a Borel set which is not in $I_{\mathbb{P}}{}^*$. We will find a T in $\mathbb{P}$ with $[T] \setminus B \in I_{\mathbb{P}}{}^*$.

Since every Borel set is $\mathbb{P}$-Baire, by Lemma 3.5 (2), B is $\mathbb{P}$-measurable. Since B is not in $I_{\mathbb{P}}{}^*$, there is a T such that $[T] \setminus B \in I_{\mathbb{P}}$, hence $[T] \setminus B \in I_{\mathbb{P}}{}^*$ by Lemma 2.13 (1), as we desired. ■

Proof of Proposition 2.17. (1) Let $\pi = f_{\dot{x}_G}$ as in the proof of Lemma 3.5. By Lemma 3.5, a set of reals A is in $I_{\mathbb{P}}{}^*$, iff $\pi^{-1}(A)$ is meager in $\mathrm{St}(\mathbb{P})$. Hence, it suffices to show that $\{c \mid \pi^{-1}(B_c) \text{ is meager}\} \in \Pi^1_2$.

We will prove the following:

$$(\star)\ \pi^{-1}(B_c) \text{ is meager} \iff (\forall M \ni c)\ \big(M\colon \text{a c.t.m. of ZFC} \\ \implies M \models \text{``}\pi^{-1}(B_c) \text{ is meager''}\big).$$

First note that the right hand side makes sense because the statement "$\mathbb{P}$ is a strongly arboreal forcing" is Π^1_2 by the assumption that $\mathbb{P}$ is provably Δ^1_2, so by downward absoluteness, this is also true in M. Since the right hand side is Π^1_2, it suffices to show the above equivalence.

The following claim is the key-point:

Claim 3.8. Let M be a countable transitive model of ZFC with $c \in M$. If $M \models$ "$\pi^{-1}(B_c)$ is meager", then for any $T \in \mathbb{P}^M$ (or $\mathbb{P} \cap M$), there is a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}(B_c)$ is meager in V.

Proof of Claim 3.8. Take any T in $\mathbb{P}^M$. Since $\mathbb{P}$ is provably Δ^1_2, $\mathbb{P}^M$, $\leq^M$ and $\perp^M$ are subsets of $\mathbb{P}$, $\leq$ and $\perp$ respectively. Hence, by strong properness, there is a $T' \leq T$ such that T' is $(M, \mathbb{P})$-generic.

We will show that T' satisfies the desired property. For that, we will use the unfolded Banach-Mazur game. Let U be a tree on $\omega \times \omega$, recursive in c such that $B_c = \mathrm{p}[U]$ holds in any transitive model of ZFC N with $c \in N$. Consider the following game G': player I and II produce a decreasing sequence $\langle {S_n}' \leq T' \mid n \in \omega \rangle$ one by one and in addition, player II produces a real $\langle y_n \mid n \in \omega \rangle$. Player II wins if $(\pi(u), y) \in [U]$ for any $u \in \bigcap_{n\in\omega} O_{{S_n}'}$. Note that we may assume that π is defined for any $u \in \bigcap_{n\in\omega} O_{{S_n}'}$ and the value of π only depends on the sequence $\langle {S_n}' \mid n \in \omega \rangle$ because we can arrange $\pi(u) = \bigcup_{n\in\omega} \mathrm{stem}({S_n}')$ by strong arborealness of $\mathbb{P}$ and Example 3.3.

Now it suffices to show that player II has a winning strategy in this game. Since $M \models$ "$\pi^{-1}(B_c)$ is meager", in M, player II has a winning strategy σ in the game G which is the same as G' except that player I can start from any condition in $\mathbb{P}$. The idea is to transfer σ to a winning strategy for player II in G' in V. Instead of writing down a winning strategy for player II in G', we will describe how to win the game G' for player II as follows:

V	I	${S_0}' \leq T'$		${S_2}'$		$\cdots$
	II		$({S_1}', y_0)$		$({S_3}', y_1)$	$\cdots$
M	I	S_0		S_2		$\cdots$
	II		(S_1, y_0)		(S_3, y_1)	$\cdots$

We will construct sequences $\langle S_n \mid n \in \omega \rangle$, $\langle {S_n}' \mid n \in \omega \rangle$, $\langle y_n \mid n \in \omega \rangle$ with the following properties:

- $\big(\langle {S_n}' \mid n \in \omega \rangle, \langle y_n \mid n \in \omega \rangle\big)$ is a run in the game G' in V,
- $\big(\langle S_n \mid n \in \omega \rangle, \langle y_n \mid n \in \omega \rangle\big)$ is a run in the game G^M in V,
- ${S_{2n}}'$ is arbitrarily chosen by player I for each n,
- player II follows σ in G^M, and
- ${S_{2n+1}}' \leq S_{2n+1}$ for each n.

Assuming we have constructed the above sequences, we prove that player II wins in the game G'. First note that G^M is a closed game for player II, hence the strategy σ remains winning in V. Therefore,

$(\pi(u), y) \in [U]$ for any $u \in \bigcap_{n\in\omega} O_{S_n}$ in V. But since ${S_{2n+1}}' \leq S_{2n+1}$ for each n, $(\pi(u), y) \in [U]$ for any $u \in \bigcap_{n\in\omega} O_{{S_n}'}$, hence player II wins the game G'.

We describe how to construct the above sequences. Suppose we have got $\langle ({S_i}', S_i, y_i) \mid i < 2n\rangle$ for some n. We will decide ${S_{2n}}'$, ${S_{2n+1}}'$, S_{2n}, S_{2n+1} and y_n. By the above properties, ${S_{2n}}'$ is arbitrarily chosen by player I and S_{2n+1}, y_n will be decided by the rest and σ. So let's decide S_{2n} and ${S_{2n+1}}'$.

Let D be the set of all possible candidates for S_{2n+1} by σ and the previous play $\langle S_i \mid i < 2n\rangle, \langle y_i \mid i < n\rangle$. Then in M, D is dense below S_{2n-1} (if it exists). Since ${S_{2n}}' \leq {S_{2n-1}}' \leq S_{2n-1}$ and T' is $(M, \mathbb{P})$-generic, $D \cap M = D$ is predense below ${S_{2n}}'$. Take an element from D which is compatible with ${S_{2n}}'$ and choose S_{2n} so that the element we have taken becomes S_{2n+1} by σ and let ${S_{2n+1}}'$ be a common extension (in V) of ${S_{2n}}'$ and S_{2n+1}. This finishes the construction of the sequences.

