Complete axiomatization of the stutter-invariant fragment of the linear time μ-calculus ⋆

Amélie Gheerbrant

ILLC, Universiteit van Amsterdam, a.gheerbrant@uva.nl

Abstract. The logic $\mu(\mathsf{U})$ is the fixpoint extension of the "Until"-only fragment of linear-time temporal logic. It also happens to be the stutter-invariant fragment of linear-time μ-calculus $\mu(\Diamond)$. We provide complete axiomatizations of $\mu(\mathsf{U})$ on the class of finite words and on the class of ω-words. We introduce for this end another logic, which we call $\mu(\Diamond_\Gamma)$, and which is a variation of $\mu(\Diamond)$ where the Next time operator is replaced by the family of its stutter-invariant counterparts. This logic has exactly the same expressive power as $\mu(\mathsf{U})$. Using already known results for $\mu(\Diamond)$, we first prove completeness for $\mu(\Diamond_\Gamma)$, which finally allows us to obtain completeness for $\mu(\mathsf{U})$.

1 Introduction

Stutter-invariancy is a property that is argued by some authors (see [8]) to be natural and desirable for a temporal logic, especially in the context of concurrent systems. Roughly, a temporal logic is stutter-invariant if it cannot detect the addition of identical copies of a state. The stutter-invariant fragment of linear-time temporal logic LTL is know to be its "Until"-only fragment $LTL(U)$ and is obtained by disallowing the use of the "Next" operator (see [10]). It has been extensively studied and it is widely used as a specification language. Nevertheless, it has been pointed out (see in particular [3]) that $LTL(U)$ fails to characterize the class of stutter-invariant ω-regular languages. In order to extend the expressive power of this framework, while retaining stutter-invariancy, some ways of extending it have been proposed. In [3], Kousha Etessami proposed for instance the logic $SI-EQLTL$, which extends $LTL(\mathsf{U})$ by means of a certain restricted type of quantification over proposition letters. He showed that $SI-EQLTL$ characterizes exactly ω-regular languages.

In this paper, we focus on another logic, which we call $\mu(\mathsf{U})$ and which is the fixpoint extension of the "Until"-only fragment of linear-time temporal logic. $\mu(\mathsf{U})$ has exactly the same expressive power as $SI-EQLTL$ (this follows from results in [5]), which implies that it also characterizes exactly stutter-invariant ω-regular languages. It is also known that it satisfies uniform interpolation (see [5]), which is a sign that $\mu(\mathsf{U})$ is a well-behaved logic. Here we further contribute to the study of the logical properties of $\mu(\mathsf{U})$ by completely axiomatizing it over the class of ω-words and over the class of finite words. We introduce for this end

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another logic, which we call $\mu(\Diamond_\Gamma)$, and which is a variation of $\mu(\Diamond)$ where the Next time operator is replaced by the family of its stutter-invariant counterparts. We use this logic as a technical tool to show completeness results for $\mu(\mathsf{U})$.

Outline of the paper: In Section 2, we recall basic facts and notions about linear-time μ-calculus $\mu(\Diamond)$. We also give a precise definition of the notion of stutter-invariancy and introduce $\mu(\mathsf{U})$, the stutter-invariant fragment of $\mu(\Diamond)$. In Section 3, we introduce the logic $\mu(\Diamond_\Gamma)$ and show that $\mu(\Diamond)$ and $\mu(\Diamond_\Gamma)$ have exactly the same expressive power on finite and ω-words. In section 4, we give axiomatizations of $\mu(\Diamond_\Gamma)$ that we respectively show to be complete on these two classes of structures. Finally, these results are put to use in Section 5, where we show similar completeness results for $\mu(\mathsf{U})$.

2 Preliminaries

In this section, we introduce the syntax and semantics of linear time μ-calculus $\mu(\Diamond)$. We also recall its axiomatization on some interesting classes of linear orders and introduce the notion of stutter-invariance.

2.1 Linear time μ-calculus

By a *propositional vocabulary* we mean a countable (possibly finite) non-empty set of propositional letters $\sigma = \{p_i \mid i \in I\}$.

Definition 1 (Syntax of $\mu(\Diamond)$). Let σ be a propositional vocabulary, and let $\mathcal{V} = \{x_1, x_2, \ldots\}$ be a disjoint countably infinite set of *propositional variables.* We inductively define the set of $\mu(\Diamond)$-formulas in vocabulary σ as follows:

$$\phi, \psi, \xi := At \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \phi \rightarrow \psi \mid \phi \vee \psi \mid \Diamond\phi \mid \mu x_i.\xi$$

where $At \in \sigma \cup \mathcal{V}$ and, in the last clause, x_i occurs only positively in ξ (i.e., within the scope of an even number of negations). We will use $\nu x_i.\xi$, $\Box\phi$, $\phi\mathsf{U}\psi$, $\mathsf{F}\phi$ as shorthand for, respectively, $\neg\mu x_i.\neg\xi(\neg x_i)$, $\neg\Diamond\neg\phi$, $\mu y.(\psi \vee (\phi \wedge \Diamond y))$ and $\mu y.(\phi \vee \Diamond y)$. We will also use $\mathsf{G}\phi$ as shorthand for $\neg\mathsf{F}\neg\phi$.

A *linear flow of time* is a structure $\mathcal{L} = (W, <)$, where W is a non-empty set of points and $<$ is a linear order on W. A *linear time σ-structure* is a structure $\mathfrak{M} = (\mathcal{L}, V)$ where $\mathcal{L} = (W, R)$ is a linear flow of time and $V : \sigma \rightarrow \wp(W)$ a valuation. Whenever $w \in W$ is a point, we call $\mathfrak{M}, w$ a *pointed σ-structure.* Linear time μ-calculus is usually considered over restricted classes of linear orders. In this paper, we will only consider it over the following classes of linear flows of time:

- $\mathbf{L}_\omega$, the class of linear orders of order type ω, i.e., flows of time $(W, <)$ that are isomorphic to $(\mathbb{N}, <)$, where $\mathbb{N}$ is the set of natural numbers with the natural ordering,
- $\mathbf{L}_{\mathit{fin}}$, the class of finite linear orders,
- the union $\mathbf{L}_\omega \cup \mathbf{L}_{\mathit{fin}}$ of these two classes

We will often refer to structures based on $\mathbf{L}_\omega$ as *ω-words* or $\mathbf{L}_\omega$-structures, to structures based on $\mathbf{L}_{fin}$ as *finite words* or $\mathbf{L}_{fin}$-structures and more generally, to structures based on $\mathbf{L}$ as $\mathbf{L}$-structures.

