‘Rolling down the River’: Saul Kripke and the Course of Modal Logic.

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Abstract. We discuss Saul Kripke’s seminal 1963 paper ’Semantical Considerations on Modal Logic’, and sketch subsequent developments in modal logic with a view to their general logical thrust.

Keywords. Modal Logic, Kripke, Invariance, Translation, Complexity.

1. Returning to the Sources: What a Famous Paper Contains

Kripke’s classical paper that forms the point of departure for this brief essay[1] has a content that is easily described to the modern reader – especially, since much of it made its way soon into the widely used textbook Hughes & Cresswell 1968 [1], a major point of entry for my generation into the area.

The author first summarizes his own earlier work on relational possible worlds semantics of propositional modal languages, where truth of a formula $\Box\varphi$ at a world s means that is true at all accessible worlds t.[2] He notes the resulting completeness theorems for deduction in classical propositional modal logics such as $T, S4$ or $S5$.[3] But the key topic of the paper is modal predicate logic with object quantifiers, the vehicle for philosophical discourse about modality. This was an active area at the time. Semantic structures seem obvious: ordered families of possible worlds endowed with object domains, and an interpretation function for predicates with respect to objects and worlds. But the issue is the truth definition for the language. Kripke mentions earlier systems by Prior and Hintikka, taking different Fregean-Austinian views on atomic statements involving objects ‘not of this world’: these

Acknowledgments. I would like to thank the anonymous reader for useful feedback, and Ştefan Minică for the LaTeX apotheosis of this paper.

[1]Semantical Considerations on Modal Logic (Kripke 1963 [18]).

[2]The semantic structures in the paper are ‘model structures’ (nowadays, ‘frames’ with a distinguished world) and ‘models’ (adding a ‘valuation’). Accessibility relations are to be reflexive: the paper has no ‘minimal modal logic K (named afterwards in honour of Kripke).

[3]In a footnote, Kripke cites Kanger and Hintikka as authors of related semantic ideas.

lack truth values.[4] While this diversity may reflect different legitimate conceptions of modal predication and quantification[5], a further fact hurts. Consistent logics for these semantics turn out to modify either the underlying modal propositional logics, or the standard inference rules of first-order predicate logic.

Kripke's own proposal takes an alternative Russelian line, giving truth values to modal formulas with any objects, inside or outside of the current world of evaluation. Boolean and modal clauses stay as for propositional systems, while quantification is only over objects existing at a world. He gives a proof system that is complete for the new semantics, summarizing his results as follows:

> The systems we have obtained have the following properties. They are a straightforward extension of the modal propositional logics, without the modifications of Prior's Q; the rule of substitution holds without restriction, unlike Hintikka's presentation; and nevertheless neither the Barcan formula nor its converse is derivable. Further, all laws of quantification theory - modified to admit the empty domain - hold. The semantic completeness theorem we gave for modal propositional logic can be extended to the new systems.

Finally, the paper discusses a number of mathematical provability interpretations for the modalities, where possible worlds stand for models of arithmetic.

2. A First Reaction: 40 Years Later

One striking feature of this much-cited, but probably not much-read paper is how few results it contains! There is no new take on propositional modal systems, there is one new proposal for modal predicate logic – but not one that has eventually commanded allegiance – and there is no trace of the subsequent discovery of mathematical and philosophical problems with the 'straightforward' models of modal predicate logic, that have persisted until today (Brauner & Ghilardi 2006 [2], van Benthem 2009 [3], Chapters 11, 26). Finally, the exploration of provability interpretations focuses precisely on a road not taken in the subsequent flowering of the field of 'Provability Logic' (Artemov 2006 [4]).