$\square_{\text{Claim 3.8}}$

Now let us prove the equivalence $(\star)$:

Suppose $\pi^{-1}(B_c)$ is meager and assume there is a countable transitive model M of ZFC with $c \in M$ such that $M \models$ "$\pi^{-1}(B_c)$ is not meager". We will derive a contradiction. Since every Borel set is $\mathbb{P}$-Baire, $\pi^{-1}(B_c)$ has the Baire property. Hence there is a $T \in \mathbb{P}^M$ such that in M, $\pi^{-1}(B_c)$ is comeager in O_T. By Claim 3.6 (a), $\pi^{-1}([T] \setminus B_c)$ is almost included in $O_T \setminus \pi^{-1}(B_c)$, hence, in M, $\pi^{-1}([T] \setminus B_c)$ is meager in $\mathrm{St}(\mathbb{P})$. Now apply the claim for $[T] \setminus B_c$. Then we get a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}([T] \setminus B_c)$ is meager. But this means that $O_{T'}$ is almost included in $\pi^{-1}(B_c)$. Since $O_{T'}$ is not meager by Lemma 3.1, $\pi^{-1}(B_c)$ is not meager, which contradicts the assumption that $\pi^{-1}(B_c)$ is meager.

For the other direction, by Fact 3.7, it suffices to show that for any T in $\mathbb{P}$, there is a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}(B_c)$ is meager. So fix any T. Then pick a countable transitive model M with $c, T \in M$. Then by Claim 3.8, there is a $T' \leq T$ such that $O_{T'} \cap \pi^{-1}(B_c)$ is meager, as we desired.

(2) Let x be $\mathbb{P}$-generic over M. Then the set $G_x = \{T \in \mathbb{P}^M \mid x \in [T]\}$ is a $\mathbb{P}^M$-generic filter over M. We show that $x \notin B_c$ when c is a Borel code in M with $B_c \in {I_{\mathbb{P}}}^*$.

Let c be such a Borel code. By (1) and the downward absoluteness for Π^1_2-formulas, $M \models$ "$B_c \in {I_{\mathbb{P}}}^*$". Let i^M be the dense embedding from $\mathbb{P}^M$ to $\Big(\big(\mathbf{B}/{I_{\mathbb{P}}}^*\big) \setminus \{0\}\Big)^M$ defined in Proposition 2.18 and $i^M_*(G_x)$ be the $\big(\mathbf{B}/{I_{\mathbb{P}}}^*\big)$-generic filter over M induced by i^M and G_x. Using the fact that ${I_{\mathbb{P}}}^*$ is a σ-ideal, it is routine to check that $B \in i^M_*(G_x)$

iff $x \in B$ for any Borel set B with a code in M. But the left hand side of the above equivalence implies $M \models$ "$B \notin I_{\mathbb{P}}{}^{*}$", hence by upward absoluteness for Σ^1_2-formulas, $B \notin I_{\mathbb{P}}{}^{*}$. Since $B_c \in I_{\mathbb{P}}{}^{*}$, $x \notin B_c$ as we desired.

(3) Let x be a quasi-$\mathbb{P}$-generic real over M and put $G_x = \{T \in \mathbb{P}^M \mid x \in [T]\}$. We show that G_x is a $\mathbb{P}^M$-generic filter over M.

We first see that G_x meets every maximal antichain of $\mathbb{P}^M$ in M. Take any maximal antichain A of $\mathbb{P}^M$ in M. Since $\mathbb{P}$ is provably ccc, A is countable in M. Now consider $B = \bigcup\{[T] \mid T \in A\}$. Then B is a Borel set with a code in M and $M \models$ "${}^{\omega}\omega \setminus B \in I_{\mathbb{P}}{}^{*}$". By (1), this is also true in V. Since x is quasi-$\mathbb{P}$-generic over M, $x \notin B^c$, i.e., x is in B. So G_x meets A.

Now we see that G_x is a filter. Take any two elements T_1, T_2 in G_x. We will find a common extension of T_1, T_2 in G_x. Consider $D = \{S \in \mathbb{P} \mid ([S] \cap [T_1] = \emptyset \text{ and } [S] \cap [T_2] = \emptyset) \text{ or } (S \leq T_1 \text{ and } [S] \cap [T_2] = \emptyset) \text{ or } (S \leq T_2 \text{ and } [S] \cap [T_2] = \emptyset) \text{ or } (S \leq T_1, T_2)\}$ in M. Then by strong arborealness of $\mathbb{P}$, D is dense in M. Hence G_x meets D. Take a condition S from $G_x \cap D$. Then only the last case in D happens because $S \in G_x \iff x \in [S]$. Hence $S \leq T_1, T_2$. Therefore, G_x is a $\mathbb{P}^M$-generic filter over M. ■

There is a close connection between forcing absoluteness for $\mathbb{P}$ and $\mathbb{P}$-Baireness:

Theorem 3.9 (Castells)**.** Let $\mathbb{P}$ be a partial order. Then the following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds, and

(2) every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-Baire.

Proof. The argument is essentially the same as in [11, Theorem 3.1]. □

4. $\mathbf{\Sigma}^1_3$-ABSOLUTENESS

Now we give a precise statement of Theorem 1.3 and prove it. Also we will prove related results.

Theorem 4.1. Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Then the following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds, and

(2) every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-measurable.

Proof. By Theorem 3.9, it suffices to show that every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-measurable iff every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-Baire. By Lemma 3.5, it is enough to see that every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-Baire assuming every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-measurable.

The following claim is the key point:

Claim 4.2. Let $\mathbb{P}$ be a strongly arboreal, proper forcing and τ be a $\mathbb{P}$-name for a real. Then for any T in $\mathbb{P}$, there is a $T' \leq T$ and a Borel function $g\colon [T'] \to \mathbb{R}$ such that $T' \Vdash \tau = g(\dot{x_G})$.

Proof of Claim 4.2. This is a combination of Proposition 2.18 in this paper and [30, Proposition 2.3.1]. □

Now take any $\mathbf{\Delta}^1_2$-set A and a Baire measurable function f from $\mathrm{St}(\mathbb{P})$ to the reals. We show that $f^{-1}(A)$ has the Baire property. It suffices to show that $\{T \mid O_T \cap f^{-1}(A)$ is meager or $O_T \setminus f^{-1}(A)$ is meager$\}$ is dense in $\mathbb{P}$.

So take any T in $\mathbb{P}$ and we will find an extension S of T with the above property. By the above claim, there is a $T' \leq T$ and a Borel function $g\colon [T'] \to \mathbb{R}$ such that $T' \Vdash \tau_f = g(\dot{x_G})$, where τ_f is the $\mathbb{P}$-name for a real defined in Lemma 3.2 (1). Hence, by Lemma 3.2 (3), $f = g \circ f_{\dot{x_G}}$ almost everywhere in $O_{T'}$. Since $g^{-1}(A)$ is $\mathbf{\Delta}^1_2$, it is $\mathbb{P}$-measurable by the assumption. By Lemma 3.5 (2), $f^{-1}_{\dot{x_G}}(g^{-1}(A)) = (g \circ f_{\dot{x_G}})^{-1}(A)$ has the Baire property. Hence $f^{-1}(A)$ has the Baire property in $O_{T'}$. In particular, there is an $S \leq T'$ such that either $O_S \cap f^{-1}(A)$ is meager or $O_S \setminus f^{-1}(A)$ is meager as we desired. ■

Theorem 4.3. Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Assume the following:

$$\{c \mid \ c \text{ is a Borel code and } B_c \in {I_{\mathbb{P}}}^*\} \in \mathbf{\Sigma}^1_2. \qquad (*)$$

Then the following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds,

(2) every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{P}$-measurable, and

(3) for any real a and $T \in \mathbb{P}$, there is a quasi-$\mathbb{P}$-generic real $x \in [T]$ over $\mathrm{L}[a]$.