Definition 2 (Semantics of $\mu(\Diamond)$). Given a $\mu(\Diamond)$-formula ϕ, a structure $\mathfrak{M} = ((W,<),V)$ and an assignment $g : \mathcal{V} \to \wp(W)$, we define a subset $[\![\phi]\!]_{\mathfrak{M},g}$ of $\mathfrak{M}$ that is interpreted as the set of points at which ϕ is true. This subset is defined by induction in the usual way. Let $ImSuc(w)$, be the set of direct successors of the point w with respect to $<$, we only recall:

$$[\![\Diamond\phi]\!]_{\mathfrak{M},g} = \{w \in W : [\![\phi]\!]_{\mathfrak{M},g} \cap ImSuc(w) \neq \emptyset\}$$

$$[\![\mu x.\phi]\!]_{\mathfrak{M},g} = \bigcap\{A \subseteq W : [\![\phi]\!]_{\mathfrak{M},g[x/A]} \subseteq A\}$$

where $g[x/A]$ is the assignment defined by $g[x/A](x) = A$ and $g[x/A](y) = g(y)$ for all $y \neq x$.

To understand the last clause, consider a $\mu(\Diamond)$-formula $\phi(x)$ and a structure $((W,<),V)$ together with a valuation g. This formula induces an operator F^ϕ taking a set $A \subseteq W$ to the set $\{v : (\mathcal{L},V,v) \models_\phi (x)\ g[x \mapsto A]\}$. $\mu(\Diamond)$ is concerned with least fixpoints of such operators. If $\phi(x)$ is positive in x, the operator F^ϕ is monotone, i.e., $x \subseteq y$ implies $F^\phi(x) \subseteq F^\phi(y)$. Monotone operators F^ϕ always have a least fixpoint, defined as the intersection of all their prefixpoints: $\bigcap\{A \subseteq W : \{v : (\mathcal{L},V,v) \models \phi(x)\ g[x \mapsto A]\} \subseteq A\}$. The formula $\mu x.\phi(x)$ denotes this least fixpoint.

If $w \in [\![\phi]\!]_{\mathfrak{M},g}$, we write $\mathfrak{M}, w \models_g \phi$ and we say that ϕ is true at $w \in \mathfrak{M}$ under the assignment g. If ϕ is a sentence, we simply write $\mathfrak{M}, w \models \phi$.

Note that the $\Diamond$ operator is interpreted as the "Next" operator of temporal logic and that the temporal operators U and F defined as shorthands have their usual meaning:

- $(\mathcal{L},V,w) \models \mathsf{F}\phi$ iff there exists w' such that $w \leq w'$ and $(\mathcal{L},V,w') \models \phi$
- $(\mathcal{L},V,w) \models \phi\mathsf{U}\psi$ iff there exists w' such that $w \leq w'$, $(\mathcal{L},V,w') \models \psi$ and for all w'' such that $w \leq w'' < w'$, $(\mathcal{L},V,w'') \models \phi$

Before we give the complete axiomatization of $\mu(\Diamond)$ on $\mathbf{L}_\omega$, $\mathbf{L}_{fin}$ and $\mathbf{L}_\omega \cup \mathbf{L}_{fin}$, let us first recall the axiomatization of the *full μ-calculus*. In the full μ-calculus, instead of considering a linear order $<$, we consider and arbitrary binary relation R on W. In this more general context, (W,R) can be an arbitrary graph and we call it a *frame*.[1] The corresponding structures are called *Kripke structures*. Let $RSuc(w) = \{w' : (w,w') \in R\}$, the semantics of $\Diamond$ is now as follows:

$$[\![\Diamond\phi]\!]_{\mathfrak{M},g} = \{w \in W : [\![\phi]\!]_{\mathfrak{M},g} \cap RSuc(w) \neq \emptyset\}$$

Definition 3. Let σ be a finite propositional vocabulary and $\phi \in \mu(\Diamond)$. We call $BV(\phi)$ and $FV(\phi)$ respectively, the set of bound variables in ϕ and the set of free variables in ϕ. The Kozen system K_μ consists of the Modus Ponens, the Substitution rule, the Necessitation rule and the following axioms and rules:

[1] Note that on arbitrary graphs, we do not introduce $\mathsf{F}\phi$, $\mathsf{G}\phi$ and $\phi\mathsf{U}\psi$ as shorthands for $\mu(\Diamond)$-formulas anymore: as we consider frames instead of linear flows of time, this would not really map the usual meaning of these temporal operators.

1. propositional tautologies,
2. $\vdash \Box\phi \leftrightarrow \neg\Diamond\neg\phi$ (dual),
3. $\vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$ (K),
4. $\vdash \phi[x/\mu x.\phi] \rightarrow \mu x.\phi$ (fixpoint axiom),
5. If $\vdash \phi[x/\psi] \rightarrow \psi$, then $\vdash \mu x.\phi \rightarrow \psi$ (fixpoint rule)

where x does not belong to $BV(\phi)$ and $FV(\psi) \cap BV(\phi) = \emptyset$.

Theorem 1. *If ϕ is a $\mu(\Diamond)$-formula, let $K_\mu + \phi$ be the smallest set which contains both K_μ and ϕ and is closed for the Modus Ponens, Substitution and Necessitation rules. The following holds:*

1. *K_μ is complete with respect to the class of Kripke structures.*
2. *$K_\mu + \Diamond\phi \leftrightarrow \Box\phi$ is complete with respect to the class of ω-words.*
3. *$K_\mu + \Diamond\phi \rightarrow \Box\phi + \mu x.\Box x$ is complete with respect to the class of finite words.*
4. *$K_\mu + \Diamond\phi \rightarrow \Box\phi$ is complete with respect to the class of finite and ω-words.*

Proof. 1 was shown in [12] and the three other completeness results might actually be derivable from it. But direct (and simpler) proofs for 2 and 3 can be found respectively in [7] and [4]. As regards 4, it follows from 2 and 3 that $K_\mu + (\Diamond\phi \leftrightarrow \Box\phi) \vee \mu x.\Box x$ is complete with respect to the class of finite words and ω-words (we can take here the disjunction of the two axioms, as $\mu x.\Box x$ doesn't contain any proposition variable or formula schema). So it is enough to show that $\vdash (\Box\phi \rightarrow \Diamond\phi) \vee \mu x.\Box x$ follows from $\mu(\Diamond) + (\Diamond\phi \rightarrow \Box\phi)$. As $\Box\bot \rightarrow \mu x.\Box x$ is valid in the full μ-calculus, it is derivable here, which yields $\vdash \Box(\phi \wedge \neg\phi) \rightarrow \mu x.\Box x$, i.e., $\vdash (\Box\phi \wedge \Box\neg\phi) \rightarrow \mu x.\Box x$ ($\Box$ distributing over conjunction), i.e., $\vdash \neg(\Box\phi \rightarrow \Diamond\phi) \rightarrow \mu x.\Box x$.