Now this is hindsight, and I do recall the liberating effect of this paper and others by Kripke, inspiring a young generation of logicians in the 1960s. Moreover, progress arises from seminal new ideas, as much as new theorems. And the paper definitely opens up a new semantic program that has kept growing since. Even so, I felt a bit like once when visiting the headwaters of the Mississippi in Ithaca State Park, Minnesota. It is hard to imagine that the mighty river originates in this placid setting. And maybe this geographical analogy is apt. The Mississippi becomes a great river because of two further features. One is its teaming up with mighty allies, in particular the Missouri. Likewise, the later development of modal logic probably owes as much to the streams of powerful new ideas that were added

[4] Options spread to truth conditions for the modality, say, construed in a three-valued format.
[5] In this connection, Kripke later speaks of different *conventions*, all tenable in modal logic.

to the paradigm by other strong logicians. Moreover, the Mississippi only becomes what it is thanks to the terrain on its path, with gradients driving it eventually to its delta. In a similar fashion, to me, it is the scientific community and its qualities of reception and nurture that determine whether ideas grow or die. The intellectual gradients of the time were right for modal logic.

3. The Historical Context: Backward, and Forward

The semantics for modal logic has a long history, and despite various authoritative surveys, it still has lots of mysteries to me. For instance, contrary to popular opinion, it just is not true that modal logic was in a ‘syntax only’ mode before the 1960s– except maybe in the hands of diehard formalists. Topological interpretations of the universal modality as an interior operation date back to Tarski in the 1930s (cf. van Benthem & Bezhanishvili 2007 [5]), and with hind-sight, we can see that this early topological semantics is even a generalization of the relational possible worlds semantics for $S4$ and its extensions, since Kripke models correspond to Alexandroff tree topologies. Indeed, the analogy extends from $S4$ to the minimal modal logic, and even below, once we move from topological to neighbourhood semantics. Topological methods were quite popular in the 1950s, and in particular, Beth 1957 [6] modeled intuitionistic logic in topological tree models. But somehow, relational models provided a simpler take that ‘fired the imagination’, perhaps also because of historical echoes to Leibnizian views of modality as truth in all possible worlds, that go back to the Middle Ages. But even their mathematical core content was around: it had been established in Jonsson & Tarski 1951 [7] on the representation theory of Boolean algebras with additional operators. Moreover, a further relevant line that was publicly known is Prior’s extensive work in the 1950s on temporal logic, where the relational interpretation for the modalities in terms of ‘earlier/later’ is precisely the point. But in temporal logic, the relational semantics is so obvious that it seems no big deal, and its ideas remained underappreciated.

4. Intermezzo: Two Directions in Modal Logic

A distinction seems relevant here. One can connect modal languages with semantic structures in two ways. In one direction, the latter are given beforehand, and one looks for logical languages that *describe essential features*, and lead, hopefully, to axiomatizable, perhaps even decidable calculi of reasoning. For instance, one can view early topological semantics as some sort of ‘proto-topology’. In this direction, no particular language or logic is sacrosanct, since one wants to get at relevant structure with whatever language is best suited. Thus, changing a modal language is quite acceptable on this approach, and indeed, temporal logics came in lots of different strengths. In the opposite direction of *giving a semantics*, however, one fixes a language and perhaps a deductive system, and asks for the design of a model class that makes the given logic come out ‘complete’. The two modes suggest different questions. Of course, eventually, the two directions meet and interact, as is the reality of research in modal logic today.

In either direction, much of the subsequent history of modal logic could not have been predicted from its beginnings in the 1960s. The story that runs from the present paper to the extensive field described in the 2006 *Handbook of Modal Logic* [10] contains many further themes, and a much larger set of 'players', with applications and motivations coming not just from philosophy and mathematics, but also computer science, linguistics, and even economic game theory.

5. Why Was the Framework Attractive, and What Made it Stay?

One basic feature of Kripke's paper is its explanation of modal notions by means of essentially a classical model-theoretic picture. Intensionality is *extensionalized* via 'multiple reference' in sets of possible worlds, i.e., an extended notion of extension. This move 'de-mythologized' intensionality, and at the same time, it also *geometrized* it, providing appealing geometrical content to known modal axioms in terms of features of accessibility relations. I still recall how illuminating it was to match laborious modal syntactic deduction in a system like $S4$ with concrete pictures of reflexive transitive orders: it was as if one could suddenly see what one was doing. These semantic moves caught on fast, and proved illuminating across a wide range of philosophical themes beyond modality, such as time, knowledge, obligation, conditionals, and so on. Thus, modal logic became the 'calculus' of a booming area of philosophical logic. At the same time, technically, possible-worlds semantics brought modal logic much closer to extensional classical logic, and thus, methodologically, the unity of the field of logic became restored: insights and techniques could now flow freely between 'classical' and 'non-classical' areas. As a result, the distinction between 'mathematical' and 'philosophical' logic becomes pretty thin – as we shall see in more detail below.