Proof. We have seen the equivalence between (1) and (2). We will show the direction from (1) to (3) and the direction from (3) to (2).

For (1) to (3), take a real a and T in $\mathbb{P}$. We will find a quasi-$\mathbb{P}$-generic real x over $\mathrm{L}[a]$ with $x \in [T]$. But by the assumption $(*)$, the statement "There is a quasi-$\mathbb{P}$-generic real x over $\mathrm{L}[a]$ with $x \in [T]$" is $\mathbf{\Sigma}^1_3$ and this is true in a generic extension $V[G]$ with $T \in G$ by the same argument as in Proposition 2.17. (Although $\mathbb{P}$ might not be provably Δ^1_2 as we assumed in Proposition 2.17, we used it only to see $M \vDash B_c \in {I_{\mathbb{P}}}^*$ when $B_c \in {I_{\mathbb{P}}}^*$ in V and this is ensured by the assumption $(*)$ and Shoenfield absoluteness without using $\mathbb{P}$ being provably Δ^1_2.) Hence by $\mathbf{\Sigma}^1_3$-forcing absoluteness, the statement is also true in V as we desired.

For (3) to (2), take any $\mathbf{\Delta}^1_2$-set A and we will show that A is $\mathbb{P}$-measurable. Take any T in $\mathbb{P}$.

Case 1: $\omega_1^{L[a]} < \omega_1^V$ for any real a.

Pick a real a with $T \in L[a]$. By the assumption, the set of all dense sets of $\mathbb{P}$ in L$[a]$ is countable in V. Hence the set of all $\mathbb{P}$-generic reals over L$[a]$ is of measure one w.r.t. $I_{\mathbb{P}}$, (i.e., the complement of that set is in $I_{\mathbb{P}}$). The rest is a standard Solovay argument to prove regularity properties in Solovay models. (Actually, every $\mathbf{\Sigma}^1_2$-set of reals is $\mathbb{P}$-measurable in this case.)

Case 2: $\omega_1^{L[a]} = \omega_1^V$ for some real a.

The argument is basically the same as in [7, Proposition 2.1]. Pick a real a with $T \in L[a]$ and such that $\omega_1^{L[a]} = \omega_1^V$ and A is $\Delta^1_2(a)$. The idea is to decompose $[T] \cap A$ and $[T] \setminus A$ into Borel sets in an absolute way between L$[a]$ and V, and a Borel set containing a quasi-$\mathbb{P}$-generic real over L$[a]$ must be ${I_{\mathbb{P}}}^*$-positive and below that Borel set we will find an extension of T as a witness for $\mathbb{P}$-measurability of A.

Since $[T] \cap A$ and $[T] \setminus A$ are $\Sigma^1_2(a)$ sets, there are Shoenfield trees U_1 and U_2 in L$[a]$ for $[T] \cap A$ and $[T] \setminus A$ respectively. From these trees, we can naturally decompose $[T] \cap A$ and $[T] \setminus A$ into ω_1-many Borel sets as in [22, 2F.1-2F.3], i.e., there are sequences $\langle c_\alpha \mid \alpha < \omega_1 \rangle$, $\langle d_\alpha \mid \alpha < \omega_1 \rangle$ of Borel codes in L$[a]$ such that $[T] \cap A = \bigcup_{\alpha<\omega_1} B_{c_\alpha}$ and $[T] \setminus A = \bigcup_{\alpha<\omega_1} B_{d_\alpha}$. The point is that the above equations are absolute between L$[a]$ and V because those two sequences only depend on U_1, U_2 and ω_1 and $\omega_1^{L[a]} = \omega_1^V$ as we assumed.

By assumption, there is a quasi-$\mathbb{P}$-generic real x over L$[a]$ with $x \in [T]$. Hence there is an $\alpha < \omega_1$ such that either $x \in B_{c_\alpha}$ or $x \in B_{d_\alpha}$. Without loss of generality, we may assume $x \in B_{c_\alpha}$. Since c_α is in L$[a]$, by the definition of quasi-$\mathbb{P}$-genericity, B_{c_α} is not in ${I_{\mathbb{P}}}^*$. Since every Borel set is $\mathbb{P}$-Baire, it is $\mathbb{P}$-measurable by Lemma 3.5 (2). Hence there is a condition T' such that $[T'] \setminus B_{c_\alpha} \in I_{\mathbb{P}}$. Since $B_{c_\alpha} \subseteq [T] \cap A$, $T' \leq T$ and $[T'] \setminus A \in I_{\mathbb{P}}$, as we desired. ■

Theorem 4.4. Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Assume

$$\{c \mid \ c \text{ is a Borel code and } B_c \in {I_{\mathbb{P}}}^*\} \in \mathbf{\Sigma}^1_2, \qquad (*)$$

and

$$I_{\mathbb{P}} \text{ is Borel generated or } I_{\mathbb{P}} = N_{\mathbb{P}}. \qquad (**)$$

Then the following are equivalent:

(1) every $\mathbf{\Sigma}^1_2$-set of reals is $\mathbb{P}$-measurable, and
(2) for any real a, $\mathbb{R} \setminus \{x \mid x \text{ is quasi-}\mathbb{P}\text{-generic over } \mathrm{L}[a]\} \in {I_{\mathbb{P}}}^*$.

Proof. For (1) to (2), take any real a and we show that $A = \{x \mid x \text{ is quasi-}\mathbb{P}\text{-generic over } \mathrm{L}[a]\}$ is of measure one w.r.t. ${I_{\mathbb{P}}}^*$. Suppose not. Then ${}^\omega\omega \setminus A \notin {I_{\mathbb{P}}}^*$. By the assumption $(*)$, ${}^\omega\omega \setminus A$ is $\mathbf{\Sigma}^1_2$. So by (1), it is $\mathbb{P}$-measurable. Hence there is a T in $\mathbb{P}$ such that $[T] \setminus ({}^\omega\omega \setminus A) = [T] \cap A \in I_{\mathbb{P}}$. We show that this cannot happen.

Case 1: $I_{\mathbb{P}}$ is Borel generated, i.e., for any N in $I_{\mathbb{P}}$ there is a Borel set $B \in I_{\mathbb{P}}$ such that $N \subseteq B$.