2.2 Stutter-invariancy

We will now introduce $\mu(\mathsf{U})$, a semantic fragment of $\mu(\Diamond)$ extending $LTL(\mathsf{U})$ (linear-time temporal logic without the Next operator, see [5]). We also provide a precise definition for the notion of stutter-invariancy and recall that, in terms of expressive power, $\mu(\mathsf{U})$ is exactly the stutter-invariant fragment of $\mu(\Diamond)$.

Definition 4 (Syntax of $\mu(\mathsf{U})$). Let σ be a propositional vocabulary, and let $\mathcal{V} = \{x_1, x_2, \ldots\}$ be a disjoint countably infinite set of *propositional variables.* We inductively define the set of $\mu(\mathsf{U})$-formulas in vocabulary σ as follows:

$$\phi, \psi, \xi := At \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \phi \rightarrow \psi \mid \phi \vee \psi \mid \phi \mathsf{U} \psi \mid \mu x_i.\xi$$

where $At \in \sigma \cup \mathcal{V}$ and, in the last clause, x_i occurs only positively in ξ (i.e., within the scope of an even number of negations). We will use $\nu x_i.\xi$, $\mathsf{F}\phi$ and $\mathsf{G}\phi$ as shorthand for, respectively, $\neg\mu x_i.\neg\xi(\neg x_i)$, $\top\mathsf{U}\phi$ and $\neg(\top\mathsf{U}\neg\phi)$.

Note that the temporal operators F and G defined as shorthand have their usual meaning. We interpret $\mu(\mathsf{U})$-formulas in the same type of structures as $\mu(\Diamond)$-formulas, i.e., structures of the form $\mathfrak{M} = (\mathcal{L}, V)$ where $\mathcal{L} \in \mathbf{L}_{fin} \cup \mathbf{L}_\omega$.

Definition 5 (Semantics of $\mu(\mathsf{U})$). Given a $\mu(\mathsf{U})$-formula ϕ, a structure $\mathfrak{M} = ((W,<),V)$ and an assignment $g : \mathcal{V} \to \wp(W)$, we define a subset $[\![\phi]\!]_{\mathfrak{M},g}$ of $\mathfrak{M}$ that is interpreted as the set of points at which ϕ is true. This subset is defined by induction in the usual way. We only recall:

$$[\![\phi \mathsf{U} \psi]\!]_{\mathfrak{M},g} = \{w \in W : \exists w' \geq w, w' \in [\![\psi]\!]_{\mathfrak{M},g} \text{ and } \forall w \leq w'' < w, w'' \in [\![\phi]\!]_{\mathfrak{M},g}\}$$

$$[\![\mu x.\phi]\!]_{\mathfrak{M},g} = \bigcap \{A \subseteq W : [\![\phi]\!]_{\mathfrak{M},g[x/A]} \subseteq A\}$$

where $g[x/A]$ is the assignment defined by $g[x/A](x) = A$ and $g[x/A](y) = g(y)$ for all $y \neq x$.

In the remaining, we always assume $\mathbf{L} \in \{\mathbf{L}_\omega, \mathbf{L}_{\mathit{fin}}, \mathbf{L}_{\mathit{fin}} \cup \mathbf{L}_\omega\}$.

Definition 6 (Stuttering). Let σ be a propositional signature, and $\mathfrak{M} = ((W,<),V,w)$, $\mathfrak{M}' = ((W',<),V',w')$ $\mathbf{L}$-structures in vocabulary σ. We say that $\mathfrak{M}'$ is a stuttering of $\mathfrak{M}$ if and only if there is a surjective function $s : W' \to W$ such that

1. $s(w') = w$
2. for every $w_i, w_j \in W'$, $w_i < w_j$ implies $s(w_i) \leq s(w_j)$
3. for every $w_i \in W'$ and $p \in \sigma, w_i \in V'(p)$ iff $s(w_i) \in V(p)$

We say that a $\mathbf{L}$-structure $\mathfrak{M}$ is *stutter-free relative to* $\mathbf{L}$ whenever for all $\mathfrak{M}'$ such that $\mathfrak{M}$ is a stuttering of $\mathfrak{M}'$, $\mathfrak{M}'$ is isomorphic to $\mathfrak{M}$.

Let for instance $\mathfrak{M}, w$ be an ω-word in vocabulary $\{p\}$ with $V(p) = W$. $\mathfrak{M}, w$ is stutter-free relative to $\mathbf{L}_\omega$, but it is not stutter-free relative to $\mathbf{L}_{\mathit{fin}} \cup \mathbf{L}_\omega$. Indeed, let $\mathfrak{M}', w'$ be a finite word in vocabulary $\{p\}$ containing one single point w'. Assume $V'(p) = \{w'\}$, then $\mathfrak{M}, w$ is a stuttering of $\mathfrak{M}', w'$ and relative to $\mathbf{L}_{\mathit{fin}} \cup \mathbf{L}_\omega$, $\mathfrak{M}', w'$ is stutter-free, while $\mathfrak{M}, w$ is not.

Definition 7 (Stutter-Invariant Class of Structures). Let σ be a propositional signature and $\mathbf{K}$ a class of $\mathbf{L}$-structures in vocabulary σ. Then $\mathbf{K}$ is a stutter-invariant class iff for every $\mathbf{L}$-structure $\mathfrak{M}$ in vocabulary σ and for every $\mathbf{L}$-stuttering $\mathfrak{M}'$ of $\mathfrak{M}$, $\mathfrak{M} \in \mathbf{K} \Leftrightarrow \mathfrak{M}' \in \mathbf{K}$.

We say that a sentence ϕ is stutter-invariant relative to $\mathbf{L}$ whenever the class of $\mathbf{L}$-structures in which ϕ is satisfied is stutter-invariant. Every $\mu(\mathsf{U})$-sentence is stutter-invariant relative to $\mathbf{L}$ (see [5]). To see that it is not possible in $\mu(\mathsf{U})$ to define $\Diamond\phi$, it is hence enough to observe that the sentence $\Diamond p$ is not stutter-invariant. Also, considering a $\mathbf{L}$-structure $\mathfrak{M}, w$, there is always a unique (up to isomorphism) $\mathfrak{M}', w'$ which is stutter-free relative to $\mathbf{L}$ and such that $\mathfrak{M}, w$ is a stuttering of $\mathfrak{M}', w'$. Observe that it follows that if a $\mu(\mathsf{U})$-formula is satisfiable in some $\mathbf{L}$-structure, it is also satisfiable in a $\mathbf{L}$-structure which is stutter-free relative to $\mathbf{L}$. Additionally, on $\mathbf{L}$, we can show that $\mu(\mathsf{U})$ is exactly the stutter-invariant fragment of $\mu(\Diamond)$:

Theorem 2. *Let ϕ be a $\mu(\Diamond)$-sentence which is stutter-invariant relative to* $\mathbf{L}$. *Then, there exists a $\mu(\mathsf{U})$-sentence ϕ^* which is equivalent to ϕ on* $\mathbf{L}$*-structures.*

Proof. The proof can be found in [5].