Still there were criticisms from the start, saying that accessibility relations were ad-hoc formal devices, and that multiple extension was too coarse-grained for true intensionality. While these are valid points in many settings, the staying power of possible-worlds modeling has become ever clearer over time. And also, however justified the worries, the other crucial historical fact is that no convincing competitor has emerged with equal power and sweep. For instance, the onslaught of situation semantics in the 1980s as an alternative paradigm has failed – and later versions of situation theory even used modal logic to bring out their key features (cf. van Benthem & Martinez 2008 [8]).[6] Likewise, mathematical criticisms of the low content of modal logic have subsided, since the mathematics of modal logics has turned out much richer than what was imagined in the 1960s.

[6] This is not to say that all discussion is over. For instance, the appealing provability interpretation of the universal modality $\Box\varphi$ is an $\exists$-*type account* saying that there exists a proof for φ, rather than the above semantic $\forall$-type account that φ is true in all relevant possible worlds.

But over the years, modal logic has undergone some major changes, affecting its role in logic as a whole. We look at a few, and try to state their essence as a contribution to ‘universal logic’, i.e., the general thrust of the field.

6. Mathematical Changes in our Understanding of Modal Logic

While modalities have traditionally been viewed as expressions that enrich a classical system, while the matching semantics moves from single situations to complex families of worlds, the modern perspective has changed considerably. Modal logics are not about richer systems than classical logic, but poorer ones! The discussion to follow starts from propositional modal logic, that has become the dominant approach in the field, partly since it high-lights the modality per se.[7]

Graph Structures One shift is that ‘possible worlds’ in their original sense are no longer the ruling paradigm. Worlds can be as diverse as information states of an agent, states of a computer, points in time, board stages in games, linguistic parse trees, or just: points in a directed graph. The term ’possible world’ is retained mainly for reasons of nostalgia and faded grandeur. Here is a better picture: modal logic is about *directed graphs*, and a reason for its broad sweep is the ubiquity of geometrical structures like this across a wide range of subjects.

Local Quantification But structure is not all. Graphs suggest an ‘internal’ description language, where one views a large total situation from some current point via accessible neighbours. Typically then, modalities express *local quantification* over these accessible points. This is more restricted than the usual quantifiers of first-order logic, that give unbounded access to arbitrary points in a model.

Standard Translation and Tandem View A powerful insight making modal logic more down-to-earth has been that Kripke’s semantics drives a straightforward *translation* of the modal language into a first-order one of the right signature over possible worlds models, by sending modalities to bounded quantifiers. Thus, a modal formula $\Box\Diamond p$ describing a world w in a model M can be read equally well, using the truth conditions as translation clauses, as a first-order formula $\forall y(Rxy \to \exists z(Ryz \wedge Pz))$. This Gestalt Switch yields a *tandem approach* to reasoning with intensional notions without having to choose between modal or first-order logic, viewing modalities as bounded quantifiers ranging over the local environment of a world in an accessibility pattern, or abstractly, a point in a graph. Thus, from being an external ‘challenger’, modal logic gets integrated into classical logic.[8]

Fine-structure and Fragments The above translation sends the basic modal language into a *fragment* of first-order logic. More generally, propositional modal languages tend to be fragments of classical logics, though not always first-order ones. What makes these fragments so insightful? For a start, the mini-language

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[7]Many modern insights have arisen from this move, letting the modalities ‘speak for themselves’ first. Later on in this paper, we will briefly discuss where *modal predicate logic* stands today.