Since $[T] \cap A \in I_{\mathbb{P}}$, there is a Borel set $B \subseteq [T]$ in $I_{\mathbb{P}}$ such that $[T] \cap A \subseteq B$. Let c be a Borel code for B. By Theorem 4.3, there is a quasi-$\mathbb{P}$-generic real x over $\mathrm{L}[a, c]$ with $x \in [T]$. Since $B \in I_{\mathbb{P}}$, $x \notin B$. But this is impossible because x is also quasi-$\mathbb{P}$-generic over $\mathrm{L}[a]$ and hence $x \in [T] \cap A \subseteq B$.

Case 2: $I_{\mathbb{P}} = N_{\mathbb{P}}$.

In this case, $[T] \cap A$ is $\mathbb{P}$-null, hence there is a $T' \leq T$ such that $[T'] \cap A = \emptyset$. By Theorem 4.3, there is a quasi-$\mathbb{P}$-generic real x over $\mathrm{L}[a]$ with $x \in [T']$. Hence $x \in [T'] \cap A$, a contradiction.

For (2) to (1), take any $\mathbf{\Sigma}^1_2$-set A. We show that A is $\mathbb{P}$-measurable. Let T be in $\mathbb{P}$. We will find an extension T' of T approximating A as in the definition of $\mathbb{P}$-measurability. If $[T] \cap A \in {I_{\mathbb{P}}}^*$, we are done. So we assume $[T] \cap A \notin {I_{\mathbb{P}}}^*$.

Case 1: $\omega_1^{\mathrm{L}[a]} < \omega_1^V$ for any real a.

As in (3) to (2) in Theorem 4.3, in this case, every $\mathbf{\Sigma}^1_2$-set of reals is $\mathbb{P}$-measurable by a standard Solovay argument.

Case 2: $\omega_1^{\mathrm{L}[a]} = \omega_1^V$ for some real a.

Let a be a real such that $[T] \cap A$ is $\Sigma^1_2(a)$ and $\omega_1^{\mathrm{L}[a]} = \omega_1^V$. Then we have a Shoenfield tree in $\mathrm{L}[a]$ for $[T] \cap A$ and we get an ω_1-many Borel decomposition of $[T] \cap A$ into Borel sets $\{B_{c_\alpha} \mid \alpha < \omega_1\}$ with $c_\alpha \in \mathrm{L}[a]$ for each α as in the proof of Theorem 4.3. Since $[T] \cap A \notin {I_{\mathbb{P}}}^*$ and the set of quasi-$\mathbb{P}$-generic reals over $\mathrm{L}[a]$ is of measure one w.r.t. ${I_{\mathbb{P}}}^*$ by (2), there is a quasi-$\mathbb{P}$-generic real x over $\mathrm{L}[a]$ with $x \in [T] \cap A$, so there is an α such that $x \in B_{c_\alpha}$.

The rest is the same as in the proof for (3) to (2) in Theorem 4.3. Since $c_\alpha \in \mathrm{L}[a]$ and x is quasi-$\mathbb{P}$-generic over $\mathrm{L}[a]$, $B_{c_\alpha} \notin {I_{\mathbb{P}}}^*$. Since Borel set is $\mathbb{P}$-measurable, there is a T' in $\mathbb{P}$ such that $[T'] \setminus B_{c_\alpha} \in I_{\mathbb{P}}$. Since $B_{c_\alpha} \subseteq [T] \cap A$, $T' \leq T$ and $[T'] \setminus A \in I_{\mathbb{P}}$, as we desired. ∎

5. $\mathbf{\Sigma}^1_4$-ABSOLUTENESS

It is natural to try to generalize the relationship up to the one between $\mathbf{\Sigma}^1_4$-forcing absoluteness and the regularity properties for $\mathbf{\Delta}^1_3$-sets of reals and $\mathbf{\Sigma}^1_3$-sets of reals. But these analogues cannot be proved in ZFC.[9] In this section, with an additional assumption (sharps for sets), we will prove the analogues of § 4.

Theorem 5.1. Let $\mathbb{P}$ be a strongly arboreal, proper, $\mathbf{\Delta}^1_2$ forcing. Suppose every set has a sharp. Then either $\mathbf{\Delta}^1_2$-determinacy holds or the following are equivalent:
(1) $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness holds, and
(2) every $\mathbf{\Delta}^1_3$-set of reals is $\mathbb{P}$-measurable.

Proof. For (1) to (2), the argument is the same as for (1) to (2) in [11, Theorem 3.1]. What we should check is that we get the absolute tree representation for $\mathbf{\Sigma}^1_3$-sets between V and $V^{\mathbb{P}}$. The rest is exactly the same.

For such tree representation, Feng-Magidor-Woodin used Shoenfield trees for $\mathbf{\Sigma}^1_2$-sets. With the help of sharps for sets, now we use Martin-Solovay trees for $\mathbf{\Sigma}^1_3$-sets. By [13, Theorem 2.1], it suffices to see that $u_2^V = u_2^{V^{\mathbb{P}}}$ for the absoluteness of Martin-Solovay trees between V and $V^{\mathbb{P}}$. But this is true assuming every set has a sharp and $\mathbb{P}$ being proper by [25, Theorem 2.1.9, Example 3.2.7].

For (2) to (1), first note that we may assume that every $\mathbf{\Delta}^1_3$-set is $\mathbb{P}$-Baire by the same argument for (2) to (1) in Theorem 4.1. The argument is the same as for in [11, Theorem 3.1]. What we need is to uniformize a $\mathbf{\Pi}^1_2$-relation by a $\mathbf{\Sigma}^1_3$-function (in [11, Theorem 3.1], Feng-Magidor-Woodin uniformized a $\mathbf{\Pi}^1_1$-relation by a $\mathbf{\Sigma}^1_2$-function). The rest is exactly the same. But such uniformization is possible assuming the failure of $\mathbf{\Delta}^1_2$-determinacy.

The author would like to thank Hugh Woodin for pointing out the following fact to him:

Theorem 5.2. Suppose every real has a sharp. Then either $\mathbf{\Delta}^1_2$-determinacy holds or $\mathbf{\Sigma}^1_3$ has the uniformization property, i.e., any $\mathbf{\Sigma}^1_3$-relation can be uniformized by a $\mathbf{\Sigma}^1_3$-function.[10]

Proof. It suffices to show that every $\mathbf{\Pi}^1_2$-relation can be uniformized by a $\mathbf{\Sigma}^1_3$-function. Suppose $\mathbf{\Delta}^1_2$-determinacy fails. Then there is a real a_0

[9] Start from L and add ω_1-many Cohen reals, then in this model, $\mathbf{\Sigma}^1_4$-forcing absoluteness for Cohen forcing holds but there is a $\mathbf{\Sigma}^1_2$-set of reals without the Baire property.

[10] Since $\mathbf{\Delta}^1_2$-determinacy implies that $\mathbf{\Pi}^1_3$ has the uniformization property, this fact states the dichotomy of the uniformization property for $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$.

such that for each real $a \geq_{\mathrm{T}} a_0$, $\Delta^1_2(a)$-determinacy fails, where $\leq_{\mathrm{T}}$ is the Turing order.