3 The logic $\mu(\Diamond_\Gamma)$

In this Section, we introduce the logic $\mu(\Diamond_\Gamma)$ and we show that, as far as expressivity is concerned, it is a fragment of $\mu(\Diamond)$. More precisely, we show that $\mu(\Diamond_\Gamma)$ has exactly the same expressive power as $\mu(\mathsf{U})$. In the last Sections, we will see that $\mu(\Diamond_\Gamma)$ can be used as a very convenient tool to show completeness results for $\mu(\mathsf{U})$.

$\mu(\Diamond_\Gamma)$ is a variation of $\mu(\Diamond)$ where instead of the regular $\Diamond$ modality, we consider the family of its stutter-invariant counterparts. For each finite set Γ of $\mu(\Diamond_\Gamma)$-formulas, we consider a $\Diamond_\Gamma$ operator which intuitively means "at the next distinct point with respect to Γ" (i.e., distinct with respect to the values it assigns to the formulas in Γ). To design this operator, we took inspiration from [3], where a "next distinct" operator was mentioned in passing. This operator was interpreted in σ-structures as our $\Diamond_\sigma$ operator. In order to obtain a well-behaved operator, we relativize it here to any finite set Γ of formulas. This gives rise to a better-behaved logic, where we can define a natural notion of substitution and where the truth of σ-formulas in σ-structures is preserved in σ^+-expansions of these structures (with $\sigma^+ \supseteq \sigma$).

We interpret $\mu(\Diamond_\Gamma)$-formulas in the same type of structures as $\mu(\Diamond)$-formulas, i.e., structures of the form $\mathfrak{M} = (\mathbf{L}, V)$ where $\mathbf{L} \in \mathbf{L}_{\mathit{fin}} \cup \mathbf{L}_\omega$. For any finite set of $\mu(\Diamond_\Gamma)$-formulas and for any points w, w', we write $w \equiv_\Gamma w'$ if w and w' satisfy the same formulas in Γ.

Definition 8. Let σ be a finite propositional signature, and let $\mathcal{V} = \{x_1, x_2, \ldots\}$ be a disjoint countably infinite stock of *propositional variables*. We inductively define the set of $\mu(\Diamond_\Gamma)$-formulas as follows:

$$\phi, \psi, \xi := At \mid \top \mid \neg\phi \mid \phi \wedge \psi \mid \phi \to \psi \mid \phi \vee \psi \mid \Diamond_\Gamma \phi \mid \mu x_i.\xi$$

where $At \in \sigma \cup \mathcal{V}$, Γ is a finite set of $\mu(\Diamond_\Gamma)$-formulas and, in the last clause, x_i occurs only positively in ξ (i.e., within the scope of an even number of negations). We use $\Box_\Gamma \phi$ and $\nu x_i.\xi(x_i)$ as shorthand for $\neg\Diamond_\Gamma\neg\phi$ and $\neg x_i \mu \neg\xi(\neg x_i)$, respectively. We interpret $\mu(\Diamond_\Gamma)$-formulas as $\mu(\Diamond)$-formulas, except that:

$$(\mathcal{T}, V, w) \models \Diamond_\Gamma \phi \text{ if } \exists w' > w \text{ such that } w \not\equiv_\Gamma w', \forall w'' \text{ with } \\ w < w'' < w', w'' \equiv_\Gamma w \text{ and } (\mathcal{T}, V, w') \models \phi$$

We write $Voc(\phi)$ for the vocabulary of ϕ and $Voc(\Gamma)$ for $\bigcup_{\phi \in \Gamma} Voc(\phi)$. Note that we include in the vocabulary of a formula *all* the proposition letters occurring in it, including those which occur in the formulas contained in the sets Γ indexing its modalities. This remark particularly matters for the notion of substitution, as whenever a formula is to be uniformly substituted for a proposition letter, the operation has to be done everywhere, including in the formulas contained in the sets indexing the modalities. Otherwise, validity would not be preserved by uniform substitution. Consider for instance $(p \wedge \Diamond_{\{p\}}\top) \to \Diamond_{\{p\}}\neg p$. It is clear that this formula is valid and that for any $\mu(\Diamond_\Gamma)$-formula ϕ, $\models (\phi \wedge \Diamond_{\{\phi\}}\top) \to \Diamond_{\{\phi\}}\neg\phi$ also holds. But it is also very clear that $\not\models (\phi \wedge \Diamond_{\{p\}}\top) \to \Diamond_{\{p\}}\neg\phi$.

Definition 9. Let $\Gamma = \{\phi_0, \dots, \phi_{n-1}\}$ be a finite set of $\mu(\Diamond_\Gamma)$-formulas. Whenever $\Gamma \neq \emptyset$, we define B_Γ as the set of all possible mappings $\Gamma \to \{\bot, \top\}$, and for each $g \in B_\Gamma$, we let β_g be the formula $\alpha_0 \wedge \ldots \wedge \alpha_{n-1}$ where $\alpha_j = \phi_j$ if $g(\phi_j) = \top$ and $\alpha_j = \neg\phi_j$ if $g(\phi_j) = \bot$. By convention, we set $\{\beta_g | g \in B_\emptyset\} = \{\bot, \top\}$.[2]

Proposition 1. *$\mu(\mathsf{U})$ and $\mu(\Diamond_\Gamma)$ have the same expressive power on the class of finite and ω-words.*

Proof. We show that a class of σ-structures based on $\mathbf{L}_{fin} \cup \mathbf{L}_\omega$, is definable by a $\mu(\mathsf{U})$-formula if and only if it is definable by a $\mu(\Diamond_\Gamma)$-formula. We define a recursive procedure transforming each formula ϕ from one language into an equivalent formula ϕ^* from the other language. We set $At^* = At$, $(\phi \wedge \psi)^* = \phi^* \wedge \psi^*$, $(\mu x.\phi)^* = \mu x.\phi^*$, $(\phi \mathsf{U} \psi)^* = \mu x.(\psi^* \vee (\phi^* \wedge \Diamond_\sigma x))$ and $(\Diamond_\Gamma \phi)^* = \bigvee_{g \neq g' \in B_\Gamma} (\beta_g \wedge \beta_g \mathsf{U}(\beta_{g'} \wedge \phi^*))$. The only interesting cases are the two last ones. As $\mu(\mathsf{U})$ and $\mu(\Diamond_\Gamma)$-formulas define only stutter-invariant classes of structures, it is enough to show that $\phi \mathsf{U} \psi$, $\mu y.(\psi^* \vee (\phi^* \wedge \Diamond_\sigma y))$ and $\Diamond_\Gamma \phi$, $\bigvee_{g \neq g' \in B_\Gamma} (\beta_g \wedge \beta_g \mathsf{U}(\beta_{g'} \wedge \phi^*))$ respectively, are satisfied in the same stutter-free σ-structures. So let $((W, <), V, w)$ be a stutter-free σ-structure (by induction hypothesis, we assume the property holds for ϕ, ϕ^* and ψ, ψ^* respectively).