[8]This co-existence was reinforced by other developments, omitted here, such as ‘*frame correspondence theory*’ for modal axioms in terms of classical properties of the accessibility relation.

of basic modal logic has proved remarkably well-chosen for its combining various desirable features. One is the ease of modal deduction, without the tedious variable management needed for first-order proof in general. Modal core patterns of reasoning 'meet the eye' at once. But perhaps deeper are the following points:

Expressive Power and Invariance The expressive power of the basic modal language turned out to match a natural structural *invariance relation* between graph models, viz. *bisimulation*, a natural back-and-forth 'process equivalence' between transitions from world to world that preserves atomic facts (Blackburn, de Rijke & Venema 2001 [9]). Bisimulation analyzes when two models are 'the same' from a modal point of view, a basic question that sets the semantic expressivity level for any well-designed logical language. And it has proven a natural level of identification for structures, not just in modal logic, but also in process theories, set theory, and other areas.[9]

Computational Complexity Next, while the expressive power of the modal language is weaker than that of first-order logic, it shows much better behaviour in terms of the computational complexity of the core tasks a logic is used for: stating properties of structures, evaluating them, and reasoning about them. This computational viewpoint has often been dismissed as a matter of 'mere implementation', but by now, there may be more awareness that *procedural fine-structure* is as fundamental an issue as expressive fine-structure. In particular, testing for modal validity is decidable in polynomial space, model checking takes polynomial time: lower than for first-order logic. Thus expressive weakness can be computational strength, and the perspective shifts once more:

'The Balance' Modal languages strike a balance between two competing forces: expressive power and computational complexity. This is a much broader theme in the field: first-order logic itself arose as a reasonably expressive fragment of second-order logic whose notion of consequence was axiomatizable (though undecidable).[10] But there are other natural compromises along these lines. In modern modal logic, many systems (cf. the 'hybrid logics' of Areces & ten Cate 2006 [11]) lie in between the basic language and full first-order logic, with extra modalities describing more graph structure.[11] This landscape of fragments of first-order logic has shown that 'small can be beautiful'. Hence, modal logics are also tools for exploring the fine-structure of complex classical systems, sometimes leading to the discovery of new classical logics in the process.[12]

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[9] The crucial issue of when two modal models represent 'the same structure' does not seem to have occurred much to philosophers, and it is still under-appreciated in philosophical logic.

[10] Our survey is by no means complete. There are many further viewpoints that capture important aspects of modal formalisms. One important more syntactic way to think of the above translation views modal languages as perspicuous *variable-free* formalisms for proof-theoretic purposes.

[11] The same is true for languages beyond first-order logic such as the *modal μ-calculus* (Bradfield & Stirling 2006 [12]). This is a decidable part of fixed-point logic with only local quantifiers.

[12] Modal logic influenced classical logic in the 'Guarded Fragment' of Andréka, van Benthem & Németi 1998 [13], a new large decidable part of first-order logic. Or see the abstract model theory with Lindström theorems for weak languages in van Benthem, ten Cate & Väänänen 2009 [14].

The above features are my take on what makes modal logic general, and a source of perspectives and insights for the 'Universal Logic' in this volume. Of course, this is a personal stance, and more could be said. A large role in changing modal logic was also played by the mathematical studies of the 1970s, on frame definability, general completeness (and incompleteness) theorems, or the interplay of model theory and algebra. All these are well-documented in the *Handbook of Modal Logic* [10], and they are very much alive today, witness the lively current interactions of modal logic and *Universal Algebra*.

7. Descriptive Expansion: Modal Patterns and Transdisciplinary Migrations

But the development of modal logic since the 1960s is not just theoretical evolution of mathematical and computational perspectives. At the same time, its basic features changed through extension of descriptive coverage, and the study of modal patterns in a widening circle of fields. This was already visible in the late 1960s, when there was a wave of philosophical logics using modal ideas in innovative ways, including epistemic logic, doxastic logic, tense logic, deontic logic, conditional logic, and of course, Kripke's own studies of modality and quantification that reverberated for decades.