Case 1: for any real $a \geq_{\mathrm{T}} a_0$, $a^\dagger$ exists.

In this case, by the result of Steel, K_a is $\mathbf{\Sigma}^1_3$-correct for any $a \geq_{\mathrm{T}} a_0$, where K_a is the Mitchell-Steel core model.[11]

For each $a \geq_{\mathrm{T}} a_0$, let $<_a$ be the canonical good $\Delta^1_3(a)$-well-ordering on the reals in K_a. Given a real b and a $\Pi^1_2(b)$-relation R, define the uniformization f as follows:

$$f(x) = y \iff y \text{ is the first } <_{\langle x, a_0, b\rangle}\text{-element with } (x, y) \in R,$$

where $\langle x, a_0, b\rangle$ is the real coding x, a_0 and b. For each $x \in \mathrm{dom}(R)$, such a y always exists because $\mathrm{K}_{\langle x, a_0, b\rangle}$ is $\mathbf{\Sigma}^1_3$-correct. So f uniformizes R and regarding the fact that $<_a$ is a good $\Delta^1_3(a)$-well-ordering in K_a for each $a \geq_{\mathrm{T}} a_0$, it is easy to see that f is $\mathbf{\Sigma}^1_3$.

Case 2: there is a real $a \geq_{\mathrm{T}} a_0$ such that $a^\dagger$ does not exist.

Then there is a real $a_1 \geq_{\mathrm{T}} a_0$ such that for any real $a \geq_{\mathrm{T}} a_1$, $a^\dagger$ does not exist. By the result of Dodd-Jensen in [10], K_a is $\mathbf{\Sigma}^1_3$-correct for any $a \geq_{\mathrm{T}} a_1$, where K_a is the Dodd-Jensen core model. The rest is the same as Case 1. ∎

∎

Theorem 5.3. Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably Δ^1_2 forcing. Suppose every set has a sharp. Then either $\mathbf{\Delta}^1_2$-determinacy holds or the following are equivalent:

(1) $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness holds,

(2) every $\mathbf{\Delta}^1_3$-set of reals is $\mathbb{P}$-measurable, and

(3) for any real a and any $T \in \mathbb{P}$, there is a quasi-$\mathbb{P}$-generic real $x \in [T]$ over K_a, where

$$\mathrm{K}_a = \begin{cases} \text{the Mitchell-Steel core model} & \text{if } a^\dagger \text{ exists,} \\ \text{the Dodd-Jensen core model} & \text{otherwise.} \end{cases}$$

Proof. In Theorem 5.1, we have seen the equivalence between (1) and (2). We show the direction from (1) to (3) and the one from (3) to (2).

For (1) to (3), all we need is that the statement "there is a quasi-$\mathbb{P}$-generic real x over K_a with $x \in [T]$" is $\mathbf{\Sigma}^1_4$ for each real a and each

[11] In [28, Theorem 7.9], Steel assumed the existence of two measurable cardinals. We can replace the lower measurable by $a^\dagger$ and the greater measurable by $a^{\dagger\#}$. (Recent development of inner model theory even allows one to omit this sharp. Jensen and Steel [17, 16] constructed K without using measurable cardinals.) For the details, see [23].

$T \in \mathbb{P}$. But this is true by Proposition 2.17 (1) and the fact that the set of reals in K_a is $\Sigma^1_3(a)$ in V.

The argument for (3) to (2) is basically the same as the one in Theorem 4.3. For simplicity, we assume the failure of Δ^1_2-determinacy, hence there is no inner model with a Woodin cardinal. The case for the failure of $\Delta^1_2(a)$-determinacy for a real a can be dealt with in the same way.

Case 1. $\omega_1^{\mathrm{K}_a} < \omega_1^V$ for any real a.

As in Theorem 4.3, we can conclude that every $\mathbf{\Delta}^1_3$-set of reals (even $\mathbf{\Sigma}^1_3$-set of reals) is $\mathbb{P}$-measurable by using $\mathbf{\Sigma}^1_3$-correctness for K_a. To see $\mathbf{\Sigma}^1_3$-correctness for K_a, we need the case distinction whether $a^\dagger$ exists or not. If $a^\dagger$ does not exist, this is due to Dodd-Jensen in [10]. When $a^\dagger$ exists, this is due to Steel.[11]

Case 2. $\omega_1^{\mathrm{K}_a} = \omega_1^V$ for some real a.

We need the absolute decomposition of $\mathbf{\Sigma}^1_3$-sets into Borel sets between K_a and V for some real a. The following result is essential; its proof was communicated to us by Ralf Schindler:

Theorem 5.4 (Schindler)**.** Suppose there is no inner model with a Woodin cardinal. Then if $u_2^{\mathrm{K}_a} < u_2^V$ for any real a, then $\omega_1^{\mathrm{K}_a} < \omega_1^V$ for any real a.

Proof. For simplicity, we only prove $\omega_1^{\mathrm{K}} < \omega_1^V$ assuming $u_2^{\mathrm{K}_a} < u_2^V$ for each real a. To derive a contradiction, we assume $\omega_1^{\mathrm{K}} = \omega_1^V$. The following is the first point:

Claim 5.5. The mouse $\mathrm{K}|\omega_1$ is universal for countable mice, i.e., $M \leq^* \mathrm{K}|\omega_1$ for any countable mouse M, where $\leq^*$ is the mouse order.

Proof of Claim 5.5. Suppose there is a countable mouse M with $M >^* \mathrm{K}|\omega_1$. Coiterate them and let $\mathcal{T}, \mathcal{U}$ be the resulting trees for M and $\mathrm{K}|\omega_1$ respectively.

Case 1: $\mathrm{lh}(\mathcal{T})$ is countable.

Since $M >^* \mathrm{K}|\omega_1$, $\mathcal{U}$ does not have a drop. But then the last model of $\mathcal{U}$ cannot be an initial segment of the last model of $\mathcal{T}$ since the length of $\mathcal{T}$ is countable, a contradiction.

Case 2: $\mathrm{lh}(\mathcal{T})$ is uncountable.

Since $M >^* \mathrm{K}|\omega_1$, $\mathcal{U}$ does not have a drop. If $\mathcal{U}$ was non-trivial, then the final model of $\mathcal{U}$ would be non-sound and could not be a proper initial segment of the final model of $\mathcal{T}$. Hence $\mathcal{U}$ is trivial and $\mathrm{K}|\omega_1$ is an initial segment of the final model of $\mathcal{T}$. But this means ω_1 is a limit of critical points of embeddings via $\mathcal{T}$, hence ω_1 is inaccessible in K, contradicting the assumption $\omega_1^{\mathrm{K}} = \omega_1^V$. □

By the same argument, we can prove that $\mathrm{K}_a|\omega_1$ is universal for countable a-mice for each real a. We now have two cases:

Case 1: There is a real a such that $a^{¶}$ does not exist.