Assume $((W, <), V, w) \models \phi \mathsf{U} \psi$. This means either that w satisfies ψ, or w satisfies ϕ and it is separated from some subsequent w' satisfying ψ by a finite sequence of points which all satisfy ϕ. As the model is stutter-free, each point in the sequence which follows w is the next distinct point from its immediate predecessor relative to the whole vocabulary σ. For suppose v is a non immediate successor of v' and v' is the next distinct point from v relative to σ. Then for every v'' satisfying $v < v'' < v'$, v and v' satisfy the same proposition letters in τ, which contradicts the stutter-freeness of the model. So, by induction hypothesis, $((W, <), V, w) \models \mu x.(\psi^* \vee (\phi^* \wedge \Diamond_\sigma x))$, because $\mu x.(\psi^* \vee (\phi^* \wedge \Diamond_\sigma x))$ states that the current state belongs to the least fixpoint which contains all the points satisfying ψ^*, together with all the points that satisfy ϕ^* and which are immediate predecessors of a point which is already in the fixpoint.

Assume $\mu x.(\psi^* \vee (\phi^* \wedge \Diamond_\sigma x))$, i.e., w belongs to the least fixpoint which contains all the points satisfying ψ^*, together with all the points that satisfy ϕ^* and which are immediate predecessors of a point which is already in the fixpoint. This means that either w satisfies ψ^*, or it satisfies ϕ^* and it is separated from some subsequent w' satisfying ψ^* by a finite sequence of successor points which all satisfy ϕ^* and by induction hypothesis, $((W, <), V, w) \models \phi U \psi$.

Assume $((W, <), V, w) \models \Diamond_\Gamma \phi$, i.e., there exists $w' > w$ such that $w \not\equiv_\Gamma w'$ and $\forall w''$ with $w < w'' < w', w'' \equiv_\Gamma w$ and $((W, <), V, w') \models \phi$. So there are $g \neq g' \in B_\Gamma$ such that $\mathfrak{M}, w \models \beta_g$ and there exists $w' > w$ with $((W, <), V, w') \models \beta_{g'} \wedge \phi^*$ and for all w'' such that $w \leq w'' < w'$, $((W, <), V, w'') \models \beta_g$. By induction hypothesis, $((W, <), V, w) \models \bigvee_{g \neq g' \in B_\Gamma} (\beta_g \wedge \beta_g \mathsf{U}(\beta_{g'} \wedge \phi^*))$.

Assume $((W, <), V, w) \models \bigvee_{g \neq g' \in B_\Gamma} (\beta_g \wedge \beta_g \mathsf{U}(\beta_{g'} \wedge \phi^*))$. So there are $g \neq g'$ such that $\beta_g \mathsf{U}(\beta_{g'} \wedge \phi^*)$, i.e., there exists w' such that $w \leq w'$, $((W, <), V, w') \models$

[2] We adopt this convention because we allowed Γ to be empty (see the instantiation of Axiom 7 where $\Gamma = \emptyset$, our convention will guaranty that $\Diamond_\emptyset \phi$, which is not satisfiable, is also inconsistent), but we could also have required that $\Gamma \neq \emptyset$.

$\beta_{g'} \wedge \phi^*$ and for all w'' such that $w \leq w'' < w'$, $((W,<),V,w'') \models \beta_g$. As $g \neq g'$, also $w \not\equiv_\Gamma w'$. By induction hypothesis, $((W,<),V,w) \models \Diamond_\Gamma \phi$.

Remark 1. It follows that $\Diamond_\Gamma$ can be used as shorthand either in $\mu(\Diamond)$ or in $\mu(\mathsf{U})$, that U can be used as shorthand in $\mu(\Diamond_\Gamma)$ and that $\mu(\Diamond_\Gamma)$ is definable as a semantic fragment of $\mu(\Diamond)$. In the remaining of the paper, this will be assumed.

Definition 10 ($\mu(\mathsf{U})$-translation of a $\mu(\Diamond_\Gamma)$-formula). Let ϕ be $\mu(\Diamond_\Gamma)$-formula, we recursively define its $\mu(\mathsf{U})$-translation $\phi_{\mu(\mathsf{U})}$ via the following procedure. $At_{\mu(\mathsf{U})} = At$, $(\neg\phi)_{\mu(\mathsf{U})} = \neg\phi_{\mu(\mathsf{U})}$, $(\phi \wedge \psi)_{\mu(\mathsf{U})} = \phi_{\mu(\mathsf{U})} \wedge \psi_{\mu(\mathsf{U})}$, $(\mu x.\phi)_{\mu(\mathsf{U})} = \mu x.\phi_{\mu(\mathsf{U})}$, and $(\Diamond_\Gamma \phi)_{\mu(\mathsf{U})} = \bigvee_{g \neq g' \in B_\Gamma} (\beta_g \wedge \beta_g \mathsf{U} (\beta_{g'} \wedge \phi_{\mu(\mathsf{U})}))$.

Proposition 2. *Let $\mathbf{L} \in \{\mathbf{L}_\omega, \mathbf{L}_{fin}, \mathbf{L}_\omega \cup \mathbf{L}_{fin}\}$. ϕ and $\phi_{\mu(\mathsf{U})}$ are equivalent on $\mathbf{L}$-structures.*

Proof. This follows from the proof of Proposition 1.

Definition 11 ($\mu(\Diamond_\Gamma)$-translation of a $\mu(\mathsf{U})$-formula). Let ϕ be $\mu(\mathsf{U})$-formula in vocabulary σ, we recursively define its $\mu(\Diamond_\Gamma)$-translation $\phi_{\mu(\Diamond_\Gamma)}$ via the following procedure. $At_{\mu(\Diamond_\Gamma)} = At$, $(\neg\phi)_{\mu(\Diamond_\Gamma)} = \neg\phi_{\mu(\Diamond_\Gamma)}$, $(\phi \wedge \psi)_{\mu(\Diamond_\Gamma)} = \phi_{\mu(\Diamond_\Gamma)} \wedge \psi_{\mu(\Diamond_\Gamma)}$, $(\mu x.\phi)_{\mu(\Diamond_\Gamma)} = \mu x.\phi_{\mu(\Diamond_\Gamma)}$, and $(\phi \mathsf{U} \psi)_{\mu(\Diamond_\Gamma)} = \mu x.(\psi_{\mu(\Diamond_\Gamma)} \vee (\phi_{\mu(\Diamond_\Gamma)} \wedge \Diamond_\sigma x))$.

Proposition 3. *ϕ and $\phi_{\mu(\Diamond_\Gamma)}$ are equivalent on $\mathbf{L}$-structures.*

Proof. This follows from the proof of Proposition 1.