But also beyond philosophy, modal patterns are ubiquitous.[13] Linguists have used modal operators for describing modal, temporal and other expressions in natural language, making them important to communication and other cognitive functions. An even broader source of modal patterns is the study of information and agency, that crosses borders between philosophy and many other disciplines. In the 1970s, economists independently rediscovered epistemic logic as a convenient perspicuous formalism for the informational reasoning about rational agency that keeps players locked into game-theoretic equilibria. Around the same time, computer scientists started using modal, temporal, and epistemic logics for a wide variety of purposes in describing processes and information flow. The resulting dynamic and temporal logics of computation, information, knowledge, belief, preference, and other crucial features of social agency are coming together today in the study of what is sometimes called *intelligent interaction*.[14]

Besides this, modal patterns have turned out crucial in ever more mathematical settings, witness non-well-founded set theory, provability logic, logics of space and space-time, or 'co-algebra' of non-well-founded infinite objects (Venema 2006 [16]). This wholly unplanned academic penetration of modal ideas since the 1960s is another reason for the remarkable staying power of the field.

[13] Freudental 1959 [15] proposed broadcasting reasoning patterns with basic modal dualities into outer space, as a way of making other intelligences aware of basics of thought on Planet Earth.

[14] Modal patterns also occur in knowledge representation, web languages, or spatially distributed computation, where again, they have often been rediscovered independently.

8. Back to Philosophy and Mathematics

While philosophers may feel that all this newer activity is a farewell to the original philosophical motivations, there is really no reason why a prodigal son might not return to the old home, bringing travel tales far beyond what his parents ever imagined. Some modal logicians are returning to epistemology, philosophy of action, and other areas of the mother discipline these days. Examples are work on belief revision, interactive epistemology, and philosophy of action inspired by computational and game-theoretic influences. And examples can be multiplied.[15]

One might think of all this as ‘applied modal logic’, though I have argued elsewhere that the term ’applied logic’ fails to do justice to what is happening today – as it suggests two falsehoods: that traditional ideas and logic systems suffice in the new areas, and that no new pure logical theory is at stake. But one striking feature of modal logics in the above areas has been the emrgence of new fundamental notions and issues, beyond anything studied up to the 1980s:

One key example is the notion of what may be called *equilibrium*, in epistemic reflection or in actual behaviour: with examples like common knowledge, game-theoretic equilibria, or iterated action. This calls for modal logics beyond the usual ones, incorporating *recursion mechanisms* with ‘fixed-points’, whose theory, in full development, is rife with open problems. Another fundamental new theme is ‘system architecture’, and in particular, *system combination*. On a naive analytical view, one should just do modal logics for various features of agency, and then throw them together to get the whole picture. But it has become clear since around 1990, that things are much more delicate. Depending on the manner of combination, even decidable modal logics may give rise to highly undecidable combinations. Thus, the behaviour of complex modal systems is one more new fundamental challenge.

Both recursion and system combination make excellent sense for logic in general – and I would put them on a par with the ‘universal’ themes in Section 6.

Back to modal predicate logic An illustration of the preceding point takes us back to the modal predicate logic originally discussed here. One reason why the latter system has been so hard to define well, avoiding model-theoretic and proof-theoretic catastrophes, is the fact that is a system combination of two modal logics,[16] whose behaviour depends critically on the mode of combination. Even so, modal predicate logic is obviously important to old and new theory and applications – and in that sense, Kripke’s paper is still highly relevant, decades later.

9. Conclusion

Is all well on the banks of the great river that Saul Kripke helped bring forth? Clearly, in its course, the river runs among diverse communities – and a common

[15] Modal logics of *preference*, another crucial ingredient of intentional goal-driven rational agency, using evaluation orders of worlds, might well make their way back into deontic logic.

[16] One can profitably view first-order logic itself as a modal logic – or more precisely, as a dynamic logic of changing assignments in suitable computational state spaces.

practical vision among them is hard to achieve. Also, major theoretical problems remain unresolved, from finding the right semantics for modal predicate logic to understanding general behaviour of system combination, the delicate balance of expressive power and computational complexity, and other issues raised here. And finally, different conceptions of modality are still alive beyond the possible worlds semantics of Kripke's paper. I mentioned alternative ∃-style proof views (Artemov 2006 [4]) – and even more radically than that, non-operator *predicate views* of modality, dismissed in the 1960s partly under the influence of Montague, may still get their field day (Halbach, Leitgeb & Welch 2003 [17]).

However that may be, I feel that even in its present state, modal logic is not just an area of interdisciplinary application, but also an excellent conceptual laboratory for pure universal logic.

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