This case was taken care of by Steel and Welch. In [27, Lemma 3.6], they assumed $u_2 = \omega_2$, which is stronger than $u_2^{\mathrm{K}_a} < u_2^V$ for each real a, and proved there is a countable mouse stronger than $\mathrm{K}|\omega_1$ w.r.t. mouse order. But assuming $\omega_1^{\mathrm{K}} = \omega_1^V$ and the non-existence of $0^{¶}$, we can run their same argument only assuming $u_2^{\mathrm{K}} < u_2^V$ and get the same conclusion. Furthermore, we can easily relativize this argument to K_a. Hence assuming $\omega_1^{\mathrm{K}} = \omega_1^V$ (even $\omega_1^{\mathrm{K}_a} = \omega_1^V$) and the non-existence of $a^{¶}$, if $u_2^{\mathrm{K}_a} < u_2^V$, then there is an a-mouse stronger than $\mathrm{K}_a|\omega_1$ w.r.t. mouse order, which contradicts the a-relativized version of Claim 5.5.

Case 2: for any real a, $a^{¶}$ exists.

This case is new. Since $u_2^{\mathrm{K}} < u_2^V$, there is a real a such that $u_2^{\mathrm{K}} < (\omega_1^+)^{\mathrm{L}[a]}$. The idea is to use $a^\dagger$ (that exists since $a^{¶}$ exists) and linearly iterate it with the lower measure in $a^\dagger$ with length ω_1. Then the height of the last model is bigger than u_2^{K} since $u_2^{\mathrm{K}} < (\omega_1^+)^{\mathrm{L}[a]}$. Now we restrict this linear iteration map to K in $a^\dagger$ constructed up to the point with the top measure. The point is this is an iteration map on it and the final model of this iteration has height bigger than u_2^{K}. Since it is a countable mouse, by Claim 5.5, we get a countable mouse in K with the same property, which yields a contradiction by a standard boundedness argument.

We will discuss this idea in detail. Let i be the linear iteration map of $a^\dagger$ derived from the iterated ultrapower starting from the lower measure in it with length ω_1. Then the target N of i has height bigger than u_2^{K} since $u_2^{\mathrm{K}} < (\omega_1^+)^{\mathrm{L}[a]}$ and the critical point of i goes to ω_1 and N has a cardinal bigger than ω_1 and $a \in N$. Let $\mathrm{K}^{a^\dagger}|\Omega$ be the K in $a^\dagger$ constructed up to Ω, the critical point of the top measure in $a^\dagger$. Then $\mathrm{K}^{a^\dagger}|\Omega$ is a mouse and we call it M.

We claim that if we restrict i to M, then it is an iteration map on M. Since i is from a linear iteration of ultrapowers via measures, by applying the result of Schindler [24] in each ultrapower in the iteration, we can prove that the restriction of i to M is an iteration with length ω_1 (which itself might be quite complicated). Moreover, the final model of this iteration has height greater than u_2^{K} because i maps Ω greater or equal to $(\omega_1^+)^{\mathrm{L}[a]}$. Let us call the tree of this iteration $\mathcal{T}$ and let M_α be the α-th iterate via $\mathcal{T}$ and $i^{\mathcal{T}}_{\alpha,\beta}\colon M_\alpha \to M_\beta$ be the induced maps for $\alpha \leq \beta \leq \omega_1$.

Since M is a countable mouse, by Claim 5.5, there is an $\alpha_0 < \omega_1$ such that $M \leq^* \mathrm{K}|\alpha_0$. We will show that $\mathrm{K}|\alpha_0$ has the same property, i.e., there is an iteration from $\mathrm{K}|\alpha_0$ with length ω_1 such that the height of the final model is greater than u_2^{K}. (Note that there might be a drop.) Coiterate $\mathrm{K}|\alpha_0$ and M and let $\pi\colon M \to N$ be the resulted map. Note that there is no drop from the M-side because $M \leq^* \mathrm{K}|\alpha_0$.

We will construct $\langle N_\alpha \mid \alpha \leq \omega_1\rangle$, $\langle \pi_\alpha\colon M_\alpha \to N_\alpha \mid \alpha \leq \omega_1\rangle$, and $\langle i^{\mathcal{U}}_{\alpha,\beta}\colon N_\alpha \to N_\beta \mid \alpha \leq \beta \leq \omega_1\rangle$ with the following properties:

(1) The diagrams below all commute,

(2) $M_\alpha \sim^* N_\alpha \sim^* M_{\alpha+1}$ for each α,

(3) N_α is the direct limit of N_β ($\beta < \alpha$) for limit α, and

(4) $i^{\mathcal{U}}_{\alpha,\alpha+1}$ and $\pi_{\alpha+1}$ are the resulted maps by the comparison between N_α and $M_{\alpha+1}$ for each α.

$$\begin{array}{ccccccc}
\mathrm{K}|\alpha_0 \rightsquigarrow N = N_0 & \xrightarrow{i^{\mathcal{U}}_{0,1}} & N_1 & \xrightarrow{i^{\mathcal{U}}_{1,2}} \cdots \longrightarrow & N_\alpha & \xrightarrow{i^{\mathcal{U}}_{\alpha,\alpha+1}} \cdots \longrightarrow & N_{\omega_1} \\
\uparrow{\scriptstyle \pi=\pi_0} & & \uparrow{\scriptstyle \pi_1} & & \uparrow{\scriptstyle \pi_\alpha} & & \uparrow{\scriptstyle \pi_{\omega_1}} \\
M = M_0 & \xrightarrow[i^{\mathcal{T}}_{0,1}]{} & M_1 & \xrightarrow[i^{\mathcal{T}}_{1,2}]{} \cdots \longrightarrow & M_\alpha & \xrightarrow[i^{\mathcal{T}}_{\alpha,\alpha+1}]{} \cdots \longrightarrow & M_{\omega_1}
\end{array}$$

The above properties uniquely specify $\langle N_\alpha \mid \alpha \leq \omega_1\rangle$, $\langle \pi_\alpha\colon M_\alpha \to N_\alpha \mid \alpha \leq \omega_1\rangle$, and $\langle i^{\mathcal{U}}_{\alpha,\beta}\colon N_\alpha \to N_\beta \mid \alpha \leq \beta \leq \omega_1\rangle$. Hence it suffices to check (1) and (2) above for this construction.

For (1), it suffices to show that $i^{\mathcal{U}}_{\alpha,\alpha+1} \circ \pi_\alpha = \pi_{\alpha+1} \circ i^{\mathcal{T}}_{\alpha,\alpha+1}$ for each α. By the Dodd-Jensen Lemma (e.g., in [32, Theorem 9.2.10]), any two simple iteration maps from a mouse to a mouse are the same. By (2) for α, π_α, $\pi_{\alpha+1}$, $i^{\mathcal{T}}_{\alpha,\alpha+1}$, and $i^{\mathcal{U}}_{\alpha,\alpha+1}$ are all simple iteration maps. Hence we get the desired commutativity. (2) follows from the fact that all the maps constructed before are simple iteration maps.