4 Complete axiomatization of $\mu(\Diamond_\Gamma)$

In this Section, we show some completeness results for the logic $\mu(\Diamond_\Gamma)$. We will use them in the next Section as a tool to obtain similar results for the logic $\mu(\mathsf{U})$.

Proposition 4. *Let ϕ be a $\mu(\Diamond_\Gamma)$-formula in vocabulary σ containing no free occurrence of the variable x. On the class of finite and ω-words, the following formulas are equivalent:*

- $\bigvee_{g \neq g' \in B_\sigma} (\neg\beta_{g'} \wedge \mu x.((\beta_{g'} \wedge \phi) \vee (\beta_g \wedge \Diamond_\sigma x))$
- $\bigvee_{g \neq g' \in B_\sigma} (\neg\beta_{g'} \wedge \mu x.((\beta_{g'} \wedge \phi) \vee (\beta_g \wedge \Diamond x))$
- $\Diamond_\sigma \phi$

Proof. Recall that U can be defined as shorthand in $\mu(\Diamond_\Gamma)$. We already noted in Section 2 and in Proposition 1 that on linear orders, the formulas $\phi \mathsf{U} \psi$, $\mu x.(\psi \vee (\phi \wedge \Diamond x))$ and $\mu x.(\psi \vee (\phi \wedge \Diamond_\sigma x))$ are equivalent. We also noted in Proposition 1 that in this context, the formulas $\Diamond_\sigma \phi$ and $\bigvee_{g \neq g' \in B_\sigma} (\neg\beta_{g'} \wedge \beta_g \mathsf{U} (\beta_{g'} \wedge \phi))$ are equivalent. The Proposition follows.

Definition 12. $K_{\mu(\Diamond_\Gamma)}$ consists of the Modus Ponens, for each Γ, the corresponding Necessitation rule (i.e., if $\vdash \phi$, then $\vdash \Box_\Gamma \phi$), the Modus Ponens, the Substitution rule and the following axioms and rules:

1. propositional tautologies,
2. $\vdash \Box_\Gamma \phi \leftrightarrow \neg\Diamond_\Gamma\neg\phi$ (dual),
3. $\vdash \Diamond_\Gamma \phi \rightarrow \Box_\Gamma\phi$ (linearity),
4. $\vdash \Box_\Gamma(\phi \rightarrow \psi) \rightarrow (\Box_\Gamma\phi \rightarrow \Box_\Gamma\psi)$ (K),
5. $\vdash \phi[x/\mu x.\phi] \rightarrow \mu x.\phi$ (fixpoint axiom),
6. If $\vdash \phi[x/\psi] \rightarrow \psi$, then $\vdash \mu x.\phi \rightarrow \psi$ (fixpoint rule)
7. $\vdash \Diamond_\Gamma\phi \leftrightarrow \bigvee_{g\neq g'\in B_\Gamma}(\beta_{g'} \wedge \mu x.((\beta_g \wedge \phi) \vee (\beta_{g'} \wedge \Diamond_\sigma x)))$, where $Voc(\Diamond_\Gamma\phi) \subseteq \sigma$ (inductive meaning of $\Diamond_\Gamma$),

for each finite set $\Gamma = \{\phi_0, \dots, \phi_{n-1}\}$ of $\mu(\Diamond_\Gamma)$-formulas and where in the three last Axioms, x does not belong to $BV(\phi)$ and $FV(\psi) \cap BV(\phi) = \emptyset$.

Lemma 1. *Axiom 7 is sound on the class of finite and ω-words.*

Proof. Let σ be a finite vocabulary, $\Gamma = \{\psi_1, \dots, \psi_m\}$ a finite set of $\mu(\Diamond_\Gamma)$-formulas and ϕ a $\mu(\Diamond_\Gamma)$-formula with $Voc(\Diamond_\Gamma\phi) \subseteq \sigma = \{p_1, \dots, p_n\}$ and $x \notin FV(\phi)$. As $\mu(\Diamond_\Gamma)$ define only stutter-invariant classes of structures, we can consider a stutter-free σ-model $\mathfrak{M}$ with $w \in \mathfrak{M}$ and it is enough to show that the following are equivalent:

1. $\mathfrak{M}, w \models \Diamond_\Gamma\phi$
2. $\mathfrak{M}, w \models \bigvee_{g\neq g'\in B_\Gamma}(\beta_{g'} \wedge \mu x.((\beta_g \wedge \phi) \vee (\beta_{g'} \wedge \Diamond_\sigma x)))$

As for Lemma 4, this follows from what was observed in Section 2 and in Proposition 1.

Theorem 3. *$K_{\mu(\Diamond_\Gamma)}$ is complete for $\mu(\Diamond_\Gamma)$ with respect to the class of ω-words and with respect to the class of finite and ω-words.*

Proof. Let ϕ be a $K_{\mu(\Diamond_\Gamma)}$-consistent formula in vocabulary σ. By Axiom 7, we can restrict our attention to σ-formulas containing only $\Box_\sigma$ modalities. Again by Axiom 7, we can define a recursive procedure transforming ϕ into a $K_{\mu(\Diamond_\Gamma)}$-equivalent formula ϕ'. We set $At' = At$, $(\neg\phi)' = \neg\phi'$, $(\phi \wedge \psi)' = \phi' \wedge \psi'$, $(\mu x.\phi)' = \mu x.\phi'$, and $(\Diamond_\sigma\phi)' = \bigvee_{g'\neq g''\in B_\sigma}(\neg\beta_g \wedge \mu y.((\beta_g \wedge \phi') \vee (\beta_{g'} \wedge \Diamond_\sigma y)))$. Consider now the $\mu(\Diamond)$-formula ϕ'', which we define as the result of removing in ϕ' all the subscripts of the modalities. We claim that ϕ'' is $K_\mu + \Diamond\phi \rightarrow \Box\phi$-consistent. For suppose not. Then, there exists a proof of $\neg\phi''$ using the axioms and rules of $K_\mu + \Diamond\phi \rightarrow \Box\phi$. Now, replace every occurrence of the operator $\Diamond$ by $\Diamond_\sigma$ in each axiom and rule used in the proof. The result is a correct $K_{\mu(\Diamond_\Gamma)}$-proof, where only correct axioms and rules of $K_{\mu(\Diamond_\Gamma)}$ are used (because the $K_\mu + \Diamond\phi \rightarrow \Box\phi$ axioms and rules can be obtained from the $K_{\mu(\Diamond_\Gamma)}$ ones simply by removing the indexes of the modalities). Additionally, this is a proof of the formula $\neg\phi'$ (as the original ϕ' can also be obtained from ϕ'' by adding the subscript σ to every $\Diamond$ in ϕ''). But this contradicts the fact that ϕ' was $K_{\mu(\Diamond_\Gamma)}$-consistent. So ϕ'' is $K_\mu + \Diamond\phi \rightarrow \Box\phi$-consistent. By Theorem 1, there is an ω-word or a finite word $\mathfrak{M}$ such that $\mathfrak{M}, w \models \phi''$ and it follows from Proposition 4 that $\mathfrak{M}, w \models \phi'$, i.e., $\mathfrak{M}, w \models \phi$. Completeness with respect to the class of ω-words follows too, because every finite word has an ω-word stuttering.

Theorem 4. *$K_{\mu(\Diamond_\Gamma)} + \mu x.\Box_\Gamma x$ is complete for $\mu(\Diamond_\Gamma)$ with respect to the class of finite words.*

Proof. We can apply the same reasoning as for the proof of Theorem 4, using completeness of $K_\mu + \Diamond\phi \rightarrow \Box\phi + \mu x.\Box x$ on finite words, instead of completeness of $K_\mu + \Diamond\phi \rightarrow \Box\phi$ on finite and ω-words.

Let $\mathfrak{M}$ be an ω-word. We say that $\mathfrak{M}$ is a *pseudo-finite* word whenever there exists a finite word $\mathfrak{M}'$ such that $\mathfrak{M}$ is a stuttering of $\mathfrak{M}'$. Note that $K_{\mu(\Diamond_\Gamma)} + \mu x.\Box_\Gamma x$ is also complete for $\mu(\Diamond_\Gamma)$ with respect to the class of finite and pseudo-finite words, as every pseudo-finite word is the stuttering of a finite word.

Remark 2. Axiom 7 *is not derivable* from the other axioms and rules. Otherwise, every $\Box_\Gamma$ would simply be interpreted as the regular $\Box$ operator of $\mu(\Diamond)$. Now, more precisely, let $K^{-7}_{\mu(\Diamond_\Gamma)}$ be the smallest set of $\mu(\Diamond_\Gamma)$-formulas which is closed under all axioms and rules in $K_{\mu(\Diamond_\Gamma)}$, except Axiom 7. Suppose Axiom 7 is derivable in $K^{-7}_{\mu(\Diamond_\Gamma)}$. Then, $K^{-7}_{\mu(\Diamond_\Gamma)}$ would be complete with respect to the class of ω-words. Therefore, as on ω-words $\models (p \wedge \Diamond_{\{p\}}\top) \rightarrow \neg p$, also in $K^{-7}_{\mu(\Diamond_\Gamma)}$, $\vdash (p \wedge \Diamond_{\{p\}}\top) \rightarrow \neg p$ and there would exists a $K^{-7}_{\mu(\Diamond_\Gamma)}$-proof of this formula. But now we could replace in that proof, every modal operator by the regular $\Diamond$ operator. This would be a correct $K_\mu + \Diamond\phi \rightarrow \Box\phi$-proof of $(p \wedge \Diamond\top) \rightarrow \neg p$. But as on $\omega - words$, $\not\models (p \wedge \Diamond\top) \rightarrow \neg p$, this contradicts the completeness of $K_{\mu(\Diamond)} + \Diamond\phi \rightarrow \Box\phi$. It follows that Axiom 7 is not derivable in $K^{-7}_{\mu(\Diamond_\Gamma)}$.

5 Complete axiomatization of $\mu(\mathsf{U})$

Recall that $LTL(\mathsf{U})$ is the fragment of $\mu(\mathsf{U})$ where the μ-operator is disallowed. In [9], the authors propose an axiomatization of $LTL(\mathsf{U})$ which is complete on the class of ω-words and finite words. In order to axiomatize $\mu(\mathsf{U})$, we extend here the Axioms and rules in [9] with the usual fixpoint rule and Axiom, together with an additional axiom accounting for the way the Until operator and the μ-operator can interact together. Using the completeness result in [9], together with the completeness of $K_{\mu(\Diamond_\Gamma)}$, this allows to derive a similar completeness Theorem for $\mu(\mathsf{U})$. Recall that, in $\mu(\mathsf{U})$, we use $\mathsf{G}\phi$ as shorthand for $\neg(\top\mathsf{U}\neg\phi)$ and $\Diamond_\tau$ as shorthand for $\bigvee_{g\neq g' \in B_\tau} \beta_g \wedge (\beta_g \mathsf{U}(\beta_{g'} \wedge \phi))$.

Definition 13. The $K_{\mu(\mathsf{U})}$ system consists of the Modus Ponens, the G Necessitation rule (i.e., if $\vdash \phi$, then $\vdash \mathsf{G}\phi$) the Substitution rule and the following axioms and rules (these rules, as well as Axioms 1 to 9, are borrowed from [9]):

1. propositional tautologies,
2. The Until operator is non strict:
   $\vdash \phi \rightarrow \bot\mathsf{U}\phi$,
3. For any consistent formula there exists a model that is a discrete linear order:
   - $\vdash \mathsf{F}\phi \rightarrow \neg\phi\mathsf{U}\phi$,
   - $\vdash \phi \wedge \mathsf{F}\phi \rightarrow \neg\psi\mathsf{U}(\phi \wedge \phi\mathsf{U}(\neg\phi\mathsf{U}\psi))$,

4. Properties that hold throughout a computation hold at the initial state: $\vdash G\phi \to \phi$,
5. Conventional logical deduction holds within individual states (K axiom):
   - $\vdash (\mathsf{G}(\phi \to \psi) \to (\phi \mathsf{U} \xi \to \psi \mathsf{U} \xi))$
   - $\vdash (\mathsf{G}(\phi \to \psi) \to (\xi \mathsf{U} \phi \to \xi \mathsf{U} \psi))$
6. Persistence of an Until formula until its second argument is satisfied: $\vdash \phi \mathsf{U} \psi \to (\phi \mathsf{U} \psi) \mathsf{U} \psi$
7. Immediacy of satisfaction of an Until formula at the current state: $\vdash \phi \mathsf{U}(\phi \mathsf{U} \psi) \to \phi \mathsf{U} \psi$
8. States of the time line are not skipped over in evaluating an Until formula: $\vdash \phi \mathsf{U} \psi \wedge \neg(\xi \mathsf{U} \psi) \to \phi \mathsf{U}(\phi \wedge \neg \xi)$
9. Models are linearly ordered: $\vdash \phi U \psi \wedge \xi U \theta \to ((\phi \wedge \xi) \mathsf{U}(\psi \wedge \theta) \vee (\phi \wedge \xi) \mathsf{U}(\psi \wedge \xi) \vee (\phi \wedge \xi) \mathsf{U}(\phi \wedge \theta))$
10. $\vdash \phi[x/\mu x.\phi] \to \mu x.\phi$, (fixpoint axiom),
11. If $\vdash \phi[x/\psi] \to \psi$, then $\vdash \mu x.\phi \to \psi$ (fixpoint rule),
12. $\vdash \mu x.(\psi \vee (\phi \wedge \Diamond_\tau x)) \leftrightarrow \phi \mathsf{U} \psi$, where $Voc(\phi) \cup Voc(\psi) \subseteq \tau$ (inductive meaning of U),

where in the three last Axioms, x does not belong to $BV(\phi) \cup BV(\psi)$ and $FV(\psi) \cap BV(\phi) = \emptyset$.