Since the height of N_{ω_1} is greater than or equal to that of M_{ω_1}, there is an iteration from $\mathrm{K}|\alpha_0$ with length ω_1 whose final model has height greater than u_2^{K}, as we desired.

Since $\mathrm{K}|\alpha_0$ is in K and α_0 is countable in K, there is a real x in K coding $\mathrm{K}|\alpha_0$. We show that the height of N_{ω_1} is less than $(\omega_1^+)^{\mathrm{L}[x]}$. In $L[x]$, we collapse ω_1^V with the forcing $\mathrm{Coll}(\omega, \omega_1^V)$. Let $g\colon \omega \to \omega_1^V$ be a generic surjection over $L[x]$. Since $K|\alpha_0$ is coded by x and the length of iteration is ω_1^V which is countable witnessed by g, by the boundedness lemma in $L[x][g]$, the height of N_{ω_1} is less than $\omega_1^{L[x][g]} = (\omega_1^+)^{\mathrm{L}[x]}$, as desired. Since x is in K, $(\omega_1^+)^{\mathrm{L}[x]} < u_2^K$ and hence the height of N_{ω_1} is less than u_2^K. But the height was greater than u_2^K. Contradiction! ■

Now by the assumption and the above fact, there is a real a such that $\omega_1^{\mathrm{K}_a} = \omega_1^V$ and $u_2^{\mathrm{K}_a} = u_2^V$. By [13, Theorem 2.1], the Martin-Solovay trees for $\mathbf{\Sigma}^1_3$-sets are absolute between K_a and V. Since the trees are on $\omega \times u_\omega$ and u_ω is absolute between K_a and V, we get the absolute decomposition of $\mathbf{\Sigma}^1_3$-sets into Borel sets between K_a and V as we desired. The rest is exactly the same as in Theorem 4.3. ■

Theorem 5.6. Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably Δ^1_2 forcing. Suppose every set has a sharp. Assume

$$I_{\mathbb{P}} \text{ is Borel generated or } I_{\mathbb{P}} = N_{\mathbb{P}}. \tag{$**$}$$

Then either $\mathbf{\Delta}^1_2$-determinacy holds or the following are equivalent:

(1) Every $\mathbf{\Sigma}^1_3$-set of reals is $\mathbb{P}$-measurable, and

(2) for any real a, $\mathbb{R} \setminus \{x \mid x \text{ is quasi-}\mathbb{P}\text{-generic over } \mathrm{K}_a\} \in {I_{\mathbb{P}}}^*$, where

$$\mathrm{K}_a = \begin{cases} \text{the Mitchell-Steel core model} & \text{if } a^\dagger \text{ exists,} \\ \text{the Dodd-Jensen core model} & \text{otherwise.} \end{cases}$$

Proof. The argument is exactly the same as Theorem 4.4 by replacing $\mathrm{L}[a]$ by K_a and using the analogous facts about K_a we have already stated. □

6. Applications

In this section, we mention two applications of our theorems to particular cases. One will be proved here and the other is in [9].

Brendle-Halbeisen-Löwe [7] proved the following:

Proposition 6.1 (Brendle-Halbeisen-Löwe)**.** Let $\mathbb{V}$ be Silver forcing. Suppose for any real a there is a quasi-$\mathbb{V}$-generic real over $\mathrm{L}[a]$. Then every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{V}$-measurable.[12]

Question 6.2 (Brendle-Halbeisen-Löwe)**.** Does the converse of Proposition 6.1 hold?[13]

We answer the above question positively:

Proposition 6.3. Assume every $\mathbf{\Delta}^1_2$-set of reals is $\mathbb{V}$-measurable. Then for any real a, there is a quasi-$\mathbb{V}$-generic real over $\mathrm{L}[a]$.

Proof. Since Silver forcing is strongly arboreal and proper, by Theorem 4.3, it suffices to show that the set of Borel codes with $B_c \in {I_{\mathbb{V}}}^*$ is $\mathbf{\Sigma}^1_2$. We use the following fact:

Fact 6.4 (Zapletal)**.** Let G be the graph on ${}^\omega 2$ connecting two binary sequences if they differ in exactly one place. Let I be the σ-ideal generated by Borel G-invariant sets (i.e., Borel sets in ${}^\omega 2$ such that any

[12]See [7, Proposition 2.1]. Regarding $I_{\mathbb{V}} = N_{\mathbb{V}}$, it is easy to check that Silver measurability in their sense coincides with our $\mathbb{V}$-measurability.

[13]See [7, Question 4].

two distinct elements of them are not connected by G). Then every analytic set is either in I or contains $[T]$ for some $T \in \mathbb{V}$.

Proof. See [30, Lemma 2.3.37]. □

We show how to use Fact 6.4 to prove Proposition 6.3. We first show that $I \subseteq I_{\mathbb{V}}$. It suffices to see that every Borel G-invariant set is in $N_{\mathbb{V}}$. Take such Borel set B. Since every Borel set is $\mathbb{V}$-measurable and $I_{\mathbb{V}} = N_{\mathbb{V}}$, for each $T \in \mathbb{V}$, there is a $T' \leq T$ such that either $[T'] \subseteq B$ or $[T'] \cap B = \emptyset$. But the former case cannot happen because $[T']$ contains many G-connected elements. Hence $[T'] \cap B = \emptyset$. Therefore B is $\mathbb{V}$-null.

With the above fact, this means every Borel set is either in I or contains $[T]$ for some $T \in \mathbb{V}$. Hence $B_c \in {I_{\mathbb{V}}}^*$ iff B_c is in I, i.e., it is the union of a countable set of G-invariant Borel sets. This is easily $\mathbf{\Sigma}^1_2$, as we desired. ■

Regarding $I_{\mathbb{V}} = N_{\mathbb{V}}$, the following is a direct consequence of Theorem 4.4 and Proposition 6.3 (or an easy consequence of [7, Lemma 3.1]):[14]

Corollary 6.5. The following are equivalent:

(1) Every $\mathbf{\Sigma}^1_2$-set of reals is $\mathbb{V}$-measurable, and

(2) for any real a, the set of quasi-$\mathbb{V}$-generic reals over $\mathrm{L}[a]$ is of measure one w.r.t $N_{\mathbb{V}}$.

Another application is for eventually different forcing by Brendle-Löwe [9]. They used Theorem 4.4 to prove that the Baire property in eventually different topology for every $\mathbf{\Sigma}^1_2$-set of reals is equivalent to the statement "ω_1 is inaccessible by reals". For the basic definitions and properties for eventually different forcing and its topology, the reader can consult [21].

7. Questions and discussions

We close this paper by raising questions and discussing them.

7.1. On $I_{\mathbb{P}}$ and ${I_{\mathbb{P}}}^*$.

Although ${I_{\mathbb{P}}}^*$ is the same as $I_{\mathbb{P}}$ for most cases as we have seen in Lemma 2.13, as in Question 2.12, we still do not know whether this is true in general. What we could wish is that this is true at least for Borel sets:

Question 7.1. Let $\mathbb{P}$ be a strongly arboreal, proper forcing. Then can we prove $B \in I_{\mathbb{P}}$ iff $B \in {I_{\mathbb{P}}}^*$ for any Borel set B?

If this is true, we do not have to mention ${I_{\mathbb{P}}}^*$ in our theorems.

[14] This answers [7, Question 3] positively.

7.2. **On the condition $(*)$ in Theorem 4.3.** It is interesting to give sufficient conditions for $\mathbb{P}$ satisfying $(*)$ in Theorem 4.3, i.e., the set of all Borel codes with $B_c \in I_{\mathbb{P}}{}^*$ is $\mathbf{\Sigma}^1_2$. These conditions could be definability conditions on $I_{\mathbb{P}}{}^*$ or directly on $\mathbb{P}$.

For the first case, we have a useful sufficient condition: we say that a σ-ideal I on the reals is $\mathbf{\Sigma}^1_2$ *on* $\mathbf{\Sigma}^1_1$ if for any analytic set $B \subseteq {}^\omega 2 \times {}^\omega\omega$, the set $\{c \mid B_c \in I\}$ is $\mathbf{\Sigma}^1_2$. It is easy to check that if $I_{\mathbb{P}}{}^*$ is $\mathbf{\Sigma}^1_2$ on $\mathbf{\Sigma}^1_1$, then $(*)$ holds. Since $I_{\mathbb{P}}$ is $\mathbf{\Sigma}^1_2$ on $\mathbf{\Sigma}^1_1$ and $I_{\mathbb{P}} = I_{\mathbb{P}}{}^*$ for most cases, $(*)$ is true for most $\mathbb{P}$.

For the second case, we ask the following:

Question 7.2. Let $\mathbb{P}$ be a strongly arboreal, strongly proper, provably Δ^1_2-forcing. Then can we prove $(*)$?

7.3. **$\mathbf{\Delta}^1_2$-determinacy and $\mathbf{\Sigma}^1_4$-forcing absoluteness.** In Theorem 5.1, we use the failure of $\mathbf{\Delta}^1_2$-determinacy to prove the equivalence between (1) and (2). But it could be that both (1) and (2) are consequences of $\mathbf{\Delta}^1_2$-determinacy. Since we have only used sharps for sets for the direction from (1) to (2), it is enough to see whether $\mathbf{\Delta}^1_2$-determinacy implies $\mathbf{\Sigma}^1_4$-forcing absoluteness:

Question 7.3. Suppose $\mathbf{\Delta}^1_2$-determinacy holds. Then can we prove $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness for each strongly arboreal, proper, provably Δ^1_2-forcing $\mathbb{P}$?

7.4. **Sharps for sets vs sharps for reals.** In Theorem 5.1, Theorem 5.3 and Theorem 5.6, we have assumed the existence of sharps for sets. It is natural to ask whether we can reduce this assumption to sharps for reals. The obstacle is whether proper forcings preserve the statement "every real has a sharp" and u_2:

Question 7.4. Suppose every real has a sharp. Let $\mathbb{P}$ be a strongly arboreal, proper, provably Δ^1_2-forcing. Then can we prove that every real has a sharp in $V^{\mathbb{P}}$ and $u_2^V = u_2^{V^{\mathbb{P}}}$?

Finally, we show that in the case of provably ccc, $\mathbf{\Sigma}^1_1$-forcings, things work perfectly:

Proposition 7.5. Let $\mathbb{P}$ be a strongly arboreal, provably ccc, $\mathbf{\Sigma}^1_1$-forcing. Then

(1) $I_{\mathbb{P}} = I_{\mathbb{P}}{}^*$.
(2) $I_{\mathbb{P}}$ is Borel generated.
(3) The condition $(*)$ holds. Moreover, $\{c \mid B_c \in I_{\mathbb{P}}{}^*\} \in \mathbf{\Delta}^1_2$.
(4) Let M be a transitive model of ZFC. Then a real x is $\mathbb{P}$-generic over M iff x is quasi-$\mathbb{P}$-generic over M.

(5) If $\mathbf{\Delta}^1_2$-determinacy holds, then so does $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness.

(6) If every real has a sharp, then every real has a sharp also in $V^{\mathbb{P}}$ and $u_2^V = u_2^{V^{\mathbb{P}}}$.

Proof. (1) is already mentioned in Lemma 2.13 (3) and (2) is immediate since $\mathbb{P}$ is ccc.

For (3), it suffices to see the following by Lemma 3.5 (1):

$$\begin{aligned}\pi^{-1}(B_c) \text{ is meager} &\iff (\exists M \ni c)\ \big(M\colon \text{a countable transitive model} \\ &\qquad \text{of ZFC and } M \vDash \text{“}\pi^{-1}(B_c) \text{ is meager”}\big) \\ &\iff (\forall M \ni c)\ \big(M\colon \text{a countable transitive model} \\ &\qquad \text{of ZFC} \implies M \vDash \text{“}\pi^{-1}(B_c) \text{ is meager”}\big),\end{aligned}$$

where $\pi = f_{\dot{x}_G}$ as before.

We only show the first equivalence. For left to right, if we take a countable elementary substructure X of $\mathcal{H}_\theta$ for enough large θ such that X has all the essential elements, then the transitive collapse of X will do the job for M in the right hand side.

For right to left, take an M with the property in the right hand side. The idea is the same as the proof of Claim 3.8 in Lemma 2.17 (1). This time, we use G, the Banach-Mazur game with a witness for $\pi^{-1}(B_c)$ starting from any element of $\mathbb{P}$, both in M and V and translate a winning strategy in G^M to the one in G.

By the assumption, in M, player II has a winning strategy σ' in G. The construction of a winning strategy for II in G in V from σ' is exactly the same as Claim 3.8. But instead of using the $(M, \mathbb{P})$-genericity for a condition T', we use the following:

Claim 7.6. Let D be a dense subset of $\mathbb{P}$ in M. Then D is predense in $\mathbb{P}$ in V.

Proof of Claim 7.6. Let D be a dense subset of $\mathbb{P}$ in M. Then since $\mathbb{P}$ is provably ccc, in M, there is a countable maximal antichain $A \subseteq D$. But since $\mathbb{P}$ is $\mathbf{\Sigma}^1_1$, the statement “a real codes a maximal antichain” is $\mathbf{\Sigma}^1_1 \wedge \mathbf{\Pi}^1_1$ and therefore A remains a maximal antichain in V. Hence D is predense in $\mathbb{P}$ in V. □

The rest is exactly the same as Claim 3.8. The argument for (4) is exactly the same as for Lemma 2.17 (2) and (3). For (5), see [25, Lemma 2.2.4]. For (6), see [25, Lemma 2.2.2, Theorem 2.2.7, Example 3.2.7]. ■

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