Lemma 2. *The $\mu(\mathsf{U})$-translations of the axioms and rules of $K_{\mu(\Diamond_\Gamma)}$ are derivable in $K_{\mu(\mathsf{U})}$.*

Proof. Except for the $\mu(\mathsf{U})$-translation of the fixpoint Axiom and of the fixpoint rule (which are both trivially derivable from $K_{\mu(\mathsf{U})}$, as they also belong to it), as well as Axiom 7, there is no explicit occurrence of the μ-operator in the $\mu(\mathsf{U})$-translation of the Axioms and rules of $K_{\mu(\Diamond_\Gamma)}$. As they are sound on the class of ω-words and finite words, by the completeness Theorem in [9], together with Proposition 2, they are derivable in $LTL(\mathsf{U})$. It follows that they are also derivable in $K_{\mu(\mathsf{U})}$, because the Axioms and rules of $K_{\mu(\mathsf{U})}$ simply extend those of $LTL(U)$.

Now consider the $\mu(\mathsf{U})$-translation of Axiom 7:

$$\bigvee_{g \neq g' \in B_\Gamma} (\beta_{g'} \wedge \beta_{g'} \mathsf{U}(\beta_g \wedge \phi))$$

$$\leftrightarrow$$

$$\bigvee_{g \neq g' \in B_\Gamma} (\beta_{g'} \wedge \mu y.((\beta_g \wedge \phi) \vee (\beta_{g'} \wedge \bigvee_{g'' \neq g''' \in B_\sigma} (\beta_{g''} \wedge \beta_{g''} \mathsf{U}(\beta_{g'''} \wedge y)))))$$

This formula is derivable from propositional tautologies, together with the substitution rule and Axiom 12 of $K_{\mu(\mathsf{U})} \vdash \mu y.(\psi \vee (\phi \wedge \Diamond_\sigma x)) \leftrightarrow \phi \mathsf{U} \psi$ (which is actually shorthand for $\vdash \mu y.(\psi \vee (\phi \wedge \bigvee_{g \neq g' \in B_\sigma} (\beta_g \wedge \beta_g \mathsf{U}(\beta_{g'} \wedge y)))) \leftrightarrow \phi \mathsf{U} \psi$). Finally, let us point out that the restriction of our axioms and rules to $LTL(\mathsf{U})$-formulas is actually slightly stronger than the axiomatization proposed in [9]. The authors chose to prefix all their modal axioms and rules by G and to allow the generalization rule only on propositional tautologies (our generalization rule is a derived rule in their framework). But our axioms and rule being sound, it is safe to use the completeness of their system as we do here.

Corollary 1. $K_{\mu(\mathsf{U})}$ *is complete for* $\mu(\mathsf{U})$ *with respect to the class of* ω*-words.*

Proof. Let ϕ be a $K_{\mu(\mathsf{U})}$-consistent formula. Now let ϕ' be the $\mu(\Diamond_\Gamma)$-translation of ϕ. By Lemma 2, ϕ' is $K_{\mu(\Diamond_\Gamma)}$-consistent and so, by Theorem 3, ϕ' is satisfied in some ω-word $\mathfrak{M}, w$. By Proposition 3, ϕ and ϕ' are equivalent on ω-words. Hence also $\mathfrak{M}, w \models \phi$.

Corollary 2. $K_{\mu(\mathsf{U})} + \mu y.\Box_\Gamma y$ *is complete for* $\mu(\mathsf{U})$ *with respect to the class of finite words.*

Proof. Similarly follows from Proposition 3, Theorem 4 and Lemma 2.

Remark 3. Let $K^{-12}_{\mu(\mathsf{U})}$ be the smallest set of $\mu(\mathsf{U})$-formulas which is closed under all axioms and rules in $K_{\mu(\mathsf{U})}$ except Axiom 12. Axiom 12 *is not derivable* in $K^{-12}_{\mu(\mathsf{U})}$. Observe that the $\mu(\Diamond)$-translation of every axiom and rule of $K^{-12}_{\mu(\mathsf{U})}$ is sound when instantiated by $\mu(\Diamond)$-formulas and that, by completeness of $\mu(\Diamond)$, their $\mu(\Diamond)$-translations are also derivable in $\mu(\Diamond)$. So if Axiom 12 was derivable in $K^{-12}_{\mu(\mathsf{U})}$, its $\mu(\Diamond)$-translation would be also derivable (and hence, valid) in K_μ. But let $\mathfrak{M}$ be a finite word in vocabulary $\{p\}$ with $W = \{w_0, w_1, w_2\}$, $w_i < w_{i+1}$ and $V(p) = w_2$. Obviously $\mathfrak{M}, w_0 \models \mu x.(p \vee (\Diamond\Diamond p \wedge \Diamond_{\{p\}} x))$, but $\mathfrak{M}, w_0 \not\models (\Diamond\Diamond p)\mathsf{U}p$, i.e., $\mathfrak{M}, w_0 \not\models \mu x.(p \vee (\Diamond\Diamond p \wedge \Diamond_{\{p\}} x)) \leftrightarrow (\Diamond\Diamond p)\mathsf{U}p$.

6 Complexity

It is known that $LTL(\mathsf{U})$ is PSPACE complete both for model checking and for satisfiability (c.f. [2]). It is also know that $\mu(\Diamond)$ is PSPACE complete both for model checking and for satisfiability (c.f. [11]). So PSPACE completeness follows for $\mu(\mathsf{U})$ in both cases. This is another argument in favor of $\mu(\mathsf{U})$: while much more expressive than $LTL(\mathsf{U})$, it has the same complexity.

7 Conclusions and future works

In this paper, we studied the logic $\mu(\mathsf{U})$. We introduced for that purpose the logic $\mu(\Diamond_\Gamma)$ as a technical tool in order to easily obtain completeness results for $\mu(\mathsf{U})$. In [5], we used an even simpler trick to show that $\mu(\mathsf{U})$ satisfies uniform interpolation. A number of other interesting logical properties of $\mu(\mathsf{U})$ remain to be investigated. In particular, we could examine counterparts of the Lös Tarski Theorem and of the Lyndon Theorem, which the full μ-calculus was shown in [1] to satisfy.

The method that we used here in order to show completeness results could also be reused in other contexts. It could for instance be applicable to the extension of $\mu(\mathsf{U})$ with past tense operators or to μ-calculus on trees (either finite or infinite). It should be noted, though, that on trees, there is still no general consensus on the appropriate notion of stuttering and that it is questionable whether the "Until only" fragment and the stutter-invariant fragment of the μ-calculus actually coincide (see [6]).

8 References

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